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INITIAL-BOUNDARY VALUE PROBLEM AND EXPONENTIAL DECAY FOR A FLEXIBLE-BEAM VIBRATION WITH GAIN ADAPTIVE DIRECT STRAIN FEEDBACK CONTROL

BAO ZHU GUO† and ZHENG HUA LUO‡

† Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, People's Republic of China; and

‡ Department of Mechanical Engineering, Nagaoka University of Technology, Kamitomioka-cho 1603-1, Nagaoka, Niigata 940-21, Japan

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1. INTRODUCTION

To control motion/vibration of flexible robot arms, there have been many finite dimensional dynamic controllers such as optimal controllers and robust controllers [1, 2]. Although dynamic controllers are designed with the expectation of increasing control performance for a well-modeled system, it is widely recognized that a high dimensional controller may not be easy to implement, that it is difficult to obtain an accurate model, and that the ignored high frequency dynamics may cause instability for the closed-loop system. It is, therefore, highly desirable to seek sensor output feedback control methods which are attractive because of their easy implementation in the sense that (1) the dynamic model is only used for stability analysis, nor for on-line controller design, that (2) it is not necessary to know the exact parameters involved in the model, and that (3) there are no spillover problems since infinite dimensional system models are deal with [3]. A simple control strategy called direct strain feedback was proposed for control of flexible robots in [4]. This control method guarantees stability of the closed-loop feedback system under arbitrary top load variations and other parameter uncertainties. However, the control performance (e.g. decay rate in transient responses) may degrade significantly due to large tip load variations which are often encountered in practical applications. In order to maintain high performance, we propose to adaptively tune the strain feedback gain, instead of keeping the feedback gain as a constant. Unlike the well-known model based adaptive control where it is required to estimate the physical parameters of control plant so as to tune adaptively parameters of the controller, a simple adaptive gain tuning method is proposed in [5]. The closed-loop system is described by the following time-dependent nonlinear beam equation

$$\begin{cases} \frac{\partial^2 y(x, t)}{\partial t^2} + \frac{EI}{\rho} \frac{\partial^4 y(x, t)}{\partial x^4} + k(t)x \frac{\partial^3 y(0, t)}{\partial t \partial x^2} + \alpha x \left(\frac{\partial^2 y(0, t)}{\partial x^2} \right)^3 = 0, & 0 < x < l, \\ y(0, t) = y'(0, t) = y''(l, t) = y'''(l, t) = 0, \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), \end{cases} \quad (1)$$

where y denotes the bending displacement of flexible arm, EI and ρ denote, respectively, the bending rigidity and the mass density per unit length of the arm, l is the length, $\alpha > 0$ is a

given constant, $k(t)$ is the integration-type gain adaptive law

$$\begin{cases} k'(t) = \alpha y''^2(0, t), \\ k(0) = k_0 \geq 0. \end{cases} \quad (2)$$

We use here and throughout y' or y_x denoting the derivative of y with respect to x and $\dot{y}$ or y_t the derivative with respect to t .

Equations (1) is a nonautonomous equation, the asymptotic stability has been proved by using the Lyapunov method in [5]. In the next section, the nonlinear semigroup approach is adopted to present the existence and uniqueness of the solution by introducing $k(t)$ as a new state variable. Section 3 is devoted to the establishment of exponential decay estimation, a stronger result, for the dynamic system (1).

2. THE WELL-POSEDNESS OF THE PROBLEM

In order to adopt nonlinear semigroup approach, we consider $k(t)$ to be a new state variable and rewrite equations (1)–(2) as

$$\begin{cases} \frac{d}{dt} [y_t(x, t) + k(t)y_{xx}(0, t)x] + \frac{EI}{\rho} y_{xxxx}(x, t) = 0, \\ y(0, t) = y'(0, t) = y''(l, t) = y'''(l, t) = 0, \\ k'(t) = \alpha y''^2(0, t), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), k(0) = k_0 \geq 0. \end{cases} \quad (3)$$

Let $\mathcal{H} = H_E^2(0, l) \times L^2(0, l) \times \mathbb{R}$ be the underlying real state Hilbert space with inner product

$$\langle (u_1(x), v_1(x), \gamma_1), (u_2(x), v_2(x), \gamma_2) \rangle = \int_0^l \left[\frac{EI}{\rho} u_1''(x)u_2''(x) + v_1(x)v_2(x) \right] dx + \frac{EI}{2\alpha\rho} \gamma_1\gamma_2,$$

where $H_E^2(0, l) = \{u(x) \mid u \in H^2(0, l), u(0) = u'(0) = 0\}$. Define a nonlinear operator $\mathfrak{G}: D(\mathfrak{G}) \subset \mathcal{C} \rightarrow \mathcal{H}$ by

$$\begin{cases} D(\mathfrak{G}) = \{(u(x), v(x), \gamma) \in H^4(0, l) \times H^2(0, l) \times \mathbb{R}^+ \mid \\ u(0) = u'(0) = u''(l) = u'''(l) = 0, v(0) = 0, v'(0) = \gamma u''(0)\}, \\ \mathfrak{G}(u(x), v(x), \gamma) = (v(x) - \gamma u''(0)x, -\frac{EI}{\rho} u'''(x), \alpha u''^2(0)). \end{cases} \quad (4)$$

Then equation (3) can be written as a nonlinear evolution equation on $\mathcal{H}$

$$\begin{cases} \frac{dY(t)}{dt} = \mathfrak{G}Y(t), \\ Y(0) = Y_0, \end{cases} \quad (5)$$

where

$$\begin{cases} Y(t) = (y(x, t), y_t(x, t) + k(t)y_{xx}(0, t)x, k(t)), \\ Y_0 = (y_0(x), y_1(x) + k_0 y_0''(0)x, k_0) \in \mathcal{C}, \end{cases}$$

$\mathcal{C} = H_E^2(0, l) \times L^2(0, l) \times \mathbb{R}^+$ is a closed convex subset of $\mathcal{H}$. Notice that the norm of state is just an energy function

$$\|Y(t)\|^2 = \int_0^l \left\{ \frac{EI}{\rho} y_{xx}^2(x, t) + [y_t(x, t) + k(t)y''(0, t)x]^2 \right\} dx + \frac{EI}{2\alpha\rho} k^2(t)$$

and, formally,

$$\begin{cases} \frac{d}{dt} \|Y(t)\|^2 = -\frac{EI}{\rho} k(t)y''^2(0, t) \leq 0, \\ \frac{d}{dt} [\|Y(t)\|^2 + \frac{EI}{2\alpha\rho} k^2(t)] = 0. \end{cases} \quad (6)$$

along the solution of equation (3).

LEMMA 1. The operator $\mathcal{A}$ defined by (4) is closed and dissipative on $\mathcal{H}$ with the domain $D(\mathcal{A})$ that is dense in $\mathcal{C}$ and

$$\mathcal{C} \subset \mathcal{R}(I - \lambda\mathcal{A}) \quad \text{for all } \lambda > 0, \quad (7)$$

where $\mathcal{R}(I - \lambda\mathcal{A})$ represents the range of operator $I - \lambda\mathcal{A}$.

Proof. The closedness of $\mathcal{A}$ is obvious. Let $U, V \in D(\mathcal{A})$, where

$$U = (u_1(x), v_1(x), \gamma_1), \quad V = (u_2(x), v_2(x), \gamma_2).$$

Then a simple calculation leads to

$$\langle \mathcal{A}U - \mathcal{A}V, U - V \rangle = -\frac{1}{2} \frac{EI}{\rho} (\gamma_1 + \gamma_2) [u_1''(0) - u_2''(0)]^2 \leq 0,$$

which proves the dissipativeness of $\mathcal{A}$. To prove (7) it is sufficient to show that for any $\lambda > 0$ and $(f(x), g(x), \gamma_0) \in \mathcal{C}$, there exists $(u(x), v(x), \gamma) \in D(\mathcal{A})$, such that

$$(i - \lambda\mathcal{A})(u(x), v(x), \gamma) = (f(x), g(x), \gamma_0),$$

i.e.

$$\lambda v(x) = u(x) + \gamma u''(0)x - f(x), \quad \gamma = \gamma_0 + \lambda \alpha u''^2(0),$$

and u satisfies

$$\begin{cases} \lambda^2 \frac{EI}{\rho} u''''(x) + u(x) + \lambda \gamma u''(0)x = f(x) + \lambda g(x), \\ u(0) = u'(0) = u''(l) = u'''(l) = 0, \\ \gamma = \gamma_0 + \lambda \alpha u''^2(0). \end{cases} \quad (8)$$

Equation (8) can be written in the operator form as

$$\begin{cases} (I + \lambda^2 \mathbb{A} + \lambda \gamma \mathbb{B})u = f + \lambda g & \text{or} & (\lambda^2 + \mathbb{A}^{-1} + \lambda \gamma \mathbb{Q})\mathbb{A}u = f + \lambda g, \\ \gamma = \gamma_0 + \lambda \alpha u''^2(0), \end{cases} \quad (9)$$

where $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{Q}$ are operators defined by (see also [4])

$$\begin{cases} \mathbb{A}\phi = \frac{EI}{\rho} \phi'''(x), & D(\mathbb{A}) = \{\phi \in L^2(0, l) \mid \phi(0) = \phi'(0) = \phi''(l) = \phi'''(l) = 0\}, \\ \mathbb{B}\phi = \phi''(0)x, & \mathbb{Q} = \mathbb{B}\mathbb{A}^{-1}. \end{cases} \quad (10)$$

$\mathbb{A}$ is an unbounded self-adjoint positive definite operator on $L^2(0, l)$ (with usual inner product $\langle \cdot, \cdot \rangle_0$ and induced norm $\|\cdot\|_0$), $\mathbb{Q}$ is an bounded self-adjoint positive definite operator on $L^2(0, l)$.

As $\langle (\lambda^2 + \mathbb{A}^{-1} + \lambda \gamma \mathbb{Q})\phi, \phi \rangle_0 \geq \lambda^2 \|\phi\|_0^2$ for all $\phi \in L^2(0, l)$ and $\gamma \geq 0$, it is seen that $\lambda^2 + \mathbb{A}^{-1} + \lambda \gamma \mathbb{Q}$ is invertible and bounded self-adjoint on $L^2(0, l)$ and $\|(\lambda^2 + \mathbb{A}^{-1} + \lambda \gamma \mathbb{Q})^{-1}\|_0 \leq 1/\lambda^2$. Thus, the equation

$$(\lambda^2 + \mathbb{A} + \lambda \gamma \mathbb{B})u = f + \lambda g$$

has a unique solution which is given by

$$u = u_\gamma = \mathbb{A}^{-1}(\lambda^2 + \mathbb{A}^{-1} + \lambda \gamma \mathbb{Q})^{-1}(f + \lambda g).$$

Clearly, u_γ is continuous in $\gamma \geq 0$ on $L^2(0, l)$ and

$$\|u_\gamma\|_0 \leq \|\mathbb{A}^{-1}\|_0 \|f + \lambda g\|_0 / \lambda^2. \quad (11)$$

On the other hand, multiplying x on both sides of the first equation of (8) and integrating from 0 to l yields

$$u_\gamma''(0) = [-\langle x, u_\gamma \rangle_0 + \langle x, f + \lambda g \rangle_0] / (\lambda^2 EI / \rho + \lambda \gamma l^3 / 3) \quad (12)$$

which together with (11) gives

$$\begin{aligned} |u_\gamma''(0)| &\leq \rho / (\lambda^2 EI) \|x\|_0 [\|u_\gamma\|_0 + \|f + \lambda g\|_0] \\ &\leq \rho / (\lambda^4 EI) \|x\|_0 [1 + \|\mathbb{A}^{-1}\|_0] \|f + \lambda g\|_0 \\ &\leq C_\lambda \|f + \lambda g\|_0, \quad \text{for all } \gamma \geq 0, \end{aligned} \quad (13)$$

where C_λ is a fixed constant depending on λ . Define

$$p(\gamma) = \gamma_0 + \lambda \alpha u_\gamma''^2(0).$$

Then $p(\gamma)$ is continuous on $\mathbb{R}^+$, and $p(\mathbb{R}^+) \subset [0, \gamma_0 + \lambda \alpha C_\lambda^2 \|f + \lambda g\|_0^2]$. Hence, $p(\gamma)$ has a fixed point γ in $\mathbb{R}^+$, $p(\gamma) = \gamma$, $\gamma \geq 0$, which implies that (8) admits a solution. So $\mathcal{R}(I - \lambda \mathcal{Q}) \supseteq \mathcal{C}$. Finally, since

$$\{\phi \in C^\infty[0, l], \phi(0) = \phi'(0) = \phi^{(n)}(0) = \phi^{(n)}(l) = 0, n \geq 2\} \times C_0^\infty[0, l] \times \mathbb{R}^+ \subset D(\mathcal{Q})$$

we see that $D(\mathcal{Q})$ is dense in $\mathcal{C}$.

From nonlinear semigroup theory (see e.g. [6, p. 120]), we have the following existence and uniqueness result.

THEOREM 1. The operator $\mathcal{Q}$ defined by (4) generates a unique nonlinear strong continuous semigroup of contractions $S(t)$ on $\mathcal{C}$. Thus equation (5) admits a unique solution

$$Y(t) = S(t)Y_0 \in C^0([0, \infty); \mathcal{C}), \quad \forall Y_0 \in \mathcal{C},$$

and strong solution

$$Y(t) = S(t)Y_0 \in C^1([0, \infty); \mathcal{C}), \quad \forall Y_0 \in D(\mathcal{Q}).$$

Hence, for every

$$y_0 \in D(\mathbb{A}), \quad y_1 \in D(\mathbb{A}^{1/2}), \quad k_0 \geq 0,$$

equation (3) admits a unique strong solution in the sense of

$$\begin{cases} y \in L^\infty([0, \infty); H^4(0, l)), & \dot{y} \in L^\infty([0, \infty); H^2(0, l)), \\ \frac{d}{dt} [\dot{y}(x, t) + k(t)y''(0, t)x] + \frac{EI}{\rho} y'''(x, t) = 0, & k'(t) = \alpha y''^2(0, t), \quad \text{for } t \geq 0 \text{ a.e. in } \mathcal{I}, \\ y(x, 0) = y_0(x), & \dot{y}(x, 0) = y_1(x), \quad k(0) = k_0. \end{cases}$$

LEMMA 2. $0 \in \mathcal{R}(\mathcal{Q})$, $(I - \mathcal{Q})^{-1}$ is compact.

Proof. $0 \in \mathcal{R}(\mathcal{Q})$ is trivial. It needs only to prove the second assertion. Let $\{V_n\} \subset \mathcal{C}$, $\|V_n\| \leq K$, be a bounded series and $U_n = (u_n(x), v_n(x), \gamma_n) \in \mathcal{C}$ satisfy $(I - \mathcal{Q})U_n = V_n$. Then by dissipativeness of $\mathcal{Q}$, $\|\mathcal{Q}U_n\| \leq \|V_n\| \leq K$ and $\|U_n\| \leq \|V_n\| \leq K$. These imply that

$$\|u_n\|_{H^4} \leq C_1, \quad \|v\|_{H^2} \leq C_1, \quad \gamma_n \leq C_1,$$

for some uniform constant C_1 . By Sobolev embedding theorem there is a subsequence of U_n , still indexed n for notational simplicity, and $U_0 = (u_0(x), v_0(x), \gamma_0) \in \mathcal{C}$ such that

$$U_n \rightarrow U_0 \text{ in the topology of } \mathcal{H}.$$

Consequently, $u_0(0) = u'_0(0) = 0$. The proof is complete.

LEMMA 3. Let $S(t)$ be the semigroup generated by $\mathcal{Q}$. Then if for some

$$Y_0 = (y_0(x), y_1(x) + k_0 y_0''(0)x, k_0) \in D(\mathcal{Q}),$$

i.e. $y_0 \in D(\mathbb{A})$, $y_1 \in D(\mathbb{A}^{1/2})$, there hold

$$\frac{d}{dt} \|S(t)Y_0\| \equiv 0, \quad \text{for all } t \geq 0,$$

then $Y_0 = (0, 0, k_0)$.

Proof. Let $Y(t) = (y(x, t), \dot{y}(x, t) + k(t)y''(0, t)x, k(t))$ be the solution of equation (3) with the initial value $Y(0) = Y_0$. In view of (6), $(d/dt)\|S(t)Y_0\| \equiv 0$ implies $k(t)y''(0, t) = 0$. If $k(t) = 0$, i.e.

$$k_0 + \int_0^t y''^2(0, \tau) d\tau = 0,$$

then $y''(0, t) = 0$, which still holds for $k(t) > 0$. Hence, one has

$$\begin{cases} y_{tt}(x, t) + \frac{EI}{\rho} y_{xxxx}(x, t) = 0, \\ y(0, t) = y'(0, t) = y''(l, t) = y'''(l, t) = 0, & y''(0, t) = 0, \\ k(t) = k_0. \end{cases} \quad (14)$$

Multiplying $y_x(x, t)$ and $xy_x(x, t)$ on both sides of (14) and then integrating with respect to x and t from 0 to l and 0 to T , respectively, we obtain

$$\begin{aligned} 3 \frac{EI}{\rho} \int_0^T \int_0^l y_{xx}^2 dx dt + \int_0^T \int_0^l y_t^2 dx dt &= l \int_0^T y_t^2(l, t) dt - 2 \int_0^T xy_x(x, t)y_t(x, t)|_0^T dx, \\ \int_0^T y_t^2(l, t) dt &= 2 \int_0^T y_x(x, t)y_t(x, t)|_0^T dx. \end{aligned}$$

By making use of the fact that

$$G(t) = \frac{EI}{\rho} \int_0^l y_{xx}^2 dx + \int_0^l y_t^2 dx = G(0)$$

in this case, we have

$$\int_0^T G(t) dt \leq K_0, \quad \text{for some constant } K_0.$$

Therefore, $G(t) \equiv 0$ or equivalently $y \equiv 0$ and $k(t) \equiv k_0$.

Theorem 1, lemmas 2, 3, equation (6) and Lasalle's invariance principle [7] implies the strong stability of the system (3) which is summarized as the following theorem.

THEOREM 2. Let y be the solution of (3). Let

$$E(t) \int_0^l \left\{ \frac{EI}{\rho} y_{xx}^2(x, t) + [y_t(x, t) + k(t)y''(0, t)x]^2 \right\} dx. \quad (15)$$

Then

$$\lim_{t \rightarrow \infty} E(t) = 0, \quad \lim_{t \rightarrow \infty} k(t) = \sqrt{\frac{\alpha\rho}{EI} E(0) + k_0^2} \quad (16)$$

and

$$k(t) \leq \sqrt{\frac{\alpha\rho}{EI} E(0) + k_0^2} \quad \text{for all } t \geq 0. \quad (17)$$

The following results are more profound than strong stability given by theorem 2 that claim that $E(t)$ also decays exponentially to zero as time goes to infinity.

THEOREM 3. Let $k_0 > 0$ and $E(t)$ be defined by (15). Then for every solution $y(x, t)$ of equation (3), there exist positive constants $K > 0$, $\mu > 0$, depending on initial condition, such that

$$E(t) \leq K e^{-\mu t} \quad \text{and} \quad |k^2(t) - \frac{\alpha \rho}{EI} E(0) - k_0^2| \leq \frac{\alpha \rho}{EI} K e^{-\mu t} \quad \text{for all } t \geq 0.$$

Proof. First assume $y_0 \in D(\mathbb{A})$, $y_1 \in D(\mathbb{A}^{1/2})$. Then y is the strong solution of equation (3) by theorem 1. Let

$$\begin{aligned} \alpha(t) = & \int_0^l y(x, t)[y_t(x, t) + k(t)xy_{xx}(0, t)] dx \\ & + 4 \int_0^l xy_x(x, t)[y_t(x, t) + k(t)xy_{xx}(0, t)] dx \\ & - 4l \int_0^l y_x(x, t)[y_t(x, t) + k(t)xy_{xx}(0, t)] dx. \end{aligned}$$

Differentiating $\alpha(t)$, one finds

$$\begin{aligned} \alpha'(t) = & -E(t) - 6 \frac{EI}{\rho} \int_0^l y_{xx}^2(x, t) dx \\ & - 5k(t)y_{xx}(0, t) \int_0^l x[y_t(x, t) + k(t)xy_{xx}(0, t)] dx \\ & + 4lk(t)y_{xx}(0, t) \int_0^l [y_t(x, t) + k(t)xy_{xx}(0, t)] dx + 4l \frac{EI}{\rho} y_{xx}^2(0, t). \end{aligned}$$

It is obviously seen that there exist positive constants D_i , $i = 1, 2, 3$, independent of initial conditions, such that

$$|\alpha(t)| \leq D_1 E(t) \quad \text{and} \quad \alpha'(t) \leq -D_2 E(t) + D_3 k^2(t) y_{xx}^2(0, t), \quad \forall t \geq 0.$$

Take $\varepsilon > 0$ such that

$$0 < \varepsilon < \min \left\{ \frac{1}{D_1}, \frac{2(\alpha \rho / EI) k_0}{D_3 \sqrt{2(\alpha \rho / EI) E(0) + k_0^2}} \right\}$$

and define $E_\varepsilon(t) = E(t) + \varepsilon \alpha(t)$. Then by noticing (17), we find

$$(1 - \varepsilon D_1)E(t) \leq E_\varepsilon(t) \leq (1 + D_1)E(t)$$

and

$$\begin{aligned} E'_\varepsilon(t) & \leq -k(t) \left[2 \frac{EI}{\rho} - \varepsilon D_3 k(t) \right] y_{xx}^2(0, t) - \frac{\varepsilon D_2}{1 + \varepsilon D_1} E_\varepsilon(t) \\ & \leq -\frac{\varepsilon D_2}{1 + \varepsilon D_1} E_\varepsilon(t). \end{aligned}$$

Hence,

$$E(t) \leq \frac{1 + \varepsilon D_1}{1 - \varepsilon D_1} E(0) \exp\left(-\frac{\varepsilon D_2}{1 + \varepsilon D_1} t\right) = K e^{-\mu t}, \quad \forall t \geq 0$$

where

$$K = \frac{1 + \varepsilon D_1}{1 - \varepsilon D_1} E(0), \quad \mu = \frac{\varepsilon D_2}{1 + \varepsilon D_1}.$$

The second part is a direct consequence of the first part by referring to (6). The general conclusions follow from the denseness of $D(\mathbb{A}) \times D(\mathbb{A}^{1/2})$ and the estimation of ε and K and μ aforementioned.

Remark 1. The decay rate μ in theorem 3 depends usually on initial condition. A natural question is what about the uniform exponential stability. When $\alpha = 0$ in equation 3, it was shown in [8] that, the dynamic system is indeed exponential stable. However, the decay rate is not proportional to feedback gain $k_0 > 0$, i.e. as k_0 goes to infinity, the decay rate will tends to zero. By this fact and the claim of (16), the uniform exponential decay may not hold to our problem, although we have not proved this claim right now.

3. THE REGULARITY OF THE SOLUTION AND EXPONENTIAL DECAY

In this section, we shall investigate the regularity and the exponential decay of the solution of equation (3), which leads to an adaptive boundary control problem.

LEMMA 4. For any $\phi \in D(\mathbb{A})$, the following inequality holds:

$$|\phi''(0)^3| \leq 3l^{5/2} \|\phi''\| \|\phi'''\|^2. \quad (18)$$

Proof. Let $\phi \in D(\mathbb{A})$, we have

$$\phi''(0)^3 = 3 \int_0^l \phi''(x) dx \int_x^l (s-x) \phi'''(s) ds \int_x^l \phi'''(s) ds.$$

The conclusion is obtained by applying the Schwartz's inequality to the above equality.

PROPOSITION 1. Assume that

$$y_1 \in D(\mathbb{A}), \quad y_0 \in D(\mathbb{A}), \quad \mathbb{A}y_0 + [k_0 y_1''(0) + \alpha[y_0''(0)]^3]x \in D(\mathbb{A}^{1/2}). \quad (19)$$

Then the solution $y(x, t)$ of equation (3) has the regularity

$$\dot{y} \in L^\infty(0, T; [D(\mathbb{A})]), \quad \ddot{y} \in L^\infty(0, T; [D(\mathbb{A}^{1/2})]). \quad (20)$$

Proof. Let $\{\phi_n\}$ be the all eigenfunctions of operator $\mathbb{A}$ which consists of an orthogonal basis on $L^2(0, l)$ and (see [3])

$$\phi_n''(0) = o(n^2), \quad \|\phi_n\| = o(1)$$

and

$$x = \sum_{n=1}^{\infty} b_n \phi_n(x) \quad \text{in } L^2(0, l), \quad b_n = o(n^{-2}).$$

Let

$$y_0(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x), \quad y_1(x) = \sum_{n=1}^{\infty} \beta_n \phi_n(x) \quad \text{in } \mathcal{H}.$$

Find an approximation solution $y_N(x, t)$ of equation (3) with following form

$$\begin{cases} y_N(x, t) = \sum_{n=1}^N g_{Nn}(t) \phi_n(x), & y_N(x, 0) = \sum_{n=1}^N \alpha_n \phi_n(x), \\ \dot{y}_N(x, 0) = \sum_{n=1}^N \beta_n \phi_n(x), \end{cases}$$

satisfying

$$\langle \ddot{y}_N(\cdot, t), \phi_n \rangle + \frac{EI}{\rho} \langle y_N'''(\cdot, t), \phi_n \rangle + [k_N(t) \dot{y}_N''(0, t) + \alpha [y_N''(0, t)]^3] \langle x, \phi_n \rangle = 0, \quad n = 1, 2, \dots, N, \quad (21)$$

where

$$k_N(t) = k_0 + \alpha \int_0^t [y_N''(0, \tau)]^2 d\tau = k_0 + \left(\alpha \int_0^t \left(\sum_{n=1}^N g_{Nn}(\tau) \phi_n''(0) \right)^2 d\tau \right). \quad (22)$$

Approximation equation (21) is equivalent to a system of ordinary differential equations

$$\begin{cases} g_{Nn}''(t) + \lambda_n g_{Nn}(t) + \left[k_N(t) \sum_{n=1}^N g_{Nn}'(t) \phi_n''(0) + \alpha \left(\sum_{n=1}^N g_{Nn}(t) \phi_n''(0) \right)^3 \right] b_n = 0, \\ g_{Nn}(0) = \alpha_n, \quad g_{Nn}'(0) = \beta_n, \quad n = 1, 2, \dots, N, \end{cases} \quad (23)$$

or

$$\ddot{y}_N(x, t) + \frac{EI}{\rho} y_N'''(x, t) + [k_N(t) \dot{y}_N''(0, t) + \alpha [y_N''(0, t)]^3] \sum_{n=1}^N b_n \phi_n(x) = 0. \quad (24)$$

Let

$$f_{Nn}(t) = g_{Nn}'(t), \quad n = 1, 2, \dots, N.$$

Then $\{f_{Nn}(t)\}$ satisfy the following equations

$$\begin{cases} f_{Nn}''(t) + \lambda_n f_{Nn}(t) + \left[k_N(t) \sum_{n=1}^N f_{Nn}'(t) \phi_n''(0) + \hat{k}_N(t) \sum_{n=1}^N f_{Nn}(t) \phi_n''(0) \right] b_n = 0, \\ f_{Nn}(0) = \beta_n, \quad f_{Nn}'(0) = \lambda_n \alpha_n + [k_0 y_N''(0, 0) + \alpha [y_N''(0, 0)]^3] b_n, \quad n = 1, 2, \dots, N, \end{cases}$$

where

$$\hat{k}_N(t) = 4\alpha [y_N''(0, t)]^2. \quad (25)$$

Let

$$\begin{aligned} \hat{E}_N(t) &= \frac{1}{2} (\|\mathbb{A} \dot{y}_N\|^2 + \|\mathbb{A}^{1/2} \ddot{y}_N\|^2) + \frac{1}{2} \hat{k}_N(t) [y_N''(0, t)]^2 \\ &= \frac{1}{2} \sum_{n=1}^N [\lambda_n^2 f_{Nn}^2(t) + \lambda_n f_{Nn}'^2(t) \|\phi_n\|^2 + \hat{k}_N(t) \left(\sum_{n=1}^N f_{Nn}(t) \phi_n''(0) \right)^2]. \end{aligned} \quad (26)$$

From lemma 4, $|\dot{y}_N''(0, t)^3| \leq 3l^{5/2} \|\dot{y}_N''(\cdot, t)\| \|\dot{y}_N'''(\cdot, t)\|^2$ and, hence,

$$\begin{aligned} \frac{d}{dt} \hat{E}_N(t) &= -k_N(t) \left[\sum_{n=1}^N f_{Nn}(t) \phi_n''(0) \right]^2 + \frac{1}{2} \hat{k}_N'(t) [\dot{y}_N''(0, t)]^2 \\ &\leq \frac{1}{2} \hat{k}_N'(t) [\dot{y}_N''(0, t)]^2 = 4\alpha y_N''(0, t) [\dot{y}_N''(0, t)]^3 \\ &\leq 12\alpha l^{5/2} \left(\frac{\rho}{EI} \right)^{5/2} y_N''(0, t) \|\mathbb{A}^{1/2} \dot{y}_N(\cdot, t)\| \|\mathbb{A} \dot{y}_N(\cdot, t)\|^2 \\ &\leq M_N \hat{E}_N(t), \end{aligned} \quad (27)$$

where

$$M_N = 24\alpha l^{5/2} \left(\frac{\rho}{EI} \right)^{7/2} \|x\| y_N \|_{L^\infty(0, T; D(\mathbb{A}))} \|\dot{y}_N\|_{L^\infty(0, T; D(\mathbb{A}^{1/2}))}.$$

Due to assumptions (19), there exists a constant $\hat{M}$, independent of N , such that $M_N \leq \hat{M}$, where $\hat{M}$ only depends on initial conditions y_0 and y_1 . Applying Gronwall's inequality to (27) yields

$$\hat{E}_N(t) \leq \hat{E}_N(0) \exp(MT). \quad (28)$$

Due to assumption (19), $\hat{E}_N(0) \rightarrow E_\infty(0)$ as $N \rightarrow \infty$, where

$$E_\infty(0) = \frac{1}{2} [\|\mathbb{A} y_1\|^2 + \|\mathbb{A}^{1/2} [\mathbb{A} y_0 + (k_0 y_1''(0) + \alpha [y_0''(0)]^3) x]\|^2 + 4\alpha [y_0''(0)]^2 [y_1''(0)]^2]$$

there exists an integer N^* such that when $N > N^*$

$$\hat{E}_N(t) \leq [1 + E_\infty(0)] \exp(MT)$$

which implies that

$$\begin{cases} \dot{y}_N \text{ is bounded in } L^\infty(0, T; [D(\mathbb{A})]), \\ \ddot{y}_N \text{ is bounded in } L^\infty(0, T; [D(\mathbb{A}^{1/2})]). \end{cases} \quad (29)$$

Letting $N \rightarrow \infty$, we have

$$\dot{y} \in L^\infty(0, T; [D(\mathbb{A})]), \quad \ddot{y} \in L^\infty(0, T; [D(\mathbb{A}^{1/2})]).$$

THEOREM 4. Let y be the solution of (3) with initial conditions satisfying (19). Let

$$F(t) = \frac{1}{2} \int_0^l \left[\frac{EI}{\rho} [y'''(x, t)]^2 + [\dot{y}''(x, t)]^2 \right] dx + \frac{\alpha}{4} y''^4(0, t). \quad (30)$$

Then there exist constants $M \geq 1$, $\omega > 0$, depending on initial condition, such that

$$F(t) \leq M e^{-\omega t} \quad \text{for all } t \geq 0. \quad (31)$$

Proof. First we show that

$$y_{xx}(0, t) \rightarrow 0, \quad \text{as } t \rightarrow \infty. \quad (32)$$

Notice that

$$\frac{d}{dt} F(t) = -\frac{EI}{\rho} k(t) \dot{y}''^2(0, t) \leq 0. \quad (33)$$

Let $E(t)$ be as (15). Let

$$\gamma(y_0, y_1) = \{(y(\cdot, t), \dot{y}(\cdot, t)) \mid t \geq 0\}. \quad (34)$$

By (33), $\gamma[y_0, y_1]$ is bounded in $D(\mathbb{A}) \times D(\mathbb{A}^{1/2})$ and, hence, is compact in $H^3(0, l) \times H^1(0, l)$. For any positive sequence $t_n \rightarrow \infty$, there exist $(\bar{y}_0, \bar{y}_1) \in D(\mathbb{A}) \times D(\mathbb{A}^{1/2})$ and a subsequence of $\{t_n\}$, still denoted by t_n , such that

$$(y(\cdot, t_n), \dot{y}(\cdot, t_n)) \rightarrow (\bar{y}_0, \bar{y}_1) \quad \text{in } D(\mathbb{A}) \times D(\mathbb{A}^{1/2}) \text{ weak topology.}$$

Meanwhile,

$$(y(\cdot, t_n), \dot{y}(\cdot, t_n)) \rightarrow (\bar{y}_0, \bar{y}_1) \quad \text{in } H^3(0, l) \times H^1(0, l).$$

Hence

$$\lim_{t \rightarrow \infty} E(t) = \int_0^l \left[[\bar{y}_1(x) + k_\infty \bar{y}_0''(0)x]^2 + \frac{EI}{\rho} [\bar{y}_0''(x)]^2 \right] dx = 0,$$

where $k_\infty = k_0 + \alpha \int_0^\infty [y''(0, \tau)]^2 d\tau < \infty$ by (17). So $\bar{y}_0''(x) \equiv 0$ and

$$y_{xx}(0, t_n) \rightarrow \bar{y}_0''(0) = 0, \quad \text{as } n \rightarrow \infty,$$

which proves (32) by the arbitrariness of $\{t_n\}$. Next, let

$$z(x, t) = y''(l - x, t). \quad (35)$$

Then

$$F(t) = \frac{1}{2} \int_0^l \left[\frac{EI}{\rho} z''^2(x, t) + \dot{z}^2(x, t) \right] dx + \frac{\alpha}{4} z^4(l, t)$$

and $z(x, t)$ satisfies

$$\begin{cases} \ddot{z}(x, t) + \frac{EI}{\rho} z''''(x, t) = 0, & 0 < x < l, \\ z(0, t) = z'(0, t) = z''(l, t) = 0, \\ \frac{EI}{\rho} z'''(l, t) = k(t) \dot{z}(l, t) + \alpha z^3(l, t), \\ z(x, 0) = y_0''(l - x), & \dot{z}(x, 0) = y_1''(l - x), \\ k'(t) = \alpha z^2(l, t), & k(0) = k_0 \geq 0. \end{cases} \quad (36)$$

Define

$$\beta(t) = \frac{1}{2} \int_0^l [z(x, t) \dot{z}(x, t) + 4xz'(x, t) \dot{z}(x, t)] dx. \quad (37)$$

Obviously,

$$|\beta(t)| \leq CF(t), \quad \forall t \geq 0, \text{ for some } C > 0. \quad (38)$$

By (32), there exists $t_0 \geq 0$, such that

$$z^2(l, t) \leq 5 \frac{EI}{\rho} (8\alpha l^3), \quad \text{for all } t \geq t_0. \quad (39)$$

A straightforward calculation yields

$$\begin{aligned} \beta'(t) = & -\frac{1}{2} \int_0^l \dot{z}^2(x, t) dx - \frac{7EI}{2\rho} \int_0^l z''^2(x, t) dx - \frac{\alpha}{2} z^4(l, t) - 2\alpha l z'(l, t) z^3(l, t) \\ & + l \dot{z}^2(l, t) - \frac{1}{2} k(t) z(l, t) \dot{z}(l, t) - 2lk(t) z'(l, t) \dot{z}(l, t). \end{aligned} \quad (40)$$

By trace theorem and (32), (39), (40), we obtain, finally,

$$\beta'(t) \leq -F(t) + C_0 \dot{z}^2(l, t), \quad \text{for all } t \geq t_0, \quad (41)$$

where

$$C_0 = \left[2l + \delta(1/2 + 2l) \sqrt{k_0^2 + \frac{\alpha\rho}{EI} E(0)} \right],$$

δ is any positive constant satisfying $\delta > (\rho/EI)(4 + l)^2 \sqrt{k_0^2 + (\alpha\rho/EI)E(0)}$. Take $\varepsilon > 0$ such that $\varepsilon < \min\{1/C, (EI/\rho)k_0/C_0\}$ and define

$$F_\varepsilon(t) = F(t) + \varepsilon\beta(t).$$

Then we have

$$(1 - C\varepsilon)F(t) \leq F_\varepsilon(t) \leq (1 + C\varepsilon)F(t)$$

and

$$\begin{aligned} F'_\varepsilon(t) &= F'(t) + \varepsilon\beta'(t) = -\frac{EI}{\rho} k(t) \dot{z}^2(l, t) + \varepsilon\beta'(t) \\ &\leq -\varepsilon F(t) - \left(\frac{EI}{\rho} k(t) - C_0\varepsilon \right) \dot{z}^2(l, t) \\ &\leq -\frac{\varepsilon}{1 + C\varepsilon} F_\varepsilon(t) - \left(\frac{EI}{\rho} k_0 - C_0\varepsilon \right) \dot{z}^2(l, t) \leq -\frac{\varepsilon}{1 + C\varepsilon} F_\varepsilon(t). \end{aligned}$$

Hence

$$F_\varepsilon(t) \leq F_\varepsilon(t_0) \exp\left(-\frac{\varepsilon}{1 + C\varepsilon} (t - t_0)\right).$$

We finally get

$$F(t) \leq \frac{1}{1 - C\varepsilon} (1 + C\varepsilon) F(t_0) \exp\left(-\frac{\varepsilon}{1 + C\varepsilon} (t - t_0)\right), \quad \forall t \geq t_0.$$

The proof is complete.

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