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A GENERAL MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL OF STOCHASTIC DIFFERENTIAL DELAY SYSTEMS*

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Abstract. In this paper, we solve an open problem and obtain a general maximum principle for a stochastic optimal control problem where the control domain is an arbitrary nonempty set and all the coefficients (especially the diffusion term and the terminal cost) contain the control and state delay. In order to overcome the difficulty of dealing with the cross term of state and its delay in the variational inequality, we propose a new method: transform a delayed variational equation into a Volterra integral equation without delay inspired by [Y. Hamaguchi, *Appl. Math. Optim.,* 87 (2023), 42], and introduce novel first-order, second-order adjoint equations via the backward stochastic Volterra integral equation theory established in [T. Wang and J. Yong, *SIAM J. Control Optim.,* 61 (2023), pp. 3608–3634]. Finally we express these two kinds of adjoint equations in more compact anticipated backward stochastic differential equation types for several special yet typical control systems.

Key words. stochastic differential delay systems, general maximum principle, backward stochastic Volterra integral equations, second-order adjoint equations, nonconvex control domain

MSC codes. 93E20, 60H20, 34K50

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1. Introduction.

The study of optimal control problems has been a hot topic for decades, and the maximum principle has been one of the main approaches to address the control problems. In 1965, Kushner (see [12]) first studied the maximum principle for the stochastic optimal control problem, where the diffusion term does not contain state and control. Since then, extensive literature has emerged to study the stochastic optimal control problems. However, either the control domain must be convex, or the diffusion term does not contain the control. In 1990, Peng (see [24]) completely solved the stochastic optimal control problem and obtained the general maximum principle, by means of *backward stochastic differential equations* (BSDEs) as adjoint equations. On the other hand, in the real world, the memory effect always exists. The increment of the control system not only depends on the current state, but also depends on the past state. Also when the controller decides to exert control, it takes some time to exercise the action. Therefore, it has profound theoretical importance and extensive application value on studying the control problems for systems with both state delay

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and control delay. Usually *stochastic differential delay equations* (SDDEs) are used to describe these delayed control systems. More details about SDDEs can be referred to in [13, 16, 19, 20].

Given a time duration $[0,T]$, for a nonempty set $U \subset \mathbb{R}^m$, not necessarily convex, a constant time delay parameter $\delta \in (0,T)$, and a constant $\lambda \in \mathbb{R}$, in this paper we consider a system of the following form:

(1.1)
$$\begin{cases} dx(t) = b\Big(t, x(t), x(t-\delta), \displaystyle\int_{-\delta}^{0} e^{\lambda\theta} x(t+\theta)d\theta, u(t), u(t-\delta)\Big)dt \\ \qquad\quad + \sigma\Big(t, x(t), x(t-\delta), \displaystyle\int_{-\delta}^{0} e^{\lambda\theta} x(t+\theta)d\theta, u(t), u(t-\delta)\Big)dW(t), t \in [0,T], \\ x(t) = \xi(t),\ u(t) = \gamma(t),\ t \in [-\delta, 0], \end{cases}$$

where $x(\cdot) \in \mathbb{R}^n$ is state and $u(\cdot) \in U$ is control. Suppose that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space and the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is generated by a d-dimensional standard Brownian motion $\{W(t)\}_{t\geq 0}$. b, σ are given random coefficients with proper dimensions. Deterministic continuous function $\xi(\cdot)$ and square integrable function $\gamma(\cdot)$ are the initial trajectories of the state and the control, respectively. We associate (1.1) with the following cost functional,

$$\begin{aligned} J(u(\cdot)) = \mathbb{E}\Big[&\int_0^T l\Big(t, x(t), x(t-\delta), \int_{-\delta}^{0} e^{\lambda\theta} x(t+\theta)d\theta, u(t), u(t-\delta)\Big)dt \\ &+ h\Big(x(T), x(T-\delta), \int_{-\delta}^{0} e^{\lambda\theta} X(T+\theta)d\theta\Big)\Big], \end{aligned} \tag{1.2}$$

where l, h are given random coefficients with proper dimensions. Define the admissible control set as follows:

$$\begin{aligned} \mathcal{U}_{ad} := \Big\{ u(\cdot) : [-\delta, T] \to \mathbb{R}^m \big| u(\cdot) \text{ is a } U\text{-valued, square-integrable, } \mathbb{F}\text{-adapted} \\ \text{process and } u(t) = \gamma(t),\ t \in [-\delta, 0] \Big\}. \end{aligned}$$

We state the optimal control problem as follows,

Problem (P)*:* Our object is to find a control $u^*(\cdot)$ over $\mathcal{U}_{ad}$ such that (1.1) is satisfied and (1.2) is minimized, i.e.,

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}_{ad}} J(u(\cdot)).$$

Any $u^*(\cdot) \in \mathcal{U}_{ad}$ that achieves the above infimum is called an *optimal control* and the corresponding solution $x^*(\cdot)$ is called the *optimal trajectory.* $(x^*(\cdot), u^*(\cdot))$ is called an *optimal pair.* Optimal control problems of stochastic differential delay systems are widely used in economics, engineering, and medicine (see [2, 17, 26, 33]), and thus have attracted more and more scholars' attention. Take an optimal consumption problem as an example, at time t let $x(t)$, $u(t)$ be the wealth, the consumption amount, respectively. It is reasonable to suppose that the wealth increment is a combination of the present value $x(t)$ plus some sliding average of previous value $\int_{-\delta}^{0} e^{\lambda\theta} x(t+\theta)d\theta$ and negative consumption amount $u(t)$. Therefore, the wealth equation satisfied by $x(\cdot)$ has the form of (1.1). The consumer always wants to find an optimal consumption strategy $u^*(\cdot)$ to maximize his terminal wealth $\mathbb{E}[X(T)]$ and consumption satisfaction

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$\mathbb{E}\int_0^T \frac{u^\gamma(t)}{\gamma}dt$, where $\gamma \in (0,1)$, $1-\gamma$ is the relative risk aversion of the consumer. Thus, the cost functional (1.2) can be chosen as $\mathbb{E}\big[-X(T)-\int_0^T \frac{u^\gamma(t)}{\gamma}dt\big]$. With different levels of consumption packages for consumers to select, the value set U of the consumption amount $u(t)$ should be limited and not necessarily convex. This typical consumption problem is a case of Problem (P), which motivates us to study the maximum principle for Problem (P).

So far, there has been extensive literature studying optimal control problems of stochastic differential delay systems. Øksendal and Sulem in [22] studied the sufficient maximum principle for the stochastic optimal control problem with convex control domain, and required the solution of a certain adjoint equation to be zero due to the lack of an Itô formula to deal with pointwise state delay terms. Chen and Wu in [1] introduced a class of anticipated BSDEs as the adjoint equations and obtained the maximum principle. Although [1] removed the "zero-solution" condition in [22], the control domain is still convex. Recently, Meng and Shi in [18] addressed the stochastic optimal control problem, allowed the control domain to be nonconvex, and gave the general maximum principle. However, the solution of some second-order adjoint equations must be zero, since at that moment there is no proper method to eliminate the cross terms of states and their delay terms. More related literature can be referred to [3, 7, 10, 21, 32, 36, 37, 38, 39].

In this paper, we consider the stochastic optimal control problem associated with (1.1), (1.2), and derive the general maximum principle with arbitrary nonempty control domain U. Differently from all the aforementioned literature, we study the optimal control problem from a new viewpoint of forward stochastic Volterra integral systems and develop some effective techniques. More precisely, inspired by [8], we first properly transform the delayed first-order variational equation into a linear forward *stochastic Volterra integral equation* (SVIE) without delay. Then, we combine it with the original first-order variational equation, lift them up, and end up with a higher-dimensional linear forward SVIE. Eventually, we adapt the arguments developed by Wang and Yong (see [30]) for optimal control problems of forward stochastic Volterra integral systems into our framework and derive the main results accordingly.

Forward Volterra integral systems were introduced by Italian mathematician Volterra (see [28]). So far there have been extensive literature on the optimal control problems of forward Volterra integral systems. However, there is very little work studying the optimal control of forward stochastic Volterra integral systems. One possible reason is that until 2002 the theory of Type-I backward SVIEs was established by Lin (see [15]). Then, in 2006 Yong (see [34]) proposed Type-II backward SVIEs and first derived the maximum principle for optimal control problems of forward stochastic Volterra integral systems with a convex control domain. Recently, Wang and Yong in [30] introduced an auxiliary process and obtained the general maximum principle, where the control domain is allowed to be nonconvex. More references can be found in [29, 31].

As far as we know, a number of papers transform the delayed control problem into a control problem of Volterra integral systems. For example, in [9], they used a proper variation of constants formula to transform equivalently the delayed quadratic optimal control problem into that of a linear Volterra integral system. Similar ideas also happened in [14] in an infinite-dimensional setting. On the other hand, there are also other methods to transform the delayed system to another system (see [4, 5, 6, 11, 27]). Among them, the delayed finite-dimensional problem was lifted up to

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an infinite-dimensional problem without delay. A limitation of such a method lies in the high regularity assumption (such as continuity and differentiability) for the coefficients when going back to the original problem. Notice that our transformations in the current paper are essentially different from the above. In addition, by our arguments on (1.1), there is no need to introduce infinite-dimensional analysis.

The innovations and contributions of this paper are as follows:

(i) The control system is very general. The control domain is not required to be convex, pointwise and distributed state delay appear not only in the state equation and the running cost, but also in the terminal cost, and pointwise control delay appears in the diffusion term and the running cost. Thus, our model can cover most control systems in the existing literature, such as [22, 1, 39, 18].

(ii) A general maximum principle is obtained. It is simple and concise, consisting of two parts: one describes the maximum condition with delay, and the other describes the maximum condition without delay. In contrast with [18], the strict zero-solution condition imposed on the adjoint equation is successfully removed.

(iii) A new method is proposed to treat cross terms of type "$x_1(t)^\top[\cdots]y_1(t)$" in the variational inequality. How to deal with these cross terms is a key yet difficult problem in obtaining the general maximum principle. Inspired by [8], we transform delayed variational equations into SVIEs without delay, and then solve this hard issue by the theory of forward, backward stochastic Volterra integral systems developed in [30].

(iv) Novel adjoint equations are introduced. The first-order adjoint equations consist of a simple BSDE and a backward SVIE, while the second-order adjoint equations consist of a simple BSDE and three coupled backward SVIEs. Then the maximum condition is first established by their solutions, even if the control domain is nonconvex and pointwise state delay appears in both the state equation and the terminal cost.

(v) The adjoint equations are expressed in more compact forms. The first-order adjoint equation is written as a set of anticipated BSDEs. The second-order adjoint equation reduces to the classical scenario when our delay system reduces to a stochastic differential system.

The rest of this paper is organized as follows. In section 2, some basic results are displayed. In section 3, the delayed variational equations are transformed into Volterra integral equations without delay, and then the adjoint equations are introduced in section 4. In section 5, the maximum principle is stated and some careful analysis on the adjoint equations are spread out. Finally, section 6 gives the concluding remarks.

2. Preliminaries. For any $A, B \in \mathbb{R}^{m\times d}$, we define by $\langle A, B\rangle = Tr[AB^\top]$ the inner product in $\mathbb{R}^{m\times d}$ with norm $|\cdot|$, and by $\mathbb{S}^n$ the set of all $n\times n$ symmetric matrices. Let $\mathbb{E}_t[\,\cdot\,] \equiv \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$ be the conditional expectation with respect to $\mathcal{F}_t$, $t \in [0,T]$, and I is the identity matrix of proper dimensions. For $t \in [0,T]$, denote by $L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ the Hilbert space consisting of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable ξ such that $\mathbb{E}|\xi|^2 < \infty$, by $L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ the Hilbert space consisting of $\mathbb{F}$-adapted process $\phi(\cdot)$ such that $\mathbb{E}\int_0^T |\phi(t)|^2 dt < \infty$, by $L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n))$ the Banach space consisting of $\mathbb{R}^n$-valued $\mathbb{F}$-adapted continuous process $\phi(\cdot)$ such that $\mathbb{E}\big[\sup_{0\leqslant t\leqslant T}|\phi(t)|^2\big] < \infty$, and by $L^2(0,T;L^2_{\mathbb{F}}(0,T;\mathbb{R}^n))$ the space consisting of $\mathbb{R}^n$-valued process $\phi(\cdot,\cdot) : [0,T]^2\times\Omega \to \mathbb{R}^n$ such that for almost all $t \in [0,T]$, $\phi(t,\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $\mathbb{E}\int_0^T\int_0^T |\phi(t,s)|^2 ds dt < \infty$.

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Consider the following SDDE,

$$\begin{cases} d\tilde{x}(t) = \tilde{b}\Big(t, \tilde{x}(t), \tilde{x}(t-\delta), \displaystyle\int_{-\delta}^{0} e^{\lambda\theta}\tilde{x}(t+\theta)d\theta\Big)dt \\ \qquad\qquad +\tilde{\sigma}\Big(t, \tilde{x}(t), \tilde{x}(t-\delta), \displaystyle\int_{-\delta}^{0} e^{\lambda\theta}\tilde{x}(t+\theta)d\theta\Big)dW(t), \quad t\in[0,T], \\ \tilde{x}(t) = \tilde{\xi}(t), \ t\in[-\delta,0], \end{cases} \tag{2.1}$$

where $\delta > 0$ is the constant delay time, $\lambda \in \mathbb{R}$ is a constant, deterministic continuous function $\tilde{\xi}(\cdot)$ is the given initial path of the state, and random coefficients $\tilde{b}, \tilde{\sigma}$ are given mappings satisfying the following:

(H1) There exists a constant $L > 0$ such that

$$\begin{aligned} &|\tilde{b}(t,x,y,z) - \tilde{b}(t,x',y',z')| + |\tilde{\sigma}(t,x,y,z) - \tilde{\sigma}(t,x',y',z')| \\ &\quad \leqslant L(|x-x'| + |y-y'| + |z-z'|) \quad \forall\, t\in[0,T],\ x,y,z,x',y',z' \in \mathbb{R}^n; \end{aligned}$$

(H2) $\sup_{0\leqslant t\leqslant T}\big(|\tilde{b}(t,0,0,0)| + |\tilde{\sigma}(t,0,0,0)|\big) < \infty$.

By standard Picard iteration method we derive the following result. Further relevant details can be found in [19].

PROPOSITION 2.1. *Suppose* (H1)–(H2) *hold. Then, the SDDE* (2.1) *admits a unique solution, and there exists a constant* $C > 0$ *such that for* $p \geqslant 2$,

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}|\tilde{x}(t)|^p\Big] \\ &\quad \leqslant C\Bigg[\sup_{-\delta\leqslant\theta\leqslant 0}|\tilde{\xi}(\theta)|^p + \mathbb{E}\Big(\int_0^T |\tilde{b}(s,0,0,0)|ds\Big)^p + \mathbb{E}\Big(\int_0^T |\tilde{\sigma}(s,0,0,0)|^2 ds\Big)^{\frac{p}{2}}\Bigg]. \end{aligned}$$

Let $\mathbb{R}^+$ be the space of real numbers not less than zero. Consider the following anticipated BSDE:

$$\begin{cases} -dY(t) = g\big(t, Y(t), Z(t), Y(t+\delta^1(t)), Z(t+\delta^2(t))\big)dt - Z(t)dW(t), t\in[0,T], \\ Y(t) = \alpha(t), \ Z(t) = \beta(t), \quad t\in[T, T+K]. \end{cases} \tag{2.2}$$

Here, terminal conditions $\alpha(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([T,T+K];\mathbb{R}^m))$ and $\beta(\cdot) \in L^2_{\mathbb{F}}(T, T+K;\mathbb{R}^{m\times d})$ are given, $\delta^1(\cdot)$ and $\delta^2(\cdot)$ are given $\mathbb{R}^+$-valued functions defined on $[0,T]$ satisfying

(H3) (i) there exists a constant $K \geqslant 0$ such that for all $s\in[0,T]$, $s+\delta^1(s) \leqslant T+K$, $s+\delta^2(s) \leqslant T+K$;

(ii) there exists a constant $M \geqslant 0$ such that for all $t\in[0,T]$ and all nonnegative and integrable function $f(\cdot)$,

$$\int_t^T f(s+\delta^1(s))ds \leqslant M\int_t^{T+K} f(s)ds, \quad \int_t^T f(s+\delta^2(s))ds \leqslant M\int_t^{T+K} f(s)ds.$$

We impose the following conditions on the generator of (2.2):

(H4) $g(s,\omega,y,z,\alpha,\beta) : \Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2_{\mathcal{F}_r}(\Omega;\mathbb{R}^m)\times L^2_{\mathcal{F}_{r'}}(\Omega;\mathbb{R}^{m\times d}) \to L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^m)$ for all $s\in[0,T]$, where $r, r' \in [s, T+K]$, and $\mathbb{E}\big[\int_0^T |g(s,0,0,0,0)|^2 ds\big] < +\infty$.

(H5) There exists a constant $C > 0$ such that for all $s\in[0,T]$, $y,\tilde{y}\in\mathbb{R}^m$, $z,\tilde{z}\in\mathbb{R}^{m\times d}$, $\alpha(\cdot),\tilde{\alpha}(\cdot)\in L^2_{\mathbb{F}}(s,T+K;\mathbb{R}^m)$, $\beta(\cdot),\tilde{\beta}(\cdot)\in L^2_{\mathbb{F}}\big(s,T+K;\mathbb{R}^{m\times d}\big)$, $r,r'\in[s,T+K]$, we have

$$\begin{aligned} &\big|g(s,y,z,\alpha(r),\beta(r')) - g(s,\tilde{y},\tilde{z},\tilde{\alpha}(r),\tilde{\beta}(r'))\big| \\ &\quad \leqslant C\big(|y-\tilde{y}| + |z-\tilde{z}| + \mathbb{E}_s\big[|\alpha(r)-\tilde{\alpha}(r)| + |\beta(r')-\tilde{\beta}(r')|\big]\big). \end{aligned}$$

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PROPOSITION 2.2 (see [25]). *Let* (H3)–(H5) *hold. Then, for any given* $\alpha(\cdot) \in L^2_{\mathbb{F}}(\Omega;\, C([T,T+K];\mathbb{R}^m))$ *and* $\beta(\cdot) \in L^2_{\mathbb{F}}(T,T+K;\mathbb{R}^{m\times d})$, (2.2) *admits a unique* $\mathcal{F}_t$*-adapted solution pair* $(Y(\cdot),Z(\cdot)) \in L^2_{\mathbb{F}}(\Omega;C([0,T+K];\mathbb{R}^m)) \times L^2_{\mathbb{F}}(0,T+K;\mathbb{R}^{m\times d})$.

Consider the following backward SVIE,

(2.3)
$$\tilde{Y}(t) = \psi(t) + \int_t^T \tilde{g}\big(t,s,\tilde{Y}(s),\tilde{Z}(t,s),\tilde{Z}(s,t)\big)ds - \int_t^T \tilde{Z}(t,s)dW(s), t\in[0,T],$$

where $\tilde{g}$ is the given function satisfying

(H6) $\tilde{g}$ is $\mathcal{B}([0,T]^2 \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}) \otimes \mathcal{F}_T$-measurable such that $s \mapsto \tilde{g}(t,s,y,z,\zeta)$ is progressively measurable for all $(t,y,z,\zeta) \in [0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^{m\times d}$, and

$$\mathbb{E}\int_0^T \left(\int_t^T |\tilde{g}(t,s,0,0,0)|ds\right)^2 dt < \infty.$$

Moreover,

$$\begin{aligned}\big|\tilde{g}(t,s,y,z,\zeta) - \tilde{g}(t,s,\bar{y},\bar{z},\bar{\zeta})\big| \leqslant L(t,s)\big(|y-\bar{y}| + |z-\bar{z}| + |\zeta-\bar{\zeta}|\big)\\ \forall\ 0\leqslant t\leqslant s\leqslant T,\ y,\bar{y}\in\mathbb{R}^m,\ z,\bar{z},\zeta,\bar{\zeta}\in\mathbb{R}^{m\times d},\ \text{a.s.},\end{aligned}$$

where L is a deterministic function such that for some $\varepsilon > 0$,

$$\sup_{t\in[0,T]}\int_t^T L(t,s)^{2+\varepsilon}ds < \infty.$$

PROPOSITION 2.3 (see [34]). *Let* (H6) *hold. Then, for any* $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$*-measurable process* $\psi(\cdot)$ *satisfying* $\mathbb{E}\int_0^T |\psi(t)|^2dt < \infty$, *the backward SVIE* (2.3) *admits a unique adapted solution* $(Y(\cdot),Z(\cdot,\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2(0,T;L^2_{\mathbb{F}}(0,T;\mathbb{R}^{m\times d}))$ *satisfying*

$$\tilde{Y}(t) = \mathbb{E}_s[\tilde{Y}(t)] + \int_s^t \tilde{Z}(t,r)dW(r),\ \text{a.e.}t\in[s,T].$$

3. A novel transformation from SDDE to SVIE. In this section, we present the variational equations to be studied, then make some interesting transformations to them. Similar transformation also appeared in [8].

Denote

$$y(t) := x(t-\delta),\quad z(t) := \int_{-\delta}^0 e^{\lambda\theta}x(t+\theta)d\theta,\quad \mu(t) := u(t-\delta). \tag{3.1}$$

Then, we can rewrite the state equation (1.1) in a more concise form as follows:

$$\begin{cases} dx(t) = b(t,x(t),y(t),z(t),u(t),\mu(t))dt\\ \qquad\quad +\sigma(t,x(t),y(t),z(t),u(t),\mu(t))dW(t),\quad t\in[0,T],\\ x(t) = \xi(t),\ u(t) = \gamma(t),\ t\in[-\delta,0]. \end{cases} \tag{3.2}$$

And the cost (1.2) becomes

$$J(u(\cdot)) = \mathbb{E}\bigg[\int_0^T l(t,x(t),y(t),z(t),u(t),\mu(t))dt + h(x(T),y(T),z(T))\bigg]. \tag{3.3}$$

Throughout the paper, we impose the following assumptions.

(A1) (i) The map $(x,y,z) \mapsto b = b(t,x,y,z,u,\mu)$, $\sigma = \sigma(t,x,y,z,u,\mu)$, $l = l(t,x,y,z,u,\mu)$, $h = h(x,y,z)$ are twice continuously differentiable in (x,y,z). They and all their derivatives f_{κ^i}, $f_{\kappa^i\kappa^\ell}$ are continuous in (x,y,z,u,μ), $i,\ell = 1,2,3$. Here $f = b,\sigma,l,h$ and $\kappa^1 := x, \kappa^2 := y, \kappa^3 := z$.

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(ii) Denote $f = b, \sigma$ and $g = l, h$. For $i, \ell = 1, 2, 3$, f_{κ^i}, $f_{\kappa^i\kappa^\ell}$, $g_{\kappa^i\kappa^\ell}$ are bounded, where $\kappa^1 = x, \kappa^2 = y, \kappa^3 = z$. There exists a constant C such that

$$|f(t,0,0,0,u,\mu)|+|g(t,0,0,0,u,\mu)|+|g_{\kappa^i}(t,0,0,0,u,\mu)|\leqslant C \ \ \forall\ u,\mu\in U,\ t\geqslant 0.$$

(iii) The initial trajectory of the state $\xi(\cdot)$ is a deterministic continuous function, and the initial trajectory of the control $\gamma(\cdot)$ is a deterministic square integrable function.

Under (A1), the SDDE (3.2) admits a unique solution by Proposition 2.1 above or Theorem 2.1 ([19, Chapter II]), hence the cost functional (3.3) is well-defined and Problem (P) is meaningful.

Since the control domain U is an arbitrary nonempty set, not necessarily convex, we then apply the spike variation technique to deal with Problem (P). Let $u^*(\cdot)$ be the optimal control and $x^*(\cdot)$ be the optimal trajectory. Let $0 < \varepsilon < \delta$, for any given $\tau \in [0, T)$, define $u^\varepsilon_\tau(t)$ for $t \in [0, T]$ as follows,

$$u^\varepsilon_\tau(t) := \begin{cases} u^*(t), & t \notin [\tau, \tau+\varepsilon], \\ v(t), & t \in [\tau, \tau+\varepsilon], \end{cases} \tag{3.4}$$

which is a perturbed admissible control of the form, where $v(\cdot)$ is any admissible control, and $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot))$ is defined similarly to (3.1).

Inspired by [24], we introduce the variational equations

$$\begin{cases} dx_1(t) = \Big[b_x(t)x_1(t) + b_y(t)y_1(t) + b_z(t)z_1(t) + \Delta b(t)\Big]dt \\ \qquad\qquad + \sum_{j=1}^{d}\Big[\sigma^j_x(t)x_1(t) + \sigma^j_y(t)y_1(t) + \sigma^j_z(t)z_1(t) + \Delta\sigma^j(t)\Big]dW^j(t), t\in[0,T], \\ x_1(t) = 0, \quad t \in [-\delta, 0], \end{cases} \tag{3.5}$$

$$\begin{cases} dx_2(t) = \Big[b_x(t)x_2(t) + b_y(t)y_2(t) + b_z(t)z_2(t) \\ \qquad\quad + \frac{1}{2}\big(x_1(t)^\top, y_1(t)^\top, z_1(t)^\top\big)\partial^2 b(t)\big(x_1(t)^\top, y_1(t)^\top, z_1(t)^\top\big)^\top\Big]dt \\ \qquad\quad + \sum_{j=1}^{d}\Big[\sigma^j_x(t)x_2(t) + \sigma^j_y(t)y_2(t) + \sigma^j_z(t)z_2(t) \\ \qquad\quad + \frac{1}{2}\big(x_1(t)^\top, y_1(t)^\top, z_1(t)^\top\big)\partial^2\sigma^j(t)\big(x_1(t)^\top, y_1(t)^\top, z_1(t)^\top\big)^\top \\ \qquad\quad + \Delta\sigma^j_x(t)x_1(t) + \Delta\sigma^j_y(t)y_1(t) + \Delta\sigma^j_z(t)z_1(t)\Big]dW^j(t), \quad t\in[0,T], \\ x_2(t) = 0, \quad t \in [-\delta, 0], \end{cases} \tag{3.6}$$

where for $t \in [0, T]$, $u^\varepsilon(t) := u^\varepsilon_\tau(t)$, $\mu^\varepsilon(t) := u^\varepsilon(t-\delta)$, $\Theta(t) := (x^*(t), y^*(t), z^*(t), u^*(t), \mu^*(t))$, $\kappa^1 := x, \kappa^2 := y, \kappa^3 := z$, and for $i, \ell = 1, 2, 3$, $f = b, \sigma^j$, $j = 1, 2, \ldots, d$,

$$\begin{cases} f_{\kappa^i}(t) := f_{\kappa^i}(t, \Theta(t)), \quad f_{\kappa^i\kappa^\ell}(t) := f_{\kappa^i\kappa^\ell}(t, \Theta(t)), \\ \Delta f(t) := f(t, x^*(t), y^*(t), z^*(t), u^\varepsilon(t), \mu^\varepsilon(t)) - f(t, \Theta(t)), \\ \Delta f_{\kappa^i}(t) := f_{\kappa^i}(t, x^*(t), y^*(t), z^*(t), u^\varepsilon(t), \mu^\varepsilon(t)) - f_{\kappa^i}(t, \Theta(t)), \end{cases} \tag{3.7}$$

for $\kappa^1_1 = x_1$, $\kappa^2_1 = y_1$, $\kappa^3_1 = z_1$,

$$\partial^2 f(t) := \begin{pmatrix} f_{xx}(t) & f_{xy}(t) & f_{xz}(t) \\ f_{yx}(t) & f_{yy}(t) & f_{yz}(t) \\ f_{zx}(t) & f_{zy}(t) & f_{zz}(t) \end{pmatrix},$$

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$$\kappa_1^i(t)^\top f_{\kappa^i\kappa^\ell}(t)\kappa_1^\ell(t) := \begin{pmatrix} \kappa_1^i(t)^\top f^1_{\kappa^i\kappa^\ell}(t)\kappa_1^\ell(t) \\ \vdots \\ \kappa_1^i(t)^\top f^n_{\kappa^i\kappa^\ell}(t)\kappa_1^\ell(t) \end{pmatrix},$$

and $y_1(\cdot), z_1(\cdot), y_2(\cdot), z_2(\cdot)$ are defined similarly to (3.1). By Proposition 2.1, under assumption (A1) the variational equations (3.5) and (3.6) admit a unique solution, respectively. In the following, we introduce some estimates whose proofs are similar to Lemmas 3.1 and 3.2 in [18].

LEMMA 3.1. *Let assumption* (A1) *hold. Then, for any* $p \geqslant 1$, *we have*

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|x^\varepsilon(t)-x^*(t)|^{2p}\right] = O(\varepsilon^p), \quad \mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|x_1(t)|^{2p}\right] = O(\varepsilon^p), \tag{3.8}$$

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|x_2(t)|^{p}\right] = O(\varepsilon^p), \quad \mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|x^\varepsilon(t)-x^*(t)-x_1(t)|^{2p}\right] = o(\varepsilon^p), \tag{3.9}$$

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|x^\varepsilon(t)-x^*(t)-x_1(t)-x_2(t)|^{p}\right] = o(\varepsilon^p). \tag{3.10}$$

Proof. We only prove the estimate (3.10), and the cases of the other estimates are similar. For simplicity, consider $n = m = d = 1$. Denote

$$\begin{array}{lll} \tilde{\mathcal{X}}(t) := x^\varepsilon(t) - x^*(t) - x_1(t), & \mathcal{X}(t) := \tilde{\mathcal{X}}(t) - x_2(t), & \tilde{\mathcal{Y}}(t) := y^\varepsilon(t) - y^*(t) - y_1(t), \\ \mathcal{Y}(t) := \tilde{\mathcal{Y}}(t) - y_2(t), & \tilde{\mathcal{Z}}(t) := z^\varepsilon(t) - z^*(t) - z_1(t), & \mathcal{Z}(t) := \tilde{\mathcal{Z}}(t) - z_2(t). \end{array}$$

Then, $\mathcal{X}(\cdot)$ satisfies the following SDDE:

(3.11)

$$\begin{cases} d\mathcal{X}(t) = \Big\{ b_x(t)\mathcal{X}(t) + b_y(t)\mathcal{Y}(t) + b_z(t)\mathcal{Z}(t) + \Delta b_x(t)(x^\varepsilon(t) - x^*(t)) \\ \quad + \Delta b_y(t)(y^\varepsilon(t) - y^*(t)) + \Delta b_z(t)(z^\varepsilon(t) - z^*(t)) + \tilde{b}_{xx}(t)\Big[|x^\varepsilon(t) - x^*(t)|^2 - |x_1(t)|^2\Big] \\ \quad + \Big[\tilde{b}_{xx}(t) - \frac{1}{2}b_{xx}(t)\Big]|x_1(t)|^2 + \tilde{b}_{yy}(t)\Big[|y^\varepsilon(t) - y^*(t)|^2 - |y_1(t)|^2\Big] \\ \quad + \Big[\tilde{b}_{yy}(t) - \frac{1}{2}b_{yy}(t)\Big]|y_1(t)|^2 + \tilde{b}_{zz}(t)\Big[|z^\varepsilon(t) - z^*(t)|^2 - |z_1(t)|^2\Big] \\ \quad + \Big[\tilde{b}_{zz}(t) - \frac{1}{2}b_{zz}(t)\Big]|z_1(t)|^2 + 2\tilde{b}_{xy}(t)\Big[\Big(x^\varepsilon(t) - x^*(t)\Big)\Big(y^\varepsilon(t) - y^*(t)\Big) - x_1(t)y_1(t)\Big] \\ \quad + 2\tilde{b}_{xz}(t)\Big[\Big(x^\varepsilon(t) - x^*(t)\Big)\Big(z^\varepsilon(t) - z^*(t)\Big) - x_1(t)z_1(t)\Big] + 2\tilde{b}_{yz}(t)\Big[\Big(y^\varepsilon(t) - y^*(t)\Big) \\ \quad \times\Big(z^\varepsilon(t) - z^*(t)\Big) - y_1(t)z_1(t)\Big] + \Big[2\tilde{b}_{xy}(t) - b_{xy}(t)\Big]x_1(t)y_1(t) \\ \quad + \Big[2\tilde{b}_{xz}(t) - b_{xz}(t)\Big]x_1(t)z_1(t) + \Big[2\tilde{b}_{yz}(t) - b_{yz}(t)\Big]y_1(t)z_1(t)\Big\}dt \\ \quad + \Big\{\sigma_x(t)\mathcal{X}(t) + \sigma_y(t)\mathcal{Y}(t) + \sigma_z(t)\mathcal{Z}(t) + \Delta\sigma_x(t)\tilde{\mathcal{X}}(t) + \Delta\sigma_y(t)\tilde{\mathcal{Y}}(t) + \Delta\sigma_z(t)\tilde{\mathcal{Z}}(t) \\ \quad + \tilde{\sigma}_{xx}(t)\Big[|x^\varepsilon(t) - x^*(t)|^2 - |x_1(t)|^2\Big] + \Big[\tilde{\sigma}_{xx}(t) - \frac{1}{2}\sigma_{xx}(t)\Big]|x_1(t)|^2 + \tilde{\sigma}_{yy}(t)\Big[|y^\varepsilon(t) \\ \quad - y^*(t)|^2 - |y_1(t)|^2\Big] + \Big[\tilde{\sigma}_{yy}(t) - \frac{1}{2}\sigma_{yy}(t)\Big]|y_1(t)|^2 + \tilde{\sigma}_{zz}(t)\Big[|z^\varepsilon(t) - z^*(t)|^2 - |z_1(t)|^2\Big] \\ \quad + \Big[\tilde{\sigma}_{zz}(t) - \frac{1}{2}\sigma_{zz}(t)\Big]|z_1(t)|^2 + 2\tilde{\sigma}_{xy}(t)\Big[\Big(x^\varepsilon(t) - x^*(t)\Big)\Big(y^\varepsilon(t) - y^*(t)\Big) - x_1(t)y_1(t)\Big] \\ \quad + 2\tilde{\sigma}_{xz}(t)\Big[\Big(x^\varepsilon(t) - x^*(t)\Big)\Big(z^\varepsilon(t) - z^*(t)\Big) - x_1(t)z_1(t)\Big] + 2\tilde{\sigma}_{yz}(t)\Big[\Big(y^\varepsilon(t) - y^*(t)\Big) \\ \quad \times\Big(z^\varepsilon(t) - z^*(t)\Big) - y_1(t)z_1(t)\Big] + \Big[2\tilde{\sigma}_{xy}(t) - \sigma_{xy}(t)\Big]x_1(t)y_1(t) \\ \quad + \Big[2\tilde{\sigma}_{xz}(t) - \sigma_{xz}(t)\Big]x_1(t)z_1(t) + \Big[2\tilde{\sigma}_{yz}(t) - \sigma_{yz}(t)\Big]y_1(t)z_1(t)\Big\}dW(t), \ t \geq 0, \\ \mathcal{X}(t) = 0, \ t \in [-\delta, 0], \end{cases}$$

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where

$$\tilde{b}_{\kappa^i\kappa^j}(t)=\int_0^1\int_0^1 \lambda b_{\kappa^i\kappa^j}(t,x^*(t)+\lambda\theta(x^\varepsilon(t)-x^*(t)),y^*(t)+\lambda\theta(y^\varepsilon(t)-y^*(t)),\\ z^*(t)+\lambda\theta(z^\varepsilon(t)-z^*(t)),u^\varepsilon(t),\mu^\varepsilon(t))d\theta d\lambda,\ i,j=1,2,3,$$

with $\kappa^1=x,\kappa^2=y,\kappa^3=z$. By the estimate of the solution to (3.11), we obtain

(3.12)

$$\begin{aligned}\mathbb{E}\sup_{0\leqslant t\leqslant T}|\mathcal{X}(t)|^p\leqslant M\mathbb{E}\bigg(\int_0^T&\Big[|x^\varepsilon(t)-x^*(t)+x_1(t)|^2|\tilde{\mathcal{X}}(t)|^2+|y^\varepsilon(t)-y^*(t)+y_1(t)|^2\\ &\times|\tilde{\mathcal{Y}}(t)|^2+|z^\varepsilon(t)-z^*(t)+z_1(t)|^2|\tilde{\mathcal{Z}}(t)|^2+|\tilde{\mathcal{X}}(t)|^2|y^\varepsilon(t)-y^*(t)|^2+|x_1(t)\tilde{\mathcal{Y}}(t)|^2\\ &+|\tilde{\mathcal{X}}(t)|^2|z^\varepsilon(t)-z^*(t)|^2+|x_1(t)\tilde{\mathcal{Z}}(t)|^2+|\tilde{\mathcal{Y}}(t)|^2|z^\varepsilon(t)-z^*(t)|^2+|y_1(t)\tilde{\mathcal{Z}}(t)|^2\\ &+\big[|\tilde{b}_{xx}(t)-\frac{1}{2}b_{xx}(t)|^2+|\tilde{\sigma}_{xx}(t)-\frac{1}{2}\sigma_{xx}(t)|^2\big]|x_1(t)|^4+\big[|\tilde{b}_{yy}(t)-\frac{1}{2}b_{yy}(t)|^2\\ &+|\tilde{\sigma}_{yy}(t)-\frac{1}{2}\sigma_{yy}(t)|^2\big]|y_1(t)|^4+\big[|\tilde{b}_{zz}(t)-\frac{1}{2}b_{zz}(t)|^2+|\tilde{\sigma}_{zz}(t)-\frac{1}{2}\sigma_{zz}(t)|^2\big]\\ &\times|z_1(t)|^4+\big[|2\tilde{b}_{xy}(t)-b_{xy}(t)|^2+|2\tilde{\sigma}_{xy}(t)-\sigma_{xy}(t)|^2\big]|x_1(t)y_1(t)|^2+\big[|2\tilde{b}_{xz}(t)\\ &-b_{xz}(t)|^2+|2\tilde{\sigma}_{xz}(t)-\sigma_{xz}(t)|^2\big]|x_1(t)z_1(t)|^2+\big[|2\tilde{b}_{yz}(t)-b_{yz}(t)|^2+|2\tilde{\sigma}_{yz}(t)\\ &-\sigma_{yz}(t)|^2\big]|y_1(t)z_1(t)|^2\Big]dt\bigg)^{\frac{p}{2}}+M\mathbb{E}\bigg(\int_0^T\big[|\Delta b_x(t)(x^\varepsilon(t)-x^*(t))|+|\Delta b_y(t)\\ &\times(y^\varepsilon(t)-y^*(t))|+|\Delta b_z(t)(z^\varepsilon(t)-z^*(t))|\big]dt\bigg)^p+M\mathbb{E}\bigg[\int_0^T\Big(|\Delta\sigma_x(t)\tilde{\mathcal{X}}(t)|^2\\ &+|\Delta\sigma_y(t)\tilde{\mathcal{Y}}(t)|^2+|\Delta\sigma_z(t)\tilde{\mathcal{Z}}(t)|^2\Big)dt\bigg]^{\frac{p}{2}},\end{aligned}$$

where M is a constant. By assumption (A1) and (3.8)–(3.9), we have

$$\begin{aligned}&\mathbb{E}\bigg(\int_0^T|x^\varepsilon(t)-x^*(t)+x_1(t)|^2|\tilde{\mathcal{X}}(t)|^2dt\bigg)^{\frac{p}{2}}\\ &\leqslant\mathbb{E}\Big\{\sup_{0\leq t\leq T}|\tilde{\mathcal{X}}(t)|^p\big[\sup_{0\leq t\leq T}|x^\varepsilon(t)-x^*(t)|^p+\sup_{0\leq t\leq T}|x_1(t)|^p\big]\Big\}=o(\varepsilon^p),\end{aligned}\tag{3.13}$$

$$\begin{aligned}&\mathbb{E}\bigg(\int_0^T|\tilde{b}_{xx}(t)-\frac{1}{2}b_{xx}(t)|^2|x_1(t)|^4dt\bigg)^{\frac{p}{2}}\\ &\leq\ \mathbb{E}\bigg[\sup_{0\leq t\leq T}|x_1(t)|^{2p}\Big(\int_0^T|\tilde{b}_{xx}(t)-\frac{1}{2}b_{xx}(t)|^2dt\Big)^{\frac{p}{2}}\bigg]=o(\varepsilon^p),\end{aligned}\tag{3.14}$$

(3.15)

$$\mathbb{E}\bigg(\int_0^T|\Delta\sigma_x(t)\tilde{\mathcal{X}}(t)|^2dt\bigg)^{\frac{p}{2}}\leqslant\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T}|\tilde{\mathcal{X}}(t)|^p\bigg(\int_0^T|\Delta\sigma_x(t)|^2dt\bigg)^{\frac{p}{2}}\bigg]=o(\varepsilon^p).$$

Noting

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|z_1(t)|^p=\mathbb{E}\sup_{0\leqslant t\leqslant T}\Big|\int_{t-\delta}^t e^{\lambda(r-t)}x_1(r)dr\Big|^p\leqslant\tfrac{1}{\lambda}\mathbb{E}\sup_{-\delta\leqslant t\leqslant T}|x_1(t)|^p,\tag{3.16}$$

we deal with all terms in (3.12) similar to (3.13)–(3.15), and derive the estimate (3.10). ∎

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LEMMA 3.2. *Let assumption* (A1) *hold. Suppose* $(x^*(\cdot), u^*(\cdot))$ *is an optimal pair,* $x^\varepsilon(\cdot)$ *is the trajectory corresponding to* $u^\varepsilon(\cdot)$ *by* (3.4)*. Then, the following variational inequality holds:*

(3.17)

$$\begin{aligned}
&J(u^\varepsilon(\cdot))-J(u^*(\cdot))\\
&\quad=\mathbb{E}\Big[h_x(T)\big[x_1(T)+x_2(T)\big]+h_y(T)\big[y_1(T)+y_2(T)\big]+h_z(T)\big[z_1(T)\\
&\qquad+\ z_2(T)\big]+\frac{1}{2}\big(x_1(T)^\top, y_1(T)^\top, z_1(T)^\top\big)\partial^2 h(T)\big(x_1(T)^\top, y_1(T)^\top, z_1(T)^\top\big)^\top\Big]\\
&\qquad+\ \mathbb{E}\int_0^T\Big[\Delta l(t)+l_x(t)[x_1(t)+x_2(t)]+l_y(t)\big[y_1(t)+y_2(t)\big]+l_z(t)\big[z_1(t)+z_2(t)\big]\\
&\qquad+\frac{1}{2}\big(x_1(t)^\top, y_1(t)^\top, z_1(t)^\top\big)\partial^2 l(t)\big(x_1(t)^\top, y_1(t)^\top, z_1(t)^\top\big)^\top\Big]dt+o(\varepsilon),
\end{aligned}$$

where $\Delta l, l_{\kappa^i}, l_{\kappa^i\kappa^\ell}, h_{\kappa^i}, h_{\kappa^i\kappa^\ell}$ *are defined similarly as in* (3.7) *for* $i,\ell=1,2,3$.

Define

$$X_1(t):=\begin{bmatrix} x_1(t)\\ y_1(t)\ \mathbf{1}_{(\delta,\infty)}(t)\\ z_1(t)\end{bmatrix},\quad X_2(t):=\begin{bmatrix} x_2(t)\\ y_2(t)\ \mathbf{1}_{(\delta,\infty)}(t)\\ z_2(t)\end{bmatrix},$$

and for $j=1,\dots,d$,

$$\begin{aligned}
A(t,s)&:=\begin{bmatrix} b_x(s) & b_y(s) & b_z(s)\\ \mathbf{1}_{(\delta,\infty)}(t-s)b_x(s) & \mathbf{1}_{(\delta,\infty)}(t-s)b_y(s) & \mathbf{1}_{(\delta,\infty)}(t-s)b_z(s)\\ I & -e^{-\lambda\delta}I & -\lambda I\end{bmatrix},\\
C^j(t,s)&:=\begin{bmatrix} \sigma_x^j(s) & \sigma_y^j(s) & \sigma_z^j(s)\\ \mathbf{1}_{(\delta,\infty)}(t-s)\sigma_x^j(s) & \mathbf{1}_{(\delta,\infty)}(t-s)\sigma_y^j(s) & \mathbf{1}_{(\delta,\infty)}(t-s)\sigma_z^j(s)\\ 0 & 0 & 0\end{bmatrix},\\
B(t,s)&:=\begin{bmatrix} \Delta b(s)\\ \mathbf{1}_{(\delta,\infty)}(t-s)\Delta b(s)\\ 0\end{bmatrix},\quad D^j(t,s):=\begin{bmatrix} \Delta\sigma^j(s)\\ \mathbf{1}_{(\delta,\infty)}(t-s)\Delta\sigma^j(s)\\ 0\end{bmatrix},\\
\bar B(t,s)&:=\begin{bmatrix} \frac{1}{2}X_1(s)^\top\partial^2 b(s)X_1(s)\\ \frac{1}{2}\ \mathbf{1}_{(\delta,\infty)}(t-s)X_1(s)^\top\partial^2 b(s)X_1(s)\\ 0\end{bmatrix},\\
\Delta\Xi^j(s)&:=\Big[\Delta\sigma_x^j(s),\Delta\sigma_y^j(s),\Delta\sigma_z^j(s)\Big],\\
\bar D^j(t,s)&:=\begin{bmatrix} \frac{1}{2}X_1(s)^\top\partial^2\sigma^j(s)X_1(s)+\Delta\Xi^j(s)X_1(s)\\ \mathbf{1}_{(\delta,\infty)}(t-s)\big[\frac{1}{2}X_1(s)^\top\partial^2\sigma^j(s)X_1(s)+\Delta\Xi^j(s)X_1(s)\big]\\ 0\end{bmatrix}.
\end{aligned}\tag{3.18}$$

Then, by (3.5)–(3.6),

(3.19)

$$X_1(t)=\int_0^t\big[A(t,s)X_1(s)+B(t,s)\big]ds+\sum_{j=1}^d\int_0^t\big[C^j(t,s)X_1(s)+D^j(t,s)\big]dW^j(s),$$

(3.20)

$$X_2(t)=\int_0^t\big[A(t,s)X_2(s)+\bar B(t,s)\big]ds+\sum_{j=1}^d\int_0^t\big[C^j(t,s)X_2(s)+\bar D^j(t,s)\big]dW^j(s).$$

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By Proposition 2.1 in [30] and assumption (A1), (3.19) and (3.20) both admit unique solutions. Therefore, the above variational inequality (3.17) can be written as

$$\begin{aligned} J(u^{\varepsilon}(\cdot)) - J(u^*(\cdot)) &= \mathbb{E}\int_0^T \Big[\bar{L}(t)[X_1(t)+X_2(t)] + \frac{1}{2}X_1(t)^\top L(t)X_1(t)\Delta l(t)\Big]dt \\ &\quad + \mathbb{E}\Big[\bar{H}[X_1(T)+X_2(T)] + \frac{1}{2}X_1(T)^\top H X_1(T)\Big] + o(\varepsilon), \end{aligned} \tag{3.21}$$

where $X_1(\cdot)$ and $X_2(\cdot)$ satisfy linear SVIEs in (3.19) and (3.20), respectively,

$$\begin{aligned} &\bar{H} := \begin{bmatrix} h_x(T) & h_y(T) & h_z(T) \end{bmatrix}, \quad \bar{L}(t) := \begin{bmatrix} l_x(t) & l_y(t) & l_z(t) \end{bmatrix}, \\ &H := \begin{bmatrix} h_{xx}(T) & h_{xy}(T) & h_{xz}(T) \\ h_{yx}(T) & h_{yy}(T) & h_{yz}(T) \\ h_{zx}(T) & h_{zy}(T) & h_{zz}(T) \end{bmatrix}, \quad L(t) := \begin{bmatrix} l_{xx}(t) & l_{xy}(t) & l_{xz}(t) \\ l_{yx}(t) & l_{yy}(t) & l_{yz}(t) \\ l_{zx}(t) & l_{zy}(t) & l_{zz}(t) \end{bmatrix}, \end{aligned}$$

and $\bar{H}$ is an $\mathbb{R}^{3n}$-valued row vector and other terms are similar. Under the above preparation, we can borrow some useful ideas from [30] where the maximum principle of optimal control problems described by SVIEs was completely solved.

4. Adjoint equations. In this section we introduce some adjoint equations to be dual with the variational equations (3.5)–(3.6).

4.1. First-order adjoint equations. We treat the terms about $X_1(\cdot)+X_2(\cdot)$ in (3.21). From [34], we introduce the first-order adjoint equation as follows:

$$\begin{cases} (a) \quad \eta(t) = \bar{H}^\top - \displaystyle\sum_{j=1}^d \int_t^T \zeta^j(s)dW^j(s), \quad t\in[0,T], \\ (b) \quad Y(t) = \bar{L}(t)^\top + A(T,t)^\top \bar{H}^\top + \displaystyle\sum_{j=1}^d C^j(T,t)^\top \zeta^j(t) + \int_t^T \Big[A(s,t)^\top Y(s) \\ \qquad\qquad + \displaystyle\sum_{j=1}^d C^j(s,t)^\top Z^j(s,t)\Big]ds - \sum_{j=1}^d \int_t^T Z^j(t,s)dW^j(s), t\in[0,T], \\ (c) \quad Y(t) = \mathbb{E}Y(t) + \displaystyle\sum_{j=1}^d \int_0^t Z^j(t,s)dW^j(s), \quad t\in[0,T]. \end{cases} \tag{4.1}$$

(4.1a) is a BSDE which admits a unique solution by Theorem 4.1 in [23]. On the other hand, (4.1b) is a linear backward SVIE, and by Proposition 2.3, it admits a unique solution that satisfies (4.1c) under assumption (A1). Notice that

$$\begin{aligned} X_1(t)+X_2(t) &= \varphi(t) + \int_0^t A(t,s)\big[X_1(s)+X_2(s)\big]ds \\ &\quad + \sum_{j=1}^d \int_0^t C^j(t,s)\big[X_1(s)+X_2(s)\big]dW^j(s), \end{aligned}$$

where

$$\varphi(t) := \int_0^t \big[\bar{B}(t,s)+B(t,s)\big]ds + \sum_{j=1}^d \int_0^t \big[\bar{D}^j(t,s)+D^j(t,s)\big]dW^j(s).$$

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Then, by the dual principle [34, Theorem 5.1], we have

$$\begin{aligned}&\mathbb{E}\int_0^T \bar{L}(t)\big[X_1(t)+X_2(t)\big]dt+\mathbb{E}\big[\bar{H}\big[X_1(T)+X_2(T)\big]\big]\\&=\mathbb{E}\int_0^T \langle\,\varphi(t),Y(t)\,\rangle\,dt+\mathbb{E}\big[\bar{H}\varphi(T)\big].\end{aligned}$$

Let for $j=1,\dots,d$,

(4.2)
$$\eta(t):=\begin{pmatrix}\eta^0(t)\\ \eta^1(t)\\ \eta^2(t)\end{pmatrix},\zeta^j(t):=\begin{pmatrix}\zeta^{0j}(t)\\ \zeta^{1j}(t)\\ \zeta^{2j}(t)\end{pmatrix},Y(t):=\begin{pmatrix}Y^0(t)\\ Y^1(t)\\ Y^2(t)\end{pmatrix},Z^j(t,s):=\begin{pmatrix}Z^{0j}(t,s)\\ Z^{1j}(t,s)\\ Z^{2j}(t,s)\end{pmatrix}.$$

Then, by (4.1) we deduce

$$\begin{aligned}&\mathbb{E}\int_0^T\big\langle\varphi(t),Y(t)\big\rangle dt+\mathbb{E}\big[\bar{H}\varphi(T)\big]=\mathbb{E}\int_0^T\int_0^t\big\langle Y(t),B(t,s)+\bar{B}(t,s)\big\rangle dsdt\\&+\sum_{j=1}^d\mathbb{E}\int_0^T\int_0^t\big\langle Z^j(t,s),D^j(t,s)+\bar{D}^j(t,s)\big\rangle dsdt\\&+\mathbb{E}\Big[\bar{H}\int_0^T\big[\bar{B}(T,s)+B(T,s)\big]ds+\sum_{j=1}^d\int_0^T\zeta^j(s)^\top\big[\bar{D}^j(T,s)+D^j(T,s)\big]ds\Big],\end{aligned}$$

which together with (3.21) yields that

(4.3)
$$\begin{aligned}J(u^\varepsilon(\cdot))-J(u^*(\cdot))=&\mathbb{E}\int_0^T\Big[\Delta l(s)+\frac{1}{2}X_1(s)^\top L(s)X_1(s)+\Big\langle\Delta b(s)+\frac{1}{2}X_1(s)^\top\partial^2 b(s)X_1(s),\\&\int_s^T Y^0(t)dt+\int_{s+\delta}^T Y^1(t)dt\mathbf{1}_{[0,T-\delta)}(s)+h_x(T)^\top+h_y(T)^\top\mathbf{1}_{[0,T-\delta)}(s)\Big\rangle+\sum_{j=1}^d\Big\langle\Delta\sigma^j(s)\\&+\frac{1}{2}X_1(s)^\top\partial^2\sigma^j(s)X_1(s)+\Delta\Xi^j(s)X_1(s),\int_{s+\delta}^T Z^{1j}(t,s)dt\mathbf{1}_{[0,T-\delta)}(s)+\int_s^T Z^{0j}(t,s)dt\\&+\zeta^{0j}(s)+\zeta^{1j}(s)\mathbf{1}_{[0,T-\delta)}(s)\Big\rangle\Big]ds+\frac{1}{2}\mathbb{E}X_1(T)^\top HX_1(T)+o(\varepsilon).\end{aligned}$$

Next we would like to write (4.3) in a more concise form, and give the main result of this subsection. To this end, for $j=1,\dots,d$, $0\leqslant t\leqslant T$, let us denote

(4.4)
$$\begin{cases}p(t):=\eta^0(t)+\eta^1(t)\ \mathbf{1}_{[0,T-\delta)}(t)+\mathbb{E}_t\Big[\displaystyle\int_t^T Y^0(s)ds+\int_{t+\delta}^T Y^1(s)ds\mathbf{1}_{[0,T-\delta)}(t)\Big],\\ q^j(t):=\zeta^{0j}(t)+\zeta^{1j}(t)\ \mathbf{1}_{[0,T-\delta)}(t)+\displaystyle\int_t^T Z^{0j}(s,t)ds+\int_{t+\delta}^T Z^{1j}(s,t)ds\mathbf{1}_{[0,T-\delta)}(t),\end{cases}$$

and $G:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ as follows:

(4.5)
$$G(t,x,y,z,p,q,u,\mu):=l(t,x,y,z,u,\mu)+\big\langle p,b(t,x,y,z,u,\mu)\big\rangle+\sum_{j=1}^d\big\langle q^j,\sigma^j(t,x,y,z,u,\mu)\big\rangle.$$

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LEMMA 4.1. *Let assumption* (A1) *hold. Suppose* $(x^*(\cdot), u^*(\cdot))$ *is an optimal pair,* $x^\varepsilon(\cdot)$ *is the trajectory corresponding to* $u^\varepsilon(\cdot)$*, given by* (3.4)*,* $(\eta(\cdot), \zeta(\cdot), Y(\cdot), Z(\cdot,\cdot))$ *is a solution to* (4.1)*. Then, the following variational inequality holds,*

$$
\begin{aligned}
&J(u^\varepsilon(\cdot)) - J(u^*(\cdot)) \\
&\quad = \mathbb{E}\int_\tau^{\tau+\varepsilon} \Delta G(t)dt + \mathbb{E}\int_{\tau+\delta}^{\tau+\delta+\varepsilon} \Delta\tilde{G}(t)dt\mathbf{1}_{[0,T-\delta)}(\tau) + \frac{1}{2}\mathscr{E}(\varepsilon) + o(\varepsilon)
\end{aligned} \tag{4.6}
$$

for all $v(\cdot) \in \mathcal{U}_{ad}$ *and* $\tau \in [0,T)$*, where*

$$
\begin{aligned}
\mathscr{E}(\varepsilon) &:= \mathbb{E}\int_0^T X_1(t)^\top \partial^2 G(t) X_1(t)dt + \mathbb{E}\big[X_1(T)^\top H X_1(T)\big], \\
\Delta G(t) &:= G(t, x^*(t), y^*(t), z^*(t), p(t), q(t), v(t), u^*(t-\delta)) \\
&\qquad - G(t, x^*(t), y^*(t), z^*(t), p(t), q(t), u^*(t), u^*(t-\delta)), \\
\Delta\tilde{G}(t) &:= G(t, x^*(t), y^*(t), z^*(t), p(t), q(t), u^*(t), v(t-\delta)) \\
&\qquad - G(t, x^*(t), y^*(t), z^*(t), p(t), q(t), u^*(t), u^*(t-\delta)).
\end{aligned} \tag{4.7}
$$

Proof. Notice that

$$
\begin{aligned}
&\mathbb{E}\int_0^T\int_0^t \Big\langle Z^{1j}(t,s),\ \mathbf{1}_{(\delta,\infty)}(t-s)\Delta\sigma^j(s)\Big\rangle dsdt = \mathbb{E}\int_0^{T-\delta} \Big\langle \int_{s+\delta}^T Z^{1j}(t,s)dt, \Delta\sigma^j(s)\Big\rangle ds \\
&\quad = \mathbb{E}\int_\tau^{\tau+\varepsilon} \Big\langle \int_{s+\delta}^T Z^{1j}(t,s)dt, \sigma^j(s, x^*(s), y^*(s), z^*(s), v(s), \mu^*(s)) - \sigma^j(s,\Theta(s))\Big\rangle ds \\
&\qquad \times \mathbf{1}_{[0,T-\delta)}(\tau) + \mathbb{E}\int_{\tau+\delta}^{\tau+\delta+\varepsilon} \Big\langle \sigma^j(s, x^*(s), y^*(s), z^*(s), u^*(s), v(s-\delta)) \\
&\qquad - \sigma^j(s,\Theta(s)), \int_{s+\delta}^T Z^{1j}(t,s)dt\Big\rangle ds\mathbf{1}_{(0,T-\delta)}(\tau+\delta),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\mathbb{E}\int_0^T \big\langle \Delta\sigma_z^j(s) z_1(s), \int_{s+\delta}^T Z^{1j}(t,s)dt\mathbf{1}_{[0,T-\delta)}(s)\big\rangle ds\right| \\
&\quad \leqslant M\left(\mathbb{E}\int_\tau^{\tau+\varepsilon}\left|\int_{s+\delta}^T Z^{1j}(t,s)dt\right|^2 ds\right)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \left(\mathbb{E}\sup_{\tau\leqslant s\leqslant\tau+\varepsilon}|z_1(s)|^2\right)^{\frac{1}{2}} \\
&\qquad + M\varepsilon^{\frac{1}{2}}\left(\mathbb{E}\sup_{\tau+\delta\leqslant s\leqslant\tau+\delta+\varepsilon}|z_1(s)|^2\right)^{\frac{1}{2}} \left(\mathbb{E}\int_{\tau+\delta}^{\tau+\delta+\varepsilon}\left|\int_{s+\delta}^T Z^{1j}(t,s)dt\right|^2 ds\right)^{\frac{1}{2}} = o(\varepsilon),
\end{aligned}
$$

where M is a constant. Then, by applying Lemma 3.1, (4.3), and (4.4), we complete the proof. ∎

4.2. Second-order adjoint equations. To treat the quadratic form in (4.6), let us borrow some ideas from [30]. Now we introduce the following systems of backward equations:

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(4.8)

$$
\begin{cases}
(a)\quad P_1(r) = H - \sum_{j=1}^{d} \int_r^T Q_1^j(\theta) dW^j(\theta), \quad 0 \leqslant r \leqslant T, \\
(b)\quad P_2(r) = A(T,r)^\top P_1(r) + \sum_{j=1}^{d} C^j(T,r)^\top Q_1^j(r) + \int_r^T \Big[A(\theta,r)^\top P_2(\theta) \\
\qquad\qquad + \sum_{j=1}^{d} C^j(\theta,r)^\top Q_2^j(\theta,r) \Big] d\theta - \sum_{j=1}^{d} \int_r^T Q_2^j(r,\theta) dW^j(\theta), 0 \leqslant r \leqslant T, \\
(c)\quad P_3(r) = \partial^2 G(r) + \sum_{j=1}^{d} C^j(T,r)^\top P_1(r) C^j(T,r) \\
\qquad\qquad + \sum_{j=1}^{d} \int_r^T \Big[C^j(T,r)^\top P_2(\theta)^\top C^j(\theta,r) + C^j(\theta,r)^\top P_2(\theta) C^j(T,r) \\
\qquad\qquad + C^j(\theta,r)^\top P_3(\theta) C^j(\theta,r) \Big] d\theta + \int_r^T \int_r^T C^j(\theta,r)^\top P_4(\theta',\theta) C^j(\theta',r) d\theta d\theta' \\
\qquad\qquad - \sum_{j=1}^{d} \int_r^T Q_3^j(r,\theta) dW^j(\theta), \quad 0 \leqslant r \leqslant T, \\
(d)\quad P_4(\theta,r) = A(T,r)^\top P_2(\theta)^\top + \sum_{j=1}^{d} C^j(T,r)^\top Q_2^j(\theta,r)^\top + A(\theta,r)^\top P_3(\theta) \\
\qquad\qquad + \sum_{j=1}^{d} C^j(\theta,r)^\top Q_3^j(\theta,r) + \int_r^T \Big[\sum_{j=1}^{d} C^j(\theta',r)^\top Q_4^j(\theta,\theta',r) \\
\qquad\qquad + A(\theta',r)^\top P_4(\theta,\theta') \Big] d\theta' - \sum_{j=1}^{d} \int_r^T Q_4^j(\theta,r,\theta') dW^j(\theta'), 0 \leqslant r \leqslant \theta \leqslant T, \\
(e)\quad P_4(\theta,r) = P_4(r,\theta)^\top, \quad Q_4(\theta,r,\theta') = Q_4(r,\theta,\theta')^\top, \quad 0 \leqslant \theta < r \leqslant T,
\end{cases}
$$

subject to the following constraints:

$$
(4.9)\quad \begin{cases}
P_2(r) = \mathbb{E}_\theta\big[P_2(r)\big] + \sum_{j=1}^{d} \int_\theta^r Q_2^j(r,\theta') dW^j(\theta'), \quad 0 \leqslant r \leqslant T, \\
P_3(r) = \mathbb{E}_\theta\big[P_3(r)\big] + \sum_{j=1}^{d} \int_\theta^r Q_3^j(r,\theta') dW^j(\theta'), \quad 0 \leqslant r \leqslant T, \\
P_4(\theta,r) = \mathbb{E}_{\theta'}\big[P_4(\theta,r)\big] + \sum_{j=1}^{d} \int_{\theta'}^{r\wedge\theta} Q_4^j(\theta,r,s) dW^j(s), 0 \leqslant \theta' \leqslant (\theta \wedge r) \leqslant T.
\end{cases}
$$

Then, we have the following result for the variational inequality (4.6).

LEMMA 4.2. *Let assumption* (A1) *hold. Suppose* $(x^*(\cdot), u^*(\cdot))$ *is an optimal pair,* $x^\varepsilon(\cdot)$ *is the trajectory corresponding to* $u^\varepsilon(\cdot)$*, given by* (3.4)*,* $(\eta(\cdot), \zeta(\cdot), Y(\cdot), Z(\cdot,\cdot))$ *is the solution to* (4.1)*,* $(p(\cdot), q(\cdot))$ *is defined by* (4.4)*. Then,* (4.8) *admits a unique adapted solution,* $(P_1(\cdot), Q_1(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T]; \mathbb{S}^{3n})) \times \big(L^2_{\mathbb{F}}(0,T; \mathbb{S}^{3n})\big)^d$*,* $(P_2(\cdot), P_3(\cdot), P_4(\cdot,\cdot)) \in L^2_{\mathbb{F}}(0,T; \mathbb{R}^{(3n)\times(3n)}) \times L^2_{\mathbb{F}}(0,T; \mathbb{S}^{3n}) \times L^2\big(0,T; L^2_{\mathbb{F}}(0,T; \mathbb{R}^{(3n)\times(3n)})\big)$*, such that* (4.9) *holds. Furthermore, the variational inequality* (4.6) *can be deduced as follows:*

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$$
\begin{aligned}
(4.10)\quad & J(u^{\varepsilon}(\cdot)) - J(u^{*}(\cdot)) \\
&= \mathbb{E}\int_{\tau}^{\tau+\varepsilon} \Delta G(t)dt + \mathbb{E}\int_{\tau+\delta}^{\tau+\delta+\varepsilon} \Delta\tilde{G}(t)dt\mathbf{1}_{[0,T-\delta)}(\tau) \\
&\quad + \frac{1}{2}\sum_{j=1}^{d}\mathbb{E}\int_{0}^{T}\Big\{D^{j}(T,t)^{\top}P_1(t)D^{j}(T,t) + \int_{t}^{T} D^{j}(\theta,t)^{\top}P_3(\theta)D^{j}(\theta,t)d\theta \\
&\quad + \int_{t}^{T}\Big[D^{j}(T,t)^{\top}P_2(\theta)^{\top}D^{j}(\theta,t) + D^{j}(\theta,t)^{\top}P_2(\theta)D^{j}(T,t)\Big]d\theta \\
&\quad + \int_{t}^{T}\int_{t}^{T} D^{j}(\theta,t)^{\top}P_4(\theta',\theta)D^{j}(\theta',t)d\theta d\theta'\Big\}dt + o(\varepsilon) \qquad \forall\ \tau\in[0,T).
\end{aligned}
$$

Proof. Note that the BSDE (4.8a) admits a unique solution. Then, by Proposition 2.3 and a similar proof of Theorem 5.2 in [30], (4.8) has a unique solution. For simplicity, we just give a sketch of the proof, a detailed proof can be referred to section 4 in [30]. In the following, without loss of generality, let $d=1$. First we introduce an auxiliary process as follows,

$$
(4.11)\qquad \mathcal{X}_1(t,r) = \int_{0}^{r}\big[A(t,s)X_1(s)+B(t,s)\big]ds + \int_{0}^{r}\big[C(t,s)X_1(s)+D(t,s)\big]dW(s)
$$

for $0\leqslant r\leqslant t\leqslant T$. Apparently, $\mathcal{X}_1(t,t)=X_1(t)$ for all $0\leqslant t\leqslant T$. Applying Lemma 3.1, we have $\sup_{0\leqslant t\leqslant T}\mathbb{E}\big[\sup_{0\leqslant r\leqslant t}|\mathcal{X}_1(t,r)|^p\big]=O(\varepsilon^{\frac{p}{2}})$. Let $\Theta(\cdot,\cdot):[0,T]^2\times\Omega\to\mathbb{R}^{(3n)\times(3n)}$ be a process such that for any $t\in[0,T]$, $\Theta(t,\cdot)\in L^2_{\mathbb{F}}(0,t;\mathbb{R}^{(3n)\times(3n)})$. Then, by the martingale representation theorem, for any $0\leqslant s\leqslant t\leqslant T$, there exists a unique $\Lambda(t,s,\cdot)\in\big(L^2_{\mathbb{F}}(0,s;\mathbb{R}^{(3n)\times(3n)})\big)^d$ satisfying

$$
(4.12)\qquad \Pi(t,s,r)\equiv\mathbb{E}_r[\Theta(t,s)]=\Theta(t,s)-\int_{r}^{s}\Lambda(t,s,\theta)dW(\theta),\ 0\leqslant r\leqslant s\leqslant t\leqslant T.
$$

Applying the Itô formula to the map $r\mapsto\mathcal{X}_1(t,r)^{\top}\Pi(t,s,r)\mathcal{X}_1(s,r)$, we obtain for $0\leqslant r\leqslant s\leqslant t$,

$$
\begin{aligned}
(4.13)\quad \mathbb{E}\big[\mathcal{X}_1(t,r)^{\top}\Theta(t,s)\mathcal{X}_1(s,r)\big] &= \mathbb{E}\big[\mathcal{X}_1(t,r)^{\top}\Pi(t,s,r)\mathcal{X}_1(s,r)\big] \\
&= \mathbb{E}\int_{0}^{r}\Big\{X_1(\theta)^{\top}\big[A(t,\theta)^{\top}\Theta(t,s)+C(t,\theta)^{\top}\Lambda(t,s,\theta)\big]\mathcal{X}_1(s,\theta) \\
&\quad + \mathcal{X}_1(t,\theta)^{\top}\big[\Theta(t,s)A(s,\theta) \\
&\quad + \Lambda(t,s,\theta)C(s,\theta)\big]X_1(\theta) + X_1(\theta)^{\top}C(t,\theta)^{\top}\Theta(t,s)C(s,\theta)X_1(\theta) \\
&\quad + D(t,\theta)^{\top}\Theta(t,s)D(s,\theta)\Big\}d\theta + o(\varepsilon).
\end{aligned}
$$

In the following, we choose a different $\Theta(\cdot,\cdot)$, $\Pi(\cdot,\cdot,\cdot)$, and $\Lambda(\cdot,\cdot,\cdot)$ to deal with the quadratic terms about $X_1(\cdot)$ in (4.7). First we deal with the term $X_1(T)^{\top}HX_1(T)$. Take $t=s=T$ and $\Theta(T,T)=H$ in (4.12). Then, from (4.8a), we have

$$
(\Pi(T,T,r),\Lambda(T,T,r))\equiv(P_1(r),Q_1(r)),\quad r\in[0,T].
$$

By (4.13), we get

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$$\begin{aligned}
&\mathbb{E}\big[X_1(T)^\top H X_1(T)\big] = \mathbb{E}\big[\mathcal{X}_1(T,T)^\top P_1(T)\mathcal{X}_1(T,T)\big] \\
&\quad = \mathbb{E}\int_0^T \Big\{ X_1(r)^\top \Big[A(T,r)^\top P_1(r) + C(T,r)^\top Q_1(r)\Big]\mathcal{X}_1(T,r) \\
&\qquad + \mathcal{X}_1(T,r)^\top \Big[P_1(r)A(T,r) + Q_1(r)C(T,r)\Big]X_1(r) + X_1(r)^\top C(T,r)^\top \\
&\qquad \times P_1(r)C(T,r)X_1(r) + D(T,r)^\top P_1(r)D(T,r)\Big\}dr + o(\varepsilon),
\end{aligned}$$

which together with (4.7) yields that

$$\begin{aligned}
\mathscr{E}(\varepsilon) = \mathbb{E}\int_0^T &\Big\{ X_1(r)^\top \Big[A(T,r)^\top P_1(r) + C(T,r)^\top Q_1(r)\Big]\mathcal{X}_1(T,r) \\
&+ \mathcal{X}_1(T,r)^\top \Big[P_1(r)A(T,r) + Q_1(r)C(T,r)\Big]X_1(r) + X_1(r)^\top \Big[\partial^2 G(r) \\
&+ C(T,r)^\top P_1(r)C(T,r)\Big]X_1(r) + D(T,r)^\top P_1(r)D(T,r)\Big\}dr + o(\varepsilon).
\end{aligned}$$

Next we deal with the term $X_1(r)^\top[\cdots]\mathcal{X}_1(T,r)$ and $\mathcal{X}_1(T,r)^\top[\cdots]X_1(r)$. Take $t=T$ in (4.12) and let

$$(\Theta(T,r),\Lambda(T,\theta,r)) \equiv (P_2(r)^\top, Q_2(\theta,r)^\top), \quad 0 \leqslant r \leqslant \theta \leqslant T.$$

Then, by (4.13) and (4.8) we obtain

$$\begin{aligned}
\mathscr{E}(\varepsilon) = \mathbb{E}\int_0^T &\Big\{ X_1(r)^\top \Big[\partial^2 G(r) + C(T,r)^\top P_1(r)C(T,r) \\
&+ \int_r^T \Big(C(T,r)^\top P_2(\theta)^\top C(\theta,r) + C(\theta,r)^\top P_2(\theta)C(T,r)\Big)d\theta\Big]X_1(r) \\
&+ \int_r^T \Big[X_1(r)^\top \Big(P_2(\theta)A(T,r) + Q_2(\theta,r)C(T,r)\Big)^\top \mathcal{X}_1(\theta,r) \\
&+ \mathcal{X}_1(\theta,r)^\top \Big(P_2(\theta)A(T,r) + Q_2(\theta,r)C(T,r)\Big)X_1(r)\Big]d\theta\Big\}dr \\
&+ \mathbb{E}\int_0^T \Big\{ \int_r^T \Big[D(T,r)^\top P_2(\theta)^\top D(\theta,r) + D(\theta,r)^\top P_2(\theta)D(T,r)\Big]d\theta \\
&+ D(T,r)^\top P_1(r)D(T,r)\Big\}dr + o(\varepsilon).
\end{aligned} \tag{4.14}$$

Finally we eliminate the terms $X_1(r)^\top[\cdots]X_1(r)$, $\mathcal{X}_1(\theta,r)^\top[\cdots]X_1(r)$, and their transpose. Take $t=s$ in (4.12) and let

$$\Theta(\theta,\theta) \equiv P_3(\theta), \quad \Lambda(\theta,\theta,r) \equiv Q_3(\theta,r), \quad 0 \leqslant r \leqslant \theta \leqslant T.$$

Then, from (4.13) we derive

$$\begin{aligned}
&\mathbb{E}\int_0^T X_1(r)^\top \Theta(r,r)X_1(r)dr \\
&\quad = o(\varepsilon) + \mathbb{E}\int_0^T\int_r^T \Big\{ X_1(r)^\top \Big[A(\theta,r)^\top \Theta(\theta,\theta) + C(\theta,r)^\top \Lambda(\theta,\theta,r)\Big]\mathcal{X}_1(\theta,r) \\
&\qquad + \mathcal{X}_1(\theta,r)^\top \Big[A(\theta,r)^\top \Theta(\theta,\theta)^\top + C(\theta,r)^\top \Lambda(\theta,\theta,r)^\top\Big]^\top X_1(r) \\
&\qquad + X_1(r)^\top C(\theta,r)^\top \Theta(\theta,\theta)C(\theta,r)X_1(r) + D(\theta,r)^\top \Theta(\theta,\theta)D(\theta,r)\Big\}d\theta dr.
\end{aligned} \tag{4.15}$$

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Let

$$\Theta(\theta,\theta')=P_4(\theta,\theta')^\top,\quad \Lambda(\theta,r,\theta')=Q_4(\theta,r,\theta')^\top,\quad 0\leqslant\theta'\leqslant r\leqslant\theta\leqslant T.$$

Then, by (4.13) we get

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_r^T \mathcal{X}_1(\theta,r)^\top\Theta(\theta,r)X_1(r)d\theta dr\\
&\quad=\mathbb{E}\int_0^T\Big\{\int_r^T\int_\theta^T X_1(r)^\top\Big[A(\theta',r)^\top\Theta(\theta',\theta)+C(\theta',r)^\top\Lambda(\theta',\theta,r)\Big]\mathcal{X}_1(\theta,r)d\theta'd\theta\\
&\qquad+\int_r^T\int_r^\theta \mathcal{X}_1(\theta,r)^\top\Big[A(\theta',r)^\top\Theta(\theta,\theta')^\top+C(\theta',r)^\top\Lambda(\theta,\theta',r)^\top\Big]^\top X_1(r)d\theta'd\theta\\
&\qquad+\int_r^T\int_\theta^T\Big[X_1(r)^\top C(\theta',r)^\top\Theta(\theta',\theta)C(\theta,r)X_1(r)\\
&\qquad+D(\theta',r)^\top\Theta(\theta',\theta)D(\theta,r)\Big]d\theta'd\theta\Big\}dr+o(\varepsilon),
\end{aligned}$$

which, with (4.6), (4.14), (4.15) imply that (4.10) holds. ∎

Remark 4.3. Notice that in [30], the authors used the following form of $P_3(\cdot)$:

$$\begin{aligned}
P_3(r)=&\partial^2G(r)+\sum_{j=1}^d C^j(T,r)^\top P_1(r)C^j(T,r)\\
&+\sum_{j=1}^d\int_r^T\Big[C^j(T,r)^\top P_2(\theta)^\top C^j(\theta,r)+C^j(\theta,r)^\top P_2(\theta)C^j(T,r)\\
&+C^j(\theta,r)^\top P_3(\theta)C^j(\theta,r)\Big]d\theta+\int_r^T\int_\theta^T\Big[C^j(\theta',r)^\top P_4(\theta',\theta)^\top C^j(\theta,r)\\
&+C^j(\theta,r)^\top P_4(\theta',\theta)C^j(\theta',r)\Big]d\theta'd\theta-\sum_{j=1}^d\int_r^T Q_3^j(r,\theta)dW^j(\theta),\quad 0\leqslant r\leqslant T.
\end{aligned}$$

In fact, by the Fubini theorem and $P_4(\theta,r)=P_4(r,\theta)^\top$ for $0\leqslant\theta<r\leqslant T$, it is the same as the third equation in (4.8).

Remark 4.4. It is worth mentioning that the first-order adjoint equation (4.1), consisting of a BSDE and a backward SVIE, is dual with the first-order and second-order variational equations (3.5)–(3.6), and the second-order adjoint equation (4.8), consisting of a BSDE and three coupled backward SVIEs, has a slightly different duality with $(x_1(t)^\top,y_1(t)^\top,z_1(t)^\top)[\cdots](x_1(t)^\top,y_1(t)^\top,z_1(t)^\top)^\top$, even though the pointwise state delay appears in the state equation and the terminal cost.

Remark 4.5. To deal with the cross term $x_1(t)^\top[\cdots]y_1(t)$ and its transpose, [18] introduced a new BSDE but required its solution to be zero. In this paper, we get rid of this strict condition. To this end, the delayed variational equations (3.5)–(3.6) are transformed into the Volterra integral equations without delay (3.19)–(3.20), so that the original delayed finite-dimensional control problem is converted into another *higher yet finite-dimensional control problem without delay.* Then the auxiliary equation (4.11) is constructed and the set of backward SVIEs (4.8) is introduced to deal with the "cross terms" $X_1(r)^\top[\cdots]X_1(r)$ which contain $x_1(t)^\top[\cdots]y_1(t)$ and $y_1(t)^\top[\cdots]x_1(t)$ without any additional conditions.

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5. General maximum principle. In this section, we obtain a general maximum principle for Problem (P) and further express first-order and second-order adjoint equations in more compact forms.

5.1. General maximum principle. First let us make an interesting analysis of the second-order adjoint equation (4.8). By the solution of $P_i(\cdot)$, $i=1,2,3,4$, define

(5.1)
$$\mathcal{P}_0(r) := \aleph(T,r)^\top P_1(T)\aleph(T,r) + \int_r^T \Big[\aleph(T,r)^\top P_2(\theta)^\top \aleph(\theta,r) + \aleph(\theta,r)^\top P_2(\theta)\aleph(T,r)\Big]d\theta + \int_r^T\int_r^T \aleph(\theta,r)^\top P_4(\theta',\theta)\aleph(\theta',r)d\theta d\theta' + \int_r^T \aleph(\theta,r)^\top P_3(\theta)\aleph(\theta,r)d\theta,$$

where

$$\aleph(t,s) := \begin{bmatrix} I & \mathbf{1}_{(\delta,\infty)}(t-s)I & 0 \end{bmatrix}^\top. \tag{5.2}$$

Then, the "quadratic" form in (4.10) with respect to D^j can be concisely expressed as the following form:

$$\frac{1}{2}\sum_{j=1}^d \mathbb{E}\int_0^T \Delta\sigma^j(t)^\top \mathcal{P}_0(t)\Delta\sigma^j(t)dt. \tag{5.3}$$

In the above expression, $\mathcal{P}_0(\cdot)$ is directly constructed by $P_1(\cdot), P_2(\cdot), P_3(\cdot)$, and $P_4(\cdot,\cdot)$ with coefficients $A(\cdot,\cdot)$, $C^j(\cdot,\cdot)$ and so on in (3.18). However, we find that the explicit expressions in (3.18) have not been fully used, and the delay structure of (5.3) needs to be characterized more clearly. This inspires us to do the following calculations. In the following, we suppose $\tau\in[0,T)$ and define

$$P_k(\cdot) := \begin{Bmatrix} P_k^{(11)}(\cdot) & P_k^{(12)}(\cdot) & P_k^{(13)}(\cdot) \\ P_k^{(21)}(\cdot) & P_k^{(22)}(\cdot) & P_k^{(23)}(\cdot) \\ P_k^{(31)}(\cdot) & P_k^{(32)}(\cdot) & P_k^{(33)}(\cdot) \end{Bmatrix}, \quad k=1,2,3,4.$$

Case I*: The term of* (P_1,Q_1). By the definition of H, we see that

$$P_1^{(i\ell)}(r) = h_{\kappa^i\kappa^\ell}(T) - \sum_{j=1}^d \int_r^T Q_{1j}^{(i\ell)}(\theta)dW^j(\theta), \quad \tau\leqslant r\leqslant T, \tag{5.4}$$

where $i,\ell=1,2,3$, and $\kappa^1:=x$, $\kappa^2:=y$, $\kappa^3:=z$. In addition,

$$\begin{aligned} &D^j(T,t)^\top P_1(t)D^j(T,t) \\ &\quad = \Delta\sigma^j(t)^\top P_1^{(11)}(t)\Delta\sigma^j(t) + \Delta\sigma^j(t)^\top \\ &\qquad \times \big[P_1^{(12)}(t) + P_1^{(21)}(t) + P_1^{(22)}(t)\big]\Delta\sigma^j(t)\mathbf{1}_{(\delta,\infty)}(T-t). \end{aligned} \tag{5.5}$$

Case II*: The term of* (P_2,Q_2). Let us look at (P_2,Q_2) in (4.8),

$$P_2^{(i\ell)}(r) = \psi_2^{(i\ell)}(r) + \int_r^T g_2^{(i\ell)}(\theta,r)d\theta - \sum_{j=1}^d \int_r^T Q_{2j}^{(i\ell)}(r,\theta)dW^j(\theta), \quad \tau\leqslant r\leqslant T,$$

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where $i,\ell=1,2,3$. Set

$$\Big\{g_2^{(i\ell)}(\theta,r)\Big\}_{i,\ell=1}^{3}:=A(\theta,r)^\top P_2(\theta)+\sum_{j=1}^{d}C^j(\theta,r)^\top Q_2^j(\theta,r),$$
$$\Big\{\psi_2^{(i\ell)}(r)\Big\}_{i,\ell=1}^{3}:=A(T,r)^\top P_1(r)+\sum_{j=1}^{d}C^j(T,r)^\top Q_1^j(r).$$

For $j=1,\dots,d$, $\ell=1,2,3$, and $\kappa^1:=x$, $\kappa^2:=y$, $\kappa^3:=z$, define for $\tau\leqslant r\leqslant T$,

$$\mathcal{G}_2^{(\ell)}(r):=h_{x\kappa^\ell}(T)+\int_r^T P_2^{(1\ell)}(\theta)d\theta+\Big[h_{y\kappa^\ell}(T)+\int_{r+\delta}^{T}P_2^{(2\ell)}(\theta)d\theta\Big]\mathbf{1}_{[0,T-\delta)}(r),$$
$$\mathcal{Q}_{2j}^{(\ell)}(r):=Q_{1j}^{(1\ell)}(r)+\int_r^T Q_{2j}^{(1\ell)}(\theta,r)d\theta+\Big[Q_{1j}^{(2\ell)}(r)+\int_{r+\delta}^{T}Q_{2j}^{(2\ell)}(\theta,r)d\theta\Big]\mathbf{1}_{[0,T-\delta)}(r),$$
$$\mathcal{K}_2^{(\ell)}(r):=P_1^{(3\ell)}(r)+\int_r^T P_2^{(3\ell)}(\theta)d\theta.$$

Then, we deduce that for $\tau\leqslant r\leqslant T$,

$$P_2^{(1\ell)}(r)=\mathbb{E}_r\Big[b_x(r)^\top\mathcal{G}_2^{(\ell)}(r)+\sum_{j=1}^{d}\sigma_x^j(r)^\top\mathcal{Q}_{2j}^{(\ell)}(r)+\mathcal{K}_2^{(\ell)}(r)\Big],$$
$$P_2^{(2\ell)}(r)=\mathbb{E}_r\Big[b_y(r)^\top\mathcal{G}_2^{(\ell)}(r)+\sum_{j=1}^{d}\sigma_y^j(r)^\top\mathcal{Q}_{2j}^{(\ell)}(r)-e^{-\lambda\delta}\mathcal{K}_2^{(\ell)}(r)\Big],$$
$$P_2^{(3\ell)}(r)=\mathbb{E}_r\Big[b_z(r)^\top\mathcal{G}_2^{(\ell)}(r)+\sum_{j=1}^{d}\sigma_z^j(r)^\top\mathcal{Q}_{2j}^{(\ell)}(r)-\lambda\mathcal{K}_2^{(\ell)}(r)\Big]. \tag{5.6}$$

For the P_2 part in (4.10), we have

$$\begin{aligned}&\mathbb{E}_t\int_t^T\Big[D^j(T,t)^\top P_2(\theta)^\top D^j(\theta,t)+D^j(\theta,t)^\top P_2(\theta)D^j(T,t)\Big]d\theta\\&\quad=\Delta\sigma^j(t)^\top\mathbb{E}_t\Big[\int_t^T\Big(P_2^{(11)}(\theta)^\top+P_2^{(11)}(\theta)+[P_2^{(12)}(\theta)^\top+P_2^{(12)}(\theta)]\mathbf{1}_{[0,T-\delta)}(t)\Big)d\theta\\&\qquad+\int_{t+\delta}^{T}\Big[P_2^{(21)}(\theta)^\top+P_2^{(21)}(\theta)+P_2^{(22)}(\theta)^\top+P_2^{(22)}(\theta)\Big]d\theta\mathbf{1}_{[0,T-\delta)}(t)\Big]\Delta\sigma^j(t).\end{aligned} \tag{5.7}$$

Case III*: The term of* (P_4,Q_4). Let us look at (P_4,Q_4) in (4.8),

$$P_4^{(i\ell)}(\theta,r)=\psi_4^{(i\ell)}(\theta,r)+\int_r^T g_4^{(i\ell)}(\theta,\theta',r)d\theta'-\sum_{j=1}^{d}\int_r^T Q_{4j}^{(i\ell)}(\theta,r,\theta')dW^j(\theta'),$$

where $\tau\leqslant r\leqslant\theta\leqslant T$, $i,\ell=1,2,3$. Define

$$\begin{aligned}\Big\{\psi_4^{(i\ell)}(\theta,r)\Big\}_{i,\ell=1}^{3}:=&A(T,r)^\top P_2(\theta)^\top+\sum_{j=1}^{d}C^j(T,r)^\top Q_2^j(\theta,r)^\top\\&+A(\theta,r)^\top P_3(\theta)+\sum_{j=1}^{d}C^j(\theta,r)^\top Q_3^j(\theta,r),\end{aligned} \tag{5.8}$$

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$$\Big\{g_4^{(i\ell)}(\theta,\theta',r)\Big\}_{i,\ell=1}^{3} := A(\theta',r)^\top P_4(\theta,\theta') + \sum_{j=1}^{d} C^j(\theta',r)^\top Q_4^j(\theta,\theta',r). \tag{5.9}$$

For $\ell = 1,2,3$, $j = 1,\cdots,d$, and $\theta \geqslant r$, define

$$\begin{aligned}
\mathcal{G}_4^{(\ell)}(\theta,r) &:= P_2^{(\ell 1)}(\theta)^\top + P_3^{(1\ell)}(\theta) + \mathbf{1}_{(\delta,\infty)}(\theta-r)P_3^{(2\ell)}(\theta) + \mathbf{1}_{(\delta,\infty)}(T-r)P_2^{(\ell 2)}(\theta)^\top \\
&\quad + \int_r^T P_4^{(1\ell)}(\theta,\theta')d\theta' + \mathbf{1}_{(\delta,\infty)}(T-r)\int_{r+\delta}^T P_4^{(2\ell)}(\theta,\theta')d\theta', \\
\mathcal{Q}_{4j}^{(\ell)}(\theta,r) &:= Q_{2j}^{(\ell 1)}(\theta,r)^\top + Q_{3j}^{(1\ell)}(\theta,r) + \mathbf{1}_{(\delta,\infty)}(\theta-r)Q_{3j}^{(2\ell)}(\theta,r) + \mathbf{1}_{(\delta,\infty)}(T-r) \\
&\quad \times Q_{2j}^{(\ell 2)}(\theta,r)^\top + \int_r^T Q_{4j}^{(1\ell)}(\theta,\theta',r)d\theta' \\
&\quad + \int_{r+\delta}^T Q_{4j}^{(2\ell)}(\theta,\theta',r)d\theta' \mathbf{1}_{(\delta,+\infty)}(T-r),
\end{aligned} \tag{5.10}$$

$$\mathcal{K}_4^{(\ell)}(\theta,r) := P_2^{(\ell 3)}(\theta)^\top + P_3^{(3\ell)}(\theta) + \int_r^T P_4^{3\ell}(\theta,\theta')d\theta'.$$

Then, for $\theta \geqslant r$, we have

$$\begin{aligned}
P_4^{(1\ell)}(\theta,r) &= \mathbb{E}_r\Big[b_x(r)^\top \mathcal{G}_4^{(\ell)}(\theta,r) + \sum_{j=1}^d \sigma_x^j(r)^\top \mathcal{Q}_{4j}^{(\ell)}(\theta,r) + \mathcal{K}_4^{(\ell)}(\theta,r)\Big], \\
P_4^{(2\ell)}(\theta,r) &= \mathbb{E}_r\Big[b_y(r)^\top \mathcal{G}_4^{(\ell)}(\theta,r) + \sum_{j=1}^d \sigma_y^j(r)^\top \mathcal{Q}_{4j}^{(\ell)}(\theta,r) - e^{-\lambda\delta}\mathcal{K}_4^{(\ell)}(\theta,r)\Big], \\
P_4^{(3\ell)}(\theta,r) &= \mathbb{E}_r\Big[b_z(r)^\top \mathcal{G}_4^{(\ell)}(\theta,r) + \sum_{j=1}^d \sigma_z^j(r)^\top \mathcal{Q}_{4j}^{(\ell)}(\theta,r) - \lambda\mathcal{K}_4^{(\ell)}(\theta,r)\Big].
\end{aligned} \tag{5.11}$$

For $\theta < r$, set

$$P_4^{(i\ell)}(\theta,r) := P_4^{(\ell i)}(r,\theta)^\top, \quad Q_4^{(i\ell)}(\theta,\theta',r) := Q_4^{(\ell i)}(\theta',\theta,r)^\top, \quad i,\ell = 1,2,3.$$

Next we look at the P_4 part in (4.10). Denote

$$\begin{aligned}
\mathcal{P}_4(t) &:= \int_t^T\int_t^T P_4^{(11)}(\theta',\theta)d\theta d\theta' + \Big(\int_{t+\delta}^T\int_t^T P_4^{(12)}(\theta',\theta)d\theta d\theta' \\
&\quad + \int_t^T\int_{t+\delta}^T P_4^{(21)}(\theta',\theta)d\theta d\theta' + \int_{t+\delta}^T\int_{t+\delta}^T P_4^{(22)}(\theta',\theta)d\theta d\theta'\Big)\mathbf{1}_{[0,T-\delta)}(t).
\end{aligned}$$

Then, we have

$$\mathbb{E}_t\Big[\int_t^T\int_t^T D^j(\theta,t)^\top P_4(\theta',\theta)D^j(\theta',t)d\theta d\theta'\Big] = \Delta\sigma^j(t)^\top \mathbb{E}_t\big[\mathcal{P}_4(t)\big]\Delta\sigma^j(t). \tag{5.12}$$

Case IV*: The term of* (P_3, Q_3). Now, let us look at (P_3, Q_3) in (4.8),

$$P_3^{(i\ell)}(r) = \psi_3^{(i\ell)}(r) + \int_r^T g_3^{(i\ell)}(\theta,r)d\theta - \sum_{j=1}^d \int_r^T Q_{3j}^{(i\ell)}(r,\theta)dW^j(\theta), \ \tau \leqslant r \leqslant T, \ i,\ell = 1,2,3.$$

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Define

$$\begin{aligned}\Big\{\psi_3^{(i\ell)}(r)\Big\}_{i,\ell=1}^3 &:= \partial^2 G(r) + \sum_{j=1}^d C^j(T,r)^\top P_1(r) C^j(T,r) \\ &\quad + \sum_{j=1}^d \int_r^T \big[C^j(T,r)^\top P_2(\theta)^\top C^j(\theta,r) + C^j(\theta,r)^\top P_2(\theta) C^j(T,r)\big] d\theta \\ &\quad + \sum_{j=1}^d \int_r^T \int_r^T C^j(\theta,r)^\top P_4(\theta',\theta) C^j(\theta',r) d\theta d\theta', \\ \Big\{g_3^{(i\ell)}(\theta,r)\Big\}_{i,\ell=1}^3 &:= \sum_{j=1}^d C^j(\theta,r)^\top P_3(\theta) C^j(\theta,r).\end{aligned}$$

Then, for $\ell = 1,2,3$, and $\kappa^1 := x$, $\kappa^2 := y$, $\kappa^3 := z$, we have

$$\begin{aligned} P_3^{(1\ell)}(r) &= G_{x\kappa^\ell}(r) + \sum_{j=1}^d \sigma_x^j(r)^\top \mathbb{E}_r\big[\mathcal{P}(r)\big] \sigma_{\kappa^\ell}^j(r), \quad \tau \leqslant r \leqslant T, \\ P_3^{(2\ell)}(r) &= G_{y\kappa^\ell}(r) + \sum_{j=1}^d \sigma_y^j(r)^\top \mathbb{E}_r\big[\mathcal{P}(r)\big] \sigma_{\kappa^\ell}^j(r), \quad \tau \leqslant r \leqslant T, \\ P_3^{(3\ell)}(r) &= G_{z\kappa^\ell}(r) + \sum_{j=1}^d \sigma_z^j(r)^\top \mathbb{E}_r\big[\mathcal{P}(r)\big] \sigma_{\kappa^\ell}^j(r), \quad \tau \leqslant r \leqslant T, \end{aligned} \tag{5.13}$$

where

$$\begin{aligned} \mathcal{P}(r) &:= h_{xx}(T) + \big[h_{yx}(T) + h_{xy}(T) + h_{yy}(T)\big] \mathbf{1}_{[0,T-\delta)}(r) \\ &\quad + \int_r^T \big[P_2^{(11)}(\theta)^\top + P_2^{(11)}(\theta)\big] d\theta + \int_r^T \big[P_2^{(12)}(\theta)^\top + P_2^{12}(\theta)\big] d\theta \mathbf{1}_{[0,T-\delta)}(r) \\ &\quad + \int_{r+\delta}^T \big[P_2^{(21)}(\theta)^\top + P_2^{(21)}(\theta) + P_2^{(22)}(\theta)^\top + P_2^{(22)}(\theta)\big] d\theta \mathbf{1}_{[0,T-\delta)}(r) \\ &\quad + \int_r^T \int_r^T P_4^{(11)}(\theta',\theta) d\theta d\theta' + \bigg\{ \int_{r+\delta}^T \int_r^T P_4^{(12)}(\theta',\theta) d\theta d\theta' \\ &\quad + \int_r^T \int_{r+\delta}^T P_4^{(21)}(\theta',\theta) d\theta d\theta' + \int_{r+\delta}^T \int_{r+\delta}^T P_4^{(22)}(\theta',\theta) d\theta d\theta' \bigg\} \mathbf{1}_{[0,T-\delta)}(r) \\ &\quad + \int_r^T P_3^{(11)}(\theta) d\theta + \int_{r+\delta}^T \big[P_3^{(21)}(\theta) + P_3^{(12)}(\theta) + P_3^{(22)}(\theta)\big] d\theta \mathbf{1}_{[0,T-\delta)}(r) \\ &= \mathcal{G}_2^{(1)}(r) + \int_r^T \mathcal{G}_4^{(1)}(\theta',r) d\theta' + \bigg[\mathcal{G}_2^{(2)}(r) + \int_{r+\delta}^T \mathcal{G}_4^{(2)}(\theta',r) d\theta'\bigg] \mathbf{1}_{[0,T-\delta)}(r). \end{aligned} \tag{5.14}$$

With $\mathcal{P}_0(\cdot)$ defined by (5.1), we point out that $\mathcal{P}(r) = \mathcal{P}_0(r)$. Next, for the P_3 part in (4.10), we have

$$\begin{aligned} \mathbb{E}_t\bigg[\int_t^T D^j(\theta,t)^\top P_3(\theta) D^j(\theta,t) d\theta\bigg] &= \Delta\sigma^j(t)^\top \mathbb{E}_t\bigg\{ \int_t^T P_3^{(11)}(\theta) d\theta \\ &\quad + \int_{t+\delta}^T \Big[P_3^{(12)}(\theta) + P_3^{(21)}(\theta) + P_3^{(22)}(\theta)\Big] d\theta \mathbf{1}_{(\delta,\infty)}(T-t) \bigg\} \Delta\sigma^j(t). \end{aligned} \tag{5.15}$$

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Based on the above preparation, now we are in a position to state the general maximum principle for Problem (P). Recall (4.5) and define the Hamiltonian function $\mathcal{H}:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{S}^n\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ as follows:

$$\begin{aligned}\mathcal{H}(\tau,x,y,z,p,q,\mathcal{P},u,\mu):=G(\tau,x,y,z,p,q,u,\mu)+\sum_{j=1}^{d}\big(\sigma^j(\tau,x,y,z,u,\mu)\\ -\sigma^j(\tau,\Theta(\tau))\big)^\top\mathcal{P}\big(\sigma^j(\tau,x,y,z,u,\mu)-\sigma^j(\tau,\Theta(\tau))\big).\end{aligned}$$

THEOREM 5.1. *Let assumption* (A1) *hold. Suppose* $(x^*(\cdot),u^*(\cdot))$ *is an optimal pair,* $(\eta(\cdot),\zeta(\cdot),Y(\cdot),Z(\cdot,\cdot))$ *is the solution to* (4.1)*,* $(p(\cdot),q(\cdot))$ *and* $\mathcal{P}(\cdot)$ *are defined by* (4.4) *and* (5.14)*,* $(P_1(\cdot),P_2(\cdot),P_3(\cdot),P_4(\cdot,\cdot))$ *is the solution to* (4.8)–(4.9)*. Then, the following maximum condition holds:*

$$\Delta\mathcal{H}(\tau)+\mathbb{E}_\tau\big[\Delta\tilde{\mathcal{H}}(\tau+\delta)\mathbf{1}_{[0,T-\delta)}(\tau)\big]\geqslant 0\quad\forall\ v\in U,\quad\text{a.e., a.s.,}\tag{5.16}$$

where

$$\begin{aligned}\Delta\mathcal{H}(\tau):=&\,\mathcal{H}(\tau,x^*(\tau),y^*(\tau),z^*(\tau),p(\tau),q(\tau),\mathcal{P}(\tau),v,\mu^*(\tau))\\ &-\mathcal{H}(\tau,x^*(\tau),y^*(\tau),z^*(\tau),p(\tau),q(\tau),\mathcal{P}(\tau),u^*(\tau),\mu^*(\tau)),\\ \Delta\tilde{\mathcal{H}}(\tau):=&\,\mathcal{H}(\tau,x^*(\tau),y^*(\tau),z^*(\tau),p(\tau),q(\tau),\mathcal{P}(\tau),u^*(\tau),v)\\ &-\mathcal{H}(\tau,x^*(\tau),y^*(\tau),z^*(\tau),p(\tau),q(\tau),\mathcal{P}(\tau),u^*(\tau),\mu^*(\tau)).\end{aligned}$$

Proof. By Lemma 4.2, (5.5), (5.7), (5.12), (5.15), and (5.14), we obtain

$$\begin{aligned}J(u^\varepsilon(\cdot))-J(u^*(\cdot))=&\,\mathbb{E}\int_\tau^{\tau+\varepsilon}\Delta G(t)dt+\mathbb{E}\int_{\tau+\delta}^{\tau+\delta+\varepsilon}\Delta\tilde{G}(t)dt\mathbf{1}_{[0,T-\delta)}(\tau)\\ &+\frac{1}{2}\sum_{j=1}^{d}\mathbb{E}\int_0^T\Delta\sigma^j(t)^\top\mathcal{P}(t)\Delta\sigma^j(t)dt+o(\varepsilon).\end{aligned}$$

Thus, similarly to the proof of Theorem 4.1 in [18], we complete the proof. ∎

Remark 5.2. Since $u(t)$ and $u(t-\delta)$ appear in the diffusion term, the second-order adjoint equations are required to deal with Problem (P). To this end, the cross terms, such as $x_1(t)^\top[\cdots]y_1(t)$, bring some essential difficulties and prompt us to provide some novel methods; see Remark 4.5. Also due to the appearance of $u(t-\delta)$ in Problem (P), the general maximum principle (5.16) includes two parts: $\mathbb{E}_\tau[\Delta\tilde{\mathcal{H}}(\tau+\delta)]$ characterizes the maximum condition with delay, while $\Delta\mathcal{H}(\tau)$ characterizes the one without delay which naturally reduces to the SDEs case in [35].

Remark 5.3. Compared with [18], (i) when the distributed delay appears in the control system, the general maximum principle of optimal control for stochastic differential delay systems can be obtained; (ii) the maximum condition (5.16) is similar to (5.6) in [18], but all the additional requirements in [18] are removed; (iii) a new set of backward SVIEs (4.8) is introduced to deal with the cross term, instead of the special BSDE (5.3) in [18].

5.2. Extensions of adjoint equations. In this subsection, we further explore the first-order and second-order adjoint equations (4.1) and (4.8). Interestingly, under some cases, (4.1) and (4.8) have more compact forms, similar to the existing literature [18, 35, 38].

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5.2.1. Extensions of first-order adjoint equations. We rewrite (4.4), and define $(\tilde{p}(\cdot),\tilde{q}(\cdot))$ as follows: for $j=1,\dots,d$,

(5.17)
$$\begin{cases} p(t):=\eta^0(t)+\eta^1(t)\ \mathbf{1}_{[0,T-\delta)}(t)+\mathbb{E}_t\Big[\displaystyle\int_t^T Y^0(s)ds+\int_{t+\delta}^T Y^1(s)ds\mathbf{1}_{[0,T-\delta)}(t)\Big],\\ q^j(t):=\zeta^{0j}(t)+\zeta^{1j}(t)\ \mathbf{1}_{[0,T-\delta)}(t)+\displaystyle\int_t^T Z^{0j}(s,t)ds+\int_{t+\delta}^T Z^{1j}(s,t)ds\mathbf{1}_{[0,T-\delta)}(t),\\ \tilde{p}(t):=\mathbb{E}_t\Big[\displaystyle\int_t^T Y^2(s)ds\Big]+\eta^2(t),\quad \tilde{q}^j(t):=\int_t^T Z^{2j}(s,t)ds+\zeta^{2j}(t). \end{cases}$$

Now we link the first-order adjoint equation (4.1) with a set of anticipated BSDEs.

THEOREM 5.4. *Let assumption* (A1) *hold. Suppose* $(x^*(\cdot),u^*(\cdot))$ *is an optimal pair,* $(\eta(\cdot),\zeta(\cdot),Y(\cdot),Z(\cdot,\cdot))$ *is the solution to* (4.1)*. Then,* $(p(\cdot),q(\cdot),\tilde{p}(\cdot),\tilde{q}(\cdot))$ *defined by* (5.17) *satisfies the following set of anticipated BSDEs:*

(5.18)
$$\begin{cases} p(t)=h_x(T)^\top+\displaystyle\int_t^T\Big\{b_x(s)^\top p(s)+\sum_{j=1}^d\sigma_x^j(s)^\top q^j(s)+l_x(s)^\top+\tilde{p}(s)\Big\}ds\\ \qquad\quad -\displaystyle\sum_{j=1}^d\int_t^T q^j(s)dW^j(s),\quad t\in[T-\delta,T],\\ p(t)=p(T-\delta)+\mathbb{E}_{T-\delta}\big[h_y(T)^\top\big]+\displaystyle\int_t^{T-\delta}\Big\{b_x(s)^\top p(s)+\sum_{j=1}^d\sigma_x^j(s)^\top q^j(s)\\ \qquad\quad +l_x(s)^\top+\tilde{p}(s)+\mathbb{E}_s\big[b_y(s+\delta)^\top p(s+\delta)+\displaystyle\sum_{j=1}^d\sigma_y^j(s+\delta)^\top q^j(s+\delta)\\ \qquad\quad +l_y(s+\delta)^\top-e^{-\lambda\delta}\tilde{p}(s+\delta)\big]\Big\}ds-\displaystyle\sum_{j=1}^d\int_t^{T-\delta}q^j(s)dW^j(s),t\in[0,T-\delta),\\ \tilde{p}(t)=h_z(T)^\top+\displaystyle\int_t^T\Big\{b_z(s)^\top p(s)+\sum_{j=1}^d\sigma_z^j(s)^\top q^j(s)+l_z(s)^\top-\lambda\tilde{p}(s)\Big\}ds\\ \qquad\quad -\displaystyle\sum_{j=1}^d\int_t^T\tilde{q}^j(s)dW^j(s),\quad t\in[0,T]. \end{cases}$$

Proof. The first two equations of (5.18) can be unified as follows:

$$\begin{aligned} p(t)=h_x(T)^\top&+\mathbb{E}_{T-\delta}\big[h_y(T)^\top\mathbf{1}_{[0,T-\delta)}(t)\big]+\int_t^T\Big\{l_x(s)^\top+b_x(s)^\top p(s)\\ &+\sum_{j=1}^d\sigma_x^j(s)^\top q^j(s)+\tilde{p}(s)+\mathbb{E}_s\big[b_y(s+\delta)^\top p(s+\delta)+\sum_{j=1}^d\sigma_y^j(s+\delta)^\top q^j(s+\delta)\\ &+l_y(s+\delta)^\top-e^{-\lambda\delta}\tilde{p}(s+\delta)\big]\mathbf{1}_{[0,T-\delta)}(s)\Big\}ds-\sum_{j=1}^d\int_t^T q^j(s)dW^j(s),t\in[0,T]. \end{aligned}$$

For simplicity, in the following, without loss of generality, let $d=1$. By (4.2) and taking the conditional expectation on both sides of (4.1), it follows that for $0\leqslant t\leqslant T$,

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$$\begin{aligned}\mathbb{E}_t\big[Y^0(t)+Y^1(t+\delta)\mathbf{1}_{[0,T-\delta)}(t)\big] &= b_x(t)^\top p(t)+\sigma_x(t)^\top q(t)+l_x(t)^\top+\tilde{p}(t)\\ &\quad+\mathbb{E}_t\big[b_y(t+\delta)^\top p(t+\delta)+\sigma_y(t+\delta)^\top q(t+\delta)\\ &\quad+l_y(t+\delta)^\top-e^{-\lambda\delta}\tilde{p}(t+\delta)\big]\mathbf{1}_{[0,T-\delta)}(t),\end{aligned}$$

and

$$Y^2(t)=b_z(t)^\top p(t)+\sigma_z(t)^\top q(t)+l_z(t)^\top-\lambda\tilde{p}(t).$$

Noting

$$\begin{aligned}&\mathbb{E}_s\int_t^T\int_t^{s+\delta}Z^1(s+\delta,r)dW(r)\mathbf{1}_{[0,T-\delta)}(s)ds\\ &=\mathbb{E}_s\int_t^T\left(\int_t^s+\int_s^{s+\delta}\right)Z^1(s+\delta,r)dW(r)\mathbf{1}_{[0,T-\delta)}(s)ds\\ &=\int_t^T\mathbb{E}_s\bigg[\int_t^s Z^1(s+\delta,r)dW(r)\mathbf{1}_{[0,T-\delta)}(s)\bigg]ds\\ &=\int_t^{T-\delta}\int_t^s Z^1(s+\delta,r)dW(r)ds\mathbf{1}_{[0,T-\delta)}(t),\end{aligned}$$

from (4.1c), one has

$$\begin{aligned}&\int_t^T\mathbb{E}_s\big[Y^0(s)+Y^1(s+\delta)\mathbf{1}_{[0,T-\delta)}(s)\big]ds\\ &=\int_t^T\mathbb{E}_s\bigg[\mathbb{E}_t\big[Y^0(s)\big]+\int_t^s Z^0(s,r)dW(r)+\mathbb{E}_t\big[Y^1(s+\delta)\mathbf{1}_{[0,T-\delta)}(s)\big]\\ &\quad+\int_t^{s+\delta}Z^1(s+\delta,r)dW(r)\mathbf{1}_{[0,T-\delta)}(s)\bigg]ds\\ &=\int_t^T\mathbb{E}_t\big[Y^0(s)+Y^1(s+\delta)\mathbf{1}_{[0,T-\delta)}(s)\big]ds\\ &\quad+\int_t^T\bigg[\int_r^T Z^0(s,r)ds+\int_r^{T-\delta}Z^1(s+\delta,r)ds\mathbf{1}_{[0,T-\delta)}(r)\mathbf{1}_{[0,T-\delta)}(t)\bigg]dW(r),\end{aligned}$$

and

$$\begin{aligned}\int_t^T Y^2(s)ds&=\int_t^T\bigg[\mathbb{E}_t\big[Y^2(s)\big]+\int_t^s Z^2(s,r)dW(r)\bigg]ds\\ &=\mathbb{E}_t\bigg[\int_t^T Y^2(s)ds\bigg]+\int_t^T\int_s^T Z^2(r,s)drdW(s).\end{aligned}$$

Recalling (4.1a), one gets

$$\begin{aligned}&\eta^0(t)+\int_t^T\mathbb{E}_t\Big[Y^0(s)+Y^1(s+\delta)\mathbf{1}_{[0,T-\delta)}(s)\mathbf{1}_{[0,T-\delta)}(t)\Big]ds+\eta^1(t)\mathbf{1}_{[0,T-\delta)}(t)\\ &+\int_t^T\bigg\{\zeta^0(r)+\int_r^T\Big[Z^0(r,s)+Z^1(r+\delta,s)\mathbf{1}_{[0,T-\delta)}(s)\mathbf{1}_{[0,T-\delta)}(r)\mathbf{1}_{[0,T-\delta)}(t)\Big]dr\\ &+\zeta^1(s)\mathbf{1}_{[0,T-\delta)}(s)\bigg\}dW(s)=h_x(T)^\top+\int_t^T\Big\{b_x(s)^\top p(s)+\sigma_x(s)^\top q(s)+l_x(s)^\top\\ &+\tilde{p}(s)+\mathbb{E}_s\Big[b_y(s+\delta)^\top p(s+\delta)\quad+\sigma_y(s+\delta)^\top q(s+\delta)+l_y(s+\delta)^\top\\ &-e^{-\lambda\delta}\tilde{p}(s+\delta)\Big]\mathbf{1}_{[0,T-\delta)}(s)\Big\}ds+\mathbb{E}_{T-\delta}\big[h_y(T)^\top\mathbf{1}_{[0,T-\delta)}(t)\big],\end{aligned}$$

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and

$$\eta^2(t)+\int_t^T \mathbb{E}_t\big[Y^2(s)\big]ds+\int_t^T\Big[\zeta^2(r)+\int_r^T Z^2(s,r)ds\Big]dW(r)$$
$$=h_z(T)^\top+\int_t^T\Big[b_z(s)^\top p(s)+\sigma_z(s)^\top q(s)+l_z(s)^\top-\lambda\tilde{p}(s)\Big]ds.$$

Thus, the proof is completed. ∎

Remark 5.5. The pointwise state delay appears in the terminal cost, thus the equation satisfied by $(p(\cdot),q(\cdot))$ is split into two parts, for $t<T-\delta$ and $t>T-\delta$. Furthermore, the set of anticipated BSDEs (5.18) can be arranged into an anticipated backward SVIE. Since the third equation of (5.18) is a linear BSDE, $p(\cdot)$ and $q(\cdot)$ can solve $\tilde{p}(\cdot)$, then along with the first two equations of (5.18), $(p(\cdot),q(\cdot))$ satisfies the following anticipated backward SVIE:

$$\begin{aligned}p(t)=&\,h_x(T)^\top+\mathbb{E}_{T-\delta}\big[h_y(T)^\top\mathbf{1}_{[0,T-\delta)}(t)\big]\\&+\frac{1}{\lambda}\Big(1-e^{\lambda((-\delta)\vee(t-T))}\Big)h_z(T)^\top+\int_t^T\Big\{b_x(s)^\top p(s)\\&+\sum_{j=1}^d\sigma_x^j(s)^\top q^j(s)+l_x(s)^\top+\mathbb{E}_s\Big[b_y(s+\delta)^\top p(s+\delta)\\&+\sum_{j=1}^d\sigma_y^j(s+\delta)^\top q^j(s+\delta)+l_y(s+\delta)^\top\Big]\mathbf{1}_{[0,T-\delta)}(s)\\&+\mathbb{E}_s\Big[\int_0^{\delta\wedge(T-s)}e^{-\lambda\theta}\Big(b_z(s+\theta)^\top p(s+\theta)\\&+\sum_{j=1}^d\sigma_z^j(s+\theta)^\top q^j(s+\theta)+l_z(s+\theta)^\top\Big)d\theta\Big]\Big\}ds\\&-\sum_{j=1}^d\int_t^T q^j(s)dW^j(s),\ t\in[0,T].\end{aligned}\tag{5.19}$$

Now the first-order adjoint equation (4.1) can be unified as the anticipated backward SVIE (5.19), and is dual with the variational equations (3.5)–(3.6), when the pointwise state delay appears in the terminal cost. This is a new finding. Moreover, when only point delay and distributed delay cases are considered in [7], (5.19) is consistent with their adjoint equation (5).

Remark 5.6. Let $h_y\equiv 0$. Then, by Theorem 5.4, $(p(\cdot),q(\cdot))$ is the unique solution to the following set of anticipated BSDEs:

$$\begin{cases}-dp(t)=\Big\{b_x(t)^\top p(t)+\sum_{j=1}^d\sigma_x^j(t)^\top q^j(t)+l_x(t)^\top+\tilde{p}(t)+\mathbb{E}_t\Big[l_y(t+\delta)^\top+b_y(t+\delta)^\top\\\quad\times p(t+\delta)+\sum_{j=1}^d\sigma_y^j(t+\delta)^\top q^j(t+\delta)-e^{-\lambda\delta}\tilde{p}(t+\delta)\Big]\mathbf{1}_{[0,T-\delta)}(t)\Big\}dt-\sum_{j=1}^d q^j(t)dW^j(t),\\-d\tilde{p}(t)=\Big\{b_z(t)^\top p(t)+\sum_{j=1}^d\sigma_z^j(t)^\top q^j(t)+l_z(t)^\top-\lambda\tilde{p}(t)\Big\}dt-\sum_{j=1}^d\tilde{q}^j(t)dW^j(t),\\p(T)=h_x(T)^\top,\quad\tilde{p}(T)=h_z(T)^\top.\end{cases}\tag{5.20}$$

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Notice that [38] assumed that the control domain is convex, and studied the sufficient maximum principle for stochastic optimal control problems with general delay. Let the noisy memory process there disappear, i.e., $X_2^u(\cdot)\equiv 0$. Then, (10)–(11) in [38] are the same as (5.20) above.

Remark 5.7. Let $h_y, h_z \equiv 0$. Then, (5.19) becomes a simple anticipated BSDE consistent with (5.1) in [18], when the distributed delay disappears in Problem (P).

5.2.2. Extensions of second-order adjoint equations. In the subsection, we study three typical control systems to display second-order adjoint equations clearly.

Case I: Stochastic optimal control problems without delay.

In this case, Problem (P) becomes a classical stochastic optimal control problem. From (5.4), (5.6), (5.11), and (5.13), $P_1^{(11)}(r)$, $P_2^{(11)}(r)$, $P_3^{(11)}(r)$, $P_4^{(11)}(\theta,r)\neq 0$, $0\leqslant r,\theta\leqslant T$, and other terms in (4.8) are all 0. Then, (5.14) becomes

$$\begin{aligned}\mathcal{P}^1(s)\equiv\mathcal{P}(s)=h_{xx}(T)+\int_s^T\Big\{&\mathbb{E}_r\Big[b_x(r)^\top\mathcal{G}_2^{(1)}(r)+\sum_{j=1}^d\sigma_x^j(r)^\top\mathcal{Q}_{2j}^{(1)}(r)\\&+\mathcal{G}_2^{(1)}(r)^\top b_x(r)+\sum_{j=1}^d\mathcal{Q}_{2j}^{(1)}(r)^\top\sigma_x^j(r)\Big]+b_x(r)^\top\mathbb{E}_r\Big[\int_r^T\mathcal{G}_4^{(1)}(\theta,r)d\theta\Big]\\&+\sum_{j=1}^d\sigma_x^j(r)^\top\int_r^T\mathcal{Q}_{4j}^{(1)}(\theta,r)d\theta+\Big[\int_r^T\mathbb{E}_r\big[\mathcal{G}_4^{(1)}(\theta,r)^\top\big]d\theta\Big]b_x(r)\\&+\sum_{j=1}^d\Big[\int_r^T\mathcal{Q}_{4j}^{(1)}(\theta,r)^\top d\theta\Big]\sigma_x^j(r)+G_{xx}(r)+\sum_{j=1}^d\sigma_x^j(r)^\top\mathbb{E}_r\big[\mathcal{P}(r)\big]\sigma_x^j(r)\Big\}dr.\end{aligned}$$

Denote

$$\bar P^1(s):=\mathbb{E}_s\big[\mathcal{P}^1(s)\big],\quad \bar Q^1(s):=\mathcal{Q}_2^{(1)}(s)+\int_s^T\mathcal{Q}_4^{(1)}(\theta',s)d\theta'. \tag{5.21}$$

Then, $(\bar P^1(\cdot),\bar Q^1(\cdot))$ satisfies the following BSDE,

$$\begin{aligned}\bar P^1(s)=h_{xx}(T)+\int_s^T\Big\{&b_x(t)^\top\bar P^1(t)+\sum_{j=1}^d\sigma_x^j(t)^\top\bar Q_j^1(t)+\bar P^1(t)^\top b_x(t)\\&+\sum_{j=1}^d\bar Q_j^1(t)^\top\sigma_x^j(t)+l_{xx}(t)+\big\langle p(t),b_{xx}(t)\big\rangle+\sum_{j=1}^d\big\langle q^j(t),\sigma_{xx}^j(t)\big\rangle\\&+\sum_{j=1}^d\sigma_x^j(t)^\top\bar P^1(t)\sigma_x^j(t)\Big\}dt-\int_s^T\sum_{j=1}^d\bar Q_j^1(t)dW^j(t),\quad s\in[0,T],\end{aligned} \tag{5.22}$$

which is consistent with (3.9) in [35].

In fact, we have

$$\mathcal{P}^1(s)=\mathcal{G}_2^{(1)}(s)+\int_s^T\mathcal{G}_4^{(1)}(\theta',s)d\theta',$$

and by (5.21),

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(5.23)

$$\mathcal{P}^1(s)=h_{xx}(T)+\int_s^T\Big\{b_x(t)^\top\bar P^1(t)+\sum_{j=1}^d\sigma_x^j(t)^\top\bar Q_j^1(t)+\bar P^1(t)^\top b_x(t)+\sum_{j=1}^d\bar Q_j^1(t)^\top\sigma_x^j(t)$$
$$+\big\langle p(t),b_{xx}(t)\big\rangle+\sum_{j=1}^d\big\langle q^j(t),\sigma_{xx}^j(t)\big\rangle+l_{xx}(t)+\sum_{j=1}^d\sigma_x^j(t)^\top\bar P^1(t)\sigma_x^j(t)\Big\}dt.$$

By the first equation of (4.9), we have

$$h_{xx}(T)=\mathbb{E}_s\big[h_{xx}(T)\big]+\sum_{j=1}^d\int_s^T Q_{1j}^{(11)}(r)dW^j(r),\quad s\in[0,T]. \tag{5.24}$$

Noting (4.9), for $k=2,3,\ i,\ell=1,2,3$, we get

$$\int_s^T P_k^{(i\ell)}(\theta)^\top d\theta=\mathbb{E}_s\bigg[\int_s^T P_k^{(i\ell)}(\theta)^\top d\theta\bigg]+\sum_{j=1}^d\int_s^T\int_s^\theta Q_{kj}^{(i\ell)}(\theta,r)^\top dW^j(r)d\theta$$
$$=\mathbb{E}_s\bigg[\int_s^T P_k^{(i\ell)}(\theta)^\top d\theta\bigg]+\sum_{j=1}^d\int_s^T\int_r^T Q_{kj}^{(i\ell)}(\theta,r)^\top d\theta dW^j(r), \tag{5.25}$$

$$\int_s^T\int_s^T P_4^{(i\ell)}(\theta',\theta)d\theta d\theta'=\int_s^T\int_s^T\mathbb{E}_s\big[P_4^{(i\ell)}(\theta',\theta)\big]d\theta d\theta'$$
$$+\sum_{j=1}^d\int_s^T\int_r^T\int_r^T Q_{4j}^{(i\ell)}(\theta',\theta,r)d\theta d\theta' dW^j(r). \tag{5.26}$$

From (5.23)–(5.26), we obtain

$$\mathcal{P}^1(s)=\bar P^1(s)+\sum_{j=1}^d\int_s^T\bar Q_j^1(t)dW^j(t),\quad s\in[0,T],$$

which implies (5.22).

Case II: Stochastic optimal control problems with control delay only.
In this case, $b_y,b_z,\sigma_y,\sigma_z,l_y,l_z,h_y,h_z=0$. From (5.4), (5.6), (5.11) and (5.13), we have

$$P_1^{(11)}(r),\ P_2^{(11)}(r),\ P_3^{(11)}(r),P_4^{(11)}(\theta,r),\ P_4^{(12)}(\theta,r)\neq0,\quad 0\leqslant r\leqslant\theta\leqslant T,$$
$$P_4^{(11)}(\theta,r),\ P_4^{(21)}(\theta,r)\neq0,\quad 0\leqslant\theta<r\leqslant T,$$

and other terms in (4.8) are all 0. From (5.8) and (5.9), we obtain

$$\psi_4^{(12)}(\theta,r)=0,\quad g_4^{(12)}(\theta,\theta',r)=b_x(r)^\top P_4^{(12)}(\theta,\theta')+\sum_{j=1}^d\sigma_x^j(r)^\top Q_{4j}^{(12)}(\theta,\theta',r)$$

and, then, for $\theta\geqslant r$,

$$P_4^{(12)}(\theta,r)=\int_r^T\big[b_x(r)^\top P_4^{(12)}(\theta,\theta')+\sum_{j=1}^d\sigma_x^j(r)^\top Q_{4j}^{(12)}(\theta,\theta',r)\big]d\theta'$$
$$-\sum_{j=1}^d\int_r^T Q_{4j}^{(12)}(\theta,r,\theta')dW^j(\theta'). \tag{5.27}$$

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On the other hand, recalling (4.9), for $\theta \geqslant r$, we have

$$P_4^{(12)}(\theta,r) = \mathbb{E}_{\theta'}\big[P_4^{(12)}(\theta,r)\big] + \sum_{j=1}^{d}\int_{\theta'}^{r} Q_{4j}^{(12)}(\theta,r,s)dW^j(s). \tag{5.28}$$

By the unique solvability of the backward SVIEs, (5.27) and (5.28) lead to that

$$P_4^{(12)}(\theta,r) = 0, \quad Q_4^{(12)}(\theta,r,\theta') = 0, \qquad \theta \geqslant r.$$

Hence, it follows that for $\theta \geqslant r$,

$$\begin{aligned}
&\mathcal{G}_4^{(2)}(\theta,r) = \int_r^T P_4^{(12)}(\theta,\theta')d\theta' = 0,\\
&\mathcal{Q}_4^{2}(\theta,r) = \int_r^T Q_4^{(12)}(\theta,\theta',r)d\theta' = \int_r^\theta Q_4^{(12)}(\theta,\theta',r)d\theta' + \int_\theta^T Q_4^{(21)}(\theta',\theta,r)^\top d\theta' = 0.
\end{aligned}$$

Then, (5.14) becomes

$$\begin{aligned}
\mathcal{P}^2(s) \equiv \mathcal{P}(s) = h_{xx}(T) + \int_s^T \Big\{ &\mathbb{E}_r\Big[b_x(r)^\top \mathcal{G}_2^{(1)}(r) + \sum_{j=1}^{d}\sigma_x^j(r)^\top \mathcal{Q}_{2j}^{(1)}(r)\\
&+ \mathcal{G}_2^{(1)}(r)^\top b_x(r) + \sum_{j=1}^{d}\mathcal{Q}_{2j}^{(1)}(r)^\top \sigma_x^j(r)\Big] + b_x(r)^\top \mathbb{E}_r\Big[\int_r^T \mathcal{G}_4^{(1)}(\theta,r)d\theta\Big]\\
&+ \sum_{j=1}^{d}\sigma_x^j(r)^\top \int_r^T \mathcal{Q}_{4j}^{(1)}(\theta,r)d\theta + \Big[\int_r^T \mathbb{E}_r\big[\mathcal{G}_4^{(1)}(\theta,r)^\top\big]d\theta\Big] b_x(r)\\
&+ \sum_{j=1}^{d}\Big[\int_r^T \mathcal{Q}_{4j}^{(1)}(\theta,r)^\top d\theta\Big]\sigma_x^j(r) + G_{xx}(r) + \sum_{j=1}^{d}\sigma_x^j(r)^\top \mathbb{E}_r\big[\mathcal{P}(r)\big]\sigma_x^j(r)\Big\}dr.
\end{aligned}$$

Denote

$$\bar{P}^2(s) := \mathbb{E}_s\big[\mathcal{P}^2(s)\big], \quad \bar{Q}^2(s) := \mathcal{Q}_2^{(1)}(s) + \int_s^T \mathcal{Q}_4^{(1)}(\theta',s)d\theta'.$$

Then, similarly to Case I, $(\bar{P}^2(\cdot),\bar{Q}^2(\cdot))$ also satisfies the BSDE (5.22).

Case III: Linear quadratic stochastic optimal control problems.

Consider the following state equation,

$$\begin{cases}
dX(t) = \big[A(t)X(t) + B(t)u(t) + \bar{B}(t)u(t-\delta)\big]dt\\
\qquad\quad + \big[\bar{C}(t)X(t-\delta) + D(t)u(t) + \bar{D}(t)u(t-\delta)\big]dW(t),\ t\in[0,T],\\
X(t) = \xi(t),\ u(t) = \eta(t),\ t\in[-\delta,0],
\end{cases}$$

with the quadratic cost functional

$$\begin{aligned}
J(u(\cdot)) = \mathbb{E}\big[\langle GX(T),X(T)\rangle + 2\langle g,X(T)\rangle\big]\\
+\mathbb{E}\int_0^T \left\langle \begin{bmatrix} Q_{00}(t) & 0 & S_{00}(t)^\top & S_{01}(t)^\top\\ 0 & Q_{11}(t) & S_{10}(t)^\top & S_{11}(t)^\top\\ S_{00}(t) & S_{10}(t) & R_{00}(t) & R_{01}(t)\\ S_{01}(t) & S_{11}(t) & R_{01}(t)^\top & R_{11}(t)\end{bmatrix}\begin{bmatrix} X(t)\\ X(t-\delta)\\ u(t)\\ u(t-\delta)\end{bmatrix}, \begin{bmatrix} X(t)\\ X(t-\delta)\\ u(t)\\ u(t-\delta)\end{bmatrix}\right\rangle dt,
\end{aligned}$$

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where $A(\cdot)$,$B(\cdot),\bar{B}(\cdot),\bar{C}(\cdot),D(\cdot),\bar{D}(\cdot),Q_{00}(\cdot),S_{00}(\cdot),S_{01}(\cdot),Q_{11}(\cdot),S_{10}(\cdot)$,$S_{11}(\cdot),R_{00}(\cdot)$, $R_{01}(\cdot),R_{11}(\cdot)$ are all deterministic functions, and $G\in\mathbb{R}^{n\times n}$, $g\in\mathbb{R}^n$. In this case, (5.4), (5.6), (5.11), and (5.13) become

$$\begin{aligned}
&P_1^{(11)}(r)=G,\quad P_2^{(11)}(r)=A(r)^\top\Big[P_1^{(11)}(r)+\int_r^T P_2^{(11)}(\theta)d\theta\Big],\quad 0\leqslant r\leqslant T,\\
&P_4^{(11)}(\theta,r)=A(r)^\top\mathcal{G}_4^{(1)}(\theta,r),\quad P_4^{12}(\theta,r)=A(r)^\top\mathcal{G}_4^{(2)}(\theta,r),\quad 0\leqslant r\leqslant\theta\leqslant T,\\
&P_4^{(11)}(\theta,r)=\mathcal{G}_4^{(1)}(r,\theta)^\top A(\theta),\quad P_4^{(21)}(\theta,r)=\mathcal{G}_4^{(1)}(r,\theta)^\top A(\theta),\quad 0\leqslant\theta<r\leqslant T,\\
&P_3^{(11)}(r)=Q_{00}(r),\quad P_3^{(22)}(r)=Q_{11}(r)+\bar{C}(r)^\top\mathcal{P}(r)\bar{C}(r),\quad 0\leqslant r\leqslant T,
\end{aligned}$$

and other terms in (4.8) are all 0. From (5.10) we have

$$\begin{aligned}
\mathcal{G}_4^{(1)}(\theta,r)&=P_2^{(11)}(\theta)^\top+P_3^{(11)}(\theta)+\int_r^T\Big[P_4^{(11)}(\theta,\theta')+\mathbf{1}_{(\delta,\infty)}(\theta'-r)P_4^{(21)}(\theta,\theta')\Big]d\theta',\\
\mathcal{G}_4^{(2)}(\theta,r)&=\mathbf{1}_{(\delta,\infty)}(\theta-r)P_3^{(22)}(\theta)+\int_r^T P_4^{(12)}(\theta,\theta')d\theta'.
\end{aligned}$$

Let $\theta-\delta\leqslant r\leqslant\theta$, $\tau+\delta\leqslant\theta\leqslant T$, and consider

$$\mathcal{G}_4^{(2)}(\theta,r)=\int_r^T P_4^{(12)}(\theta,\theta')d\theta'=\int_r^\theta P_4^{(12)}(\theta,\theta')d\theta'=\int_r^\theta A(\theta')^\top\mathcal{G}_4^{(2)}(\theta,\theta')d\theta'.$$

Then, we have

$$\mathcal{G}_4^{(2)}(\theta,r)=0,\quad \theta-\delta\leqslant r\leqslant\theta,\quad \tau+\delta\leqslant\theta\leqslant T.$$

Hence, (5.14) becomes

$$\begin{aligned}
\mathcal{P}^3(s)\equiv\mathcal{P}(s)=G+\int_s^T\Big\{&A(r)^\top\mathcal{G}_2^{(1)}(r)+\mathcal{G}_2^{(1)}(r)^\top A(r)\\
&+A(r)^\top\Big[\int_r^T\mathcal{G}_4^{(1)}(\theta,r)d\theta+\int_{r+\delta}^T\mathcal{G}_4^{(2)}(\theta,r)d\theta\Big]\\
&+\Big[\int_r^T\mathcal{G}_4^{(1)}(\theta,r)^\top d\theta+\int_{r+\delta}^T\mathcal{G}_4^{(2)}(\theta,r)^\top d\theta\Big]A(r)+Q_{00}(r)\Big\}dr\\
&+\int_{s+\delta}^T\big[Q_{11}(r)+\bar{C}(r)^\top\mathcal{P}(r)\bar{C}(r)\big]dr.
\end{aligned}$$

Similarly to Case I, $\mathcal{P}^3(\cdot)$ satisfies the following ordinary differential equation:

$$\begin{cases}
-\dot{\mathcal{P}}^3(s)=A(s)^\top\mathcal{P}^3(s)+\mathcal{P}^3(s)^\top A(s)+Q_{00}(s)+\big[Q_{11}(s+\delta)\\
\qquad\qquad+\bar{C}(s+\delta)^\top\mathcal{P}^3(s+\delta)\bar{C}(s+\delta)\big]\mathbf{1}_{[0,T-\delta)}(s),\quad \text{a.e. } s\in[0,T],\\
\mathcal{P}^3(T)=G.
\end{cases}\tag{5.29}$$

Remark 5.8. For Case I, when the delay disappears in the control system, (5.19) satisfied by $(p(\cdot),q(\cdot))$, becomes (3.8) in [35]; (5.22) satisfied by $\mathcal{P}(\cdot)$, becomes (3.9) in [35], and so Theorem 5.1 reduces to Theorem 3.2 in [35]. For Cases II and III, (5.19), (5.22), and (5.29) are consistent with (5.1) and (5.2) in [18], respectively, thus Theorem 5.1 reduces to Theorem 5.1 in [18].

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6. Concluding remarks. In this paper, a stochastic optimal control problem is considered and the control domain is allowed to be nonconvex. The pointwise state delay, distributed state delay, and pointwise control delay can appear in the diffusion term and the terminal cost. Via the theory of backward stochastic Volterra integral systems, we transform delayed variational equations into Volterra integral equations without delay, introduce some new second-order adjoint equations, and derive a general maximum principle, without any additional conditions. Finally, to express adjoint equations more compactly, we discuss them in detail for three typical control systems.

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