What Kinds of System Can Be Used as Tracking-Differentiator*  

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Abstract: A survey of several kinds differentiators is given firstly. The comparison shows the tracking-differentiator (TD) permits a weaker condition on the stability of the systems, which can be used to construct differentiators. Moreover, TD sets some weaker constraints on the input signal. Then the paper gives the proofs for two kinds of systems that can be used to construct TD: one is the semi-globally finite time stable autonomous system, a more general kind is the one whose certain time-scale transform is uniformly semi-globally practically asymptotically stable.  

Key Words: Tracking-differentiator, Filippov Solution, Semi-globally Finite Time Stable, Uniformly Semi-globally Practically Asymptotically Stable

1 INTRODUCTION

Differentiation of signals is commonly used in control algorithms. Many approaches for designing differentiator have been proposed, such as the high-gain observer based differentiator [1], the linear time-derivative trackers [2]- [3], the robust exact differentiation [4] - [5] and the finite-time-convergent differentiator [6], etc. Compared with the above mentioned differentiators, the tracking differentiator (TD), which was first proposed by J.Q. Han and W. Wang [7], sets a weaker condition on the stability of the systems to be constructed for TD. However, a rigorous proof for the general tracking differentiator is still open, though the filtering properties and stability of some special kinds of tracking differentiator have been discussed [8]- [12].

In this paper, a comparison discussion for several kinds of differentiator is firstly given. The comparison is made mainly on three aspects: the properties of the system that can be used as differentiators, the constraints on the inputs, and the properties of the outputs. Then the paper gives the proofs for two kinds of systems that can be used to construct TD: one is the semi-globally finite time stable autonomous continuous/discontinuous system, a more general kind is the one whose certain time-scale transform is uniformly semi-globally practically asymptotically stable.

The paper is organized as follows. The comparison for some kinds of differentiators is presented in Section II. The proofs for two kinds of systems used to construct TD are given in Section III and IV respectively. The last section is the conclusion.

2 COMPARISON FOR SOME KINDS OF DIFFERENTIATORS

Each kind of the differentiators discussed will be characterized in the following aspects:

- the structures and properties of the system;  
- the constraints on the input signal;  
- the properties of the output signal.

Remark 1: For some differentiators, \( y^{(i-1)}(t), i \in n - 1 \) can be non-smooth. In this case, we will give an indication that \( y^{(i)}(t) \) denotes the generalized derivative.

1) Tracking-differentiator (TD) [7] [8]:

- The systems used as TD [7]:

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\vdots
\dot{z}_{n-1} &= z_n \\
\dot{z}_n &= r^h(z_1 - y(t), \frac{z_2}{r}, \ldots, \frac{z_n}{r^{n-1}})
\end{align*}
\]

It is assumed that the original system

\[
\begin{align*}
x_1 &= x_2 \\
\vdots
x_{n-1} &= x_n \\
x_n &= h(x_1, x_2, \ldots, x_n)
\end{align*}
\]

have solution in the sense of Filippov [14] and satisfies

\[
\lim_{t \to \infty} x_i(t) = 0, \quad i \in n.
\]

- The constraints on the input:

\[
y(t) \text{ is bounded and integrable in } t \in [t_0, t_0 + L] \text{ for each } t_0 \geq 0 \text{ and } L \geq 0.
\]

- The property of the output:

\[
\lim_{r \to \infty} \int_{t_0}^{t_0 + L} |z_1(t) - y(t)| dt = 0
\]

which means that \( z_1(t) \) can be sufficiently close to \( y(t), t \in [t_0, t_0 + T] \) when \( r \) is large enough. Thus \( z_i(t), i \in n \) can be used as an estimation of \( y^{(i-1)}(t), t \in [t_0, t_0 + T] \), where \( y^{(i-1)}(t) \) is the generalized derivative if \( y^{(i-2)}(t) \) is non-smooth.

2) High-gain observer based differentiator (HGOD) [1]:

Assume \( y(t) \) is the input. To keep consistence, all differentiators are discussed under the condition that the highest order of the derivative to be estimated is \( n - 1 \), i.e, the signals to be estimated are \( y^{(i)}(t), i \in n - 1 \).

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The systems used as HGOD:

\[
\dot{z} = Az + H(y(t) - Cz)
\]  \hspace{1cm} (4)

where 

\[
z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad H = \begin{bmatrix} \alpha_1 \\ \frac{\alpha_2}{T^2} \\ \vdots \\ \frac{\alpha_n}{T^n} \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
\]

ε is a small positive parameter and the positive constants αi are chosen such that the roots of 

\[
s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n = 0
\]

have negative real parts. Obviously, HGOD employs a linear time-invariant (LTI) high-gain observer which is exponential stable.

- The constraints on the input:
  \( y^{(n)}(t) \) is bounded.

- The property of the output:
  There exists a finite time \( T_1(\varepsilon) \) such that 
  \[ |z_i(t) - y^{(i-1)}(t)| \leq O(\varepsilon), \quad t \geq T_1, \ i \in \mathbb{N}. \]

3) **Linear time-derivative trackers (LTDT):**

[2]- [3] proposed two forms of linear time-derivative tracker. Since their properties are similar, we introduce one of them.

- The systems used as LTDT:
  \[
  \dot{z} = Az + H^{-1}(t)C^T(y(t) - Cz) \\
  H(t) = -\gamma H(t) - A^T H(t) - H(t)A + CT. 
  \]  \hspace{1cm} (5)

where \( \gamma > 0 \) can be chosen sufficient large. LTDT (5) is constructed by a linear time-invariant (LTI) observer which is proven to be exponential stable [2];

- The constraints on the input:
  \( y^{(i)}(t), i \in \mathbb{N} \) are all bounded.

- The property of the output:
  There exists a finite time \( T_2(\gamma) \) such that 
  \[ |z_i(t) - y^{(i-1)}(t)| \leq \frac{\eta_i}{\gamma}, \quad t \geq T_2, \ i \in \mathbb{N} \]

where \( \eta_i \) is a constant which depends upon the dimension \( n \) and \( y^{(i)} \).

4) **Robust exact differentiation (RED)** [4]- [5]:

- The systems used as RED:
  \[
  \dot{z}_1 = -\lambda_1 |z_1 - \tilde{y}(t)|^{\frac{n-1}{2}} \text{sign}(z_1 - \tilde{y}(t)) + z_2 \\
  \dot{z}_2 = -\lambda_2 |z_1 - \tilde{y}(t)|^{\frac{n-2}{2}} \text{sign}(z_1 - \tilde{y}(t)) + z_3 \\
  \vdots \\
  \dot{z}_{n-1} = -\lambda_{n-1} |z_1 - \tilde{y}(t)|^{\frac{2}{2}} \text{sign}(z_1 - \tilde{y}(t)) + z_n \\
  \dot{z}_n = -\lambda_n \text{sign}(z_1 - \tilde{y}(t)) 
  \]  \hspace{1cm} (6)

where \( \lambda_i > 0 \) is the parameter to be determined. RED (6) is a non-smooth feedback system which is finite time stable.

- The constraints on the input:
  \( \dot{y}(t) = y(t) + w(t) \) satisfies the condition that the \( (n-1) \)th derivative of the base signal \( y(t) \) has a known Lipschitz constant \( L_y > 0 \) and the \( w(t) \) is an unknown function which is bounded and Lebesgue-measurable.

- The property of the output:
  If \( w(t) = 0 \), then there exists a finite time \( T_3 \) such that 
  \[ |z_i(t) - y^{(i-1)}(t)| \leq \mu_i \varepsilon^{n-i}/n, \quad t \geq T_3, \ i \in \mathbb{N}. \]

4) **Finite time convergent differentiator (FTCD)** [6]:

- The systems used as FTCD:
  \[
  \begin{cases}
  \dot{z}_1 = z_2 \\
  \vdots \\
  \dot{z}_{n-1} = z_n \\
  \varepsilon^n z_n = f(z_1 - y(t), \varepsilon z_2, \ldots, \varepsilon^{n-1} z_n).
  \end{cases} 
  \]  \hspace{1cm} (7)

It is assumed that the original system 

\[
\begin{cases}
\dot{x}_1 = x_2 \\
\vdots \\
\dot{x}_{n-1} = x_n \\
\dot{x}_n = f(x_1, x_2, \ldots, x_n)
\end{cases}
\]  \hspace{1cm} (8)

is finite time stable with some Lipschitz Lyapunov function \( V \), and there exist \( \rho_i \in (0, 1], i \in \mathbb{N} \) and a nonnegative constant \( \sigma \) such that 

\[ |f(x_1, x_2, \ldots, x_n) - f(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)| \leq \sigma \sum_{i=1}^{n} |x_i - \tilde{x}_i|^{\rho_i}. \]

FTCD (7) and TD (1) have the similar form. However, FTCD requires (8) have the properties of continuity and finite time stability. Thus both the discontinuous fast control system and the linear exponential stable systems, which can be used as TD [7]- [8], are excluded in FTCD.

- The constraints on the input:
  \( y^{(i)}(t), i \in \mathbb{N} \) are all bounded and \( y^{(n-1)}(t) \) can be general derivative, that is \( y^{(n-1)}(t) \) may not exists only at finite instants \( t_j, j \in \mathbb{K} \), but both the left derivative \( y^{n-1}_-(t_j) \) and the right derivative \( y^{n-1}_+(t_j) \) exist.

- The property of the output:
  There exist \( \Gamma > n \) and a finite time \( T_4(\varepsilon) \) such that 
  \[ |z_i(t) - y^{(i-1)}(t)| = O(\varepsilon^{\Gamma-n-1}), \quad t_{j-1} + \varepsilon T_4 \leq t \leq t_j, \ i \in \mathbb{N}, \ j \in \mathbb{K}. \]

where \( t_0 = 0 \) and \( t_{k+1} = \infty \).

A summary for above differentiators is given in Tab. 1 and Tab. 2:

- The tables show that:
  - The original system of TD has the most weak stability. Therefore, the systems that can be designed as HGOD, RED and FTCD can also be used as TD.
The systems to be used as differentiator

<table>
<thead>
<tr>
<th>Structure</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>TD</td>
<td>Nonlinear</td>
</tr>
<tr>
<td>HGOD</td>
<td>LTI</td>
</tr>
<tr>
<td>LTDT</td>
<td>LTV</td>
</tr>
<tr>
<td>RED</td>
<td>Non-smooth</td>
</tr>
<tr>
<td>FTCD</td>
<td>Continuous</td>
</tr>
</tbody>
</table>

The Filippov solution of the original system (2) converges to zero

Properties of the output signal

<table>
<thead>
<tr>
<th>Constraints on the input signal</th>
<th>Properties of the output signal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y(t) ) is bounded and integrable in any bounded interval</td>
<td>( \lim_{t \rightarrow T} \int_{t_0}^{t_0 + T} | z_i(t) - y(t) | dt = 0 )</td>
</tr>
<tr>
<td>( y^{(i)}(t) ) is bounded</td>
<td>(</td>
</tr>
<tr>
<td>( y^{(i)}(t), i \in \mathbb{N} ) are bounded</td>
<td>(</td>
</tr>
<tr>
<td>( y^{(i-1)}(t) ) satisfies some Lipschitz Condition</td>
<td>( z_i(t) = \frac{y^{(i-1)}(t)}{\delta}, \quad t \geq T_3 )</td>
</tr>
<tr>
<td>( y^{(i)}(t), i \in \mathbb{N} - 1 ) and ( y^{(i-1)}(t) ) (generalized derivative) are bounded</td>
<td>( z_i(t) - y^{(i-1)}(t) = O(\varepsilon^{1-n+1}), \quad t_{j-1} + \varepsilon T_4 \leq t \leq t_j )</td>
</tr>
</tbody>
</table>

\( m(t) \) almost everywhere in each bounded closed domain \( \mathcal{W} \subset \mathcal{G} \).

Denote the solution of (9) with the initial condition \( x(t_0) = x_0 \) as \( x(t; t_0, x_0) \). Next we extend the definition of the uniformly semi-globally finite time stable, which is proposed in [15] for the continuous system, to the discontinuous system (9).

**Definition 1.** The origin of system (9) with \( f(t, 0) = 0 \) is uniformly semi-globally finite time stable in the open domain \((0, \infty) \times \mathcal{W} \subset \mathbb{R}^n \) if it is Lyapunov stable and satisfies: for each bounded open domain \( \mathcal{F} \subset \mathcal{W} \), there exists a settling time \( T \geq 0 \) such that \( \forall (t_0, x_0) \in [0, \infty) \times \mathcal{F} \), all solutions \( x(t; t_0, x_0) \) satisfy:

\[
\begin{align*}
\lim_{t \rightarrow T_0 + T} x(t; t_0, x_0) &= 0 \\
x(t; t_0, x_0) &= 0, \quad t > t_0 + T.
\end{align*}
\]

**Lemma 1.** ([14], Chapter 2) Assume \( f(t, x) \) and \( f'(t, x) \) satisfy the conditions C1-C2 in the open domain \( \mathcal{G} \). If all solutions of the system (9) with the initial condition \( t_0 \in [a, b], (t_0, x_0) \in \mathcal{G} \) exist for \( t \in [a, b] \) and their graphs \( (t, x(t)) \in \mathcal{G} \), then for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for any \( t_0' \in [a, b], x_0' \) and \( f'(t, x) \) which satisfy the conditions:

\[
|t_0' - t_0| \leq \delta, \quad \| x_0' - x_0 \| \leq \delta, \quad f'(t, x) \subset \{ f(t, x^\delta) \}^\delta, \quad \frac{\| x_0^\delta - x_0 \|}{\delta} \leq \delta, \quad \frac{\| x_0^\delta - x_0 \|}{\delta} \leq \delta,
\]

where \( x^\delta \) stands for the \( \delta \) neighborhood of \( x \) and \( [f(t, x^\delta)]^\delta \) is defined for each solution of the system \( x \in \mathcal{G} \) exists for \( t \in [a, b] \) and there exists a solution \( x(t; t_0, x_0) \) satisfying:

\[
\max_{t \in [a, b]} \| x'(t; t_0', x_0') - x(t; t_0, x_0) \| \leq \varepsilon.
\]

From Lemma 1, it is easy to prove that there exists an uniform \( \delta \) for all \( (t_0, x_0) \in [a, b] \times \mathcal{W} \), where \( \mathcal{W} \) is a compact and closed set.
Lemma 2. Assume \(f(t,x)\) and \(f'(t,x)\) satisfy the conditions C1-C2 in \(\mathcal{G}\). If all solutions of the system (9) with the initial condition \(t_0 \in [a,b]\), \((t_0,x_0) \in \mathcal{G}\) exist for \(t \in [a,b]\) and their graphs \((t,x(t)) \in \mathcal{G}\), then for any \(\epsilon > 0\), there exists \(\delta > 0\) such that for any \((t_0,x_0) \in [a,b] \times \mathcal{W}\), \(t_0 \in [a,b]\), \(x_0\) and \(f'(t,x)\) satisfying the conditions:

\[
|t_0 - t_0| \leq \delta, ||x_0 - x_0|| \leq \delta, f'(t, x) \subset [f(t, x)]^\delta,
\]

every solution of the system

\[
\dot{x}^* = f'(t, x^*), \quad x^*(t_0) = x_0^*
\] (11)

exists for \(t \in [a,b]\) and there exists a solution \(x(t; t_0, x_0)\) satisfying:

\[
\max_{a \in [a,b]} ||x^*(t; t_0, x_0) - x(t; t_0, x_0)|| \leq \epsilon.
\]

Proof. We use the reduction to absurdity. Suppose there exist \(\epsilon > 0\) and \(\delta_1 \rightarrow 0\). For each \(\delta_1\), there exist \((t_i, x_i) \in [a,b] \times \mathcal{W}\), \(t_i \in [a,b]\), \(x_i \in \mathcal{G}\) and \(f'(t, x)\) satisfying the conditions:

\[
|t_i - t_i| \leq \delta_1, ||x_i - x_i|| \leq \delta_1, f'(t, x) \subset [f(t, x)]^\delta_1.
\]

Moreover, at least one solution \(x^*(t; t_i^*, x_i^*)\) of the system

\[
\dot{x}^* = f'(t, x^*), \quad x^*(t_i^*) = x_i^*
\] (12)

either cannot be extended to \([a,b]\) or for every solution \(x(t; t_i, x_i)\) of the system (9), \(\exists \delta_2 \in [a,b]\) such that

\[
||x^*(t; t_i^*, x_i^*) - x(t; t_i, x_i)|| > \epsilon.
\]

Then there must exist subsequence \((t_k, x_k) \rightarrow (t_0, x_0) \in [a,b] \times \mathcal{W}\) and \((t_k^*, x_k^*) \rightarrow (t_0, x_0)\). According to Lemma 1, for sufficient large \(k\), \(x^*(t; t_k^*, x_k^*)\) exists for \(t \in [a,b]\) and there exists a solution \(x(t; t_0, x_0)\) satisfying:

\[
\max_{a \in [a,b]} \| x^*(t; t_k^*, x_k^*) - x(t; t_0, x_0) \| \leq \epsilon/2,
\]

\[
\max_{a \in [a,b]} \| x(t; t_k, x_k) - x(t; t_0, x_0) \| \leq \epsilon/2.
\]

Thus

\[
\max_{a \in [a,b]} \| x^*(t; t_k^*, x_k^*) - x(t; t_k, x_k) \| \leq \epsilon.
\]

which leads to a contradiction. Q.E.D.

In the following analysis, let \(h(x_1, \ldots, x_n)\) always satisfy the conditions C1-C2 in the domain of interest and let \(y(t) = y_1(t)\) satisfy

Assumption 1:

1. \(y_1(t) = y_2(t), t \in [0,\infty) \setminus \{t_i\}_{i=1}^{\infty}: y_2(t_i^+)\) (right limit) and \(y_2(t_i^-)\) (left limit) exist;
2. \(\min_{t} |t_i + t_i| > c_1 > 0\) and \(\max_{t} |y_1(t_i^-) - y_1(t_i^-)| < c_2 < \infty;\)
3. \(y_1(t_i^+)\) and \(y_1(t_i^-)\) are bounded. \(y_2(t)\) is bounded at any bounded interval of \([0,\infty),\)

Since we will study \(\int_{t_0}^{t_0 + L} |z_1(t) - y_1(t)| dt\), which is independent of \(y_1(t), t \in [t_i - c_1, t_i + c_1]\), for simplicity in the following analysis, let \(y_1(t) = y_1(t_i^+)\). And similarly, let \(y_2(t) = y_2(t_i^-)\). Define

\[
s = \frac{t}{\sigma}, \quad s_i = \frac{t_i}{\sigma}, \quad \sigma = \frac{1}{r} \in (0, \infty).
\]

Taking the transformation

\[
\begin{align*}
x_1^+(s) &= z_1(s) - y_1(t) \\
x_2^+(s) &= x_3(s) \\
&\cdots \\
x_n^+(s) &= x_n(s) \\
x_0^+(s) &= \sigma^{n-1} x_n(t).
\end{align*}
\] (13)

for the system (1), there is

\[
\begin{align*}
\dot{x}_1^+(s) &= x_2^+(s) - \sigma y_2(t) \\
\dot{x}_2^+(s) &= x_3(s) \\
&\cdots \\
\dot{x}_{n-1}^+(s) &= x_n(s) \\
\dot{x}_n^+(s) &= h(x_1^+(s), x_2^+(s), \ldots, x_n^+(s)).
\end{align*}
\] (14)

We concern the solutions of (14) in each interval \(s \in [s_i - \frac{L}{\sigma}, s_i + \frac{L}{\sigma}]\), i.e., \(t \in [t_i - c_1, t_i + c_1]\) corresponding with the initial condition:

\[
x^+(s_i) = \begin{bmatrix} z_1(t_i - c_1) - y_1(t_i - c_1) \\ \sigma^{n-1} z_n(t_i - c_1) \end{bmatrix} = \begin{bmatrix} z_1(t_i - c_1) - y_1(t_i - c_1) \\ \sigma^{n-1} z_n(t_i - c_1) \end{bmatrix}.
\]

Theorem 1. Suppose the system (2) is uniformly semi-globally finite time stable in the open domain \((0, \infty) \times \mathcal{W}\), where \(\mathcal{W} \supseteq \dot{B}(0, c_2) = \{x ||x|| \leq c_2\}\), then for any \([z_1(t_0), z_2(t_0), \ldots, z_n(t_0)]^T \in \mathcal{W}, t_0 \geq 0,\) the system (1) satisfies:

\[
\lim_{t \to \infty} \int_{t_0}^{t} |z_1(t) - y_1(t)| dt = 0.
\]

Proof. We will give the proof for the case of \(n = 2\), and the proof for the case of \(n > 2\) is similar. Since \(\min_{t} |t_i - t_i| > c_1 > 0\), we can find \(k > 0\) such that \(t_k - t_0 \leq t_0 + L\) and \(t_k \geq t_0 + L\). Hence

\[
\int_{t_0}^{t_0 + L} |z_1(t) - y_1(t)| dt = \int_{t_0}^{t_1} |z_1(t) - y_1(t)| dt + \cdots + \int_{t_{k-1}}^{t_0 + L} |z_1(t) - y_1(t)| dt.
\] (15)

Consider the first term in the right side of (15). It is obvious that \([x_2^+ - \sigma y_2, h(x_1^+, x_2^+)]\) satisfies the conditions C1-C2. For \(z_1(t_0) - y_1(t_0), z_2(t_0)\) \(\in \mathcal{W}\), there exist \(\sigma_2 > 0\) and a bounded open domain \(\mathcal{Y} \subseteq \mathcal{Y} \subseteq \mathcal{W}, \mathcal{Y} \supset B(0, c_2),\) such that when \(\sigma \in (0, \sigma_2)\),

\[
x^+(s_0) = [z_1(n_0) - y_1(n_0), \sigma z_2(n_0)]^T \in \mathcal{Y}.
\]
Denote the solutions of
\[
\begin{cases}
\dot{x}_1^s(t) = x_2^s(t) - \sigma y_2(t) \\
\dot{x}_2^s(t) = h(x_1^s(t), x_2^s(t)),
\end{cases}
\]
for \(x^s(x, s, 0, \sigma, \sigma)\), i.e., \(x^s(0, 0, \sigma, \sigma)\). And denote the solutions of
\[
\begin{cases}
\dot{x}_1(x) = x_2(x) \\
\dot{x}_2(x) = h(x_1(x), x_2(x)),
\end{cases}
\]
as \(x(x, 0, \sigma, \sigma)\), i.e., \(x(0, 0, \sigma, \sigma)\).

Thus
\[
\begin{align*}
\int_{0}^{t} & |\dot{y}_1(t) - y_1(t)| dt \\
& \leq \int_{0}^{t} |\dot{x}_1^s(t) - x_1(t, 0, \sigma, \sigma)| dt
\end{align*}
\]

Since (17) is semi-globally finite-time stable, there exist \(\alpha > 0\) and \(T > 0\) such that
\[
\|x(t, t_0, \sigma)\| \leq \alpha, \quad t \geq t_0.
\]

Let \(\sigma_2 = \min \{\sigma_1, \frac{\sigma}{2}, \frac{\sigma}{4}\}\). Then \(\forall \sigma \in (0, \sigma_2)\), there is
\[
x(0, 0, \sigma, \sigma) = 0, \quad t_0 + \sigma T \leq t_0 + \varepsilon \leq t \leq t_1.
\]

Next, the solution \(x^s(t, t_0, \sigma)\) of system (18) in \((t_0, t_1)\) is analyzed. Devide the time region \([t_0, t_1] \cup [t_1, t_0 + \sigma T + T]\) into
\[
[t_0, \frac{t_0 - \varepsilon}{\sigma} + T], [\frac{t_0 - \varepsilon}{\sigma} + T, \frac{t_0 - \varepsilon}{\sigma} + T + 2T], [...], [\frac{t_0 - \varepsilon}{\sigma} + m T, \frac{t_0 - \varepsilon}{\sigma}],
\]
where the integer \(m\) satisfies \(m T \geq t - \frac{t_0}{\sigma}\).

Let
\[
\begin{align*}
f^*(s, x^s(s), \sigma) &= \left[x_2^s(s) - \sigma y_2(s) \right] \\
h^*(x_1(s), x_2(s)) &= h(x_1(s), x_2(s)).
\end{align*}
\]

Define \(t' = s - \frac{t_0}{\sigma} = \frac{t}{\sigma} - \frac{t_0}{\sigma}\) then the solution of system
\[
\dot{x}^*(t') = f^*(t' + t_0, \frac{t_0}{\sigma}, x^*(t'), \sigma), \quad x^*(0) = x_0, t' \in [0, T] (22)
\]
satisfies
\[
x^*(t'; 0, x_0, \sigma) = x^*(t; t_0, x_0, \sigma, \sigma), \quad t \in [\frac{t_0}{\sigma} + T, \frac{t_0}{\sigma} + T + T].
\]

Furthermore, there is
\[
x(t'; 0, x_0, \sigma) = x(t; \frac{t_0}{\sigma}, x_0, \sigma, \sigma), \quad t \in [\frac{t_0}{\sigma} + T, \frac{t_0}{\sigma} + T + T].\]

According to Assumption 1, there exists \(c_3 > 0\), such that
\[
|y_2(t)| \leq c_3, \quad t \leq t_0 + L.
\]

Therefore, for each \(\delta, \sigma^* = \min \{\sigma_1, \sigma_2\}\), \(\forall \sigma \in (0, \sigma^*)\), there is,
\[
|f^*(t' + t_0, \frac{t_0}{\sigma}, x, \sigma) - f(x)| = |\sigma y_2(t_0 + \sigma t')| \leq \delta, \quad t' \in [0, T].
\]

According to Lemma 2, \(\forall \varepsilon^*, \exists \delta > 0\) such that every solution of the system (22) exists for \(t' \in [0, T]\) and there exists a solution \(x(t'; 0, x_0, \sigma)\) satisfying
\[
\max_{0 \leq t' \leq T} |\dot{x}^*(t'; 0, x_0, \sigma) - x(t'; 0, x_0, \sigma)| \leq \varepsilon^*.
\]

From (23) and (24), we have
\[
\max_{0 \leq t' \leq (t_0 + \sigma T + T)} |x^*(t'; 0, x_0, \sigma) - x(t; t_0, x_0, \sigma)| \leq \varepsilon^*.
\]

Moreover, from (20),
\[
|x^*(t_0 + T, x_0, \sigma)| \leq \varepsilon^*.
\]

Denote
\[
x_{20} = x^*(t_0 + T, x_0, \sigma), \quad x_{020} = x(t_0, x_0, \sigma).
\]

Similarly, the solution of the system
\[
\dot{x}^*(t'') = f^*(t'', t_0, \sigma, x^*(t''), \sigma), \quad x^*(0) = x_{20}, t'' \in [0, T] (29)
\]
satisfies
\[
\dot{x}^*(t''; 0, x_{20}, \sigma, \sigma) = x^*(t; t_0, x_{20}, \sigma, \sigma), \quad t \in [t_0, t_0 + T].
\]

And there is
\[
x(t''; 0, x_{20}) = x(t; t_0 + T, x_0, \sigma, \sigma), \quad t \in [t_0 + T, t_0 + T + T].
\]

According to Lemma 2, for the same \(\varepsilon^*\) and \(\delta\), every solution of the system (29) exists for \(t'' \in [0, T]\) and there exists a solution \(x(t'', 0, x_{20})\) satisfying
\[
\max_{0 \leq t'' \leq T} ||\dot{x}^*(t''; 0, x_{20}, \sigma) - x(t''; 0, x_{20})|| \leq \varepsilon^*.
\]

Since the system (17) is finite time stable, for \(\varepsilon\) in (18), \(\exists \delta > 0\) such that when \(\|x_0\| \leq \delta\),
\[
\begin{array}{ll}
\|x(t'; 0, x_0)\| = \|x(t; t_0, x_0, \sigma, \sigma)\| \leq \varepsilon, \quad t \in [\frac{t_0}{\sigma} + T, \frac{t_0}{\sigma} + T + T].
\end{array}
\]
\[
\|x(T; 0, x_0)\| = \|x(t_0 + T, x_0, \sigma, \sigma)\| = 0.
\]

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Set $\varepsilon^* = \min\{\xi, \delta_1\}$, then from (30)-(33), there is
\[
\begin{aligned}
&\max_{t_0 + \varepsilon \sigma \leq \varepsilon, t_0} \| x^*(t, t_0) \| \leq \varepsilon, \\
&\| x^*(t, t_0) - x(t_0) \| \leq \delta_1.
\end{aligned}
\] (34)

Similarly, let
\[
\begin{aligned}
x_t = x_t(t + (i - 1)T, t_0) \in \mathcal{T},
\end{aligned}
\] where $i = 3, 4, ..., m + 1$. Then when $i = 3, 4, ..., m$,
\[
\| x(t) \| \leq \delta_1, \quad \max_{t_0 \leq t \leq t_0 + \varepsilon \sigma} \| x^*(t, t_0) - x(t_0) \| \leq \varepsilon.
\]

Therefore, the third term of (18) satisfies
\[
\int_{t_0}^{t_1} |x_1(t) - x_2(t)| dt \leq (t_1 - t_0) \varepsilon
\]
which leads to
\[
\int_{t_0}^{t_1} |y_1(t) - y_2(t)| dt \leq \varepsilon (t_1 - t_0 + \alpha). \quad (35)
\]

Next, the second term of the right side of (15) is studied by the system
\[
\begin{aligned}
\{ x_1(t) &= x_2(t) - \sigma_2 y_2(t), \\
x_2(t) &= h(x_1(t), x_2(t)) \\
, x_1(s) &= \begin{bmatrix} z_1(t_1) - y_1(t_1) \\ \sigma_2 z(t_1) \end{bmatrix}
\end{aligned}
\] (36)

Since $y_1(t_1) = y_1(t_1^*)$, there is
\[
\begin{aligned}
| x_1(s) | &= | z_1(t_1) - y_1(t_1) | = | z_1(t_1^*) - y_1(t_1^*) | \\
&\leq | z_1(t_1^*) - y_1(t_1^*) | + | y_1(t_1^*) - y_1(t_1^*) | \\
&= | x_2(t_1^*) \| + | y_1(t_1^*) - y_1(t_1^*) |
\end{aligned}
\]
and
\[
| x_2(t_1^*) | = | \sigma_2 z(t_1) | = | \sigma_2 z(t_1^*) | = | x_2(t_1^*) \|, \quad (37)
\]

which gives the result:
\[
\begin{aligned}
\lim_{t \to \infty} \int_{t_0}^{t_1} |z_1(t) - y_1(t)| dt &\leq \varepsilon (t_1 - t_0 + \alpha).
\end{aligned}
\]

Remark 3: (37) suggests that the smaller $\varepsilon$ or $c_1$, a larger $r$ in TD (1) will be used. And the smaller $T$ or $c_3$, a smaller $r$ in TD (1) can be used. In other words, if the time periods between the discontinuous points in $y(t)$ is small, or the magnitude of $y(t)$ is large, a larger $r$ will give a better performance. On the other hand, if the original system (2) can converge to zero in a short time, TD (1) can perform better.

Remark 4: It is shown that both the fast control form TD and the non-smooth feedback form TD proposed in [7]-[10] are constructed by the semi-globally finite time stable systems.

4 TD AND UNIFORMLY SEMI-GLOBALLY PRACTICALLY ASYMPTOTICALLY STABLE SYSTEMS

Consider the following system
\[
\dot{x}^*(t, x^*, \sigma) = f(t, x^*, \sigma) \quad (38)
\]
where $f(t, x^*, \sigma) \forall \sigma \in (0, \infty)$ satisfies the conditions C1)-C2) for $(t, x^*) \in [0, \infty) \times \mathcal{Y}$ and $\mathcal{Y}$ is a domain in $\mathcal{R}^n$. Denote the solution of (38) with $x^*(t_0) = x_0$ as $x^*(t, t_0, x_0, \sigma)$.

Next we extend the definition of the uniformly semi-globally practically asymptotically stable, proposed in [16] for the continuous system, to the discontinuous system (38).

Definition 2: The origin of the system (38) is uniformly semi-globally practically asymptotically stable in $\mathcal{Y}$ if for $\forall (t_0, x_0) \in [0, \infty) \times \mathcal{Y}$, all solutions $x(t; t_0, x_0, \sigma)$ exist in $(t_0, \infty) \times \mathcal{Y}$ and satisfy:

1. (Uniform boundedness) For every bounded open neighborhood $\mathcal{Y}_1 \subset \mathcal{Y}$, there exist a compact set $\mathcal{Y}_2 \subset \mathcal{Y}$ and $\sigma^*$ such that for $\forall (t_0, x_0) \in [0, \infty) \times \mathcal{Y}_1$ and $\forall \sigma \in (0, \sigma^*)$,
\[
x^*(t; t_0, x_0, \sigma) \in \mathcal{Y}_2, \quad \forall t \geq t_0.
\]
2. (Uniformly practical stability) For every bounded open neighborhood $\mathcal{Y}_2 \subset \mathcal{Y}$, there exist a compact set $\mathcal{Y}_2 \subset \mathcal{Y}$ and $\sigma^*$ such that for $\forall (t_0, x_0) \in [0, \infty) \times \mathcal{Y}_2$ and $\forall \sigma \in (0, \sigma^*)$,
\[
x^*(t; t_0, x_0, \sigma) \in \mathcal{Y}_2, \quad \forall t \geq t_0.
\]
3. (Uniformly practical convergence) For every bounded open neighborhood \( \gamma_3 \subset \gamma_3 \subset \mathbb{W} \) and \( \gamma_3 \subset \mathbb{W} \), there exist \( \sigma^* \) and \( T^* > 0 \) such that for \( \forall (t_0, x_0) \in [0, \infty) \times \gamma_3 \) and \( \forall \sigma \in (0, \sigma^*) \),

\[
x'(t; t_0, x_0, \sigma) \in \mathbb{W}, \quad \forall t \geq t_0 + T^*.
\]

**Theorem 2.** Suppose the system (14) is uniformly semi-globally practically asymptotically stable in the open domain \( (0, \infty) \times \mathbb{W} \) and \( \mathcal{B}(0, c_2) \subset \mathbb{W} \subset \mathbb{R}^n \), then for any \( [z_1(t_0) - y_1(t_0), z_2(t_0), \ldots, z_n(t_0)]^T \in \mathbb{W} \), the system (1) satisfies:

\[
\lim_{t \to \infty} \int_{t_0}^{t} |z_1(t) - y_1(t)| dt = 0.
\]

**Proof:** The proof is similar to that of Theorem 1 and we also only give the proof for the case of \( n = 2 \).

Consider the first term in the right-side of (15):

\[
\int_{t_0}^{t} |z_1(t) - y_1(t)| dt = \int_{t_0}^{t} |x_1(t; t_0, x_0, \sigma)| dt
\]

\[
= \int_{t_0}^{t_0 + \varepsilon} |x_1(t; t_0, x_0, \sigma)| dt + \int_{t_0 + \varepsilon}^{t} |x_1(t; t_0, x_0, \sigma)| dt.
\]

Using definition 2, we can get for each \( \varepsilon \), there exists \( \sigma_1 > 0 \), \( T^* > 0 \) and \( \alpha > 0 \) such that when \( \sigma \in (0, \sigma_1) \), all the solutions of

\[
\begin{align*}
x_1'(s) &= x_2'(s) - \sigma y_2(t) \\
x_2'(s) &= h(x_1(s), x_2(s))
\end{align*}
\]

satisfy

\[
\left\{ \begin{array}{l}
\|x'(t; t_0, x_0, \sigma)\| \leq \alpha, \\
\|x'(t; t_0, x_0, \sigma)\| \leq \varepsilon,
\end{array} \right.
\] (40)

Similar to Theorem 1, for \( \forall [z_1(t_0) - y_1(t_0), z_2(t_0)]^T \in \mathbb{W} \), there exist \( \sigma_1 > 0 \) and a bounded open domain \( \mathcal{Y} \subset \gamma_3 \subset \mathbb{W} \), \( \mathcal{Y} \supset \mathcal{B}(0, c_2) \), such that when \( \sigma \in (0, \sigma_1) \)

\[
x'(s_0) = [z_1(t_0) - y_1(t_0), \sigma z_2(t_0)]^T \in \mathcal{Y}.
\]

Denote \( \sigma^* \triangleq \min \{ \sigma_1, \frac{c}{T^*} \} \), then combination of (39) and (41) shows for \( \sigma \in (0, \sigma^*) \)

\[
\int_{t_0}^{t} |z_1(t) - y_1(t)| dt \leq \varepsilon (x + t_1 - t_0).
\] (42)

Consider the second term in the right side of (15) by analyzing the system (36). The same as Theorem 1, since \( x'(t_0) \in \mathcal{Y} \), repeating the analysis between (40) and (42), yields, for \( \sigma \in (0, \sigma^*) \),

\[
\int_{t_1}^{t_2} |z_1(t) - y_1(t)| dt \leq \varepsilon (x + t_2 - t_1).
\]

Since the right-hand of (15) has finite terms, there is \( \forall \varepsilon > 0 \), \( \exists \sigma^* > 0 \), such that \( \forall \sigma \in (0, \sigma^*) \),

\[
\lim_{t \to \infty} \int_{t_0}^{t} |z_1(t) - y_1(t)| dt \leq \varepsilon (L + k \alpha)
\]

where

\[
\sigma^* = \min \{ \sigma_1, \frac{c}{T^*} \}.
\]

It follows as

\[
\lim_{t \to \infty} \int_{t_0}^{t} |z_1(t) - y_1(t)| dt = 0.
\]

Q.E.D.

**Remark 5:** Theorem 2 sets the constraints directly on the system (1) rather than the original system (2). However, the similar conclusions on how to choose \( r \) can be obtained as those in Remark 3.

Form Theorem 2, it is easy to get the following corollary.

**Corollary 1.** Suppose the system (14) is uniformly semi-globally practically asymptotically stable in the open domain \( (0, \infty) \times \mathbb{W} \) and \( \mathcal{B}(0, c_2) \subset \mathbb{W} \subset \mathbb{R}^n \), then for any \( [z_1(t_0) - y_1(t_0), z_2(t_0), \ldots, z_n(t_0)]^T \in \mathbb{W} \), the system (1) satisfies:

\[
\lim_{t \to \infty} \int_{t_0}^{t} |z_1(t) - y_1(t)| dt = 0.
\]

**Remark 6:** If the original system (2) is semi-globally finite time stable in \( (0, \infty) \times \mathbb{W} \) and \( y(t) \) satisfies Assumption 1, then the system (1) is uniformly semi-globally practically asymptotically stable in \( (0, \frac{L}{\alpha + k}) \times \mathbb{W} \). Therefore, Theorem 1 can also be proved by Corollary 1.

**Remark 7:** According to Corollary 1, some kinds of systems, which are not finite time stable, can also be used to construct TD. For example, the linear TD proposed in [8]:

\[
\begin{align*}
\dot{z}_1(t) &= z_2(t) \\
\dot{z}_2(t) &= -rk_1 z_1(t) + rk_2 z_2(t)
\end{align*}
\]

where \([0 \quad 1 \quad -k_1 \quad -k_2]\) is Hurwize.

5 CONCLUSION

Several kinds of differentiators is compared in the paper. From the comparison, it is shown that TD permits a weaker condition on the stability of the systems, which can be used as differentiators and sets weaker constraints on input signal. Moreover, the proofs are given for two kinds of system that can be used to construct TD: the system (2) is semi-globally finite time stable or the system (1) is uniformly semi-globally practically asymptotically stable.

REFERENCES


