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Output feedback stabilization and estimation of the region of attraction for nonlinear systems: a vector control Lyapunov function perspective

Dabo Xu, Xinghu Wang, Yiguang Hong, Zhong-Ping Jiang, and Shengyuan Xu

***Abstract*—This technical note deals with (local) asymptotic stabilization of uncertain nonlinear systems by output feedback control. Our study provides a new design technique from a vector control Lyapunov function (VCLF) perspective. A comparison principle is developed and as a major consequence, an effective estimation method of the region of attraction (ROA) is established. Application to a class of minimum-phase strict-feedback nonlinear systems gives rise to a constructive stabilization design as well as an estimate of the resulting ROA.**

***Index Terms*—Nonlinear systems; control Lyapunov function; vector Lyapunov function; comparison principle; region of attraction.**

I. Introduction

Vector Lyapunov functions (VLFs) have been effectively promoted as a viable alternative to the well-known scalar Lyapunov functions for the analysis and synthesis of linear and nonlinear systems; see [2], [15], [16] and some recent results in [12], [13], [18], [25]. The effectiveness has been extensively explored for various complex systems, particularly for multivariable or large-scale systems; see, e.g., [17], [19]. One major motivation for developing the VLF theory and methods is to circumvent the computational difficulties arising from building up scalar Lyapunov functions by taking full advantage of its less rigid requirements and flexible mechanism on the system components; see [16] for more details. Recently, as important enhancements of the fundamental Lyapunov's direct method and control Lyapunov functions (see [1], [6], [20]), extensions of VLF to control systems known under the name of "vector control Lyapunov functions" (for short, VCLFs) have been introduced for controlling nonlinear systems; see [12], [18], [27]. In particular, the authors of [18] addressed the state-feedback design based on VCLF theory while the authors of [12] studied the necessary and sufficient conditions on the existence of certain VCLFs for smooth global stabilization. For output feedback local stabilization control, a sufficient condition in terms of VCLFs (more precisely, output control vector Lyapunov functions) was given in [27] as a generalization of the main results developed in [22], [23] using scalar control Lyapunov functions.

Regarding the local or regional asymptotic stabilization problem, another basic yet challenging question is to establish suitable estimates for the associated ROA; see [10, pp. 122]. It is well-known that computing the exact ROA is extremely difficult or impossible in general. In the past literature, for some special scenarios, one may compute its subsets as practical estimates or approximations by using suitable computational techniques; see, e.g., [5], [7], [24] and references therein. In the present study, our main objective is to develop a new design technique based on VCLFs for output feedback local stabilization control as well as a specific estimate for the ROA of the closed-loop system.

Toward this end, we first present a result on a general comparison principle that connects the existence of VLFs to stability and at the same time gives an effective, VLF-based estimate of the ROA. In sharp contrast to the scalar Lyapunov functions methodology, we then show that both the asymptotic stabilization and the estimation of the ROA can be done based on VCLFs for nonlinear systems. In virtue of certain dissipative components of a nonlinear system, the computational difficulties arising from the construction of scalar Lyapunov functions could be reduced or circumvented. Thus it may provide at least an interesting alternative to the problem. In comparison with previous output feedback stabilization results of [22], [23], [27], the present study provides a general comparison result as well as an explicit estimate of the ROA. Besides the aforementioned contributions, we also develop a new design technique based on a defined VCLF. In particular, a constructive design is presented for a class of minimum-phase nonlinear systems.

The rest of this technical note is organized as follows. Section II formulates the problem. Section III presents a general comparison result and a design technique based on VCLFs. Section IV discusses an interesting application of the general theorem to a class of minimum-phase, uncertain nonlinear systems in normal form. To illustrate the efficiency of the proposed design, a numerical example is given in Section V. Section VI closes the technical note with a few concluding remarks.

Throughout this technical note, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. $\mathbb{R}^n_+$ denotes the set of vectors with componentwise nonnegative entries and in particular $\mathbb{R}_+$ denotes the set of nonnegative real numbers. For a pair of vectors $x, x' \in \mathbb{R}^n$, $x \prec x'$ (or $x \preceq x'$ respectively) means $x_i < x'_i$ (or $x_i \leq x'_i$ respectively) for each $1 \leq i \leq n$. A function $f : \mathbb{R}^n \to \mathbb{R}_+$ is said to be positive definite if $f(x) > 0$ for $x \neq 0$ and $f(0) = 0$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$ is of class $\mathcal{K}$ if it is continuous and strictly increasing; it is of class $\mathcal{K}_\infty$ if in addition $\alpha(s) \to \infty$ as $s \to \infty$. For a given function $\kappa \in \mathcal{K}$, denote $\mathcal{O}(\kappa) := \{\alpha \in \mathcal{K} : \limsup_{s\to 0^+} \frac{\alpha(s)}{\kappa(s)} < \infty\}$. For two functions $f : \mathbb{R}^{\bar{n}} \to \mathbb{R}^l$, $g : \mathbb{R}^n \to \mathbb{R}^{\bar{n}}$, $f \circ g(x) := f(g(x))$. Given a C^1 (continuously differentiable) function $W : \mathbb{R}^n \to \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, $L_f W(x)$ refers to the Lie derivative $L_f W(x) = \frac{\partial W}{\partial x} f(x)$. $\mathbf{1}_{\bar{n}}$ denotes the $\bar{n}$-dimensional column vector with unity entries. For a real matrix $Q = [q_{ij}]_{m\times n}$, we use $|Q|$ to denote its componentwise absolute, i.e., $|Q| = [|q_{ij}|]_{m\times n}$. For any column vectors $x_1, \cdots, x_r$, $\mathrm{col}(x_1, \cdots, x_r)$ is used to denote $[x_1^\intercal, \cdots, x_r^\intercal]^\intercal$.

II. Definitions and Problem Formulation

Consider nonlinear systems described by

$$\dot{x} = f(x, u, \mu(t)),\ \ y = h(x) \tag{1}$$

with the state $x \in \mathbb{R}^n$, the control input $u \in \mathbb{R}^m$, the output $y \in \mathbb{R}^p$, and the external/parametric time-varying uncertainties $\mu := \mu(t)$ contained in a known compact set $\mathbb{D} \subset \mathbb{R}^{n_\mu}$. Suppose that for some given open set $\mathcal{U} \subset \mathbb{R}^m$ and

$$\mathcal{S}_\rho = \{x \in \mathbb{R}^n : \|x\| < \rho,\ \rho > 0\}, \tag{2}$$

the vector fields f, h are smooth on $\mathcal{S}_\rho \times \mathcal{U} \times \mathbb{D}$ and $\mathcal{S}_\rho$ respectively, and $f(0, 0, \mu) = 0$ for all μ, $h(0) = 0$.

For the purpose of stability analysis, associated with (1), its unforced system is defined by

$$\dot{x} = f_0(x, \mu),\ \ f_0(x, \mu) := f(x, 0, \mu),\ \ (x, \mu) \in \mathcal{S}_\rho \times \mathbb{D}. \tag{3}$$

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D. Xu is with School of Automation, Nanjing University of Science and Technology, Nanjing 210094, China, e-mail: xu.dabo@gmail.com. X. Wang is with Department of Automation, University of Science and Technology of China, Hefei 230027, China, e-mail: xinghuw@ustc.edu.cn. Y. Hong is with Key Laboratory of Systems and Control, Chinese Academy of Sciences, Beijing 100190, China, e-mail: yghong@iss.ac.cn. Z.-P. Jiang is with Tandon School of Engineering, New York University, Brooklyn, NY 11201, USA, e-mail: zjiang@nyu.edu. S. Xu is with School of Automation, Nanjing University of Science and Technology, Nanjing 210094, China, e-mail: syxu@njust.edu.cn.

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Furthermore, in company with the system (3) and as its comparison system, a reduced-order positively invariant system

$$\dot{z} = \mathscr{F}(z),\ z \in \mathbb{R}_+^{\bar{n}},\ \bar{n} \le n \tag{4}$$

will be considered for some continuous and *quasimonotone nondecreasing*[1] function $\mathscr{F}(z)$. For the sake of conciseness and avoiding technical complexity arising in configuring a suitable comparison system (4), we shall focus on linear comparison systems in the present study, i.e., $\mathscr{F}(z) = \Lambda z$. Recall from [8] that a real square matrix Q is called a Z-matrix if its off-diagonal entries of Q are nonpositive. A Z-matrix Q is an M-matrix if all its eigenvalues lie in the open right half plane. A matrix $Q = [q_{ij}]_{n\times n}$ is *essentially nonnegative* if $q_{ij} \ge 0$ for all $i \ne j$ and $1 \le i, j \le n$. Clearly, if $-\Lambda$ is an M-matrix, then the linear mapping $\mathscr{F}(z) = \Lambda z$ is quasimonotone nondecreasing and moreover Λ is Hurwitz.

The concerned stability notion and the ROA estimate for the system (3) and its comparison system (4) are defined below, cf. [15, Definition 1.1.1] and [15, Definition 1.6.1].

Definition 2.1: The trivial solution $x = 0$ of the system (3) is said to be *(robustly) asymptotically stable* with the ROA containing $\mathcal{B} \subset \mathbb{R}^n$ if both the following conditions (S1) and (S2) are satisfied:

$$\begin{aligned}
&\text{(S1)}\ \forall \epsilon > 0,\ \exists \delta_0 = \delta_0(\epsilon)\\
&\qquad \text{s.t.}\ \ \|x(0)\| \le \delta_0 \implies \|x(t)\| < \epsilon,\ \forall t \ge 0;\\
&\text{(S2)}\ \forall \epsilon > 0,\ \exists T = T(\epsilon)\\
&\qquad \text{s.t.}\ \ x(0) \in \mathcal{B} \implies \|x(t)\| < \epsilon,\ \forall t \ge T
\end{aligned}$$

where $x(t) := x(t, x(0), \mu(t))$ is any trajectory of (3) starting from $x(0)$ with $\mu(t) \in \mathbb{D}$.

Analogous to the above conditions (S1) and (S2), the asymptotic stability of $z = 0$ of the comparison system (4) with the ROA containing $\mathcal{Z}_+ \subset \mathbb{R}_+^{\bar{n}}$ is defined (with empty $\mathbb{D}$ in question) by replacing $\mathcal{B}$, $x(0)$, $\|x(0)\|$, and $\|x(t)\|$ by $\mathcal{Z}_+$, $z(0)$ (the initial region of (4) is primarily limited in $\mathbb{R}_+^{\bar{n}}$), $\sum_{i=1}^{\bar{n}} z_i(0)$, and $\sum_{i=1}^{\bar{n}} z_i(t)$, respectively. ∎

For stability analysis of the system (3), as a general notion than the basic (scalar) Lyapunov functions, our concerned VLF is given in the spirit of [15, Theorem 1.6.1]; also see [2], [16].

Definition 2.2: The system (3) is said to have a VLF triple $\{V, \Lambda, \mathcal{S}_\rho\}$ where $V : \mathbb{R}^n \to \mathbb{R}^{\bar{n}}$ is C^1 and $\Lambda \in \mathbb{R}^{\bar{n}\times\bar{n}}$ is essentially nonnegative and Hurwitz, if the function $V_s(x) = \mathbf{1}_{\bar{n}}^{\intercal} V(x)$ is positive definite and $L_{f_0} V(x, \mu) \preceq \Lambda V(x)$ for all $(x, \mu) \in \mathcal{S}_\rho \times \mathbb{D}$. When $\bar{n} = 1$, it becomes the well-known (scalar) Lyapunov function. ∎

Now, let us return to the system (1) and further denote that

$$\begin{aligned}
H(\mathcal{S}_\rho) &= \big\{y \in \mathbb{R}^p : y = h(x), x \in \mathcal{S}_\rho\big\},\\
\widetilde{H}(y) &= \big\{x \in \mathcal{S}_\rho : h(x) = y\big\}.
\end{aligned} \tag{5}$$

Definition 2.3: The system (1) has a VCLF quadruple $\{V, \Lambda, \mathcal{S}_\rho, \mathcal{U}\}$, if there is a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_+^{\bar{n}}$ and a matrix $\Lambda \in \mathbb{R}^{\bar{n}\times\bar{n}}$, which is essentially nonnegative and Hurwitz, satisfying

(i) the function $V_s(x) = \mathbf{1}_{\bar{n}}^{\intercal} V(x)$ is positive definite;

(ii) for each $y \in H(\mathcal{S}_\rho)\backslash\{0\}$, there is a point $u \in \mathcal{U}$ such that

$$\max_{\mu\in\mathbb{D}}\{L_f V(x, u, \mu)\} \prec \Lambda V(x),\ \forall x \in \widetilde{H}(y); \tag{6}$$

(iii) for all $x \in \widetilde{H}(0)$,

$$\max_{\mu\in\mathbb{D}}\{L_f V(x, 0, \mu)\} \preceq \Lambda V(x).$$

∎

[1] Recall from [15] that a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ is *quasimonotone nondecreasing* in x if for each $i : 1 \le i \le n$, the following implication condition holds:

$$\forall x', x'' \in \mathbb{R}^n \text{ satisfying } x'_i = x''_i,\ x' \preceq x'' \implies f_i(x') \le f_i(x'').$$

In this technical note, we will study the following output feedback control problem with estimation of the ROA.

Problem 1: Consider the system (1). Find, if possible, an output feedback controller of the form

$$u = u_c(y) \tag{7}$$

so that the closed-loop system composed of (1) and (7) is asymptotically stable at the origin together with an estimate $\mathcal{B} \subset \mathcal{S}_\rho$ for its resulting ROA. ∎

III. A New Design Technique via VCLFs

The main objective of this section is to address a comparison principle in terms of VLFs and propose a new technique for solving the problem of asymptotic stabilization using VCLFs.

A. Invariance Analysis of Comparison Systems

As said previously, the comparison system is assumed to take the linear form

$$\dot{z} = \Lambda z,\ z \in \mathbb{R}_+^{\bar{n}},\ \bar{n} \le n \tag{8}$$

where $\Lambda \in \mathbb{R}^{\bar{n}\times\bar{n}}$ is an essentially nonnegative matrix. The following result is an extension of an invariance analysis for the usual scalar comparison system; see, for instance, [24, Proposition 2.1]

Proposition 3.1: Suppose the trivial solution $z = 0$ of the system (8) is asymptotically stable. Then it has an invariant polyhedra

$$\mathcal{Z}_+(\ell, \omega) = \{z \in \mathbb{R}_+^{\bar{n}} : 0 \preceq z \preceq \ell\omega\} \tag{9}$$

for some $\omega \in \mathbb{R}_+^{\bar{n}}$ and an arbitrary $\ell > 0$. ∎

Proof: First note that $z = 0$ of (8) is asymptotically stable and Λ is Hurwitz. By [8, Condition 2.5.3.12], there is a vector $\omega \in \mathbb{R}_+^{\bar{n}}$ such that

$$\Lambda\omega \prec 0. \tag{10}$$

Secondly, let

$$Q = \Lambda + \lambda_{\max} I_{\bar{n}},\ \lambda_{\max} = \max_{1\le i\le\bar{n}} |\Lambda_{ii}| \tag{11}$$

with $\Lambda := [\Lambda_{ij}]_{\bar{n}\times\bar{n}}$. It can be easily shown that

$$Q \succeq 0_{\bar{n}\times\bar{n}},\ [-\lambda_{\max} I_{\bar{n}} + Q] I_{\bar{n}} = I_{\bar{n}}\Lambda$$

and for any scalar $\ell > 0$,

$$\ell[-\lambda_{\max} I_{\bar{n}} + Q]\omega \prec 0.$$

Moreover, according to [3, Proposition 2], the following symmetric polyhedra

$$\mathcal{R}(G, \omega) = \{z \in \mathbb{R}^{\bar{n}} : -\ell\omega \preceq Gz \preceq \ell\omega\} \tag{12}$$

with $(G, \omega) \in \mathbb{R}^{\bar{n}\times\bar{n}} \times \mathbb{R}_+^{\bar{n}}$ and $\ell > 0$, is invariant for (8) if and only if there is a matrix $Q \in \mathbb{R}^{\bar{n}\times\bar{n}}$ and a real number $\lambda_{\max} > 0$ such that both the following conditions are satisfied:

$$\text{(i)}\ \ [-\lambda_{\max} I_{\bar{n}} + Q]G = G\Lambda;\quad \text{(ii)}\ \ \ell[-\lambda_{\max} I_{\bar{n}} + |Q|]\omega \preceq 0. \tag{13}$$

Therefore, to verify the conditions of (13), we may let $G = I_{\bar{n}}$ and $Q, \lambda_{\max}$ by (11) that renders $\mathcal{R}(G, \omega)$ an invariant polyhedra.

Finally, note that the system (8) is positively invariant and the intersection

$$\mathcal{R}(G, \omega) \cap \mathbb{R}_+^{\bar{n}}$$

is still positively invariant for the system (8) that yields the polyhedra (9). Hence, the positive invariance property of (9) is concluded that ends the proof. □

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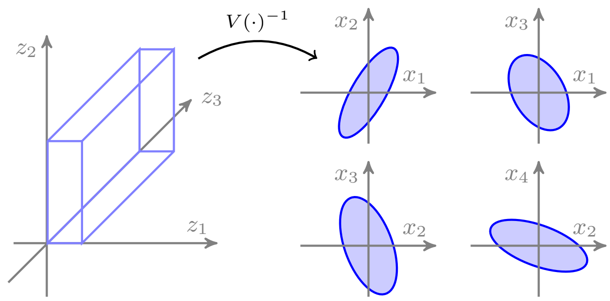

$\mathcal{Z}_+(\ell,\omega) \in \mathbb{R}^3_+$ $\qquad$ $\mathcal{B}(l^*,\omega) \in \mathbb{R}^4$

Fig. 1. Illustrative views for an estimate of the ROA shaped from an invariant polyhedra.

B. Estimating the ROA via VLFs

The underlying comparison principle of a vector version is given below as an interesting supplement to [15, Theorem 1.6.1]. Notably, a specific estimate of the ROA will also be provided. Fig. 1 illustrates the idea of such an ROA approximation.

Lemma 3.1: Suppose that the system (3) has a VLF triple $\{V, \Lambda, \mathcal{S}_\rho\}$. Then its trivial solution $x = 0$ is asymptotically stable with the ROA containing the compact set

$$\mathcal{B}(l^*,\omega) = \{x \in \mathcal{S}_\rho : V(x) \preceq l^*\omega\} \tag{14}$$

for a pair $(l^*,\omega) \in \mathbb{R}_+ \times \mathbb{R}^{\bar{n}}_+$. ∎

Proof: The proof for asymptotic stability per Definition 2.1 can be referred to [15, Theorem 1.6.1] and is thus omitted. It remains to prove that (14) is indeed an estimate of the ROA.

Firstly, since $V_s(x) = \mathbf{1}_{\bar{n}}^\intercal V(x)$ is positive definite, there are some functions $\alpha_1, \alpha_2 \in \mathcal{K}$ such that

$$\alpha_2(\|x\|) \le V_s(x) \le \alpha_1(\|x\|),\ \forall x \in \mathcal{S}_\rho. \tag{15}$$

By Proposition 3.1, we obtain an invariant polyhedra $\mathcal{Z}_+(\ell,\omega)$ given by (9) for a fixed number $\ell > 0$ satisfying

$$\ell < \frac{\alpha_2(\rho)}{\mathbf{1}_{\bar{n}}^\intercal \omega}. \tag{16}$$

Then, we have a number $0 < l^* \le \ell$ satisfying

$$\begin{aligned} &\{x \in \mathbb{R}^n : V(x) \preceq l^*\omega\} \\ &\subseteq \{x \in \mathbb{R}^n : \mathbf{1}_{\bar{n}}^\intercal V(x) \le \ell \mathbf{1}_{\bar{n}}^\intercal \omega\} \\ &\subseteq \{x \in \mathbb{R}^n : \|x\| \le \alpha_2^{-1}(\ell \mathbf{1}_{\bar{n}}^\intercal \omega)\} \subseteq \mathcal{S}_\rho. \end{aligned} \tag{17}$$

Thus, by the invariance property, we have

$$\begin{aligned} \forall z(0) \in \mathcal{Z}_+(\ell,\omega) \implies\ & z(t) := z(t, z(0)) \in \mathcal{Z}_+(\ell,\omega), \\ & \mathbf{1}_{\bar{n}}^\intercal z(t) \le \ell \mathbf{1}_{\bar{n}}^\intercal \omega,\ \forall t \ge 0. \end{aligned}$$

Next, we state that

$$\begin{aligned} &\forall x(0) \in \mathcal{B}(l^*,\omega) \implies x(t) := x(t, x(0), \mu(t)) \ \text{ satisfies} \\ &\|x(t)\| < \rho,\ V(x(t)) \preceq z(t),\ \forall t \ge 0 \end{aligned} \tag{18}$$

where $z(t)$ is the state trajectory of (8) starting from $z(0) = V(x(0))$. Let us show (18) by contradiction. Suppose (18) is not true and there exists $x(0) \in \mathcal{B}(l^*,\omega)$ and a time $t_1 > 0$ such that $\|x(t_1)\| = \rho$. It leads to a contradiction

$$\begin{aligned} \alpha_2(\rho) = \alpha_2(\|x(t_1)\|) &\le V_s(x(t_1)) \\ &\le \mathbf{1}_{\bar{n}}^\intercal z(t_1, z(0)) \le \ell \mathbf{1}_{\bar{n}}^\intercal \omega < \alpha_2(\rho). \end{aligned}$$

Finally, by (15) and (18), it implies

$$\alpha_2(\|x(t)\|) \le V_s(x(t)) \le \mathbf{1}_{\bar{n}}^\intercal z(t),\ \forall t \ge 0.$$

Moreover, since Λ is Hurwitz, $\lim_{t\to\infty} z(t) = 0$ and consequently $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Remark 3.1: Regarding effective estimation of the ROA, the most often used approaches in literature would be those conducted in terms of scalar Lyapunov functions. Relevant to the study of Lemma 3.1, an interesting result would be the method in [26] for large-scale interconnected systems of the form

$$\dot{x}_i = f_i(x_i) + \sum_{j=1, j\ne i}^{\bar{n}} g_{ij}(x_j, t),\ x_i \in \mathbb{R}^{n_i},\ i = 1, \cdots, \bar{n} \tag{19}$$

with the state $x = \mathrm{col}(x_1, \cdots, x_{\bar{n}}) \in \mathbb{R}^n$. In this literature, suppose that every isolated or uncoupled subsystem $\dot{x}_i = f_i(x_i)$ is at least locally exponentially stable at the origin with a known Lyapunov function $v_i(x_i)$, and moreover, these Lyapunov functions $v_i(x_i)$, $i = 1, \cdots, \bar{n}$, may confirm a linear comparison system $\dot{z} = \Lambda z$ for the system (19) under certain conditions imposed on the interconnection terms g_{ij} in (19). Then, for a *prescribed* scalar Lyapunov function $V_C \circ v(x) \in \mathbb{R}$ with $v(x) = \mathrm{col}(v_1(x_1), \cdots, v_{\bar{n}}(x_{\bar{n}}))$ in terms of

$$V_C \circ v(x) = v(x)^\intercal H v(x) \text{ or } \sum_{i=1}^{\bar{n}} r_i |v_i| \text{ or } \max_{1\le i\le \bar{n}} |v_i(x_i)|/p_i \tag{20}$$

for a positive definite matrix H and constants $r_i, p_i > 0$, an idea in [26] to establish an estimate of the ROA is performed by searching for a set

$$\Theta_l = \{x \in \mathbb{R}^n : V_C \circ v(x) < l_1\} \tag{21}$$

subject to

$$\sup_{x\in\Theta_l,\ x\ne 0} -\frac{1}{V_C \circ v(x)} [\partial V_C / \partial v] \Lambda v(x) = l_2 \tag{22}$$

for constants $l_1, l_2 > 0$. In particular, the cases in (20) on restricting the mapping $V_C : \mathbb{R}^{\bar{n}} \to \mathbb{R}$ may be investigated separately, leading to distinct improved estimates of (21); see [26, Theorem 4] and the other relevant theorems thereof for more details. The functions in (20) were essentially introduced to characterize trial scalar Lyapunov function candidates.

In comparison with [26], Lemma 3.1 offers an estimate in a quite different manner that is essentially shaped from a positively invariant polyhedra of the comparison system by virtue of Proposition 3.1. Such positively invariant sets for the comparison system are essentially exploited in terms of VLFs. Also, this new approach leads to an estimate characterized by the set (14) and is applicable to a relatively more general setting of (3) than that of the structured system (19). Indeed, the VLFs in Lemma 3.1 have a general form allowing interconnections. Constructing scalar Lyapunov functions, as the above $V_C \circ v(x)$, is actually not required in the proposed method. Hence, our study offers at least an interesting alternative to the problem. ∎

Remark 3.2: In view of the estimate (14) with constraints (10) and (16), we observe that further improvement of this estimation would not be so suited as those of scalar Lyapunov function based approaches such as that of (21) subject to (22) in Remark 3.1.

Nevertheless, we may proceed with a simpler optimization problem of maximizing the estimate (14) of the ROA for the system (3) with a VLF triple $\{V, \Lambda, \mathcal{S}_\rho\}$. For this purpose, we may first select $\omega \succeq 0$ satisfying (10) and moreover $\mathbf{1}^\intercal \omega = 1$. Then, maximization of the set in (14) is reduced to simply calculate the lowest upper bound of $\ell > 0$ satisfying (16) w.r.t. the above selected $\omega \succeq 0$. For this specific optimization problem, it should be noted that there is no loss of any generality to introduce the requirement of $\mathbf{1}^\intercal \omega = 1$. In the present result, ω and ℓ can be understood as the shape and size of the specific estimate (14), respectively. ∎

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C. Asymptotic Stabilization via VCLFs

Based on Definition 2.3, we are ready to state a general result of this technical note.

Theorem 3.1: Problem 1 for the system (1) is solvable if it has a VCLF quadruple $\{V, \Lambda, S_\rho, \mathcal{U}\}$. Moreover, (14) provides an estimate of the ROA. ∎

The proof of Theorem 3.1 is straightforward by the foregoing Lemma 3.1 and a practical modification of [27, Theorem 1]. It is noted that the preceding Theorem 3.1 can be shown by partition of unity which is crucial to establish a general result on solvability of the problem. For a brief explanation of the use of partition of unity, by the condition (ii) of Definition 2.3, for each $y \in H(\mathcal{S}_\rho)\backslash\{0\}$, there is a point $u \in \mathcal{U}$ assuring the inequality (6). Then, in the same spirit of [27, Lemma 3], it is possible to show that the property (6) also holds in a proper superset of $\widetilde{H}(y)$ relating to a small neighbourhood of the point y under the same control u. Moreover, we are capable of selecting a countable number of such open neighbourhoods to cover the region of interest and in turn, smoothing these corresponding controls by means of the technique of partition of unity. It leads to a smooth output feedback controller except possibly at the origin. Furthermore, to assure a continuous controller, a mild condition of the so-called *small control property* (see [1, Theorem 5.2] or [20, Theorem 1]) can be imposed in addition to the condition of Theorem 3.1.

Hence, the method by Theorem 3.1 might rely on partition of unity whose calculation is not constructive in general. Nonetheless, for some special classes of nonlinear systems or for large-scale systems, such difficulties could be circumvented in virtue of their components or subsystems satisfying certain dissipativity properties. In this direction, an exemplary scenario is shown in the next section.

IV. Application to Strict-Feedback Nonlinear Systems

For developing a constructive feedback design based on Theorem 3.1, we consider a class of systems (1) transformable into a strict-feedback normal form (see [9])

$$\dot{\zeta} = \phi(\zeta, \xi, \mu), \ \dot{\xi} = A\xi + Bg(\zeta, \xi, \mu) + Bu \tag{23}$$

with $(\zeta, \xi) \in \mathbb{R}^{n_\zeta} \times \mathbb{R}^r$, $n_\zeta + r = n$, $r > 1$, $u \in \mathbb{R}$ and (A, B) taking the Brunovsky normal form. For the system (23), we suppose $\mathcal{U} = \mathbb{R}$ and the functions ϕ and g are well defined on $\mathcal{S}_\rho \times \mathbb{D}$ where

$$\mathcal{S}_\rho = \{\mathrm{col}(\zeta, \xi) \in \mathbb{R}^n : \|\mathrm{col}(\zeta, \xi)\| < \rho, \ \rho > 0\} \tag{24}$$

with $\phi(0, 0, \mu) = 0$, $g(0, 0, \mu) = 0$ for all $\mu(t)$. Further suppose that the state ζ as a dynamic uncertainty (see [11]) is not measurable and only ξ is available for feedback design and is thus considered as the measured output. To carry out output feedback design, we need a minimum-phase condition, cf. [14].

Condition 4.1: For the ζ-subsystem of (23), there is a C^1 function $V_1 : \mathbb{R}^{n_\zeta} \to \mathbb{R}$ such that, for all $\mathrm{col}(\zeta, \xi) \in \mathcal{S}_\rho$,

$$\begin{aligned} &\underline{\alpha}_1(\|\zeta\|) \le V_1(\zeta) \le \bar{\alpha}_1(\|\zeta\|), \\ &\dot{V}_1|_{(23)} \le -\alpha_1 \circ V_1(\zeta) + \gamma_1(\xi)\|\xi\|^2 \end{aligned} \tag{25}$$

for $\underline{\alpha}_1, \bar{\alpha}_1 \in \mathcal{K} \cap \mathcal{O}(s^2), \alpha_1 \in \mathcal{K} \cap \mathcal{O}(s)$ and $\gamma_1(\cdot) \ge 1$. ∎

For technical convenience, as in [14], we further define $\theta = \mathrm{col}(\xi_1, \xi_2, \cdots, \xi_{r-1})$, $\vartheta = \xi_r + \sum_{i=1}^{r-1} \lambda_i \xi_i$,

$$M = \begin{bmatrix} 0 & I \\ -\lambda_1, & -\lambda_2, \cdots, -\lambda_{r-1} \end{bmatrix}, \ N^\top = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}_{1\times(r-1)},$$

and choose $(\lambda_1, \cdots, \lambda_{r-1})$ such that M is Hurwitz. Then we obtain an equivalent system of (23) described by

$$\dot{\zeta} = \bar{g}_0(\zeta, \theta, \vartheta, \mu), \ \dot{\theta} = M\theta + N\vartheta, \ \dot{\vartheta} = \bar{g}(\zeta, \theta, \vartheta, \mu) + u \tag{26}$$

where $\bar{g}_0(\zeta, \theta, \vartheta, \mu) = \phi(\zeta, \Delta, \mu)$, $\Delta := \mathrm{col}\big(\theta_1, \theta_2, \cdots, \theta_{r-1}, \vartheta - \sum_{i=1}^{r-1} \lambda_i \theta_i\big)$, and $\bar{g}(\zeta, \theta, \vartheta, \mu) = g(\zeta, \Delta, \mu) + \lambda_{r-1}\xi_r + \cdots + \lambda_1 \xi_2$.

From now on, we focus on the output feedback stabilizing feedback for the system (26) with the output given by $y = \vartheta$. Before presenting the main theorem of this section, we denote

$$x := \mathrm{col}(\zeta, \theta, \vartheta), \ \mathcal{S}_{\bar{\rho}} = \big\{x \in \mathbb{R}^n : \|x\| < \bar{\rho}, \ \bar{\rho} > 0\big\}$$

for a real number $\bar{\rho} > 0$ that is fixed such that

$$\begin{aligned} \mathcal{S}_{\bar{\rho}} \subset \big\{&\mathrm{col}(\zeta, \theta, \vartheta) \in \mathbb{R}^n : \theta_i = \xi_i \text{ for } 1 \le i \le r-1, \\ &\vartheta = \xi_r + \sum\nolimits_{i=1}^{r-1} \lambda_i \xi_i, \mathrm{col}(\zeta, \xi) \in \mathcal{S}_\rho\big\} \end{aligned} \tag{27}$$

and for constants $\bar{\gamma}_{12}, \bar{\gamma}_{13}, \bar{\gamma}_{31}, \bar{\gamma}_{32} > 0$ and a function $\gamma_{33}(\cdot) \ge 1$,

$$\begin{aligned} &-\alpha_1 \circ V_1(\zeta) \le -a_1 V_1(\zeta), \ \gamma_1(\Delta)\|\Delta\|^2 \le \bar{\gamma}_{12}\|\theta\|^2 + \bar{\gamma}_{13}\vartheta^2, \\ &\vartheta \bar{g}(\zeta, \theta, \vartheta, \mu) \le \bar{\gamma}_{31}\|\zeta\|^2 + \bar{\gamma}_{32}\|\theta\|^2 + \gamma_{33}(\vartheta)\vartheta^2, \forall (x, \mu) \in \mathcal{S}_{\bar{\rho}} \times \mathbb{D}. \end{aligned} \tag{28}$$

Here $\alpha_1(\cdot), \gamma_1(\cdot)$ are given in (25).

Remark 4.1: It should be noted that Condition 4.1 merely implies local input-to-state stability in the sense of [21] for the dynamic uncertainty characterized by ζ-subsystem of (23). The relevant *semiglobal* or *global* stabilization (see [10, page 473]) may fail in general for such nonlinear systems; see [4], [11], [14] for more details. Hence, it is reasonable to carry out local or regional stabilization control for the system (23) under Condition 4.1. ∎

Before stating the main result of this section, we need two technical lemmas.

Lemma 4.1: Consider a Z-matrix Ω taking a block form

$$\Omega = \begin{bmatrix} kI_{n_1} + \Omega^a & \Omega^b \\ \Omega^c & \Omega^d \end{bmatrix} \tag{29}$$

where $k \in \mathbb{R}$, $\Omega^a \in \mathbb{R}^{n_1 \times n_1}$, $\Omega^d \in \mathbb{R}^{n_2 \times n_2}$, and Ω^b, Ω^c are of appropriate dimensions. Then the statement

$$\exists k^\star > 0 \ \text{ s.t. } \ \Omega \text{ is an M-matrix for each } k \ge k^\star$$

is true if and only if Ω^d is an M-matrix. ∎

Proof: "$\Longrightarrow$" Suppose there is a real number $k^\star > 0$ making Ω an M-matrix for each $k \ge k^\star$. Then the diagonal entries of Ω are positive and there is a diagonal matrix D with positive diagonal entries such that ΩD is strictly row diagonally dominant (see [8, Condition 2.5.3.13]). Then, we may let

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \ D_2 = \mathrm{diag}(l_1, \cdots, l_{n_2}),$$

$$\Omega D = \begin{bmatrix} kD_1 + \Omega^a D_1 & \Omega^b D_2 \\ \Omega^c D_1 & \Omega^d D_2 \end{bmatrix}.$$

In the above, notice that $\Omega^d D_2$ is strictly row diagonally dominant. Hence, by using [8, Condition 2.5.3.13], Ω^d is an M-matrix.

"$\Longleftarrow$" Suppose that Ω^d is an M-matrix. Note that the diagonal entries of Ω^d are positive, and there exists a diagonal matrix $D_2 = \mathrm{diag}(l_1, \cdots, l_{n_2})$ with positive diagonal entries such that $\Omega^d D_2$ is strictly row diagonally dominant. Thus, we have

$$l_i \Omega^d_{ii} > \sum\nolimits_{j \ne i} l_j |\Omega^d_{ij}|, \ i = 1, \cdots, n_2.$$

Let

$$l_0 = \min_{1 \le i \le n_2} \frac{l_i \Omega^d_{ii} - \sum_{j \ne i} l_j |\Omega^d_{ij}|}{2\sum_{j=1}^{n_1} |\Omega^c_{ij}|},$$

$$k^\star = \max_{1 \le i \le n_1} \frac{\sum_{j=1}^{n_2} |\Omega^b_{ij}| l_i + \sum_{j=1}^{n_1} \Omega^a_{ij} + 1}{l_0}.$$

Define $D = \mathrm{diag}(l_0 I_{n_1}, D_2)$. It can be shown that for each $k > k^\star$, the matrix ΩD is strictly row diagonally dominant. Thus, by [8,

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Condition 2.5.3.13] again, Ω is an M-matrix for each $k > k^\star$. The proof is complete. □

Lemma 4.2: Consider the same Z-matrix Ω of the block form (29) with an M-matrix Ω^d. Then for any $\omega \in \mathbb{R}_+^{n_1+n_2}$ satisfying $\begin{bmatrix}\Omega^c & \Omega^d\end{bmatrix}\omega \succ 0$, there is a number $k^* > 0$ such that

$$\begin{bmatrix} kI_{n_1} + \Omega^a & \Omega^b \\ \Omega^c & \Omega^d \end{bmatrix} \omega \succ 0, \ \forall k \geq k^*. \tag{30}$$

∎

The proof of Lemma 4.2 is straightforward by [8, Condition 2.5.3.12] and is thus omitted.

Theorem 4.1: Suppose that Condition 4.1 is satisfied for the system (23). Then associated with (26), there is a well defined VCLF quadruple $\{V, \Lambda, \mathcal{S}_{\bar\rho}, \mathbb{R}\}$ per Definition 2.3 for a real number $\bar\rho > 0$. Moreover, a controller

$$u_c(\vartheta) = \begin{cases} 0, & \text{if } \vartheta = 0; \\ -\frac{\nu(\vartheta)+\sqrt{\nu(\vartheta)^2+\psi(\vartheta)\vartheta^2}}{\vartheta}, & \text{otherwise} \end{cases} \tag{31}$$

inspired by Sontag's formula together with an estimate $\mathcal{B} \subset \mathcal{S}_{\bar\rho}$ of the ROA for the closed-loop system can be given, where $\nu(\vartheta)$ and $\psi(\vartheta)$ are design functions. ∎

Proof: The proof is divided into three parts leading to a construction of (31) and its resulting estimate $\mathcal{B} \subset \mathcal{S}_{\bar\rho}$.

Part 1: Configure a comparison system. First, let

$$V_2(\theta) = \theta^\intercal P\theta, \ V_3(\vartheta) = \frac{1}{2}\vartheta^2 \tag{32}$$

for a positive definite matrix P satisfying $PM + M^\intercal P = -I$. It can be seen that $V_2(\theta)$ satisfies

$$\dot V_2|_{(26)} \leq -\theta^\intercal\theta + 2\theta^\intercal PN\vartheta.$$

Denote the spectral radius of P by λ_P. In the above, note that

$$2\theta^\intercal PN\vartheta \leq \epsilon_1\|\theta^\intercal PN\|^2 + \epsilon_1^{-1}\vartheta^2, \ \ -\theta^\intercal\theta \leq -\lambda_P^{-1}\theta^\intercal P\theta$$

with $\epsilon_1 > 0$ to be determined by (34), and moreover

$$\|\theta^\intercal PN\|^2 \leq c_1\theta^\intercal P\theta$$

for some constant $c_1 > 0$. Thus we have

$$\dot V_2|_{(26)} \leq -\bar\sigma V_2(\theta) + 2\epsilon_1^{-1}V_3(\vartheta) \tag{33}$$

where $\bar\sigma := \lambda_P^{-1} - \epsilon_1 c_1$ and $\epsilon_1 > 0$ is such that

$$\lambda_P^{-1} - \epsilon_1 c_1 > 0. \tag{34}$$

Also, for the function $V_3(\vartheta)$ and using (28), there are constants $\gamma_{31}, \gamma_{32} > 0$ such that, for all $(x, \mu) \in \mathcal{S}_{\bar\rho} \times \mathbb{D}$,

$$\dot V_3|_{(26)} \leq \vartheta u + \gamma_{33}(|\vartheta|)\vartheta^2 + \gamma_{31}V_1(\zeta) + \gamma_{32}V_2(\theta). \tag{35}$$

Next, by Condition 4.1 and in view of (28), we have constants $a_1, \gamma_{12}, \gamma_{13} > 0$ such that, for all $(x, \mu) \in \mathcal{S}_{\bar\rho} \times \mathbb{D}$,

$$\dot V_1|_{(26)} \leq -a_1V_1(\zeta) + \gamma_{12}V_2(\theta) + \gamma_{13}V_3(\vartheta). \tag{36}$$

Ultimately, we obtain a VCLF candidate

$$V(x) = \begin{bmatrix} V_1(\zeta) & V_2(\theta) & V_3(\vartheta) \end{bmatrix}^\intercal \tag{37}$$

and a candidate comparison system (8) with $\bar n = 3$ and

$$\Lambda = \begin{bmatrix} -a_1 & \gamma_{12} & \gamma_{13} \\ 0 & -\bar\sigma & 2\epsilon_1^{-1} \\ \gamma_{31} & \gamma_{32} & -k \end{bmatrix} \tag{38}$$

for a design parameter $k > 0$. By Lemma 4.1, we are able to further pick $k > 0$ so that the above matrix Λ is Hurwitz. Clearly, the matrix Λ also is essentially nonnegative.

Part 2: Verify the conditions in Definition 2.3. First, it can be seen that the sum function

$$V_s(x) = V_1(\zeta) + V_2(\theta) + V_3(\vartheta) \tag{39}$$

is positive definite, which verifies the condition (i) in Definition 2.3. Moreover, from the dissipation inequalities (36), (33) and (35), it satisfies

$$\dot V|_{(26)} \preceq \Lambda V(x) + \begin{bmatrix} 0, & 0, & \vartheta u + \gamma_{33}(|\vartheta|)\vartheta^2 + kV_3(\vartheta) \end{bmatrix}^\intercal$$

which manifests the conditions (ii) and (iii) in Definition 2.3. Thus, the quadruple $\{V, \Lambda, \mathcal{S}_{\bar\rho}, \mathbb{R}\}$ given by (37), (38) and (27) has been verified to be a VCLF quadruple. Consequently, by Theorem 3.1, the problem is solvable and in particular, an estimate of the ROA can be given by Lemmas 3.1 and 4.2.

Part 3: Construct a specific controller. Similar with [27, Theorem 6], one controller of the form (31) can be given by Sontag's formula (see [6], [20]) with $\nu(\vartheta) = \gamma_{33}(|\vartheta|)\vartheta^2 + kV_3(\vartheta)$ and for any continuous function $\psi(\vartheta) \geq 0$. The proof is complete. □

Remark 4.2: The outlined approach by Theorem 4.1 provides a two-step procedure for establishing (31) under Condition 4.1. The first step is to seek a comparison system by selecting the parameter k so that (38) is Hurwitz according to Lemma 4.1 and to calculate an estimate of the ROA assured according to Lemmas 3.1 and 4.2. The second step is to construct a controller (31) that validates the prescribed estimate. ∎

Remark 4.3: It is worth noting that a similar stabilization problem for the system (23) has been addressed in [14]. By contrast, what makes Theorem 4.1 distinguished is two-fold besides the different conditions and control goals. On one hand, the method under investigation is based on a general comparison result of Theorem 3.1 that essentially does not rely on scalar CLFs. If not, a carefully weighted summation (for example $\sum_{i=1}^3 p_iV_i(\zeta)$ as a sum-type construction) of the components of (37) or some others is required for conventional Lyapunov analysis; see, for a rigorous demonstration, [16, Example 1.2]. By contrast, the possible difficulty of building up such a scalar function could be circumvented in virtue of VLFs and certain invariance analysis. On the other hand, the controller of the form (31) is presented that contains the usual linear controller as a special case. Similar with the argument of [18, Theorem 6.1], it may assure certain inverse optimality in terms of VLFs. Furthermore, it can be shown that the small control property is satisfied for the problem of this section, i.e., a function $\psi(\vartheta) \geq 0$ can be chosen such that (31) is continuous. As a result, Theorem 4.1 provides an effective approach to asymptotic stabilization of minimum-phase strict-feedback nonlinear systems and offers much flexibility for the problem. ∎

Remark 4.4: We shall point out that, in some special situations, the ROA and its estimate may be achieved arbitrarily large by adjusting the controller. For a brief illustration, focusing on the setting of this section, we investigate some other scenario to draw an interesting extension of the main theorem but in the case when Condition 4.1 is strengthened to the following one: for any radius $\rho > 0$ of the set $\mathcal{S}_\rho$ in (24), Condition 4.1 holds and moreover, $\underline{\alpha}_1, \bar\alpha_1, \alpha_1 \in \mathcal{K}_\infty$. It is stated that the relevant *semiglobal stabilization* for the system (23) can be approached by means of the proposed VCLFs as Theorem 4.1 together with the ROA estimation as Lemma 3.1. The above statement is assured, not shown here in detail, by the fact that for any compact set $X \subset \mathbb{R}^n$ (to be made a valid estimate of the ROA for the closed-loop system), there is a controller (31) for the system (26) with state $x = \mathrm{col}(\zeta, \theta, \vartheta) \in \mathbb{R}^n$ w.r.t. some large set $\mathcal{S}_{\bar\rho}$ as (27) rendering the closed-loop system an estimate $\mathcal{B} \supseteq X$ of the ROA. ∎

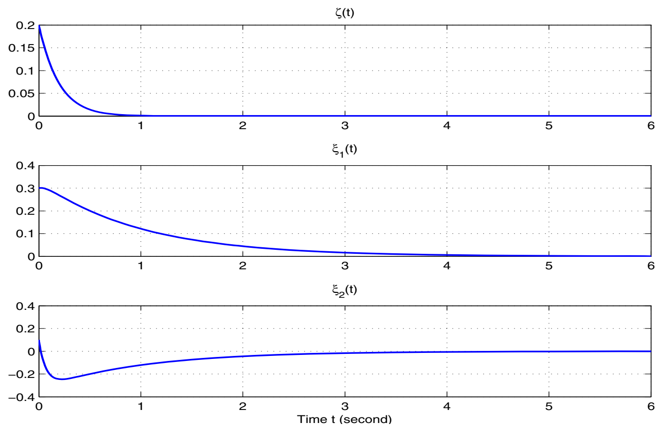

Fig. 2. State response of the closed-loop system.

V. Example

To illustrate the result by Theorem 4.1, consider a nonlinear system of the form (23) given by

$$\dot{\zeta} = \zeta(\xi_1 - \mu_1) + \mu_2\zeta^3,$$
$$\dot{\xi}_1 = \xi_2,\ \ \dot{\xi}_2 = \xi_1 + \mu_3\xi_2^2\sin(\zeta) + \xi_2\zeta + u \tag{40}$$

with $\mu_1 = 5 + \sin(t)$, $\mu_2 = \cos(t)$ and $\mu_3 = 2\cos(t)$. For example, suppose that $\rho = 1$ in (24). To verify Condition 4.1, let $V_1(\zeta) = \frac{1}{2}\zeta^2$, which satisfies, for all $\mathrm{col}(\zeta, \xi) \in \mathcal{S}_\rho$,

$$\begin{aligned}\dot{V}_1|_{(40)} &= -\mu_1\zeta^2 + \zeta^2\xi_1 + \mu_2\zeta^4 \\ &\le -\mu_1\zeta^2 + \frac{1}{2}\zeta^4 + \frac{1}{2}\xi_1^2 + \mu_2\zeta^4 \\ &\le -(\mu_1 - \frac{1}{2} - \mu_2)\zeta^2 + \frac{1}{2}\xi_1^2 \\ &\le -5V_1(\zeta) + \frac{1}{2}\xi_1^2.\end{aligned}$$

Hence, the stabilization can be done by Theorem 4.1.

For a controller design, according to the proof of Theorem 4.1, we may choose $\theta = \xi_1$, $\vartheta = \xi_2 + \lambda_1\xi_1$, $M = -\lambda_1$, $N = 1$, and $\lambda_1 = 1$ which leads to $P = \frac{1}{2}$ for (32), $(c_1, \epsilon_1, \bar{\sigma}) = (\frac{1}{2}, 1, \frac{3}{2})$ for (34), and $\Lambda = [-5, 1, 0; 0, -\frac{3}{2}, 2; 2, 2, -k]$ for (38). With the above obtained matrix Λ and according to Lemma 4.1, it is not difficult to pick k so that Λ is Hurwitz. In this case, for determining an estimate of the resulting ROA, choose $\omega = \mathrm{col}(1, 2, 1)$ by Lemma 4.2 such that $\Lambda\omega \prec 0$ and consequently choose $l^* = 1$ according to (16) and (17). Finally, for this example, note that a linear continuous controller can be designed. A simulation result is shown in Fig. 2 with a controller $u_c(\vartheta) = -15\vartheta$ and an initial condition $(0.2, 0.3, 0.1)$.

VI. Conclusion

A new design technique has been developed based on VCLFs for solving output feedback asymptotic stabilization of nonlinear systems and for generating an estimate of the ROA for the closed-loop system. An interesting application of the result has been shown leading to a constructive design. Our future work will be directed at the investigation of distributed or decentralized controller design for multi-agent or large-scale interconnected nonlinear systems (cf. an interesting result in [13]). Another issue deserving further investigating for practical applications is to explore some specific designs for enlarging or improving estimates of the ROA.

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