SIAM J. CONTROL OPTIM.
Vol. 63, No. 3, pp. 1736–1767
© 2025 Society for Industrial and Applied Mathematics

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

NECESSARY AND SUFFICIENT CONDITIONS OF OPEN-LOOP AND CLOSED-LOOP SOLVABILITY FOR DELAYED STOCHASTIC LQ OPTIMAL CONTROL PROBLEMS*

WEIJUN MENG†, JINGTAO SHI‡, JI-FENG ZHANG§, AND YANLONG ZHAO¶

Abstract. In this paper, a linear quadratic optimal control problem driven by a stochastic differential delay system is investigated, where both state delay and control delay can appear in the state equation, especially in the diffusion term. Three kinds of solvability for the delayed control problem are proposed—open-loop solvability, the closed-loop representation of open-loop optimal control, and closed-loop solvability—and their necessary and sufficient conditions are obtained. The delayed control problem is transformed into an infinite dimensional optimal control problem without delay but with a new control operator. Some novel auxiliary equations are constructed to overcome the difficulties caused by the new control operator, because state delay and control delay coexist, and some stochastic analysis tools are lacking in the study of the above three kinds of solvability. The open-loop solvability is ensured by the solvability of a constrained forward-backward stochastic evolution system and a convexity condition, or by the solvability of an anticipated-backward stochastic differential delay system and a convexity condition; the closed-loop representation of the open-loop optimal control is given via a coupled matrix-valued Riccati equation; the closed-loop solvability is ensured by the solvability of an operator-valued Riccati equation or a coupled matrix-valued Riccati equation.

Key words. linear quadratic control, time delay, open-loop solvability, closed-loop solvability, Riccati equation

MSC codes. 93C25, 49K15, 49K27, 49N10

DOI. 10.1137/23M1625482

1. Introduction.

Many problems can be regarded as optimal control problems in the fields of economy, finance, aerospace, network communication, and so on (see [3, 5, 7]). In the real world, the development of certain phenomena depends not only on the present state, but also on the past state trajectories. After a controller exerts control, it takes some time to have a practical effect on the control systems. Meanwhile, the development of control systems is affected by some uncertainties. Therefore, how to obtain the optimal control of stochastic control systems with both state delay and control delay has become the core problem of control theory.

*Received by the editors December 19, 2023; accepted for publication (in revised form) January 19, 2025; published electronically May 13, 2025.
https://doi.org/10.1137/23M1625482
Funding: This work is supported by the National Natural Science Foundation of China (62025306, T2293770, 12471419, 12271304, 62433020), the CAS Project for Young Scientists in Basic Research (YSBR-008), and the Shandong Provincial Natural Science Foundations (ZR2024ZD35, ZR2022JQ01).
†School of Mathematics & Statistics, Nanjing University of Science and Technology, Nanjing 210094, China (mengwj@mail.sdu.edu.cn).
‡School of Mathematics, Shandong University, Jinan 250100, China (shijingtao@sdu.edu.cn).
§Corresponding author. School of Automation and Electrical Engineering, Zhongyuan University of Technology, Zhengzhou 450007; State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190; and school of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100149, China (jif@iss.ac.cn).
¶State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190; and school of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100149, China (ylzhao@amss.ac.cn).

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Delayed control systems have wide background and applications (see [3, 7, 9, 13, 14, 24, 26]). For example, we consider a pension fund model introduced in [7] and modify it to take into account the time of implementing the portfolio strategy. Suppose that the manager can invest in two assets: a risky asset (e.g., stock) and a riskless asset (e.g., bond). Then, the wealth equation is as follows:

$$\begin{cases} dx(t) = \big[rx(t) + \sigma\lambda u(t-\delta)\big]dt - \big[q + f(x(t) - x(t-\delta))\big]dt \\ \qquad\quad +\sigma u(t-\delta)dW(t), 0 \leqslant t \leqslant T, \\ x(\theta) = \varphi(\theta), \quad u(\theta) = \psi(\theta),\ \theta \in [-\delta, 0], \end{cases}$$

where $x(\cdot)$ is the fund wealth, $u(\cdot)$ is the amount of money invested in the risky asset, $r \geqslant 0$ is the instantaneous return rate of the riskless asset, $\mu \geqslant r$ is the instantaneous rate of expected return of the risky asset, and $\sigma > 0$ is the instantaneous rate of volatility. Assume that μ can be expressed by the relation $\mu = r + \sigma\lambda$, where $\lambda \geqslant 0$ is the instantaneous risk premium of the market. Compared with the classical self-financing portfolio model, $u(t-\delta)$ considers the time of implementing the portfolio strategy, and the difference $q + f(x(t) - x(t-\delta))$ represents the external cashflows of contributions and benefits which enter the dynamics of the fund. The portfolio strategy $u(t-\delta)$ at time $t-\delta$ is executed at time t, when the asset prices and the fund wealth have already changed. q is the difference between the exiting cashflow of the aggregate benefits, paid by the fund as a minimum guarantee to its members in retirement, and the entering cashflow, paid by the members who are adhering to the fund. f is a constant, and the term $f(x(t) - x(t-\delta))$ represents the dividends to members when the investment is profitable or the replenishment of cash flow when the investment is loss-making. $\varphi(\cdot)$ is the initial wealth or the fund donated at $[-\delta, 0]$, and $\psi(\cdot)$ is the initial investment strategy according to $\varphi(\cdot)$. The manager wants to achieve the expected return a, that is, he would like to minimize the following objective functional:

$$J(\varphi(\cdot), \psi(\cdot); u(\cdot)) = \mathbb{E}\big[|x(T) - a|^2\big].$$

In the above, we use a single delay to describe the time of implementing the portfolio strategy. In fact, in fields such as biology, physics, and medicine, a single delay cannot adequately describe the dynamics of a system, and multiple pointwise delays and distributed delay have to be used (see [13, 14, 24]), because the time required for plants and animals to grow and mature varies significantly, the transport and diffusion rates of substances are also different, and sometimes these delay effects show smooth changes in time, rather than instantaneous responses.

Motivated by these practical examples, we would like to study stochastic linear quadratic optimal control problems with both state delay and control delay. In the 18th century, Euler, Bernoulli, Lagrange, Laplace, and Poisson first considered delay systems when studying various geometric problems. For deterministic delayed optimal control problems, Delfour in [6] solved a linear quadratic optimal control problem with pointwise and distributed state delay by the product space approach. Later, Vinter and Kwong in [32] reformulated a linear differential delay system with distributed control delay as an evolution system with bounded control operators by the structural state method. Ichikawa in [12] studied an optimal control problem with pointwise control delay by the extended state method. Subsequently, massive research results have been produced, such as [1, 2]. *Stochastic differential delay equations* (SDDEs) are usually used to describe the dynamics of delayed stochastic systems; also refer

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

to [25, 26]. So far, optimal control problems of stochastic differential delay systems have been extensively studied. When only state delay appears in control systems, Flandoli in [8] transformed the delayed optimal control problem into an abstract one in Hilbert space, then derived the optimal feedback. Liang, Xu, and Zhang in [19] applied the method of completion of squares to obtain the feedback of the optimal control. When only control delay appears in control systems, Wang and Zhang in [33] described equivalently the stochastic control systems with input delay by an abstract model without delay in a Hilbert space, then derived the feedback of the optimal control. Zhang and Xu in [36] gave the solvability condition of the optimal control and the analytical controller based on a modified Riccati differential equation. For more literature, readers can refer to [7, 11, 23] (for stochastic optimal control problems with state delay only) and [3, 11, 34] (for stochastic optimal control problems with control delay only). However, when state delay and control delay both appear in control systems, most of the literature has only studied the maximum principle for the optimal control and does not provide the feedback of the optimal control (see [5, 9, 17, 35]).

Recently, Sun and Yong in [29] first found that there is a significant difference between open-loop and closed-loop saddle points for a stochastic linear quadratic two-person zero-sum differential game. As a continuation of the work of [29], Sun, Li, and Yong in [28] studied the open-loop and closed-loop solvability for stochastic linear quadratic optimal control problems and established the equivalence between the strongly regular solvability of the Riccati equation and the uniform convexity of the cost functional. Ni et al. in [27] considered a stochastic linear quadratic problem with transmission delay and characterized its solvability by Riccati-like equations and linear matrix equality-inequalities. As for related problems in an infinite time horizon, Sun and Yong in [30] discussed a stochastic linear quadratic optimal control problem with constant coefficients and researched the open-loop and closed-loop solvability. Li, Shi, and Yong in [18] presented a systematic theory for two-person non-zero-sum differential games of mean-field type stochastic differential systems with quadratic performance in an infinite time horizon. In the aspect of infinite dimensional problems, Lü generalized [28] to a stochastic linear quadratic optimal control problem governed by a stochastic evolution system in [20] and gave two strict assumptions. Later Lü in [21] dropped them, gave the closed-loop solvability for a linear quadratic optimal control problem driven by a mean-field type stochastic evolution system, and improved the main results in [20] noticeably.

This paper investigates a stochastic linear quadratic optimal control problem involving both state delay and control delay. The optimal control consists of three parts at least: the first one is proportional to the current value of the state, the second one involves an integral of the state trajectory over the past time interval, and the third one involves an integral of the control trajectory over the past time interval. The structure of the optimal control is so complex, how can we define the closed-loop solvability for the delayed stochastic optimal control problem? After the appropriate definitions are introduced, how do we characterize the closed-loop solvability?

The contributions and innovations in this paper are summarized as follows:

- A very general model is studied. Both state delay and control delay can appear in the state equation and the cost functional, especially in the diffusion term. When the original delayed system is transformed into an infinite dimensional control system without delay, the new control operators appear and cannot be dealt with using the existing methods (see [8, 15, 16, 19, 33, 36]). Thus, some new approaches are constructed to overcome the above difficulties.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

- Three kinds of solvability are proposed: the open-loop solvability, the closed-loop representation of the open-loop optimal control, and the closed-loop solvability for the original delayed stochastic optimal control problem. To characterize them, an equivalent optimal control problem without delay is constructed, and then the open-loop and closed-loop solvability are defined.
- Some necessary and sufficient conditions for the above three kinds of solvability are derived.
  (a) The open-loop solvability is ensured by the solvability of a constrained forward-backward stochastic evolution system and a convexity condition. A novel backward equation is introduced as an adjoint equation, since the new control operators make the transformed problem not a standard infinite dimensional stochastic optimal control problem, and its existence and uniqueness are proved by an equivalent backward stochastic evolution equation (SEE). Moreover, a clearer equivalence condition is deduced by going back to the original delayed control problem.
  (b) The closed-loop representation of the open-loop optimal control is given through a coupled matrix-valued Riccati equation. The transformed stochastic optimal control problem with the new control operators cannot be approximated by infinite dimensional control problems with bounded control operators, due to the lack of stochastic analytic tools. An integral operator-valued Riccati equation is constructed to overcome the difficulties caused by the new control operators, and inspired by this, the above coupled matrix-valued Riccati equation is obtained.
  (c) The closed-loop solvability is ensured by the solvability of a differential operator-valued Riccati equation. This is the first result for the closed-loop solvability of delayed stochastic optimal control problems. The difficulties are overcome through the introduction of the closed-loop strategy in decoupling forward delayed state equations and backward advanced adjoint equations, and sufficient conditions for the solvability of the Riccati equation are also provided. In addition, a clearer characterization of the closed-loop solvability is displayed by a coupled matrix-valued Riccati equation when going back to the original delayed control problem.

This paper is organized as follows. Section 2 formulates the optimal control problem for a stochastic differential delay system. Section 3 transforms it into an infinite dimensional control problem without delay. Section 4 derives necessary and sufficient conditions for the open-loop solvability. Section 5 presents the closed-loop representation of the open-loop optimal control. Section 6 ensures the closed-loop solvability under certain conditions. Finally section 7 gives some concluding remarks.

2. Problem formulation. Suppose $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space and the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ is generated by a one-dimensional standard Brownian motion $\{W(t)\}_{t\geqslant 0}$. $\mathbb{E}_t[\cdot]$ denotes the conditional expectation with respect to $\mathcal{F}_t$, i.e., $\mathbb{E}_t[\cdot] \equiv \mathbb{E}[\ \cdot\ |\mathcal{F}_t]$. First we define the following spaces which will be used in this paper. Let F be a closed convex subset of $\mathbb{R}^n$, and let E be a real Banach space. Then, $L^\infty(F;E)$ denotes the Banach space consisting of E-valued functions $\phi(\cdot)$ such that $\sup_{t\in F}||\phi(t)||_E < \infty$, $H^1(F;E)$ denotes the Sobolev space consisting of square integrable functions with square integrable distributional derivatives $D_t\phi$, $L^2_{\mathbb{F}}(\Omega; C(F;E))$ denotes the Banach space consisting of E-valued $\mathbb{F}$-adapted continuous processes $\phi(\cdot)$ such that $\mathbb{E}[\sup_{t\in F}||\phi(t)||^2_E] < \infty$, and $L^2_{\mathbb{F}}(F;E)$ denotes the Hilbert space consisting of $\mathbb{F}$-adapted processes $\phi(\cdot)$ such that $\mathbb{E}\int_F ||\phi(t)||^2_E dt < \infty$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

When $F=[a,b]\subseteq\mathbb{R}$, we simply denote $L^2(a,b;E)$ for $L^2([a,b];E)$ and other spaces are similar.

Let $||\cdot||_{H^1}$ and $\langle\cdot,\cdot\rangle_{H^1}$ denote the norm and the inner product in the Sobolev space $H^1(F;E)$, similar to other spaces. For simplicity, $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the norm and the inner product in the Euclidean space. E' denotes the dual space of E, and the symbol $\langle\cdot,\cdot\rangle_{E',E}$ is referred to as the duality pairing between E' and E. Given two real Hilbert spaces U_1 and U_2, $\mathscr{L}(U_1,U_2)$ denotes the real Banach space of all continuous linear maps; when $U_1=U_2$, we write $\mathscr{L}(U_1)$ in place of $\mathscr{L}(U_1,U_2)$. Φ^* denotes the adjoint operator of $\Phi\in\mathscr{L}(U_1,U_2)$. $\mathbb{S}^n$ is the space of all $n\times n$ symmetric matrices, I is the identity matrix with appropriate dimension or the identity map, and $\mathcal{R}$ is the operator range or the matrix range, if no ambiguity exists. The superscript † represents the Moore–Penrose inverse of vectors or matrices.

In this section, we formulate the stochastic optimal control problem.

For given finite time duration $T>0$ and given constant time delay $\delta>0$, let $A(d\theta)$ be $\mathbb{R}^{n\times n}$-valued finite measure on $[-\delta,0]$ as follows:

$$\int_{[-\delta,0]}A(d\theta)\tilde{\varphi}(\theta):=\sum_{i=0}^{N}A_i\tilde{\varphi}(\theta_i)+\int_{-\delta}^{0}A^0(\theta)\tilde{\varphi}(\theta)d\theta \tag{2.1}$$

with any square integrable function $\tilde{\varphi}(\cdot)$, and $-\delta=\theta_N<\theta_{N-1}<\cdots<\theta_1<\theta_0=0$. A_i and A^0 represent the pointwise delay and the distributed delay, respectively. $B(d\theta)$ and $D(d\theta)$ are similar to (2.1), involving B_i, $B^0(\cdot)$ and D_i, $D^0(\cdot)$, respectively. The term about $C(d\theta)$ has the following form:

$$\int_{[-\delta,0]}C(d\theta)\tilde{\varphi}(\theta):=C_0\tilde{\varphi}(0)+\int_{-\delta}^{0}C^0(\theta)\tilde{\varphi}(\theta)d\theta. \tag{2.2}$$

For given $s\in[0,T)$, consider the controlled linear SDDE

$$\begin{cases} dX(t)=\displaystyle\int_{[-\delta,0]}\Big(A(d\theta)X_t(\theta)+B(d\theta)u_t(\theta)\Big)dt \\ \qquad\qquad +\displaystyle\int_{[-\delta,0]}\Big(C(d\theta)X_t(\theta)+D(d\theta)u_t(\theta)\Big)dW(t),\ t\in[s,T], \\ X(s)=x,\quad X(t)=\varphi(t-s),\quad t\in[s-\delta,s), \\ u(t)=\psi(t-s),\quad t\in[s-\delta,s], \end{cases} \tag{2.3}$$

along with the cost functional as follows:

$$\begin{aligned} J(s,x,\varphi(\cdot),\psi(\cdot);u(\cdot))=\mathbb{E}\Big\{\int_s^T\Big[&\int_{[-\delta,0]^2}\langle Q(t,d\theta d\theta')X_t(\theta),X_t(\theta')\rangle \\ &+2\langle S(t,d\theta d\theta')X_t(\theta),u_t(\theta')\rangle+\langle R(t,d\theta d\theta')u_t(\theta),u_t(\theta')\rangle\Big]dt \\ &+\int_{[-\delta,0]^2}\langle G(d\theta d\theta')X_T(\theta),X_T(\theta')\rangle\Big\}. \end{aligned} \tag{2.4}$$

Here, $X(\cdot)$ is the state and $u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ is the control. x is the initial state, and $\varphi(\cdot)\in L^2(-\delta,0;\mathbb{R}^n)$ and $\psi(\cdot)\in L^2(-\delta,0;\mathbb{R}^m)$ are the initial trajectories of the state and the control, respectively. $X_t(\cdot):=X(t+\cdot)$ and $u_t(\cdot):=u(t+\cdot)$ represent the past trajectories of the state and the control. In the cost functional

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

(2.4), $Q(t,d\theta d\theta')$ and $S(t,d\theta d\theta')$ are also finite measures, involving $Q_{00}(\cdot),Q_{10}(\cdot,\cdot)$, $Q_{11}(\cdot,\cdot,\cdot)$ and $S_{00}(\cdot),S_{01}(\cdot,\cdot),S_{10}(\cdot,\cdot),S_{11}(\cdot,\cdot,\cdot)$, respectively:

$$\begin{aligned}
&\int_{[-\delta,0]^2}\langle Q(t,d\theta d\theta')\tilde\varphi(\theta),\tilde\varphi(\theta')\rangle\\
&\quad:=\int_{[-\delta,0]^2}\big\langle Q_{11}(t,\theta,\theta')\tilde\varphi(\theta),\tilde\varphi(\theta')\big\rangle d\theta' d\theta\\
&\qquad+\big\langle Q_{00}(t)\tilde\varphi(0),\tilde\varphi(0)\big\rangle+2\int_{-\delta}^0\big\langle Q_{10}(t,\theta)^\top\tilde\varphi(\theta),\tilde\varphi(0)\big\rangle d\theta\ \forall\tilde\varphi\in L^2(-\delta,0;\mathbb{R}^n),\\
&\int_{[-\delta,0]^2}\langle S(t,d\theta d\theta')\tilde\varphi(\theta),\tilde\psi(\theta')\rangle\\
&\quad:=\big\langle S_{00}(t)\tilde\varphi(0),\tilde\psi(0)\big\rangle\\
&\qquad+\int_{-\delta}^0\big\langle S_{01}(t,\theta)\tilde\varphi(\theta),\tilde\psi(0)\big\rangle d\theta+\int_{-\delta}^0\big\langle S_{10}(t,\theta)^\top\tilde\psi(\theta),\tilde\varphi(0)\big\rangle d\theta\\
&\qquad+\int_{[-\delta,0]^2}\big\langle S_{11}(t,\theta,\theta')\tilde\varphi(\theta),\tilde\psi(\theta')\big\rangle d\theta' d\theta\ \forall\tilde\varphi\in L^2(-\delta,0;\mathbb{R}^n),\tilde\psi\in L^2(-\delta,0;\mathbb{R}^m);
\end{aligned}$$

$R(t,d\theta d\theta')$ and $G(d\theta d\theta')$ are similar to $Q(t,d\theta d\theta')$, involving $R_{00}(\cdot),R_{10}(\cdot,\cdot),R_{11}(\cdot,\cdot,\cdot)$ and $G_{00},G_{10}(\cdot),G_{11}(\cdot,\cdot)$. In the above, $A_i,C_0,G_{00}\in\mathbb{R}^{n\times n},B_i,D_i\in\mathbb{R}^{n\times m},i=0,\dots,N$, $A^0(\cdot),B^0(\cdot),C^0(\cdot),D^0(\cdot),Q_{00}(\cdot),Q_{10}(\cdot,\cdot),Q_{11}(\cdot,\cdot,\cdot),S_{00}(\cdot),S_{01}(\cdot,\cdot),S_{10}(\cdot,\cdot)$, $S_{11}(\cdot,\cdot,\cdot),R_{00}(\cdot),R_{10}(\cdot,\cdot),R_{11}(\cdot,\cdot,\cdot),G_{10}(\cdot),G_{11}(\cdot,\cdot)$ are matrix-valued functions of appropriate dimensions.

Let us assume the following:

(A1) The coefficients of the state equation (2.3) satisfy the following assumptions:

$$A^0(\cdot),C^0(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times n}),\quad B^0(\cdot),D^0(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times m}).$$

(A2) The coefficients of the cost functional (2.4) satisfy the following assumptions:

$$\begin{aligned}
&Q_{00}(\cdot)\in L^\infty(0,T;\mathbb{S}^n),\quad Q_{10}(\cdot,\cdot)\in L^\infty([0,T]\times[-\delta,0];\mathbb{R}^{n\times n}),\\
&Q_{11}(\cdot,\cdot,\cdot)\in L^\infty([0,T]\times[-\delta,0]\times[-\delta,0];\mathbb{R}^{n\times n}),\quad S_{00}(\cdot)\in L^\infty(0,T;\mathbb{R}^{m\times n}),\\
&S_{01}(\cdot,\cdot)\in L^\infty([0,T]\times[-\delta,0];\mathbb{R}^{m\times n}),\quad S_{10}(\cdot,\cdot)\in L^\infty([0,T]\times[-\delta,0];\mathbb{R}^{m\times n}),\\
&S_{11}(\cdot,\cdot,\cdot)\in L^\infty([0,T]\times[-\delta,0]\times[-\delta,0];\mathbb{R}^{m\times n}),\quad R_{00}(\cdot)\in L^\infty(0,T;\mathbb{S}^m),\\
&R_{10}(\cdot,\cdot)\in L^\infty([0,T]\times[-\delta,0];\mathbb{R}^{m\times m}),R_{11}(\cdot,\cdot,\cdot)\in L^\infty([0,T]\times[-\delta,0]\times[-\delta,0];\mathbb{R}^{m\times m}),\\
&G_{10}(\cdot,\cdot)\in L^\infty([0,T]\times[-\delta,0];\mathbb{R}^{n\times n}),\quad G_{11}(\cdot)\in L^2(-\delta,0;\mathbb{R}^{n\times n}),\quad G_{00}\in\mathbb{S}^n,\\
&Q_{11}(t,\theta,\theta')^\top=Q_{11}(t,\theta',\theta),\ R_{11}(t,\theta,\theta')^\top=R_{11}(t,\theta',\theta),\ G_{11}(\theta,\theta')^\top=G_{11}(\theta',\theta).
\end{aligned}$$

We choose the product space $\mathfrak{M}:=\mathbb{R}^n\times L^2(-\delta,0;\mathbb{R}^n)$ as the space of initial data, which is a Hilbert space endowed with inner product and norm

$$\begin{aligned}
\langle x,y\rangle_{\mathfrak{M}}:=\langle x^0,y^0\rangle+\int_{-\delta}^0\langle x^1(\theta),y^1(\theta)\rangle d\theta,\quad\text{and}\quad ||x||_{\mathfrak{M}}:=\langle x,x\rangle_{\mathfrak{M}}^{\frac12},\\
\forall x=\begin{pmatrix}x^0\\x^1\end{pmatrix},\quad y=\begin{pmatrix}y^0\\y^1\end{pmatrix},\quad x^0,y^0\in\mathbb{R}^n,x^1,y^1\in L^2(-\delta,0;\mathbb{R}^n).
\end{aligned}$$

Under assumptions (A1)–(A2), for any initial data $(s,x,\varphi,\psi)\in[0,T)\times\mathfrak{M}\times L^2(-\delta,0;\mathbb{R}^m)$ and any admissible control $u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$, by the Picard iteration method or by Theorem 2.1 [25, Chapter II], the SDDE (2.3) admits a unique solution $X(\cdot)\equiv X(\cdot\ ;s,x,\varphi,\psi,u(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([s,T];\mathbb{R}^n))$, therefore the cost functional (2.4) is meaningful.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Problem (P). For any $(s,x,\varphi,\psi)\in[0,T)\times\mathfrak{M}\times L^2(-\delta,0;\mathbb{R}^m)$, find a $\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ such that (2.3) is satisfied and

$$J(s,x,\varphi(\cdot),\psi(\cdot);\bar{u}(\cdot))=\inf_{u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)}J(s,x,\varphi(\cdot),\psi(\cdot);u(\cdot)):=V(s,x,\varphi(\cdot),\psi(\cdot)).$$

Any $\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ that achieves the above infimum is called an *optimal control* for the initial data (s,x,φ,ψ), and the corresponding solution $\bar{X}(\cdot)\equiv X(\cdot\,;s,x,\varphi,\psi,\bar{u}(\cdot))$ is called the *optimal state.* The function $V(\cdot,\cdot,\cdot,\cdot)$ is called the *value function* of Problem (P).

3. Problem transformation. In this section, inspired by [6] and [12], we study Problem (P) by a control problem without delay, containing a new control operator.

Define the C_0-semigroup $\Phi(\cdot)$ as follows:

$$\begin{aligned}&\Phi(t):\mathfrak{M}\longrightarrow\mathfrak{M},\\&\xi\mapsto\begin{pmatrix}x(t)\\x_t(\cdot)\end{pmatrix}\quad\forall\xi:=\begin{pmatrix}x\\\varphi\end{pmatrix}\in\mathfrak{M},\end{aligned}\tag{3.1}$$

where $x(\cdot)\equiv x(\cdot\,;s,x,\varphi)$ is the solution to the following equation:

$$\begin{cases}\dot{x}(t)=\displaystyle\int_{[-\delta,0]}A(d\theta)x_t(\theta),\ \text{a.e. } t\in[0,T],\\ x(0)=x,\ x(t)=\varphi(t),\ t\in[-\delta,0),\end{cases}$$

with $x_t(\cdot):=x(t+\cdot)$. The generator of $\Phi(\cdot)$ is defined as

$$\begin{aligned}&\tilde{A}:\mathscr{D}(\tilde{A})\longrightarrow\mathfrak{M},\\&\xi\mapsto\begin{pmatrix}\int_{[-\delta,0]}A(d\theta)\varphi(\theta)\\ \dot{\varphi}(\cdot)\end{pmatrix}\quad\forall\xi\in\mathscr{D}(\tilde{A}),\end{aligned}\tag{3.2}$$

and its domain is $\mathscr{D}(\tilde{A})=\big\{\xi=(x^\top,\varphi^\top)^\top\in\mathfrak{M}|\ \varphi(\cdot)\in H^1(-\delta,0;\mathbb{R}^n),x=\varphi(0)\big\}$. As mentioned in [6], $\mathscr{D}(\tilde{A})$ is dense in $\mathfrak{M}$ and is a Banach space endowed with the norm $||\xi||_{\mathscr{D}(\tilde{A})}:=||\varphi(\cdot)||_{H^1}$. Denote $\mathfrak{L}:=L^2(-\delta,0;\mathbb{R}^m)$ and define the following operators:

(3.3)

$$\begin{aligned}&\tilde{B}:\mathfrak{L}\longrightarrow\mathfrak{M}\qquad\tilde{D}:\mathfrak{L}\longrightarrow\mathfrak{M}\qquad\tilde{C}:\mathfrak{M}\longrightarrow\mathfrak{M}\\&\psi\mapsto\begin{pmatrix}\int_{[-\delta,0]}B(d\theta)\,\psi(\theta)\\0\end{pmatrix},\ \psi\mapsto\begin{pmatrix}\int_{[-\delta,0]}D(d\theta)\,\psi(\theta)\\0\end{pmatrix},\\&\begin{pmatrix}x\\\varphi\end{pmatrix}\mapsto\begin{pmatrix}C_0x+\int_{-\delta}^0C^0(\theta)\,\varphi(\theta)d\theta\\0\end{pmatrix}.\end{aligned}$$

Then, $\tilde{C}\in\mathscr{L}(\mathfrak{M})$, but $\tilde{B},\tilde{D}\notin\mathscr{L}(\mathfrak{L},\mathfrak{M})$. Thus, we can write (2.3) in $\mathbb{R}^n$ as the following SEE in $\mathfrak{M}$:

$$\begin{cases}d\mathbf{X}(t)=[\tilde{A}\mathbf{X}(t)+\tilde{B}u_t]dt+[\tilde{C}\mathbf{X}(t)+\tilde{D}(t)u_t]dW(t),\qquad t\in[s,T],\\ \mathbf{X}(s)=\xi=\begin{pmatrix}x\\\varphi\end{pmatrix},\ u(t)=\psi(t-s),\quad t\in[s-\delta,s].\end{cases}\tag{3.4}$$

By Theorem 3.14 in [22], the SEE (3.4) has a unique solution. If we regard $\mathbf{X}(\cdot)$ as the new state, then (3.4) does not contain state delay. Before dealing with control delay, we give the following result to illustrate the equivalence of (2.3) and (3.4).

LEMMA 3.1. *Let* (A1)–(A2) *hold. For all* $\xi\in\mathfrak{M}$, $\psi(\cdot)\in\mathfrak{L}$, $u(\cdot)\in L^2(s,T;\mathbb{R}^m)$, *assume that* $X(\cdot)$ *is the solution to* (2.3). *Then,* $\mathbf{X}(\cdot)$ *defined as* $\mathbf{X}(t):=\left(\begin{smallmatrix}X(t)\\X_t(\cdot)\end{smallmatrix}\right)$ *is the mild solution to* (3.4), *i.e.,*

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

(3.5)
$$\mathbf{X}(t)=\Phi(t-s)\xi+\int_s^t\Phi(t-r)\tilde{B}u_r dr+\int_s^t\Phi(t-r)\big[\tilde{C}\mathbf{X}(r)+\tilde{D}u_r\big]dW(r),t\in[s,T].$$

Furthermore, there exists a constant $M>0$ such that

$$\mathbb{E}\Big[\sup_{s\leqslant t\leqslant T}||\mathbf{X}(t)||^2_{\mathfrak{M}}\Big]\leq M\Big[|x|^2+\int_{-\delta}^0(|\varphi(\theta)|^2+|\psi(\theta)|^2)d\theta+\mathbb{E}\int_s^T|u(r)|^2dr\Big].$$

The proof is similar to Theorem 2.3 in [8] and thus is omitted here.

Remark 3.2. From Lemma 3.1, the SDDE (2.3) is equivalent to the SEE (3.5). When $C_0,C^0(\theta),D_i,D^0(\theta)$ depend on t, Lemma 3.1 holds, $i=0,\dots,N$. When (2.2) contains multiple pointwise delays, Lemma 3.1 also holds; see pages 941–943 in [8].

Next we deal with control delay, introducing the semigroup of left translation:

$$\mathcal{L}(t):\mathfrak{L}\longrightarrow\mathfrak{L},$$
$$\big[\mathcal{L}(t)Y\big](\theta):=\begin{cases}\begin{cases}Y(t+\theta), & -\delta\leq\theta\leq-t,\\ 0, & -t<\theta\leq 0,\end{cases} & \text{if } t\leq\delta,\\ 0, \quad -\delta\leq\theta\leq 0, & \text{if } t>\delta.\end{cases}\tag{3.6}$$

Its generator is given by $\mathcal{A}:\mathscr{D}(\mathcal{A})\longrightarrow\mathfrak{L}$, $\mathcal{A}Y:=\frac{dY}{d\theta}$ $\forall Y\in\mathscr{D}(\mathcal{A})$. The domain $\mathscr{D}(\mathcal{A})=\big\{Y\in H^1(-\delta,0;\mathbb{R}^m)|\ Y \text{ is absolutely continuous and } Y(0)=0\big\}$ is a Banach space endowed with the norm $||\cdot||_{H^1}$. Denote $V:=H^1(-\delta,0;\mathbb{R}^m)$, let V' be the dual of V, and consider the following evolution equation:

$$\mathbf{Y}_t=\mathcal{L}(t-s)\psi+\int_s^t\mathcal{L}(t-r)\Delta u(r)dr,\quad t\in[s,T],\tag{3.7}$$

with the bounded linear operator $\Delta:\mathbb{R}^m\longrightarrow V'$, $\langle\Delta u,w\rangle_{V',V}:=\langle u,w(0)\rangle$ for all $u\in\mathbb{R}^m$, $w\in V$. Then, by Lemma 1.1 in [12], (3.7) is well-defined and

$$\mathbf{Y}_t(\theta)=\begin{cases}\begin{cases}u(t+\theta), & s-t<\theta\leq 0,\\ \psi(\theta+t-s), & -\delta\leq\theta\leq s-t,\end{cases} & \text{if } t-s\leq\delta,\\ u(t+\theta), \quad -\delta\leq\theta\leq 0, & \text{if } t-s>\delta.\end{cases}\tag{3.8}$$

By (3.8), we get $\mathbf{Y}_t(\theta)=u_t(\theta)$ for almost everywhere $\theta\in[-\delta,0]$ and all $t\in[s,T]$.

Therefore, (3.5) can be written as the following formula, equivalent to (2.3):

$$\begin{cases}\mathbf{X}(t) &=\Phi(t-s)\xi+\int_s^t\Phi(t-r)\tilde{B}\mathbf{Y}_r dr\\ &\quad+\int_s^t\Phi(t-r)\big[\tilde{C}\mathbf{X}(r)+\tilde{D}\mathbf{Y}_r\big]dW(r),t\in[s,T],\\ \mathbf{Y}_t &=\mathcal{L}(t-s)\psi+\int_s^t\mathcal{L}(t-r)\Delta u(r)dr,\quad t\in[s,T].\end{cases}\tag{3.9}$$

Denote $\mathfrak{Z}:=\mathfrak{M}\times\mathfrak{L}$, for any $z=\binom{\xi}{\psi}$, $z_1=\binom{\xi_1}{\psi_1}$ and $z_2=\binom{\xi_2}{\psi_2}\in\mathfrak{Z}$, $\|z\|_{\mathfrak{Z}}:=\big[\|\xi\|^2_{\mathfrak{M}}+\|\psi\|^2_{\mathfrak{L}}\big]^{\frac{1}{2}}$, $\langle z_1,z_2\rangle_{\mathfrak{Z}}:=\langle\xi_1,\xi_2\rangle_{\mathfrak{M}}+\langle\psi_1,\psi_2\rangle_{\mathfrak{L}}$. Define the following C_0-semigroup:

$$\mathbf{T}(t):\mathfrak{Z}\longrightarrow\mathfrak{Z},$$
$$\mathbf{T}(t)\begin{pmatrix}\xi\\ \psi\end{pmatrix}:=\begin{bmatrix}\Phi(t)\xi+\int_0^t\Phi(t-r)\tilde{B}\mathcal{L}(r)\psi dr\\ \mathcal{L}(t)\psi\end{bmatrix},$$

and $\mathbf{Z}_0:=\binom{\xi}{\psi}$, $\mathbf{Z}(\cdot):=\binom{\mathbf{X}(\cdot)}{\mathbf{Y}_\cdot}$, $\mathbf{B}:=\binom{0}{\Delta}$, $\mathbf{C}:=\begin{pmatrix}\tilde{C} & \tilde{D}\\ 0 & 0\end{pmatrix}$. Then, (3.9) can be written as

$$\mathbf{Z}(t)=\mathbf{T}(t-s)\mathbf{Z}_0+\int_s^t\mathbf{T}(t-r)\mathbf{B}u(r)dr+\int_s^t\mathbf{T}(t-r)\mathbf{C}\mathbf{Z}(r)dW(r).\tag{3.10}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Noting $\mathbf{C} \notin \mathscr{L}(\mathfrak{Z})$, $\mathbf{B}$ maps $\mathbb{R}^m$ to $\mathfrak{M} \times V'$ out $\mathfrak{Z}$, thus $\mathbf{B} \notin \mathscr{L}(\mathbb{R}^m, \mathfrak{Z})$, the above integration is not defined in $\mathfrak{Z}$, (3.10) is just a formal expression, and it actually means (3.9).

Now we have transformed the original delayed state equation (2.3) into the new state equation (3.10) (or (3.9)), containing neither state delay nor control delay.

Next we rewrite the cost functional (2.4) by $\mathbf{Z}(\cdot)$ and $u(\cdot)$, but before that we define some bounded linear operators. Recalling $\mathfrak{L} := L^2(-\delta, 0; \mathbb{R}^m)$, we also denote $L^2(-\delta, 0; \mathbb{R}^n)$ by $\mathfrak{L}$ for ease of writing, and the dimension depends on the specific situation. Denote $\tilde{\kappa}_{00}(t)\tilde{x} = \kappa_{00}(t)\tilde{x}$, $\tilde{\kappa}_{01}(t)\tilde{\varphi} := \int_{-\delta}^{0} \kappa_{01}(t,\theta)\tilde{\varphi}(\theta)d\theta$, $(\tilde{\kappa}_{10}(t)\tilde{x})(\cdot) := \kappa_{10}(t,\cdot)\tilde{x}$, $(\tilde{\kappa}_{11}(t)\tilde{\varphi})(\cdot) := \int_{-\delta}^{0} \kappa_{11}(t,\theta,\cdot)\tilde{\varphi}(\theta)d\theta$, for any $\tilde{x} \in \mathbb{R}^d, \tilde{\varphi} \in \mathfrak{L}, d = n, m$, where $\kappa = Q, S, R, G, Q_{01}(t,\theta) = Q_{10}(t,\theta)^\top, R_{01}(t,\theta) = R_{10}(t,\theta)^\top, G_{01}(\theta) = G_{10}(\theta)^\top$. Then, $\tilde{Q}_{01}(t)^* = \tilde{Q}_{10}(t), \tilde{R}_{01}(t)^* = \tilde{R}_{10}(t), \tilde{G}_{01}^* = \tilde{G}_{10}$. Notice that $\tilde{S}_{01}(t)^* = \tilde{S}_{10}(t)$ is not always true. Let

$$\tilde{Q}(t) := \begin{bmatrix} \tilde{Q}_{00}(t) & \tilde{Q}_{01}(t) \\ \tilde{Q}_{10}(t) & \tilde{Q}_{11}(t) \end{bmatrix}, \tilde{S}(t) := \begin{bmatrix} \tilde{S}_{00}(t) & \tilde{S}_{01}(t) \\ \tilde{S}_{10}(t) & \tilde{S}_{11}(t) \end{bmatrix},$$
$$\tilde{R}(t) := \begin{bmatrix} \tilde{R}_{00}(t) & \tilde{R}_{01}(t) \\ \tilde{R}_{10}(t) & \tilde{R}_{11}(t) \end{bmatrix}, \tilde{G} := \begin{bmatrix} \tilde{G}_{00} & \tilde{G}_{01} \\ \tilde{G}_{10} & \tilde{G}_{11} \end{bmatrix}.$$

Then, we rewrite the cost functional (2.4) as follows:

$$\begin{aligned} J(s,x,\varphi(\cdot),\psi(\cdot);u(\cdot)) = \mathbb{E}\int_s^T \Big[&\big\langle \tilde{Q}(t)\mathbf{X}(t), \mathbf{X}(t)\big\rangle + 2\Big\langle \tilde{S}(t)\mathbf{X}(t), \big(u(t)\mathbf{Y}_t\big)\Big\rangle \\ &+ \Big\langle \tilde{R}(t)\big(u(t)\mathbf{Y}_t\big), \big(u(t)\mathbf{Y}_t\big)\Big\rangle\Big]dt + \mathbb{E}\big\langle \tilde{G}\mathbf{X}(T), \mathbf{X}(T)\big\rangle. \end{aligned} \tag{3.11}$$

In the above, $\langle\cdot,\cdot\rangle$ has the different meaning.

Define

$$\tilde{S}_0(t) := \big[\tilde{S}_{00}(t) \quad \tilde{S}_{01}(t)\big],\ \tilde{S}_1(t) := \big[\tilde{S}_{10}(t) \quad \tilde{S}_{11}(t)\big],\ \mathbf{S}(t) := \big[\tilde{S}_0(t) \quad \tilde{R}_{01}(t)\big],$$
$$\mathbf{Q}(t) := \begin{bmatrix} \tilde{Q}(t) & \tilde{S}_1(t)^* \\ \tilde{S}_1(t) & \tilde{R}_{11}(t) \end{bmatrix}, \quad \mathbf{G} := \begin{bmatrix} \tilde{G} & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{R}(t) := \tilde{R}_{00}(t).$$

Then, we rewrite (3.11) like this:

$$\begin{aligned} J(s,\mathbf{Z}_0;u(\cdot)) = \mathbb{E}\int_s^T \Big[&\big\langle \mathbf{Q}(t)\mathbf{Z}(t), \mathbf{Z}(t)\big\rangle_{\mathfrak{Z}} + 2\big\langle \mathbf{S}(t)\mathbf{Z}(t), u(t)\big\rangle \\ &+ \big\langle \mathbf{R}(t)u(t), u(t)\big\rangle\Big]dt + \mathbb{E}\big\langle \mathbf{G}\mathbf{Z}(T), \mathbf{Z}(T)\big\rangle_{\mathfrak{Z}}, \end{aligned} \tag{3.12}$$

and thus we transform Problem (P) into a linear quadratic problem associated with (3.10) (or (3.9)) and (3.12), and we formulate it specifically as follows.

Problem (EP). For any $(s, \mathbf{Z}_0) \in [0,T) \times \mathfrak{Z}$, find a $\bar{u}(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ such that (3.10) (or (3.9)) is satisfied and

$$J(s,\mathbf{Z}_0;\bar{u}(\cdot)) = \inf_{u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)} J(s,\mathbf{Z}_0;u(\cdot)) := V(s,\mathbf{Z}_0). \tag{3.13}$$

Similarly, any $\bar{u}(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ that achieves the above infimum is called an *optimal control* for the initial pair $(s,\mathbf{Z}_0)$, and the corresponding solution $\bar{\mathbf{Z}}(\cdot)$ is called the *optimal state*. The function $V(\cdot,\cdot)$ is called the *value function* of Problem (EP).

Remark 3.3. By (3.7), (3.8), (3.9), and Remark 3.2, Problem (P) is equivalent to Problem (EP). When $C_0, C^0(\theta), D_i, D^0(\theta)$ depend on t, the equivalence also holds,

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

$i=0,\dots,N$. We transform the delayed finite dimensional Problem (P) into the infinite dimensional Problem (EP) without delay, containing the new control operator $\mathbf{B}$. It is worth mentioning that the unboundedness of $\mathbf{B}$ is as high as that studied by [15, 16], but its domain does not have a relation to that of the semigroup generator. Therefore, the existing approaches in the literature do not apply. In the next section, we will take some new methods to address the unboundedness of the control operator.

4. Open-loop solvability. In this section, we define the open-loop solvability for Problem (P) by the transformed Problem (EP) and ensure it by the solvability of a constrained forward-backward stochastic evolution system and a convexity condition. Finally we turn back to the original Problem (P) and explore its open-loop solvability.

First we give the definition of the open-loop solvability for Problem (P).

DEFINITION 4.1. *Problem* (P) *is said to be*

(i) (uniquely) open-loop solvable at initial data $(s,x,\varphi,\psi)\in[0,T]\times\mathfrak{Z}$ *if there exists a (unique)* $\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ *satisfying* (3.13)*;*

(ii) (uniquely) open-loop solvable at some $s\in[0,T)$ *if for any* $(x,\varphi,\psi)\in\mathfrak{Z}$*, there exists a (unique)* $\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ *satisfying* (3.13)*;*

(iii) (uniquely) open-loop solvable on $[s,T)$ *if it is (uniquely) open-loop solvable at all* $t\in[s,T)$.

Next we give the necessary and sufficient condition of the open-loop solvability.

THEOREM 4.2. *Let* (A1)–(A2) *hold. For any given initial data* $(s,x,\varphi,\psi)\in[0,T)\times\mathfrak{Z}$*,* $\bar{u}(\cdot)$ *is an open-loop optimal control of Problem* (P) *if and only if the following two conditions hold:*

(i) (Stationarity condition)

$$\tilde{S}_0(t)\bar{\mathbf{X}}(t)+\tilde{R}_{01}(t)\bar{\mathbf{Y}}_t+\tilde{R}_{00}(t)\bar{u}(t)+[p_2(t)](0)=0,\quad \text{a.e. a.s.,} \tag{4.1}$$

where $(\bar{\mathbf{X}}(\cdot),\bar{\mathbf{Y}}_\cdot,p_1(\cdot),k_1(\cdot),p_2(\cdot),k_2(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{M}))\times L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{L}))\times L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{M}))\times L^2_{\mathbb{F}}(s,T;\mathfrak{M})\times L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{L}))\times L^2_{\mathbb{F}}(s,T;\mathfrak{L})$ *is the solution to the following forward-backward SEE:*

(4.2)

$$\begin{cases}
\text{(a) } \bar{\mathbf{X}}(t)=\Phi(t-s)\xi+\displaystyle\int_s^t\Phi(t-r)\tilde{B}\bar{\mathbf{Y}}_r dr\\
\qquad\qquad+\displaystyle\int_s^t\Phi(t-r)\Big(\tilde{C}\bar{\mathbf{X}}(r)+\tilde{D}\bar{\mathbf{Y}}_r\Big)dW(r),\quad t\in[s,T],\\
\text{(b) } \bar{\mathbf{Y}}_t=\mathcal{L}(t-s)\psi+\displaystyle\int_s^t\mathcal{L}(t-r)\Delta\bar{u}(r)dr,\quad t\in[s,T],\\
\text{(c) } p_1(t)=\Phi(T-t)^*\tilde{G}\bar{\mathbf{X}}(T)+\displaystyle\int_t^T\Phi(r-t)^*\big[\tilde{C}^*k_1(r)+\tilde{Q}(r)\bar{\mathbf{X}}(r)+\tilde{S}_0(r)^*\bar{u}(r)\\
\qquad\qquad+\tilde{S}_1(r)^*\bar{\mathbf{Y}}_r\big]dr-\displaystyle\int_t^T\Phi(r-t)^*k_1(r)dW(r),\quad t\in[s,T],\\
\text{(d) } [p_2(t)](\theta)=\displaystyle\int_t^{T\wedge(t+\delta+\theta)}\big[\tilde{S}_1(r)\bar{\mathbf{X}}(r)+\tilde{R}_{01}(r)^*\bar{u}(r)+\tilde{R}_{11}(r)\bar{\mathbf{Y}}_r\big](t+\theta-r)dr\\
\qquad\qquad+\displaystyle\int_{[-\delta,0]}\Big(B(d\beta)^\top[p_1(t+\theta-\beta)]^0+D(d\beta)^\top[k_1(t+\theta-\beta)]^0\Big)\mathbf{1}_{[t+\theta-T,\theta]}(\beta)\\
\qquad\qquad-\displaystyle\int_t^{T\wedge(t+\delta+\theta)}[k_2(r)](t+\theta-r)dW(r),\quad t\in[s,T],\ \theta\in[-\delta,0],
\end{cases}$$

where $\xi=\big(x^\top,\ \varphi^\top\big)^\top$*.* $[p_1(r)]^0$*,* $[k_1(r)]^0\in\mathbb{R}^n$ *denote the* $\mathbb{R}^n$ *components of* $p_1(r)$ *and* $k_1(r)$*.*

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

(ii) (Convexity condition)

$$J(s,0;u^0(\cdot)) \geqslant 0 \ \forall u^0(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m), \tag{4.3}$$

where $(\mathbf{X}^0(\cdot),\mathbf{Y}^0_\cdot)$ *is the solution to the following integral equation:*

$$\begin{cases} \mathbf{X}^0(t) = \displaystyle\int_s^t \Phi(t-r)\tilde{B}\mathbf{Y}^0_r dr + \int_s^t \Phi(t-r)\Big(\tilde{C}\mathbf{X}^0(r) + \tilde{D}\mathbf{Y}^0_r\Big)dW(r), \quad t\in[s,T], \\ \mathbf{Y}^0_t = \displaystyle\int_s^t \mathcal{L}(t-r)\Delta u^0(r)dr, \qquad t\in[s,T]. \end{cases}$$

Proof. We split the proof into three steps as follows.

Step 1: For given $\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$, show that the forward-backward SEE (4.2) admits a unique solution.

By Theorem 4.10 in [22], $(p_1(\cdot),k_1(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{M}))\times L^2_{\mathbb{F}}(s,T;\mathfrak{M})$. It remains to prove that (4.2)(d) admits a unique solution $(p_2(\cdot),k_2(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{L}))\times L^2_{\mathbb{F}}(s,T;\mathfrak{L})$ for given $(\bar{\mathbf{X}}(\cdot),\bar{\mathbf{Y}}_\cdot,p_1(\cdot),k_1(\cdot))$. Notice that $\tilde{B},\tilde{D}\in\mathscr{L}(\mathscr{D}(\mathcal{A}),\mathfrak{M})$. Then, for any $\kappa\in\mathscr{D}(\mathcal{A})$,

$$\begin{aligned} &\left\langle \int_t^T \mathcal{L}(r-t)^*\tilde{B}^* p_1(r)dr,\kappa\right\rangle_{\langle\mathscr{D}(\mathcal{A})',\mathscr{D}(\mathcal{A})\rangle} \\ &\quad= \int_t^T \left\langle \tilde{B}^*p_1(r),\mathcal{L}(r-t)\kappa\right\rangle_{\langle\mathscr{D}(\mathcal{A})',\mathscr{D}(\mathcal{A})\rangle} dr \\ &\quad= \int_t^T \langle p_1(r),\tilde{B}\mathcal{L}(r-t)\kappa\rangle_{\mathfrak{M}} dr \\ &\quad= \int_{-\delta}^0 \left\langle \int_{[-\delta,0]} B(d\theta)^\top [p_1(r+t-\theta)]^0 \mathbf{1}_{[t+r-T,r]}(\theta),\kappa(r)\right\rangle dr, \end{aligned}$$

it follows that

$$\left(\int_t^T \mathcal{L}(r-t)^*\tilde{B}^*p_1(r)dr\right)(\theta) = \int_{[-\delta,0]} B(d\beta)^\top [p_1(t+\theta-\beta)]^0 \mathbf{1}_{[t+\theta-T,\theta]}(\beta). \tag{4.4}$$

Similarly, we have

$$\left(\int_t^T \mathcal{L}(r-t)^*\tilde{D}^*k_1(r)dr\right)(\theta) = \int_{[-\delta,0]} D(d\beta)^\top [k_1(t+\theta-\beta)]^0 \mathbf{1}_{[t+\theta-T,\theta]}(\beta). \tag{4.5}$$

By (3.6) and Lemma 3.3 in [7], (4.2)(d) is equivalent to the following backward SEE:

$$\begin{aligned} \tilde{p}_2(t) = &\int_t^T \mathcal{L}(r-t)^*\Big[\tilde{B}^*p_1(r)+\tilde{D}^*k_1(r)+\tilde{S}_1(r)\bar{\mathbf{X}}(r)+\tilde{R}_{11}(r)\bar{\mathbf{Y}}_r+\tilde{R}_{01}(r)^*\bar{u}(r)\Big]dr \\ &-\int_t^T \mathcal{L}(r-t)^*\tilde{k}_2(r)dW(r),\ t\in[s,T]. \end{aligned} \tag{4.6}$$

Next we would like to prove that (4.6) admits a unique solution $(\tilde{p}_2(\cdot),\tilde{k}_2(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([s,T];\mathfrak{L}))\times L^2_{\mathbb{F}}(s,T;\mathfrak{L})$, and we only need to prove the existence. Denote

$$\begin{aligned} \tilde{p}_2(t) := \mathbb{E}_t\Big[\int_t^T \mathcal{L}(r-t)^*\Big(&\tilde{B}^*p_1(r)+\tilde{D}^*k_1(r)+\tilde{S}_1(r)\bar{\mathbf{X}}(r) \\ &+\tilde{R}_{11}(r)\bar{\mathbf{Y}}_r+\tilde{R}_{01}(r)^*\bar{u}(r)\Big)dr\Big]. \end{aligned}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Then, we have

$$\tilde{p}_2(t) = \mathbb{E}_t\Big[\int_{[-\delta,0]} \Big(D(d\beta)^\top [k_1(t+\cdot-\beta)]^0 + B(d\beta)^\top [p_1(t+\cdot-\beta)]^0\Big)\mathbf{1}_{[t+\cdot-T,\cdot]}(\beta)\Big] + \mathbb{E}_t\Big[\int_t^T \mathcal{L}(r-t)^*\Big(\tilde{S}_1(r)\bar{\mathbf{X}}(r) + \tilde{R}_{11}(r)\bar{\mathbf{Y}}_r + \tilde{R}_{01}(r)^*\bar{u}(r)\Big)dr\Big] := \mathbf{I}(t) + \mathbf{II}(t).$$

Let $L^2(s,T;L^2_{\mathbb{F}}(s,T;\mathfrak{L}))$ be the Banach space of all strongly $\mathscr{B}([s,T])\otimes\mathscr{B}([s,T])\otimes\mathcal{F}_T$-measurable functions $h : [s,T]^2\times\Omega\to\mathfrak{L}$, satisfying that for $r\in[s,T]$, $h(r,\cdot)$ is $\mathbb{F}$-adapted and $\mathbb{E}\int_s^T\int_s^T ||h(r,\beta)||^2_{L^2}d\beta dr<\infty$. Notice that $\tilde{S}_1(\cdot)\bar{\mathbf{X}}(\cdot)+\tilde{R}_{11}(\cdot)\bar{\mathbf{Y}}_\cdot + \tilde{R}_{01}(\cdot)^*\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathfrak{L})$. Then, by Corollary 2.149 in [22], there exists $h(\cdot,\cdot)\in L^2(s,T;L^2_{\mathbb{F}}(s,T;\mathfrak{L}))$ such that

$$\mathbf{II}(t) = \int_t^T \mathcal{L}(r-t)^*\Big\{\Big(\tilde{S}_1(r)\bar{\mathbf{X}}(r) + \tilde{R}_{11}(r)\bar{\mathbf{Y}}_r + \tilde{R}_{01}(r)^*\bar{u}(r)\Big) - \int_t^r h(r,\beta)dW(\beta)\Big\}dr, t\in[s,T],$$

which yields

$$\mathbf{II}(t) = \int_t^T \mathcal{L}(r-t)^*\Big(\tilde{S}_1(r)\bar{\mathbf{X}}(r) + \tilde{R}_{11}(r)\bar{\mathbf{Y}}_r + \tilde{R}_{01}(r)^*\bar{u}(r)\Big)dr - \int_t^T \mathcal{L}(r-t)^*\int_r^T \mathcal{L}(\beta-r)^* h(\beta,r)d\beta dW(r).$$

For $\mathbf{I}(t)$, we have

(4.7)

$$[\mathbf{I}(t)](\theta) = \mathbb{E}_t\Bigg[\sum_{i=0}^N \Big(D_i^\top [k_1(t+\theta-\theta_i)]^0 + B_i^\top [p_1(t+\theta-\theta_i)]^0\Big)\mathbf{1}_{[t+\theta-T,\theta]}(\theta_i) + \int_{-\delta}^0 \Big(D^0(\beta)^\top [k_1(t+\theta-\beta)]^0 + B^0(\beta)^\top [p_1(t+\theta-\beta)]^0\Big)\mathbf{1}_{[t+\theta-T,\theta]}(\beta)d\beta\Bigg].$$

Since $[p_1(\cdot)]^0, [k_1(\cdot)]^0\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^n)$, by Corollary 2.149 in [22], there exists $\tilde{h}(\cdot,\cdot)$, $\tilde{\tilde{h}}(\cdot,\cdot)\in L^2(s,T;L^2_{\mathbb{F}}(s,T;\mathbb{R}^n))$ such that for almost everywhere $\tau\in[t,T]$,

$$[p_1(\tau)]^0 = \mathbb{E}_t[p_1(\tau)]^0 + \int_t^\tau \tilde{h}(\tau,r)dW(r),\quad [k_1(\tau)]^0 = \mathbb{E}_t[k_1(\tau)]^0 + \int_t^\tau \tilde{\tilde{h}}(\tau,r)dW(r),$$

which and (4.7) yield that for almost everywhere $\theta\in[-\delta,0]$,

$$\begin{aligned}
&[\mathbf{I}(t)](\theta)\\
&= \Bigg[\sum_{i=0}^N \Big(D_i^\top [k_1(t+\theta-\theta_i)]^0 + B_i^\top [p_1(t+\theta-\theta_i)]^0\Big)\mathbf{1}_{[t+\theta-T,\theta]}(\theta_i)\\
&\quad + \int_{-\delta}^0 \Big(D^0(\beta)^\top [k_1(t+\theta-\beta)]^0 + B^0(\beta)^\top [p_1(t+\theta-\beta)]^0\Big)\mathbf{1}_{[t+\theta-T,\theta]}(\beta)d\beta\Bigg]\\
&\quad - \int_t^{T\wedge(t+\delta+\theta)} \Bigg[\sum_{i=0}^N \Big(B_i^\top \tilde{h}(t+\theta-\theta_i,r) + D_i^\top \tilde{\tilde{h}}(t+\theta-\theta_i,r)\Big)\mathbf{1}_{[t+\theta-T,t+\theta-r]}(\theta_i)\\
&\quad + \int_{-\delta}^0 \Big(D^0(\beta)^\top \tilde{\tilde{h}}(t+\theta-\beta,r) + B^0(\beta)^\top \tilde{h}(t+\theta-\beta,r)\Big)\mathbf{1}_{[t+\theta-T,t+\theta-r]}(\beta)d\beta\Bigg]dW(r).
\end{aligned}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Define

$$[\tilde{k}(r)](\theta) := \sum_{i=0}^{N} \Big(D_i^\top \tilde{\tilde{h}}(r+\theta-\theta_i, r) + B_i^\top \tilde{h}(r+\theta-\theta_i, r) \Big) \mathbf{1}_{[r+\theta-T,\theta]}(\theta_i) + \int_{-\delta}^{0} \Big(D^0(\beta)^\top \tilde{\tilde{h}}(r+\theta-\beta, r) + B(\beta)^\top \tilde{h}(r+\theta-\beta, r) \Big) \mathbf{1}_{[r+\theta-T,\theta]}(\beta) d\beta.$$

Then, by (4.4) and (4.5), we obtain

$$\mathbf{I}(t) = \int_t^T \mathcal{L}(r-t)^* \Big(\tilde{D}^* k_1(r) + \tilde{B}^* p_1(r) \Big) dr - \int_t^T \mathcal{L}(r-t)^* \tilde{k}(r) dW(r).$$

Let

$$\tilde{k}_2(r) := \int_r^T \mathcal{L}(\beta - r)^* h(\beta, r) d\beta + \tilde{k}(r).$$

Then, $(\tilde{p}_2(\cdot), \tilde{k}_2(\cdot))$ satisfies (4.6). Notice that $(p_1(\cdot), k_1(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([s,T]; \mathfrak{M})) \times L^2_{\mathbb{F}}(s,T;\mathfrak{M})$, $h(\cdot,\cdot) \in L^2(s,T; L^2_{\mathbb{F}}(s,T;\mathfrak{L}))$, and $\tilde{h}(\cdot,\cdot), \tilde{\tilde{h}}(\cdot,\cdot) \in L^2(s,T; L^2_{\mathbb{F}}(s,T;\mathbb{R}^n))$. Then, we have $(\tilde{p}_2(\cdot), \tilde{k}_2(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([s,T];\mathfrak{L})) \times L^2_{\mathbb{F}}(s,T;\mathfrak{L})$.

Step 2: Prove the necessity of Theorem 4.2.

Applying (3.3) and Theorem 3.3 in [10], we have

$$\mathbb{E} \int_s^T \Big(\langle p_1(t), \tilde{B} \mathbf{Y}_t^0 \rangle_{\mathfrak{M}} + \langle k_1(t), \tilde{D} \mathbf{Y}_t^0 \rangle_{\mathfrak{M}} \Big) dt = \mathbb{E} \int_s^T \langle \mathbf{X}^0(t), \tilde{Q}(t) \bar{\mathbf{X}}(t) + \tilde{S}_1(t)^* \bar{\mathbf{Y}}_t + \tilde{S}_0(t)^* \bar{u}(t) \rangle_{\mathfrak{M}} dt + \mathbb{E} \langle \tilde{G} \bar{\mathbf{X}}(T), \mathbf{X}^0(T) \rangle_{\mathfrak{M}}.$$

Noting for any $f(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathfrak{L})$, we have

$$\mathbb{E} \int_s^T \big\langle \mathbf{Y}_t^0, f(t) \big\rangle_{L^2} dt = \mathbb{E} \int_s^T \Big\langle u^0(t), \int_t^{T \wedge (t+\delta)} [f(r)](t-r) dr \Big\rangle dt. \tag{4.8}$$

By $k_2(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathfrak{L})$, we deduce

$$\mathbb{E} \int_{-\delta}^0 \int_{s+\theta}^s |[k_2(t-\theta)](\theta)|^2 dt d\theta + \mathbb{E} \int_{-\delta}^0 \int_s^{T+\theta} |[k_2(t-\theta)](\theta)|^2 dt d\theta = \mathbb{E} \int_{-\delta}^0 \int_{s+\theta}^{T+\theta} |[k_2(t-\theta)](\theta)|^2 dt d\theta = \mathbb{E} \int_s^T \int_{-\delta}^0 |[k_2(r)](\theta)|^2 d\theta dr < \infty,$$

which implies that

$$\mathbb{E} \int_t^{T \vee (t+\delta)} |[k_2(r)](t-r)|^2 dr < \infty, \quad \text{a.e. } t \in [s,T].$$

Thus, we obtain

$$\mathbb{E} \int_s^T \big\langle u^0(t), \int_t^{T \wedge (t+\delta)} [k_2(r)](t-r) dW(r) \big\rangle dt = \mathbb{E} \int_s^T \big\langle u^0(t), \mathbb{E}_t \int_t^{T \wedge (t+\delta)} [k_2(r)](t-r) dW(r) \big\rangle dt = 0. \tag{4.9}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

By the definition of $\tilde{B}$, we derive

$$\begin{aligned}\mathbb{E}\int_s^T \langle p_1(t), \tilde{B}\mathbf{Y}_t^0\rangle_{\mathfrak{M}} dt &= \mathbb{E}\int_s^T \Big\langle [p_1(t)]^0, \sum_{i=0}^N B_i u^0(t+\theta_i) \\ &\qquad \times \mathbf{1}_{(s-t,0]}(\theta_i) + \int_{-\delta}^0 B^0(\beta)u^0(t+\beta)\mathbf{1}_{(s-t,0]}(\beta)d\beta \Big\rangle dt \\ &= \mathbb{E}\int_s^T \Big\langle u^0(t), \int_{[-\delta,0]} B(d\beta)^\top [p_1(t-\beta)]^0 \mathbf{1}_{[t-T,0]}(\beta) \Big\rangle dt. \end{aligned} \tag{4.10}$$

By the definition of $\tilde{D}$, we obtain

$$\mathbb{E}\int_s^T \langle k_1(t), \tilde{D}\mathbf{Y}_t^0\rangle_{\mathfrak{M}} dt = \mathbb{E}\int_s^T \Big\langle u^0(t), \int_{[-\delta,0]} D(d\beta)^\top [k_1(t-\beta)]^0 \mathbf{1}_{[t-T,0]}(\beta) \Big\rangle dt. \tag{4.11}$$

By (4.8)–(4.11) and applying the convex variation technique in Theorem 4.1 in [29], we complete the proof of necessity.

Step 3: Prove the sufficiency of Theorem 4.2.

In fact, sufficiency is implied by the proof of necessity, thus we complete the proof. ∎

Remark 4.3. Since the new control operator $\mathbf{B}$ in (3.10) makes the transformed Problem (EP) not a standard infinite dimensional stochastic optimal control problem, a novel equation (4.2)(d) is introduced as an adjoint equation of (4.2)(b). For the deterministic system, the solvability of (4.2)(d) is natural and does not need to be proved separately, while in the stochastic system, due to the backward structure, its solution contains two components $p_2(\cdot)$ and $k_2(\cdot)$, so an additional proof is required. From the above proof, for a.e. $\theta \in [-\delta, 0]$, it is equivalent to the backward SEE (4.6) in $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^m)$, consisting of $\mathbb{R}^m$-valued $\mathcal{F}_T$-measurable random variables ξ such that $\mathbb{E}|\xi|^2 < \infty$. Moreover, when $C_0, C^0(\theta), D_i, D^0(\theta)$ depend on t, Theorem 4.2 still holds, $i = 0, \ldots, N$.

Inspired by (4.2) and (4.6), we go back to the original delayed control problem and characterize the open-loop solvability for Problem (P).

THEOREM 4.4. *Let* (A1)–(A2) *hold and* $G_{10}, G_{11} = 0$. *For any given initial data* $(s, x, \varphi, \psi) \in [0, T) \times \mathfrak{Z}$, $\bar{u}(\cdot)$ *is an open-loop optimal control of Problem* (P) *if and only if the following two conditions hold:*

(i) (Stationarity condition)

$$\begin{aligned}&\mathcal{M}(t) + S_{00}(t)\bar{X}(t) + R_{00}(t)\bar{u}(t) \\ &\quad + \int_{-\delta}^0 \Big[R_{10}(t,\theta)^\top \bar{u}(t+\theta) + S_{01}(t,\theta)\bar{X}(t+\theta)\Big] d\theta = 0, \text{ a.e. a.s.,}\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}\mathcal{M}(t) := \mathbb{E}_t\Big[\int_t^{T\wedge(t+\delta)} \Big(&S_{10}(r,t-r)\bar{X}(r) + R_{10}(r,t-r)\bar{u}(r) + \int_{-\delta}^0 \big[R_{11}(r,\theta,t-r)\bar{u}(r+\theta) \\ &+ S_{11}(r,\theta,t-r)\bar{X}(r+\theta)\big] d\theta + B^0(t-r)^\top \mathfrak{P}(r) + D^0(t-r)^\top \mathfrak{Q}(r)\Big) dr \\ &+ \mathbf{1}_{[0,T+\theta_i]}(t) \sum_{i=0}^N \Big(B_i^\top \mathfrak{P}(t-\theta_i) + D_i^\top \mathfrak{Q}(t-\theta_i)\Big)\Big],\end{aligned}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

with $(\bar{X}(\cdot),\mathfrak{P}(\cdot),\mathfrak{Q}(\cdot))$ *satisfying the following anticipated-backward SDDE:*

(4.13)
$$
\begin{cases}
d\bar{X}(t)=\displaystyle\int_{[-\delta,0]}\Big(A(d\theta)\bar{X}_t(\theta)+B(d\theta)\bar{u}_t(\theta)\Big)dt\\
\qquad\quad+\displaystyle\int_{[-\delta,0]}\Big(C(d\theta)\bar{X}_t(\theta)+D(d\theta)\bar{u}_t(\theta)\Big)dW(t),\quad t\in[s,T],\\
-d\mathfrak{P}(t)=\Big\{\displaystyle\sum_{i=0}^{N}A_i^\top\mathbb{E}_t[\mathfrak{P}(t-\theta_i)]\mathbf{1}_{[0,T+\theta_i)}(t)+C_0(t)^\top\mathfrak{Q}(t)+Q_{00}(t)\bar{X}(t)\\
\qquad\quad+S_{00}(t)^\top\bar{u}(t)+\displaystyle\int_{-\delta}^{0}\Big(S_{10}(t,\theta)^\top\bar{u}(t+\theta)+Q_{10}(t,\theta)^\top\bar{X}(t+\theta)\Big)d\theta\\
\qquad\quad+\displaystyle\int_{(t-T)\vee(-\delta)}^{0}\mathbb{E}_t\Big(A^0(\theta)^\top\mathfrak{P}(t-\theta)+C^0(\theta)^\top\mathfrak{Q}(t-\theta)+Q_{10}(t-\theta,\theta)\\
\qquad\quad\times\bar{X}(t-\theta)+S_{01}(t-\theta,\theta)^\top\bar{u}(t-\theta)+\displaystyle\int_{-\delta}^{0}[Q_{11}(t-\theta,\theta',\theta)\bar{X}(t-\theta+\theta')\\
\qquad\quad+S_{11}(t-\theta,\theta,\theta')^\top\bar{u}(t-\theta+\theta')]d\theta'\Big)d\theta\Big\}dt-\mathfrak{Q}(t)dW(t),\ t\in[s,T],\\
\bar{X}(s)=x,\ \bar{X}(t)=\varphi(t-s),\ t\in[s-\delta,s),\ \bar{u}(t)=\psi(t-s),\ t\in[s-\delta,s],\\
\mathfrak{P}(T)=G_{00}\bar{X}(T).
\end{cases}
$$

(ii) (Convexity condition)

$$J(s,0,0,0;u^0(\cdot))\geqslant 0\ \forall u^0(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m),$$

where $X^0(\cdot)$ *satisfies the following SDDE:*

$$
\begin{cases}
dX^0(t)=\displaystyle\int_{[-\delta,0]}\Big(A(d\theta)X_t^0(\theta)+B(d\theta)u_t^0(\theta)\Big)dt\\
\qquad\quad+\displaystyle\int_{[-\delta,0]}\Big(C(d\theta)X_t^0(\theta)+D(d\theta)u_t^0(\theta)\Big)dW(t),\quad t\in[s,T],\\
X^0(t)=0,\ u^0(t)=0,\quad t\in[s-\delta,s].
\end{cases}
$$

Proof. Using the convex variational technique and applying the Itô formula to $\langle\mathfrak{P}(\cdot),\,X^0(\cdot)\rangle$, the proof is completed, similar to the proof of Theorem 4.1 in [29]. ∎

Remark 4.5. (i) From (4.1) and (4.12), an interesting thing is that if $[p_1(t)]^0=\mathfrak{P}(t)$, $[k_1(t)]^0=\mathfrak{Q}(t)$ for all $t\in[s,T]$, then $\mathcal{M}(t)=\mathbb{E}_t([p_2(t)](0))=[p_2(t)](0)$, thus the stationarity conditions (4.1) and (4.12) are consistent. (ii) Theorem 4.4 is derived similarly, when the coefficients of the state equation (2.3) are time-variant. (iii) Let delay disappear in Problem (P). Then, Theorem 4.4 reduces to Theorem 2.3.2 in [31] when $b,\sigma,g,q,\rho=0$ there. (iv) Let Problem (P) only contain pointwise delay and A_i, $B_i,D_i=0,i=1,\dots,N-1$. Then, the second equation of (4.13) is similar to (12) in [4].

5. Closed-loop representation of open-loop optimal control. In this section, we study the solvability of an integral operator-valued Riccati equation, inspired

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

by which we give the closed-loop representation of the open-loop optimal control for Problem (P) by introducing a coupled matrix-valued Riccati equation.

DEFINITION 5.1. *An open-loop optimal control $\bar{u}(\cdot)$ of Problem* (P) *is said to admit a closed-loop representation if there exists $\bar{K}(\cdot) \in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$ such that for any initial data $(x,\varphi,\psi) \in \mathfrak{Z}$, the function*

$$\bar{u}(t) := \bar{K}(t)\bar{\mathbf{Z}}(t), \quad t \in [s,T],$$

is an open-loop optimal control of Problem (P) *for $(x,\varphi,\psi) \in \mathfrak{Z}$, where $\bar{\mathbf{Z}}(\cdot)$ is the solution to the following closed-loop system with $\mathbf{Z}_0 := (x^\top, \varphi^\top, \psi^\top)^\top$:*

(5.1)
$$\bar{\mathbf{Z}}(t) = \mathbf{T}(t-s)\mathbf{Z}_0 + \int_s^t \mathbf{T}(t-r)\mathbf{B}\bar{K}(r)\bar{\mathbf{Z}}(r)dr + \int_s^t \mathbf{T}(t-r)\mathbf{C}\bar{\mathbf{Z}}(r)dW(r), t \in [s,T].$$

For any $z \in \mathfrak{Z}$, consider the following integral operator-valued Riccati equation:

$$\begin{aligned} P(t)z = {} & \mathbf{T}(T-t)^*\mathbf{G}\mathbf{T}(T-t)z + \int_t^T \mathbf{T}(r-t)^*\Big[\mathbf{C}^*P(r)\mathbf{C} + \mathbf{Q}(r) \\ & - (\mathbf{B}^*P(r))^*\mathbf{R}(r)^{-1}(\mathbf{B}^*P(r))\Big]\mathbf{T}(r-t)zdr, \quad t \in [s,T], \end{aligned} \tag{5.2}$$

where $\mathbf{B}^* := (0, \Delta^*)$. The following theorem guarantees its solvability.

THEOREM 5.2. *Suppose all coefficients of Problem* (P) *are continuous and $\mathbf{C} \in \mathscr{L}(\mathfrak{Z})$. Assume that there exists a constant $\mu > 0$ such that $R_{00} \geqslant \mu$. Then, the integral operator-valued Riccati equation* (5.2) *admits a unique solution in the class of strongly continuous self-adjoint operators.*

Proof. In the following, denote $||\cdot||_{\mathscr{L}(\mathfrak{Z})}$, $||\cdot||_{\mathfrak{Z}}$ by $||\cdot||$ for simplicity. First we show that there exists $T_0 \in [0, T-s]$ such that (5.2) admits a unique solution on $[T-T_0, T]$.

Let $\mathcal{B}(l) := \{P(\cdot) : [T-T_0,T] \to \mathscr{L}(\mathfrak{Z}) | P(\cdot)$ is a strongly continuous self-adjoint operator, $\sup_{t\in[T-T_0,T]}||P(t)|| \leqslant l\}$. Consider the mapping: $\mathscr{T} : \mathcal{B}(l) \to \mathcal{B}(l)$, $\tilde{P}(\cdot) \mapsto P(\cdot)$, and $P(\cdot) = \mathscr{T}(\tilde{P}(\cdot))$ satisfies the following integral equation for any $z \in \mathfrak{Z}$, $t \in [T-T_0, T]$:

$$\begin{aligned} P(t)z = {} & \mathbf{T}(T-t)^*\mathbf{G}\mathbf{T}(T-t)z + \int_t^T \mathbf{T}(r-t)^*\big[\mathbf{C}^*\tilde{P}(r)\mathbf{C} + \mathbf{Q}(r) \\ & - (\mathbf{B}^*P(r))^*\mathbf{R}(r)^{-1}(\mathbf{B}^*P(r))\big]\mathbf{T}(r-t)zdr. \end{aligned} \tag{5.3}$$

Then, we'll complete the proof of this part in two steps.

Step 1: Show that $\mathscr{T}$ is well-defined.

Define $\tau := T - T_0$, and consider the following optimal control problem:

$$\begin{cases} \tilde{Z}(t) = \mathbf{T}(t-\tau)z_0 + \displaystyle\int_\tau^t \mathbf{T}(t-r)\mathbf{B}u(r)dr, \quad t \in [\tau,T],\ z_0 \in \mathfrak{Z}, \\ \displaystyle\min_{u(\cdot)\in L^2(\tau,T;\mathbb{R}^m)} \tilde{J}(\tau,z_0;u(\cdot)) = \int_\tau^T \Big[\big\langle \big(\mathbf{C}^*\tilde{P}(t)\mathbf{C} + \mathbf{Q}(t)\big)\tilde{Z}(t), \tilde{Z}(t)\big\rangle \\ \qquad\qquad + \big\langle \mathbf{R}(t)u(t), u(t)\big\rangle\Big]dt + \langle \mathbf{G}\tilde{Z}(T), \tilde{Z}(T)\rangle. \end{cases}$$

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Then, similar to Theorem 2.3 in [12], the optimal control is $\bar{\bar{u}}(t) = -\mathbf{R}(t)^{-1}\mathbf{B}^* P(t)\bar{\bar{Z}}(t)$, and the value function is $\tilde{V}(\tau, z_0) = \langle P(\tau)z_0, z_0\rangle$, where $P(\cdot)$ satisfies (5.3). Moreover, similar to Lemma 2.6 in [12], (5.3) is equivalent to the following equation:

(5.4)
$$\begin{cases} P(t)z = \mathbf{T}(T-t)^*\mathbf{G}\mathbf{T}_\infty(T,t)z + \displaystyle\int_t^T \mathbf{T}(r-t)^*\Big(\mathbf{C}^*\tilde{P}(r)\mathbf{C} + \mathbf{Q}(r)\Big)\mathbf{T}_\infty(r,t)z dr, \\ \mathbf{T}_\infty(r,t)z = \mathbf{T}(r-t)z - \displaystyle\int_t^r \mathbf{T}(r-\beta)\mathbf{B}\mathbf{R}(\beta)^{-1}\mathbf{B}^* P(\beta)\mathbf{T}_\infty(\beta,t)z d\beta, \tau \leqslant t \leqslant r \leqslant T. \end{cases}$$

Let $P_0(\cdot)$ be the solution to the following integral equation:

$$\begin{aligned} P_0(t)z = {} & \mathbf{T}(T-t)^*\mathbf{G}\mathbf{T}(T-t)z \\ & + \int_t^T \mathbf{T}(r-t)^*\Big(\mathbf{C}^*\tilde{P}(r)\mathbf{C} + \mathbf{Q}(r)\Big)\mathbf{T}(r-t)z dr, t \in [\tau, T]. \end{aligned}$$

Then, we have $P(t) \leqslant P_0(t)$. Thus, we obtain

(5.5)
$$||P(t)|| \leqslant ||\mathbf{G}||\gamma'^2(e^{2\gamma T_0} \vee 1) + \gamma'^2 T_0(e^{2\gamma T_0} \vee 1)\left(\sup_{s\leqslant r\leqslant T} ||\mathbf{Q}(r)|| + l||\mathbf{C}||^2\right), t \in [\tau, T],$$

where $\gamma' \geqslant 1$ and $\gamma \in \mathbb{R}$ satisfy that $||\mathbf{T}(t)|| \leqslant \gamma' e^{\gamma t}$ for all $t \in [s, T]$. Choose large enough l and small enough T_0 such that $\gamma'^2 T_0(e^{2\gamma T_0} \vee 1)||\mathbf{C}||^2 < \frac{1}{2}$, and

$$l > 2\gamma'^2(e^{2\gamma T_0} \vee 1)(T_0 + 2)\left(||\mathbf{G}|| + \sup_{s\leqslant r\leqslant T} ||\mathbf{Q}(r)||\right).$$

Then, we have $\sup_{\tau\leqslant t\leqslant T} ||P(t)|| < l$, thus $\mathscr{T}$ is well-defined.

Step 2: Show that $\mathscr{T}$ is a contraction mapping.

Denote $\hat{\tilde{P}}(\cdot) = \tilde{P}_1(\cdot) - \tilde{P}_2(\cdot)$, $\hat{P}(\cdot) = P_1(\cdot) - P_2(\cdot)$, and $\hat{\mathbf{T}}_\infty(\cdot,\cdot) = \mathbf{T}^1_\infty(\cdot,\cdot) - \mathbf{T}^2_\infty(\cdot,\cdot)$. Then, we get

$$||\mathbf{T}_\infty(r,t)|| \leqslant M(T_0), \quad \tau \leqslant t \leqslant r \leqslant T.$$

Here and after, $M(T_0)$ is a generic constant, depending on $\mu, T_0, |B_i|, \sup_{\theta\in[-\delta,0]} |B^0(\theta)|, ||\mathbf{G}||$, $\sup_{s\leqslant r\leqslant T} ||\mathbf{Q}(r)||, ||\Phi||, ||\mathcal{L}||, ||\mathbf{C}||, l$. And $M(T_0)$ increases as T_0 increases. By (5.4) we have

$$\sup_{\tau\leqslant t\leqslant T} ||\hat{\mathbf{T}}_\infty(t,\tau)||^2 \leqslant M(T_0)\int_\tau^T ||\hat{P}(r)||^2 dr, \ \sup_{\tau\leqslant t\leqslant T} ||\hat{P}(t)||^2 \leqslant M(T_0) \sup_{\tau\leqslant t\leqslant T} ||\hat{\tilde{P}}(t)||^2.$$

Choose T_0 such that $M(T_0)$ in the above inequality satisfies

$$M(T_0) < 1. \tag{5.6}$$

Then, $\mathscr{T}$ is a contraction mapping on $[T - T_0, T]$, thus there exists T_0 such that (5.2) admits a unique solution on $[T - T_0, T]$.

Finally, we aim to show that (5.2) admits a unique solution on the whole interval $[s, T]$.

For any $z \in \mathfrak{Z}$ and $t \in [T - T_0, T]$, consider

$$\bar{P}_0(t)z = \mathbf{T}(T-t)^*\mathbf{G}\mathbf{T}(T-t)z + \int_t^T \mathbf{T}(r-t)^*\Big(\mathbf{C}^*\bar{P}_0(r)\mathbf{C} + \mathbf{Q}(r)\Big)\mathbf{T}(r-t)z dr.$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Then, for $t \in [T-T_0, T]$, $||P(t)|| \leqslant ||\bar{P}_0(t)|| \leqslant \tilde{l}$, where $\tilde{l}$ depends on $|B_i|$, $\sup_{\theta\in[-\delta,0]} |B^0(\theta)|$, $\sup_{r\in[s,T]} ||\mathbf{Q}(r)||, ||\mathbf{G}||, ||\Phi||, ||\mathcal{L}||, T_0, ||\mathbf{C}||$. On $[T-T_0-T_1, T-T_0]$, consider the mapping $\mathscr{T}$, and in this case, $\mathbf{G}$ is replaced by $P(T-T_0)$ in the above part. Choose small enough T_1 and large enough l such that $\gamma'^2 T_1(e^{2\gamma T_1} \vee 1)\|\mathbf{C}\|^2 < \frac{1}{2}$ and $l > 2\gamma'^2(e^{2\gamma T_1} \vee 1)(T_1+2)(\tilde{l} + \sup_{r\in[s,T]} ||\mathbf{Q}(r)||)$. Then, similar to (5.5), we get for $t \in [T-T_0-T_1, T-T_0]$,

$$||P(t)|| \leqslant ||P(T-T_0)||(e^{2\gamma T_1} \vee 1)\gamma'^2 + \gamma'^2 T_1(e^{2\gamma T_1} \vee 1)\left(\sup_{s\leqslant r\leqslant T} ||\mathbf{Q}(r)|| + l||\mathbf{C}||^2\right) < l,$$

thus $\mathscr{T}$ is well-defined on $[T-T_1-T_0, T-T_0]$. Similar to (5.6), let $M(T_1) < 1$. Then, $\mathscr{T}$ is a contraction mapping on $[T-T_1-T_0, T-T_0]$. Repeating the above steps, (5.2) admits a unique solution on $[s,T]$, which completes the proof of Theorem 5.2. ∎

In the rest of this section, we consider Problem (P) with the following state equation instead of (2.3):

$$\begin{cases} dX(t) = \left[\sum_{i=0}^{N} A_i X(t+\theta_i) + \int_{-\delta}^{0} A^0(\theta)X(t+\theta)d\theta + \sum_{i=0}^{N} B_i u(t+\theta_i) \right. \\ \qquad \left. + \int_{-\delta}^{0} B^0(\theta)u(t+\theta)d\theta\right]dt + \left[C_0 X(t) + \int_{-\delta}^{0} C^0(\theta)X(t+\theta)d\theta \right. \\ \qquad \left. + D_0 u(t) + \int_{-\delta}^{0} D^0(\theta)u(t+\theta)d\theta\right]dW(t), \quad t \in [s,T], \\ X(s) = x, X(t) = \varphi(t-s), t \in [s-\delta, s), u(t) = \psi(t-s), t \in [s-\delta, s]. \end{cases} \tag{5.7}$$

Inspired by (5.2), let $P_{00}(t)\xi = [P(t)\binom{\xi}{0}]^0, P_{01}(t)\psi = [P(t)\binom{0}{\psi}]^0, P_{10}(t)\xi = [P(t)\binom{\xi}{0}]^1$, $P_{11}(t)\psi = [P(t)\binom{0}{\psi}]^1$. Then, under some proper conditions on the coefficients, $(P_{00}(\cdot), P_{01}(\cdot), P_{10}(\cdot), P_{11}(\cdot))$ satisfies the following differential operator-valued Riccati equation:

$$\begin{cases} (a)\ \dot{P}_{00}(t) = -\tilde{A}^* P_{00}(t) - P_{00}(t)\tilde{A} - \tilde{C}^* P_{00}(t)\tilde{C} - \mathbf{Q}(t) \\ \qquad\qquad + \big(\Delta^* P_{10}(t)\big)^* R_{00}(t)^{-1}\big(\Delta^* P_{10}(t)\big), \\ (b)\ \dot{P}_{01}(t) = -\tilde{A}^* P_{01}(t) - P_{00}(t)\tilde{B} - P_{01}(t)\mathcal{A} - \tilde{C}^* P_{00}(t)\tilde{D} \\ \qquad\qquad + \big(\Delta^* P_{10}(t)\big)^* R_{00}(t)^{-1}\big(\Delta^* P_{11}(t)\big), \\ (c)\ \dot{P}_{10}(t) = -\tilde{B}^* P_{00}(t) - \mathcal{A}^* P_{10}(t) - P_{10}(t)\tilde{A} - \tilde{D}^* P_{00}(t)\tilde{C} \\ \qquad\qquad + \big(\Delta^* P_{11}(t)\big)^* R_{00}(t)^{-1}\big(\Delta^* P_{10}(t)\big), \\ (d)\ \dot{P}_{11}(t) = -\tilde{B}^* P_{01}(t) - \mathcal{A}^* P_{11}(t) - P_{10}(t)\tilde{B} - P_{11}(t)\mathcal{A} - \tilde{D}^* P_{00}(t)\tilde{D} \\ \qquad\qquad + \big(\Delta^* P_{11}(t)\big)^* R_{00}(t)^{-1}\big(\Delta^* P_{11}(t)\big), \\ P_{00}(T) = \tilde{G}, P_{01}(T) = 0, P_{10}(T) = 0, P_{11}(T) = 0. \end{cases} \tag{5.8}$$

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Next we decompose (5.8), adjust some terms in the equations for $P_{00}(\cdot)$, $P_{01}(\cdot)$, $P_{11}(\cdot)$, and introduce the following Riccati equations. Denote $\mathfrak{R}(t) := R_{00}(t) + D_0^\top E_0(t) D_0$. Then, inspired by (5.8)(a), for almost everywhere $t \in [s,T], \theta, \alpha \in [-\delta, 0]$, introduce the coupled matrix-valued Riccati equation:

(5.9)
$$
\begin{cases}
\dot{E}_0(t) = -A_0^\top E_0(t) - E_0(t)A_0 - E_1(t,0) - E_1(t,0)^\top - C_0^\top E_0(t) C_0 \\
-Q_{00}(t) + \Big(E_3(t,0) + S_{00}(t) + B_0^\top E_0(t) + D_0^\top E_0(t) C_0\Big)^\top \\
\times \mathfrak{R}(t)^{-1} \Big(E_3(t,0) + S_{00}(t) + B_0^\top E_0(t) + D_0^\top E_0(t) C_0\Big), \\
\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta}\right) E_1(t,\theta) = -E_1(t,\theta)A_0 - E_2(t,\theta,0) \\
-\left[\sum_{i=1}^{N-1} A_i \hat{\delta}(\theta - \theta_i) + A^0(\theta)\right]^\top E_0(t) \\
-Q_{10}(t,\theta) - C^0(\theta)^\top E_0(t) C_0 + \Big[E_4(t,0,\theta) + S_{01}(t,\theta) + B_0^\top E_1(t,\theta)^\top \\
+ D_0^\top E_0(t) C^0(\theta)\Big]^\top \mathfrak{R}(t)^{-1} \Big[E_3(t,0) + S_{00}(t) + B_0^\top E_0(t) + D_0^\top E_0(t) C_0\Big], \\
\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \alpha}\right) E_2(t,\theta,\alpha) = -\left[A^0(\theta) + \sum_{i=1}^{N-1} A_i \hat{\delta}(\theta - \theta_i)\right]^\top E_1(t,\alpha)^\top \\
-E_1(t,\theta)\left[A^0(\alpha) + \sum_{i=1}^{N-1} A_i \hat{\delta}(\alpha - \theta_i)\right] - C^0(\theta)^\top E_0(t) C^0(\alpha) - Q_{11}(t,\alpha,\theta) \\
+\Big[E_4(t,0,\theta) + S_{01}(t,\theta) + B_0^\top E_1(t,\theta)^\top + D_0^\top E_0(t) C^0(\theta)\Big]^\top \\
\times \mathfrak{R}(t)^{-1} \Big[E_4(t,0,\alpha) + S_{01}(t,\alpha) + B_0^\top E_1(t,\alpha)^\top + D_0^\top E_0(t) C^0(\alpha)\Big], \\
\quad E_0(T) = G_{00},\ E_1(T,\theta) = G_{10}(\theta),\ E_1(t,-\delta) = A_N^\top E_0(t), \\
\quad E_2(T,\theta,\alpha) = G_{11}(\alpha,\theta), E_2(t,-\delta,\alpha) = A_N^\top E_1(t,\alpha)^\top, E_2(t,\theta,-\delta) = E_1(t,\theta) A_N.
\end{cases}
$$

Similarly, inspired by (5.8)(b), introduce the coupled matrix-valued Riccati equation:

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

(5.10)
$$
\begin{cases}
\Big(\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial\theta}\Big)E_3(t,\theta)=-\Big[\displaystyle\sum_{i=1}^{N-1}B_i\hat{\delta}(\theta-\theta_i)+B^0(\theta)\Big]^\top E_0(t)-D^0(\theta)^\top E_0(t)C_0\\
-E_4(t,\theta,0)-S_{10}(t,\theta)+\Big[E_5(t,0,\theta)+R_{10}(t,\theta)^\top+B_0^\top E_3(t,\theta)^\top\\
+D_0^\top E_0(t)D^0(\theta)\Big]^\top\mathfrak{R}(t)^{-1}\times\Big[E_3(t,0)+S_{00}(t)+B_0^\top E_0(t)^\top+D_0^\top E_0(t)C_0\Big]\\
-E_3(t,\theta)A_0,\Big(\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial\theta}-\dfrac{\partial}{\partial\alpha}\Big)E_4(t,\theta,\alpha)=-\Big[B^0(\theta)+\displaystyle\sum_{i=1}^{N-1}B_i\hat{\delta}(\theta-\theta_i)\Big]^\top E_1(t,\alpha)^\top\\
-E_3(t,\theta)\Big[A^0(\alpha)+\displaystyle\sum_{i=1}^{N-1}A_i\hat{\delta}(\alpha-\theta_i)\Big]-D^0(\theta)^\top E_0(t)C^0(\alpha)-S_{11}(t,\alpha,\theta)\\
+\Big[E_5(t,0,\theta)\\
+R_{10}(t,\theta)^\top+B_0^\top E_3(t,\theta)^\top+D_0^\top E_0(t)D^0(\theta)\Big]^\top\mathfrak{R}(t)^{-1}\Big[E_4(t,0,\alpha)+S_{01}(t,\alpha)\\
+B_0^\top E_1(t,\alpha)^\top+D_0^\top E_0(t)C^0(\alpha)\Big],\\
E_3(T,\theta)=0,\ E_3(t,-\delta)=B_N^\top E_0(t),\\
E_4(T,\theta,\alpha)=0,\ E_4(t,-\delta,\alpha)=B_N^\top E_1(t,\alpha)^\top,\ E_4(t,\theta,-\delta)=E_3(t,\theta)A_N.
\end{cases}
$$

Inspired by (5.8)(d), introduce the following matrix-valued Riccati equation:

(5.11)
$$
\begin{cases}
\Big(\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial\theta}-\dfrac{\partial}{\partial\alpha}\Big)E_5(t,\theta,\alpha)=-\Big[\displaystyle\sum_{i=1}^{N-1}B_i\hat{\delta}(\theta-\theta_i)+B^0(\theta)\Big]^\top E_3(t,\alpha)^\top\\
\quad-E_3(t,\theta)\Big[\displaystyle\sum_{i=1}^{N-1}B_i\hat{\delta}(\alpha-\theta_i)+B^0(\alpha)\Big]-D^0(\theta)^\top E_0(t)D^0(\alpha)-R_{11}(t,\alpha,\theta)\\
\quad+\Big[E_5(t,0,\theta)+D_0^\top E_0(t)D^0(\theta)+R_{10}(t,\theta)^\top+B_0^\top E_3(t,\theta)^\top\Big]^\top\\
\quad\times\mathfrak{R}(t)^{-1}\Big[E_5(t,0,\alpha)+D_0^\top E_0(t)D^0(\alpha)+R_{10}(t,\alpha)^\top+B_0^\top E_3(t,\alpha)^\top\Big],\text{a.e. } t,\alpha,\theta,\\
E_5(T,\theta,\alpha)=0,\ E_5(t,-\delta,\alpha)=B_N^\top E_3(t,\alpha)^\top,\ E_5(t,\theta,-\delta)=E_3(t,\theta)B_N,
\end{cases}
$$

where $\hat{\delta}(\cdot)$ is the delta function, i.e., $\hat{\delta}(\theta)=0$ for $\theta\neq 0$ and $\int_{-\infty}^{\infty}\hat{\delta}(\theta)d\theta=1$. Then, we can derive the closed-loop representation of open-loop optimal control for Problem (P).

THEOREM 5.3. *Suppose all coefficients of Problem* (P) *are continuous and* $\mathfrak{R}>0$. *Let continuous functions* $E_0(t)$, $E_1(t,\theta)$, $E_2(t,\theta,\alpha)$, $E_3(t,\theta)$, $E_4(t,\theta,\alpha)$, $E_5(t,\theta,\alpha)$, $t\in[s,T]$, $\theta,\alpha\in[-\delta,0]$, *satisfy the coupled matrix-valued Riccati equations* (5.9)–

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

(5.11), *and* $E_0(t) = E_0(t)^\top$, $E_2(t,\theta,\alpha) = E_2(t,\alpha,\theta)^\top$, $E_5(t,\theta,\alpha) = E_5(t,\alpha,\theta)^\top$. *For any given initial data* $(s,x,\varphi,\psi) \in [0,T) \times \mathfrak{Z}$, *denote*

(5.12)
$$\begin{aligned}\bar{K}(t)\begin{pmatrix} x \\ \varphi \\ \psi \end{pmatrix} = -\mathfrak{R}(t)^{-1}\Big\{ &\Big[E_3(t,0) + B_0^\top E_0(t) + D_0^\top E_0(t)C_0 + S_{00}(t)\Big]x \\ &+ \int_{-\delta}^0 \Big[E_4(t,0,\theta) + B_0^\top E_1(t,\theta)^\top + S_{01}(t,\theta) + D_0^\top E_0(t)C^0(\theta)\Big]\varphi(\theta)d\theta \\ &+ \int_{-\delta}^0 \Big[E_5(t,0,\theta) + B_0^\top E_3(t,\theta)^\top + R_{10}(t,\theta)^\top + D_0^\top E_0(t)D^0(\theta)\Big]\psi(\theta)d\theta\Big\}.\end{aligned}$$

Then, the closed-loop representation of the open-loop optimal control for Problem (P) *with the state equation* (5.7) *is as follows:*

$$\bar{u}(t) = \bar{K}(t)\bar{\mathbf{Z}}(t), \quad \text{a.e. a.s.}, \tag{5.13}$$

where $\bar{\mathbf{Z}}(\cdot)$ *satisfies* (5.1), *and the value function has the following form:*

$$\begin{aligned}V(s,x,\varphi(\cdot),\psi(\cdot)) = {}&\langle E_0(s)x,x\rangle + 2\int_{-\delta}^0 \langle \varphi(\theta), E_1(s,\theta)x\rangle d\theta \\ &+ \int_{-\delta}^0\int_{-\delta}^0 \langle E_2(s,\theta,\alpha)\varphi(\alpha),\varphi(\theta)\rangle d\theta d\alpha + 2\int_{-\delta}^0 \langle \psi(\theta), E_3(s,\theta)x\rangle d\theta \\ &+ 2\int_{-\delta}^0\int_{-\delta}^0 \langle \psi(\theta), E_4(s,\theta,\alpha)\varphi(\alpha)\rangle d\alpha d\theta \\ &+ \int_{-\delta}^0\int_{-\delta}^0 \langle E_5(s,\theta,\alpha)\psi(\alpha),\psi(\theta)\rangle d\theta d\alpha.\end{aligned}$$

Proof. Problem (P) is equivalent to Problem (EP) as noted in Remark 3.3, thus

(5.14)
$$\begin{aligned}\bar{u}(t) = -\mathfrak{R}(t)^{-1}\Big\{ &\big[E_3(t,0) + B_0^\top E_0(t) + D_0^\top E_0(t)C_0 + S_{00}(t)\big]\bar{X}(t) \\ &+ \int_{-\delta}^0 \big[E_4(t,0,\theta) + B_0^\top E_1(t,\theta)^\top + S_{01}(t,\theta) + D_0^\top E_0(t)C^0(\theta)\big]\bar{X}(t+\theta)d\theta \\ &+ \int_{-\delta}^0 \big[E_5(t,0,\theta) + B_0^\top E_3(t,\theta)^\top + R_{10}(t,\theta)^\top \\ &+ D_0^\top E_0(t)D^0(\theta)\big]\bar{u}(t+\theta)d\theta\Big\}, \text{ a.e. a.s.}\end{aligned}$$

Then, by (2.4), we only need to prove that

$$J(s,x,\varphi(\cdot),\psi(\cdot);\bar{u}(\cdot)) \leqslant J(s,x,\varphi(\cdot),\psi(\cdot);u(\cdot)) \quad \forall u(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m).$$

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Define

$$\begin{aligned}\Gamma(t) :=& \big\langle E_0(t)X(t), X(t)\big\rangle + 2\int_{-\delta}^{0}\big\langle X(t+\theta), E_1(t,\theta)X(t)\big\rangle d\theta\\ &+\int_{-\delta}^{0}\int_{-\delta}^{0}\big\langle E_2(t,\theta,\alpha)X(t+\alpha), X(t+\theta)\big\rangle d\theta d\alpha\\ &+2\int_{-\delta}^{0}\big\langle u(t+\theta), E_3(t,\theta)X(t)\big\rangle d\theta\\ &+2\int_{-\delta}^{0}\int_{-\delta}^{0}\big\langle u(t+\theta), E_4(t,\theta,\alpha)X(t+\alpha)\big\rangle d\alpha d\theta\\ &+\int_{-\delta}^{0}\int_{-\delta}^{0}\big\langle E_5(t,\theta,\alpha)u(t+\alpha), u(t+\theta)\big\rangle d\theta d\alpha.\end{aligned}$$

Then, by (5.7), (5.9)–(5.11) and applying the Itô formula, we obtain

$$\begin{aligned}&J(s,x,\varphi(\cdot),\psi(\cdot);u(\cdot))\\ &=\Gamma(s)+\mathbb{E}\int_s^T\Big\langle\mathfrak{R}(t)\Big(u(t)+\mathfrak{R}(t)^{-1}\Big\{\big[E_3(t,0)+B_0^\top E_0(t)+D_0^\top E_0(t)C_0+S_{00}(t)\big]X(t)\\ &\quad+\int_{-\delta}^{0}\big[E_4(t,0,\theta)+B_0^\top E_1(t,\theta)^\top+S_{01}(t,\theta)+D_0^\top E_0(t)C^0(\theta)\big]X(t+\theta)d\theta\\ &\quad+\int_{-\delta}^{0}\big[E_5(t,0,\theta)+B_0^\top E_3(t,\theta)^\top+R_{10}(t,\theta)^\top+D_0^\top E_0(t)D^0(\theta)\big]u(t+\theta)d\theta\Big\}\Big),\\ &\qquad u(t)+\mathfrak{R}(t)^{-1}\Big\{\big[E_3(t,0)+B_0^\top E_0(t)+D_0^\top E_0(t)C_0+S_{00}(t)\big]X(t)\\ &\quad+\int_{-\delta}^{0}\big[E_4(t,0,\theta)+B_0^\top E_1(t,\theta)^\top+S_{01}(t,\theta)+D_0^\top E_0(t)C^0(\theta)\big]X(t+\theta)d\theta\\ &\quad+\int_{-\delta}^{0}\big[E_5(t,0,\theta)+B_0^\top E_3(t,\theta)^\top+R_{10}(t,\theta)^\top+D_0^\top E_0(t)D^0(\theta)\big]u(t+\theta)d\theta\Big\}\Big\rangle dt,\end{aligned}$$

which completes the proof. ∎

Remark 5.4. Now we study the solvability of the coupled matrix-valued Riccati equations (5.9)–(5.11). Assume that $A_i, B_i = 0$, $i = 1,\dots,N-1$, and D_0, G_{00}, G_{10}, $G_{11} = 0$. Then, (5.9)–(5.11) admit unique solutions. Here we just provide a sketch of the proof, and we refer to [1] for full details of each step.

Step 1: Consider the integral forms of the coupled matrix-valued Riccati equations (5.9)–(5.11). Then, there exists $\tau > 0$ such that (5.9)–(5.11) admit unique solutions for $T-\tau \leqslant t \leqslant T$, $-\delta \leqslant \theta, \alpha \leqslant 0$. In fact, denote by M the upper bound of all coefficients of Problem (P), and for any given $l > 0$, define

$$\begin{aligned}\mathcal{B}(l) :=& \big\{(E_0(\cdot), E_1(\cdot,\cdot), E_2(\cdot,\cdot,\cdot), E_3(\cdot,\cdot), E_4(\cdot,\cdot,\cdot), E_5(\cdot,\cdot,\cdot)) \in C([T-\tau,T];\mathbb{S}^n)\\ &\times C([T-\tau,T]\times[-\delta,0];\mathbb{R}^{n\times n})\times C([T-\tau]\times[-\delta,0]^2;\mathbb{R}^{n\times n})\\ &\times C([T-\tau,T]\times[-\delta,0];\mathbb{R}^{m\times n})\\ &\times C([T-\tau,T]\times[-\delta,0]^2;\mathbb{R}^{m\times n})\times C([T-\tau,T]\times[-\delta,0]^2;\mathbb{R}^{m\times m});\\ &\sup_{\substack{t\in[T-\tau,T]\\ \theta,\alpha\in[-\delta,0]}}\big\{|E_0(t)|+|E_1(t,\theta)|+|E_2(t,\theta,\alpha)|\\ &+|E_3(t,\theta)|+|E_4(t,\theta,\alpha)|+|E_5(t,\theta,\alpha)|\big\}\leqslant l\big\}.\end{aligned}$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Consider the mapping $\mathscr{T} : \mathcal{B}(l) \longrightarrow \mathcal{B}(l)$, $(E_0(\cdot), E_1(\cdot,\cdot), E_2(\cdot,\cdot,\cdot), E_3(\cdot,\cdot), E_4(\cdot,\cdot,\cdot), E_5(\cdot,\cdot,\cdot)) \mapsto (\tilde{E}_0(\cdot), \tilde{E}_1(\cdot,\cdot), \tilde{E}_2(\cdot,\cdot,\cdot), \tilde{E}_3(\cdot,\cdot), \tilde{E}_4(\cdot,\cdot,\cdot), \tilde{E}_5(\cdot,\cdot,\cdot))$, where $\tilde{E}_0(\cdot), \tilde{E}_1(\cdot,\cdot)$ and $\tilde{E}_2(\cdot,\cdot,\cdot)$ satisfy the integral form of (5.9):

(5.15)

$$\begin{aligned}\tilde{E}_0(t) = \int_t^T \Big[& A_0^\top E_0(s) + E_0(s)A_0 + E_1(s,0) + E_1(s,0)^\top + C_0^\top E_0(s)C_0 + Q_{00}(s) \\ & - \big(E_3(s,0) + S_{00}(s) + B_0^\top E_0(s)\big)^\top R_{00}(s)^{-1}\big(E_3(s,0) + S_{00}(s) + B_0^\top E_0(s)\big)\Big] ds,\end{aligned}$$

$$\begin{aligned}\tilde{E}_1(t,\theta) = {} & A_N^\top \tilde{E}_0(t+\theta+\delta)\mathbf{1}_{[-\delta,T-t-\delta)}(\theta) + \int_t^{(t+\theta+\delta)\wedge T} \Big\{ - E_1(r,t+\theta-r)A_0 \\ & + E_2(r,t+\theta-r,0) + A^0(t+\theta-r)^\top E_0(r) + Q_{10}(r,t+\theta-r) \\ & + C^0(t+\theta-r)^\top E_0(r)C_0 - \big[E_4(r,0,t+\theta-r) + S_{01}(r,t+\theta-r) \\ & + B_0^\top E_1(r,t+\theta-r)^\top\big]^\top R_{00}(r)^{-1}\big[E_3(r,0) + S_{00}(r) + B_0^\top E_0(r)\big]\Big\} dr, \end{aligned} \tag{5.16}$$

and

(5.17)

$$\begin{aligned}\tilde{E}_2(t,\theta,\alpha) = {} & A_N^\top \tilde{E}_1(t+\theta+\delta,\alpha-\theta-\delta)^\top \mathbf{1}_{[-\delta,T-t-\delta)}(\theta) \\ & + \int_t^{(t+\theta+\delta)\wedge T} \Big\{ A^0(t+\theta-r)^\top E_1(r,t+\alpha-r)^\top + E_1(r,t+\theta-r)A^0(t+\alpha-r) \\ & + C^0(t+\theta-r)^\top E_0(r)C^0(t+\alpha-r) + Q_{11}(r,t+\alpha-r,t+\theta-r) \\ & - \big[E_4(r,0,t+\theta-r) + S_{01}(r,t+\theta-r) + B_0^\top E_1(r,t+\theta-r)^\top\big]^\top R_{00}(r)^{-1} \\ & \times \big[E_4(r,0,t+\alpha-r) + S_{01}(r,t+\alpha-r) + B_0^\top E_1(r,t+\alpha-r)^\top\big]\Big\} dr, \alpha \geqslant \theta,\end{aligned}$$

and for $\alpha < \theta$, $\tilde{E}_2(t,\theta,\alpha) = \tilde{E}_2(t,\alpha,\theta)^\top$. Notice that the forms of (5.10) and (5.11) are similar to (5.9). Then, the equations for $\tilde{E}_3(\cdot,\cdot), \tilde{E}_4(\cdot,\cdot,\cdot)$, and $\tilde{E}_5(\cdot,\cdot,\cdot)$ can be constructed similarly to (5.16) and (5.17). Hence there exists a $\tau > 0$ (depending only on M, l) such that $\mathscr{T}$ is a contraction mapping. By the fixed point theorem, the coupled matrix-valued Riccati equations (5.9)– (5.11) admit unique solutions.

Step 2: Let $(E_0(\cdot), E_1(\cdot,\cdot), E_2(\cdot,\cdot,\cdot), E_3(\cdot,\cdot), E_4(\cdot,\cdot,\cdot), E_5(\cdot,\cdot,\cdot))$ be the continuous solution to (5.9)–(5.11) for $T - \tau \leqslant t \leqslant T$ and $\theta, \alpha \in [-\delta, 0]$. Then, $E_0(\cdot), E_1(\cdot,\cdot)$, $E_2(\cdot,\cdot,\cdot), E_3(\cdot,\cdot)$, $E_4(\cdot,\cdot,\cdot), E_5(\cdot,\cdot,\cdot)$ satisfy Lipschitz conditions. In fact, choose $|h|$ small enough, denote

$$\begin{aligned}\mathcal{M}(t) := \sup_{\theta,\alpha\in[-\delta,0]} \Big\{ & |E_1(t,\theta) - E_1(t,\theta+h)| + |E_2(t,\theta,\alpha) - E_2(t,\theta+h,\alpha)| + |E_3(t,\theta) \\ & - E_3(t,\theta+h)| + |E_4(t,\theta,\alpha) - E_4(t,\theta+h,\alpha)| + |E_5(t,\theta,\alpha) - E_5(t,\theta+h,\alpha)| \\ & + |E_2(t,\theta,\alpha) - E_2(t,\theta,\alpha+h)| + |E_4(t,\theta,\alpha) - E_4(t,\theta,\alpha+h)| + |E_5(t,\theta,\alpha) \\ & - E_5(t,\theta,\alpha+h)|\Big\}.\end{aligned}$$

Then, similar to (5.15)–(5.17), there exists $M' > 0$ (depending only on M, τ) such that

$$\mathcal{M}(t) \leqslant M' \int_t^T \mathcal{M}(r)dr + O(h).$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Let $h \to 0$. Then, $E_0(\cdot), E_1(\cdot,\cdot), E_2(\cdot,\cdot,\cdot), E_3(\cdot,\cdot), E_4(\cdot,\cdot,\cdot), E_5(\cdot,\cdot,\cdot)$ satisfy Lipschitz conditions.

Step 3: Extend the solution from $[T-\tau, T]$ to $[s, T]$. Then, (5.9)–(5.11) admit unique solutions on $[s, T]$. For example, on $[T-\tau-\tilde{\tau}, T-\tau]$, we substitute l with $2l$ in Step 1, where $\tilde{\tau}$ is the new step size. Next, we show that $E_0(\cdot), E_1(\cdot,\cdot), E_2(\cdot,\cdot,\cdot), E_3(\cdot,\cdot)$, $E_4(\cdot,\cdot,\cdot), E_5(\cdot,\cdot,\cdot)$ satisfy Lipschitz conditions on $[T-\tau-\tilde{\tau}, T-\tau]$ in Step 2. Finally, we repeat Step 1 and Step 2 until we derive the solution on the whole interval $[s, T]$.

Remark 5.5. By the coupled matrix-valued Riccati equations (5.9)–(5.11), we obtain the closed-loop representation (5.14)—a new state feedback form. Let Problem (P) become the deterministic case, i.e., the diffusion term disappears in (5.7). Then, (5.9)–(5.11) are similar to (2.33)–(2.38) in [12]. Moreover, Theorem 5.3 is derived similarly, when the coefficients of the state equation (5.7) are time-variant.

6. Closed-loop solvability. In this section, we study a stochastic optimal control problem which involves only state delay not control delay. The general case is open, due to some technical reasons, up to now. By an equivalent transformed control problem, we define the closed-loop solvability for the original delayed control problem, and ensure it by the solvability of a differential operator-valued Riccati equation.

First we reformulate the optimal control problem as follows. Now the state equation (2.3) becomes the following SDDE:

$$\begin{cases} dX(t) = \left[\displaystyle\int_{[-\delta,0]} A(d\theta)X_t(\theta) + B_0 u(t)\right] dt \\ \quad + \left[\int_{[-\delta,0]} C(d\theta)X_t(\theta) + D_0 u(t)\right] dW(t), t \in [s,T], \\ X(s) = x, \quad X(t) = \varphi(t-s),\ t \in [s-\delta, s), \end{cases} \tag{6.1}$$

where $\int_{[-\delta,0]} A(d\theta)\tilde{\varphi}(\theta)$ and $\int_{[-\delta,0]} C(d\theta)\tilde{\varphi}(\theta)$ are defined by (2.1) and (2.2), for any $\tilde{\varphi} \in \mathfrak{L}$. The cost functional (2.4) becomes

(6.2)

$$\begin{aligned} &J(s,x,\varphi(\cdot);u(\cdot)) \\ &= \mathbb{E}\int_s^T \Big[\big\langle Q_{00}(t)X(t), X(t)\big\rangle + 2\int_{-\delta}^0 \big\langle Q_{10}(t,\theta)^\top X(t+\theta), X(t)\big\rangle d\theta \\ &\quad + \int_{[-\delta,0]^2} \big\langle Q_{11}(t,\theta,\theta')X(t+\theta), X(t+\theta')\big\rangle d\theta' d\theta + 2\big\langle S_{00}(t)X(t), u(t)\big\rangle \\ &\quad + 2\int_{-\delta}^0 \big\langle S_{01}(t,\theta)X(t+\theta), u(t)\big\rangle d\theta + \big\langle R_{00}(t)u(t), u(t)\big\rangle\Big] dt + \mathbb{E}\Big[\big\langle G_{00}X(T), X(T)\big\rangle \\ &\quad + 2\int_{-\delta}^0 \big\langle G_{10}(\theta)^\top X(T+\theta), X(T)\big\rangle d\theta \\ &\quad + \int_{[-\delta,0]^2} \big\langle G_{11}(\theta,\theta')X(T+\theta), X(T+\theta')\big\rangle d\theta' d\theta\Big]. \end{aligned}$$

We restate the control problem studied in this section as follows.

Problem ($\tilde{\mathbf{P}}$). For any $(s,x,\varphi) \in [0,T) \times \mathfrak{M}$, find a $\bar{u}(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ such that (6.1) is satisfied and

$$J(s,x,\varphi(\cdot);\bar{u}(\cdot)) = \inf_{u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)} J(s,x,\varphi(\cdot);u(\cdot)) := V(s,x,\varphi(\cdot)).$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

As in section 3, we transform the delayed state equation (6.1) in $\mathbb{R}^n$ into one in $\mathfrak{M}$ without delay. Now the transformed state equation (3.10) becomes

$$\begin{aligned}\mathbf{X}(t)=\Phi(t-s)\xi+\int_s^t\Phi(t-r)\tilde{B}u(r)dr\\+\int_s^t\Phi(t-r)\Big(\tilde{C}\mathbf{X}(r)+\tilde{D}u(r)\Big)dW(r),t\in[s,T],\end{aligned}\tag{6.3}$$

where $\xi:=\binom{x}{\varphi}$, $\Phi(\cdot),\tilde{C}$ are defined as (3.1) and (3.3), $\tilde{B},\tilde{D}$ are redefined as $\tilde{B}:\mathbb{R}^m\to\mathfrak{M},\ u\mapsto\binom{B_0u}{0}$, and $\tilde{D}:\mathbb{R}^m\to\mathfrak{M},\ u\mapsto\binom{D_0u}{0}$, for any $u\in\mathbb{R}^m$. The cost (3.11) becomes

$$\begin{aligned}J(s,\xi;u(\cdot))=J(s,x,\varphi(\cdot);u(\cdot))=\mathbb{E}\bigg\{\int_s^T\Big[\big\langle\tilde{Q}(t)\mathbf{X}(t),\mathbf{X}(t)\big\rangle_{\mathfrak{M}}\\+2\big\langle\tilde{S}_0(t)\mathbf{X}(t),u(t)\big\rangle+\big\langle\tilde{R}_{00}(t)u(t),u(t)\big\rangle\Big]dt+\big\langle\tilde{G}\mathbf{X}(T),\mathbf{X}(T)\big\rangle_{\mathfrak{M}}\bigg\}.\end{aligned}\tag{6.4}$$

Then we restate Problem (EP) and define the closed-loop solvability for Problem ($\tilde{\mathrm{P}}$).

Problem ($\widetilde{\mathbf{EP}}$). For any $(s,\xi)\in[0,T)\times\mathfrak{M}$, find a $\bar{u}(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ such that (6.3) is satisfied and

$$J(s,\xi;\bar{u}(\cdot))=\inf_{u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)}J(s,\xi;u(\cdot)):=V(s,\xi).$$

DEFINITION 6.1. *Any $K(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$ is called a* closed-loop strategy of Problem ($\tilde{\mathrm{P}}$) *on $[s,T]$. For any $K(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$ and $(x,\varphi)\in\mathfrak{M}$, let $\xi\equiv\binom{x}{\varphi}$, $\mathbf{X}(\cdot)\equiv\mathbf{X}(\cdot\ ;s,\xi,K(\cdot))$ be the solution to the following equation:*

$$\begin{aligned}\mathbf{X}(t)=\Phi(t-s)\xi+\int_s^t\Phi(t-r)\tilde{B}K(r)\mathbf{X}(r)dr\\+\int_s^t\Phi(t-r)\Big[\tilde{C}\mathbf{X}(r)+\tilde{D}K(r)\mathbf{X}(r)\Big]dW(r),\end{aligned}\tag{6.5}$$

and

$$u(t)=K(t)\mathbf{X}(t),\quad t\in[s,T].$$

Then, $(\mathbf{X}(\cdot),u(\cdot))$ is called the outcome pair *of $K(\cdot)$ on $[s,T]$ corresponding to the initial trajectory (x,φ); $\mathbf{X}(\cdot)$, $u(\cdot)$ are called the corresponding* closed-loop state *and* closed-loop outcome control*, respectively.*

DEFINITION 6.2. *A closed-loop strategy $\bar{K}(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$ is said to be* optimal *on $[s,T]$ if*

$$J\big(s,\xi;\bar{K}(\cdot)\bar{\mathbf{X}}(\cdot)\big)\le\ J\big(s,\xi;u(\cdot)\big)\quad\forall u(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m),\ \forall\xi=\begin{pmatrix}x\\\varphi\end{pmatrix}\in\mathfrak{M},$$

where $\bar{\mathbf{X}}(\cdot)$ is the closed-loop state corresponding to $(\bar{K}(\cdot),x,\varphi)$. If there (uniquely) exists an optimal closed-loop strategy on $[s,T]$, Problem ($\tilde{\mathrm{P}}$) *is said to be* (uniquely) closed-loop solvable on $[s,T]$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Introduce the following linear operator-valued equation:

$$\begin{cases} \dot{P}(t)+P(t)(\tilde{A}+\tilde{B}\bar{K}(t))+(\tilde{A}+\tilde{B}\bar{K}(t))^*P(t)+(\tilde{C}+\tilde{D}\bar{K}(t))^*P(t)(\tilde{C}+\tilde{D}\bar{K}(t)) \\ \quad +\tilde{Q}(t)+\bar{K}(t)^*\tilde{R}_{00}(t)\bar{K}(t)+\bar{K}(t)^*\tilde{S}_0(t)+\tilde{S}_0(t)^*\bar{K}(t)=0,\ t\in[s,T], \\ P(T)=\tilde{G}. \end{cases} \tag{6.6}$$

Then, we explore the necessary conditions of closed-loop solvability for Problem $(\tilde{\mathrm{P}})$.

THEOREM 6.3. *Let* (A1)–(A2) *hold. Suppose* $\bar{K}(\cdot)$ *is the optimal closed-loop strategy of* Problem $(\tilde{\mathrm{P}})$ *on* $[s,T]$. *Then,*

$$\begin{aligned} &\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\geq 0, \quad \text{a.e.}, \\ &\big[\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\big]\bar{K}(t)+\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{S}_0(t)=0, \quad \text{a.e.}, \end{aligned} \tag{6.7}$$

where $P(\cdot)$ *satisfies* (6.6).

Proof. For any $v(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$ and $t\in[s,T]$, consider the following SEE:

$$\begin{cases} dz(t)=\big[\tilde{A}z(t)+\tilde{B}\bar{K}(t)z(t)+\tilde{B}v(t)\big]dt+\big[\tilde{C}z(t)+\tilde{D}\bar{K}(t)z(t)+\tilde{D}v(t)\big]dW(t), \\ z(s)=\xi, \end{cases} \tag{6.8}$$

where $\tilde{A}$ is defined as (3.2). Then, applying the Itô formula to $\langle P(\cdot)z(\cdot),z(\cdot)\rangle$ (substituting $\tilde{A}$ with its Yosida approximation $\tilde{A}_\lambda$, and letting $\lambda\to\infty$), we obtain

$$\begin{aligned} &J(s,\xi;\bar{K}(\cdot)z(\cdot)+v(\cdot)) \\ &\quad=\mathbb{E}\langle P(s)\xi,\xi\rangle+\mathbb{E}\int_s^T\big[\big\langle\big(\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\big)v(t),v(t)\big\rangle \\ &\qquad+2\big\langle\big(\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{R}_{00}(t)\bar{K}(t)+\tilde{S}_0(t)+\tilde{D}^*P(t)\tilde{D}\bar{K}(t)\big)z(t),v(t)\big\rangle\big]dt. \end{aligned}$$

Since $\bar{K}(\cdot)$ is the optimal closed-loop strategy, we have

$$\begin{aligned} &\mathbb{E}\int_s^T\Big[2\Big\langle\Big(\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{R}_{00}(t)\bar{K}(t)+\tilde{S}_0(t)+\tilde{D}^*P(t)\tilde{D}\bar{K}(t)\Big)z(t),v(t)\Big\rangle \\ &\quad+\Big\langle\Big(\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\Big)v(t),v(t)\Big\rangle\Big]dt\geqslant 0\ \forall v(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m). \end{aligned} \tag{6.9}$$

In the following, we aim to prove that

$$\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\geqslant 0, \text{ a.e.} \tag{6.10}$$

Suppose there exists $\Omega_0\subseteq[s,T]$ and $|\Omega_0|>\frac{1}{l}$, for some $l>0$, such that $\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}<0$ on Ω_0. Without loss of generality, assume that there exists $\beta>0$ such that $\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\leqslant-\beta I$. Then, we can choose a sequence of Borel measurable sets $\{\Omega_k\}$ such that $\Omega_k\subseteq\Omega_0$ and $|\Omega_k|=\frac{1}{l+k}$. Let $\xi=0$, $v_k=(\sqrt{k},0,\dots,0)^\top I_{\Omega_k}(t)$, and $z_k(\cdot)$ be the corresponding solution to (6.8). Then, we have

$$\sup_{s\leqslant t\leqslant T}\mathbb{E}|z_k(t)|^2\leqslant M\mathbb{E}\int_s^T|v_k(t)|^2dt=\frac{k}{k+l}M\leqslant M;$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

here and after, M is a generic constant. By (6.9), we have

$$\begin{aligned}0 \leqslant \varlimsup_{k\to\infty} \mathbb{E}\int_s^T \big\langle \big(\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\big)v_k(t), v_k(t)\big\rangle dt + 2\varlimsup_{k\to\infty}\mathbb{E}\int_s^T \big\langle \big(\tilde{B}^*P(t)\\ + \tilde{D}^*P(t)\tilde{C} + \tilde{R}_{00}(t)\bar{K}(t) + \tilde{S}_0(t) + \tilde{D}^*P(t)\tilde{D}\bar{K}(t)\big)z_k(t), v_k(t)\big\rangle dt\\ \leqslant -\beta\frac{k}{k+l} + M\sqrt{\frac{k}{k+l}}\left(\int_{\Omega_k} ||\bar{K}(t)||^2_{\mathscr{L}(\mathfrak{M},\mathbb{R}^m)}dt\right)^{\frac{1}{2}} \to -\beta, \text{ as } k\to\infty,\end{aligned}$$

which is a contradiction! Thus, (6.10) holds. It remains to prove the second equality in (6.7). $\bar{K}(\cdot)$ is the optimal closed-loop strategy of Problem $(\tilde{\mathrm{P}})$ on $[s,T]$, thus it is also optimal on $[r,T]$ for any $r\in(s,T]$, then (6.9) holds for any $r\in(s,T]$. Choose $\xi\in\mathfrak{M}, v_j(t)=\frac{1}{j}v(t), v(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$, and let $z_j(\cdot)$ be the solution to the following SEE:

$$\begin{cases} dz_j(t) = \big[\tilde{A}z_j(t) + \tilde{B}\bar{K}(t)z_j(t) + \tilde{B}v_j(t)\big]dt\\ \quad + \big[\tilde{C}z_j(t) + \tilde{D}\bar{K}(t)z_j(t) + \tilde{D}v_j(t)\big]dW(t), t\in[r,T],\\ z_j(r) = \xi. \end{cases}$$

Then, by (6.9), for all $v(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$, we derive

$$\begin{aligned}\varlimsup_{j\to\infty}\mathbb{E}\int_r^T \big\langle \big(\tilde{B}^*P(t) + \tilde{D}^*P(t)\tilde{C} + \tilde{R}_{00}(t)\bar{K}(t) + \tilde{S}_0(t)\\ + \tilde{D}^*P(t)\tilde{D}\bar{K}(t)\big)z_j(t), v(t)\big\rangle dt \geqslant 0.\end{aligned} \tag{6.11}$$

Consider the following SEE:

$$\begin{cases} d\tilde{z}(t) = \Big(\tilde{A}\tilde{z}(t) + \tilde{B}\bar{K}(t)\tilde{z}(t)\Big)dt + \Big(\tilde{C}\tilde{z}(t) + \tilde{D}\bar{K}(t)\tilde{z}(t)\Big)dW(t), \quad t\in[r,T],\\ \tilde{z}(r) = \xi. \end{cases}$$

Then, we have

$$\sup_{r\leqslant t\leqslant T}\mathbb{E}|z_j(t)-\tilde{z}(t)|^2 \leqslant \mathbb{E}\int_r^T |v_j(t)|^2dt \to 0, \text{ as } j\to\infty,$$

which and (6.11) imply that

$$\mathbb{E}\int_r^T \Big\langle \Big(\tilde{B}^*P(t) + \tilde{D}^*P(t)\tilde{C} + \tilde{R}_{00}(t)\bar{K}(t) + \tilde{S}_0(t) + \tilde{D}^*P(t)\tilde{D}\bar{K}(t)\Big)\tilde{z}(t), v(t)\Big\rangle dt \geqslant 0$$

for any $v(\cdot)\in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$. Choose $v(t)=v\mathbf{1}_{[r,r+\varepsilon]}(t)$, $v\in\mathbb{R}^m$. Then, we deduce

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_r^{r+\varepsilon}\Big\langle \Big(\tilde{B}^*P(t) + \tilde{D}^*P(t)\tilde{C} + \tilde{R}_{00}(t)\bar{K}(t) + \tilde{S}_0(t) + \tilde{D}^*P(t)\tilde{D}\bar{K}(t)\Big)\xi, v\Big\rangle dt = 0$$

for any $\xi\in\mathfrak{M},\ v\in\mathbb{R}^m$. By the arbitrariness of ξ and v, the second equality of (6.7) holds. Hence we complete the proof. ∎

Next we give the sufficient conditions of the closed-loop solvability for Problem $(\tilde{\mathrm{P}})$.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

THEOREM 6.4. *Let* (A1)–(A2) *hold. Suppose* $\tilde{R}_{00}+\tilde{D}^*P\tilde{D}\geq 0$, $\mathcal{R}(\tilde{B}^*P+\tilde{D}^*P\tilde{C}+\tilde{S}_0)\subseteq\mathcal{R}(\tilde{R}_{00}+\tilde{D}^*P\tilde{D})$, *with* $P(\cdot)$ *satisfying the Riccati equation* (6.6). *Here*

$$\begin{aligned}\bar{K}(t)=&-\Big(\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\Big)^\dagger\Big[\tilde{B}^*P(t)+\tilde{S}_0(t)+\tilde{D}^*P(t)\tilde{C}\Big]\\&+\Big[I-\Big(\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\Big)^\dagger\Big(\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}\Big)\Big]\theta(t),\ \text{a.e.}\end{aligned}\tag{6.12}$$

for any $\theta(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$. *Suppose* $\bar{K}(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$. *Then, it is the optimal closed-loop strategy of* Problem $(\tilde{\mathrm{P}})$, *and the value function is as follows:*

$$V(s,\xi)=\big\langle P(s)\xi,\xi\big\rangle_{\mathfrak{M}}.\tag{6.13}$$

Proof. Denote $\mathcal{M}(t):=\tilde{R}_{00}(t)+\tilde{D}^*P(t)\tilde{D}$. Then, by (6.12) and $\mathcal{R}(\tilde{B}^*P+\tilde{D}^*P\tilde{C}+\tilde{S}_0)\subseteq\mathcal{R}(\mathcal{M})$, we derive

$$\mathcal{M}(t)\bar{K}(t)+\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{S}_0(t)=0,\quad\text{a.e.}\tag{6.14}$$

By (6.3) and applying the Itô formula to $\langle P(\cdot)\mathbf{X}(\cdot),\mathbf{X}(\cdot)\rangle$, we obtain

$$\begin{aligned}J(s,\xi;u(\cdot))=\mathbb{E}\Big\{&\langle P(s)\xi,\xi\rangle+\int_s^T\Big[\langle\mathcal{M}(t)u(t),u(t)\rangle-\langle\mathcal{M}(t)\bar{K}(t)\mathbf{X}(t),\bar{K}(t)\mathbf{X}(t)\rangle\\&-2\Big\langle\Big(\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{S}_0(t)\Big)\mathbf{X}(t),\bar{K}(t)\mathbf{X}(t)\Big\rangle\\&+2\Big\langle\Big(\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{S}_0(t)\Big)\mathbf{X}(t),u(t)\Big\rangle\Big]dt\Big\}.\end{aligned}$$

Noting (6.14), we have

$$\begin{aligned}&-\langle\mathcal{M}(t)\bar{K}(t)\mathbf{X}(t),\bar{K}(t)\mathbf{X}(t)\rangle-2\Big\langle\Big(\tilde{B}^*P(t)+\tilde{D}^*P(t)\tilde{C}+\tilde{S}_0(t)\Big)\mathbf{X}(t),\bar{K}(t)\mathbf{X}(t)\Big\rangle\\&=-\big\langle\,\mathcal{M}(t)\,\bar{K}(t)\,\mathbf{X}(t),\bar{K}(t)\,\mathbf{X}(t)\big\rangle+2\big\langle\mathcal{M}(t)\,\bar{K}(t)\,\mathbf{X}(t),\bar{K}(t)\,\mathbf{X}(t)\big\rangle\\&=\langle\mathcal{M}(t)\,\bar{K}(t)\,\mathbf{X}(t),\bar{K}(t)\,\mathbf{X}(t)\rangle,\end{aligned}$$

which yields

$$J(s,\xi;u(\cdot))=\mathbb{E}\left\{\langle P(s)\xi,\xi\rangle+\int_s^T\big\langle\mathcal{M}(t)\big(u(t)-\bar{K}(t)\mathbf{X}(t)\big),u(t)-\bar{K}(t)\mathbf{X}(t)\big\rangle dt\right\}.$$

Thus, we complete the proof. ∎

We summarize the above discussion and characterize the closed-loop solvability for Problem $(\tilde{\mathrm{P}})$.

THEOREM 6.5. *Let* (A1)–(A2) *hold. Then,* $\bar{K}(\cdot)$ *is the optimal closed-loop strategy of* Problem $(\tilde{\mathrm{P}})$ *on* $[s,T]$ *if and only if*

(i) $\bar{K}(\cdot)$ *is given by* (6.12), *where* $P(\cdot)$ *satisfies the differential operator-valued Riccati equation* (6.6),
(ii) $\tilde{R}_{00}+\tilde{D}^*P\tilde{D}\geq 0$, $\mathcal{R}(\tilde{B}^*P+\tilde{D}^*P\tilde{C}+\tilde{S}_0)\subseteq\mathcal{R}(\tilde{R}_{00}+\tilde{D}^*P\tilde{D})$,
(iii) $\bar{K}(\cdot)\in L^2(s,T;\mathscr{L}(\mathfrak{M},\mathbb{R}^m))$.

In this case, the value function is given by (6.13).

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Remark 6.6. In Theorem 6.5, we give some sufficient conditions for the solvability of the Riccati equation (6.6). Moreover, we overcome the difficulties of decoupling forward delayed state equations and backward advanced adjoint equations, by introducing the closed-loop strategy and the auxiliary equation (6.6). When $B_0, C_0, D_0, C^0(\theta)$ depend on t, Theorem 6.5 is derived similarly.

Inspired by (6.6), recall that $\hat{\delta}(\cdot)$ is the delta function, denote $\mathfrak{R}(t) := R_{00}(t) + D_0^\top \mathcal{E}_0(t) D_0$, and for almost everywhere $t \in [s,T]$, $\theta, \alpha \in [-\delta, 0]$, introduce the following coupled matrix-valued Riccati equation:

(6.15)
$$
\begin{cases}
\dot{\mathcal{E}}_0(t) + A_0^\top \mathcal{E}_0(t) + \mathcal{E}_0(t) A_0 + \mathcal{E}_1(t,0) + \mathcal{E}_1(t,0)^\top + C_0^\top \mathcal{E}_0(t) C_0 + Q_{00}(t) \\
\quad - \Big[S_{00}(t) + B_0^\top \mathcal{E}_0(t) + D_0^\top \mathcal{E}_0(t) C_0 \Big]^\top \mathfrak{R}(t)^\dagger \Big[S_{00}(t) + B_0^\top \mathcal{E}_0(t) + D_0^\top \mathcal{E}_0(t) C_0 \Big] = 0, \\
\Big(\dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial \theta} \Big) \mathcal{E}_1(t,\theta) + A_0^\top \mathcal{E}_1(t,\theta) + \mathcal{E}_2(t,0,\theta) + \mathcal{E}_0(t) \Big[\sum_{i=1}^{N-1} A_i \hat{\delta}(\theta - \theta_i) + A^0(\theta) \Big] \\
\quad + Q_{10}(t,\theta)^\top + C_0^\top \mathcal{E}_0(t) C^0(\theta) - \Big[S_{00}(t) + B_0^\top \mathcal{E}_0(t) + D_0^\top \mathcal{E}_0(t) C_0 \Big]^\top \mathfrak{R}(t)^\dagger \\
\quad \times \Big[S_{01}(t,\theta) + B_0^\top \mathcal{E}_1(t,\theta) + D_0^\top \mathcal{E}_0(t) C^0(\theta) \Big] = 0, \\
\Big(\dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial \theta} - \dfrac{\partial}{\partial \alpha} \Big) \mathcal{E}_2(t,\theta,\alpha) + \Big[\sum_{i=1}^{N-1} A_i \hat{\delta}(\theta - \theta_i) + A^0(\theta) \Big]^\top \mathcal{E}_1(t,\alpha) + \mathcal{E}_1(t,\theta)^\top \Big[A^0(\alpha) \\
\quad + \sum_{i=1}^{N-1} A_i \hat{\delta}(\alpha - \theta_i) \Big] + C^0(\theta)^\top \mathcal{E}_0(t) C^0(\alpha) + Q_{11}(t,\alpha,\theta) - \Big[S_{01}(t,\theta) + B_0^\top \mathcal{E}_1(t,\theta) \\
\quad + D_0^\top \mathcal{E}_0(t) C^0(\theta) \Big]^\top \mathfrak{R}(t)^\dagger \Big[S_{01}(t,\alpha) + B_0^\top \mathcal{E}_1(t,\alpha) + D_0^\top \mathcal{E}_0(t) C^0(\alpha) \Big] = 0, \\
\mathcal{E}_0(T) = G_{00}, \quad \mathcal{E}_1(T,\theta) = G_{10}(\theta)^\top, \quad \mathcal{E}_1(t,-\delta) = \mathcal{E}_0(t) A_N, \\
\mathcal{E}_2(T,\theta,\alpha) = G_{11}(\alpha,\theta), \mathcal{E}_2(t,-\delta,\alpha) = A_N^\top \mathcal{E}_1(t,\alpha), \mathcal{E}_2(t,\theta,-\delta) = \mathcal{E}_1(t,\theta)^\top A_N.
\end{cases}
$$

Then, we go back to the original delayed control problem ($\tilde{\mathrm{P}}$) and give a clear characterization of its closed-loop solvability.

THEOREM 6.7. *Suppose all coefficients of Problem* ($\tilde{\mathrm{P}}$) *are continuous and* $\mathfrak{R} \geq 0$. *Let* $\mathcal{E}_0(t)$, $\mathcal{E}_1(t,\theta)$, $\mathcal{E}_2(t,\theta,\alpha)$, $t \in [s,T]$, $\theta, \alpha \in [-\delta, 0]$, *be continuous functions satisfying* (6.15), *and* $\mathcal{E}_0(t) = \mathcal{E}_0(t)^\top$, $\mathcal{E}_2(t,\theta,\alpha) = \mathcal{E}_2(t,\alpha,\theta)^\top$. *Moreover,*

$$
\begin{aligned}
&\big(B_0^\top \mathcal{E}_0(t) + S_{00}(t) + D_0^\top \mathcal{E}_0(t) C_0 \big) x + \int_{-\delta}^0 \Big(B_0^\top \mathcal{E}_1(t,\theta) \\
&\qquad + D_0^\top \mathcal{E}_0(t) C^0(\theta) + S_{01}(t,\theta) \Big) \varphi(\theta) d\theta \in \mathcal{R}(\mathfrak{R}(t)) \forall x \in \mathbb{R}^n, \varphi \in \mathfrak{L}.
\end{aligned} \tag{6.16}
$$

Let $\bar{K}(\cdot) \in L^2(s,T; \mathscr{L}(\mathfrak{M}, \mathbb{R}^m))$ *be given by*

$$
\begin{aligned}
\bar{K}(t)\xi = -\mathfrak{R}(t)^\dagger \Big[&\Big(B_0^\top \mathcal{E}_0(t) + S_{00}(t) + D_0^\top \mathcal{E}_0(t) C_0 \Big) x \\
&+ \int_{-\delta}^0 \Big(B_0^\top \mathcal{E}_1(t,\theta) + D_0^\top \mathcal{E}_0(t) C^0(\theta) + S_{01}(t,\theta) \Big) \varphi(\theta) d\theta \Big] \\
&+ \big[I - \mathfrak{R}(t)^\dagger \mathfrak{R}(t) \big] \theta(t) \xi, \ \theta(\cdot) \in L^2(s,T; \mathscr{L}(\mathfrak{M}, \mathbb{R}^m)) \forall \xi = \begin{pmatrix} x \\ \varphi \end{pmatrix}.
\end{aligned} \tag{6.17}
$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Then, $\bar{K}(\cdot)$ is the optimal closed-loop strategy for Problem ($\tilde{\mathrm{P}}$)*, and the value function is as follows:*

$$V(s,x,\varphi(\cdot)) = \big\langle \mathcal{E}_0(s)x, x\big\rangle + 2\int_{-\delta}^{0} \big\langle \mathcal{E}_1(s,\theta)\varphi(\theta), x\big\rangle d\theta + \int_{[-\delta,0]^2} \big\langle \mathcal{E}_2(s,\theta,\alpha)\varphi(\alpha), \varphi(\theta)\big\rangle d\alpha d\theta.$$

Proof. For any $u(\cdot) \in L^2_{\mathbb{F}}(s,T;\mathbb{R}^m)$, let $X(\cdot)$ be the state satisfying (6.1). Define

$$\Gamma(t) := \langle \mathcal{E}_0(t)X(t), X(t)\rangle + 2\int_{-\delta}^{0} \langle \mathcal{E}_1(t,\theta)X(t+\theta), X(t)\rangle d\theta + \int_{[-\delta,0]^2} \langle \mathcal{E}_2(t,\theta,\alpha)X(t+\alpha), X(t+\theta)\rangle d\alpha d\theta.$$

Then, by (6.15)–(6.17), with some computations we derive

$$\begin{aligned} &d\Gamma(t) + \big\langle Q_{00}(t)X(t)\,, X(t)\big\rangle + 2\int_{-\delta}^{0} \big\langle\, Q_{10}(t,\theta)^\top X(t+\theta), X(t)\,\big\rangle d\theta \\ &\quad + \int_{[-\delta,0]^2} \big\langle Q_{11}(t,\theta,\theta')X(t+\theta), \\ &X(t+\theta')\big\rangle d\theta' d\theta + 2\big\langle S_{00}(t)X(t)\,, u(t)\big\rangle + 2\int_{-\delta}^{0} \big\langle S_{01}(t,\theta)X(t+\theta)\,, u(t)\big\rangle d\theta \\ &\quad + \big\langle R_{00}(t)u(t)\,, u(t)\big\rangle \\ &= \Big\langle \mathfrak{R}(t)\Big[u(t) - \bar{K}(t)\begin{pmatrix} X(t) \\ X_t \end{pmatrix}\Big], u(t) - \bar{K}(t)\begin{pmatrix} X(t) \\ X_t \end{pmatrix}\Big\rangle, \quad \text{a.e. } t\in[s,T]. \end{aligned}$$

Integrating both sides of which from s to T, we complete the proof. ∎

COROLLARY 6.8. *Suppose all coefficients of Problem* ($\tilde{\mathrm{P}}$) *are continuous. Let $\mathcal{E}_0(t)$, $\mathcal{E}_1(t,\theta)$, $\mathcal{E}_2(t,\theta,\alpha)$, $t\in[s,T]$, $\theta,\alpha\in[-\delta,0]$, be continuous functions satisfying the coupled matrix-valued Riccati equation* (6.15)*, and $\mathfrak{R}(t) = R_{00}(t) + D_0^\top \mathcal{E}_0(t)D_0 > 0$. Let continuous functions $E_0(t)$, $E_1(t,\theta)$, $E_2(t,\theta,\alpha)$, $E_3(t,\theta)$, $E_4(t,\theta,\alpha), E_5(t,\theta,\alpha)$, $t\in[s,T]$, $\theta,\alpha\in[-\delta,0]$, satisfy the coupled matrix-valued Riccati equations* (5.9)–(5.11)*. Then, $\mathcal{E}_0(t) = E_0(t), \mathcal{E}_1(t,\theta) = E_1(t,\theta)^\top, \mathcal{E}_2(t,\theta,\alpha) = E_2(t,\theta,\alpha), E_3(t,\theta,\alpha), E_4(t,\theta,\alpha), E_5(t,\theta,\alpha) = 0$, and the closed-loop outcome control of Problem* ($\tilde{\mathrm{P}}$) *is as follows:*

$$\bar{u}(t) = \bar{K}(t)\bar{\mathbf{X}}(t), \tag{6.18}$$

where $\bar{K}(\cdot)$ is defined by (6.17) *and $\bar{\mathbf{X}}(\cdot)$ is the solution to* (6.5)*. In this case,* (6.18) *is the same as the closed-loop representation of the open-loop optimal control* (5.13)*.*

Remark 6.9. Similar to Remark 5.4, let $A_i, D_0, G_{00}, G_{10}, G_{11} = 0, i = 1,\dots,N-1$. Then, (6.15) admits a unique solution. Theorem 6.5 ensures the closed-loop solvability for Problem ($\tilde{\mathrm{P}}$) by the solvability of the differential operator-valued Riccati equation (6.6). Furthermore, by the coupled matrix-valued Riccati equation (6.15),

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Theorem 6.7 explicitly represents the optimal closed-loop strategy $\bar{K}(\cdot)$ using the coefficients of the original delayed control systems. When delay disappears in Problem $(\tilde{\mathrm{P}})$, Theorem 6.7 is similar to the sufficient part of Theorem 2.4.3 in [31]. When the coefficients of the state equation (6.1) are time-variant, Theorem 6.7 also holds.

7. Concluding remarks. This paper studies the linear quadratic optimal control problem for a delayed stochastic system with both state delay and control delay in the diffusion term. We transform it into an infinite dimensional problem without delay, ensuring the open-loop solvability through a constrained forward-backward stochastic evolution system and a convexity condition. We also provide a closed-loop representation using a coupled matrix-valued Riccati equation and ensure the closed-loop solvability via a differential operator-valued Riccati equation, ultimately clarifying the original delayed optimal control problem.

Acknowledgments. The authors thank the editor and two anonymous reviewers for their constructive and insightful comments for improving the quality of this work. The authors also acknowledge Professor Jiongmin Yong at the University of Central Florida for his valuable suggestions and helpful discussions.

REFERENCES

[1] Y. Alekal, P. Brunovsky, D. Chyung, and E. Lee, *The quadratic problem for systems with time delays*, IEEE Trans. Automat. Control, 16 (1971), pp. 673–687, https://doi.org/10.1109/TAC.1971.1099824.

[2] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter, *Representation and Control of Infinite Dimensional Systems*, 2nd ed., Birkhäuser, Boston, 2007.

[3] R. Carmona, J.-P. Fouque, S. M. Mousavi, and L.-H. Sun, *Systemic risk and stochastic games with delay*, J. Optim. Theory Appl., 179 (2018), pp. 366–399, https://doi.org/10.1007/s10957-018-1267-8.

[4] L. Chen and Z. Wu, *Maximum principle for the stochastic optimal control problem with delay and application*, Automatica, 46 (2010), pp. 1074–1080, https://doi.org/10.1016/j.automatica.2010.03.005.

[5] L. Chen and Z. Wu, *Stochastic optimal control problem in advertising model with delay*, J. Syst. Sci. Complex, 33 (2020), pp. 968–987, https://doi.org/10.1007/s11424-020-8185-1.

[6] M. C. Delfour, *State theory of linear hereditary differential systems*, J. Math. Anal. Appl., 60 (1977), pp. 8–35, https://doi.org/10.1016/0022-247X(77)90044-0.

[7] S. Federico, *A stochastic control problem with delay arising in a pension fund model*, Finance Stoch., 15 (2011), pp. 421–459, https://doi.org/10.1007/s00780-010-0146-4.

[8] F. Flandoli, *Solution and control of a bilinear stochastic delay equation*, SIAM J. Control Optim., 28 (1990), pp. 936–949, https://doi.org/10.1137/0328052.

[9] F. Gozzi and C. Marinelli, *Stochastic optimal control of delay equations arising in advertising models*, in Stochastic Partial Differential Equations and Applications-VII, Lecture Notes in Pure and Appl. Math. 245, G. Da Prato and L. Tubaro, eds., Chapman & Hall/CRC, London, 2006, pp. 133–148.

[10] Y. Hu and S. Peng, *Maximum principle for semilinear stochastic evolution control systems*, Stoch. Stoch. Rep., 33 (1990), pp. 159–180, https://doi.org/10.1080/17442509008833671.

[11] J. Huang, X. Li, and J. Shi, *Forward-backward linear quadratic stochastic optimal control problem with delay*, Systems Control Lett., 61 (2012), pp. 623–630, https://doi.org/10.1016/j.sysconle.2012.02.010.

[12] A. Ichikawa, *Quadratic control of evolution equations with delays in control*, SIAM J. Control Optim., 20 (1982), pp. 645–668, https://doi.org/10.1137/0320048.

[13] V. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*, Springer Dordrecht, 1992.

[14] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, *Control of Systems with Aftereffect*, AMS, Providence, RI, 1996.

[15] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories* I*: Abstract Parabolic Systems*, Cambridge University Press, Cambridge, UK, 2000.

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.

[16] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories* II*: Abstract Hyperbolic-Like Systems over a Finite Time Horizon*, Cambridge University Press, Cambridge, UK, 2000.
[17] N. Li, G. Wang, and Z. Wu, *Linear-quadratic optimal control for time-delay stochastic system with recursive utility under full and partial information*, Automatica, 121 (2020), 109169, https://doi.org/10.1016/j.automatica.2020.109169.
[18] X. Li, J. Shi, and J. Yong, *Mean-field linear-quadratic stochastic differential games in an infinite horizon*, ESAIM Control Optim. Calc. Var., 27 (2021), 81, https://doi.org/10.1051/cocv/2021078.
[19] X. Liang, J. Xu, and H. Zhang, *Solution to stochastic LQ control problem for Itô systems with state delay or input delay*, Systems Control Lett., 113 (2018), pp. 86–92, https://doi.org/10.1016/j.sysconle.2018.01.012.
[20] Q. Lü, *Well-posedness of stochastic Riccati equations and closed-loop solvability for stochastic linear quadratic control problems*, J. Differential Equations, 267 (2019), pp. 180–227, https://doi.org/10.1016/j.jde.2019.01.008.
[21] Q. Lü, *Stochastic linear quadratic optimal control problems for mean-field stochastic evolution equations*, ESAIM Control Optim. Calc. Var., 26 (2020), 127, https://doi.org/10.1051/cocv/2020081.
[22] Q. Lü and X. Zhang, *Mathematical Control Theory for Stochastic Partial Differential Equations*, Springer, Cham, 2021.
[23] T. Ma, J. Xu, and H. Zhang, *Explicit solution to forward and backward stochastic differential equations with state delay and its application to optimal control*, Control Theory Technol., 20 (2022), pp. 303–315, https://doi.org/10.1007/s11768-022-00106-x.
[24] N. MacDonald, *Time Lags in Biological Models*, Lecture Notes in Biomath. 27, Springer-Verlag, Berlin, 1978.
[25] S. E. A. Mohammed, *Stochastic Functional Differential Equations*, Pitman, Boston, 1984.
[26] S. E. A. Mohammed, *Stochastic differential equations with memory: Theory, examples and applications*, in Stochastic Analysis and Related Topics 6, Progr. Probab. 42, L. Decreusefond, J. Gjerde, B. Øksendal, and A. S. Üstünel, eds., Birkhäuser, Boston, 1998, pp. 1–77.
[27] Y.-H. Ni, K.-F. C. Yiu, H. Zhang, and J.-F. Zhang, *Delayed optimal control of stochastic LQ problem*, SIAM J. Control Optim., 55 (2017), pp. 3370–3407, https://doi.org/10.1137/16M1100897.
[28] J. Sun, X. Li, and J. Yong, *Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems*, SIAM J. Control Optim., 54 (2016), pp. 2274–2308, https://doi.org/10.1137/15M103532X.
[29] J. Sun and J. Yong, *Linear quadratic stochastic differential games: Open-loop and closed-loop saddle points*, SIAM J. Control Optim., 52 (2014), pp. 4082–4121, https://doi.org/10.1137/140953642.
[30] J. Sun and J. Yong, *Stochastic linear quadratic optimal control problems in infinite horizon*, Appl. Math. Optim., 78 (2018), pp. 145–183, https://doi.org/10.1007/s00245-017-9402-8.
[31] J. Sun and J. Yong, *Stochastic Linear-Quadratic Optimal Control Theory: Open-Loop and Closed-Loop Solutions*, Springer Briefs Math., Springer, Cham, 2020.
[32] R. B. Vinter and R. H. Kwong, *The infinite time quadratic control problem for linear systems with state and control delays: An evolution equation approach*, SIAM J. Control Optim., 19 (1981), pp. 139–153, https://doi.org/10.1137/0319011.
[33] H. Wang and H. Zhang, *LQ control for Itô-type stochastic systems with input delays*, Automatica, 49 (2013), pp. 3538–3549, https://doi.org/10.1016/j.automatica.2013.09.018.
[34] J. Xu, J. Shi, and H. Zhang, *A leader-follower stochastic linear quadratic differential game with time delay*, Sci. China Inf. Sci., 61 (2018), 112202, https://doi.org/10.1007/s11432-017-9293-4.
[35] F. Zhang, *Stochastic maximum principle for optimal control problems involving delayed systems*, Sci. China Inf. Sci., 64 (2021), 119206, https://doi.org/10.1007/s11432-019-2826-3.
[36] H. Zhang and J. Xu, *Control for Itô stochastic systems with input delay*, IEEE Trans. Automat. Control, 62 (2017), pp. 350–365, https://doi.org/10.1109/TAC.2016.2551371.

Downloaded 05/13/25 to 219.230.100.50 . Redistribution subject to SIAM license or copyright; see https://epubs.siam.org/terms-privacy

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.