

Nash Equilibrium Computation in Subnetwork Zero-Sum Games With Switching Communications

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Abstract—In this paper, we investigate a distributed Nash equilibrium computation problem for a time-varying multi-agent network consisting of two subnetworks, where the two subnetworks share the same objective function. We first propose a subgradient-based distributed algorithm with heterogeneous stepsizes to compute a Nash equilibrium of a zero-sum game. We then prove that the proposed algorithm can achieve a Nash equilibrium under uniformly jointly strongly connected (UJSC) weight-balanced digraphs with homogenous stepsizes. Moreover, we demonstrate that for weighted-unbalanced graphs a Nash equilibrium may not be achieved with homogenous stepsizes unless certain conditions on the objective function hold. We show that there always exist heterogeneous stepsizes for the proposed algorithm to guarantee that a Nash equilibrium can be achieved for UJSC digraphs. Finally, in two standard weight-unbalanced cases, we verify the convergence to a Nash equilibrium by adaptively updating the stepsizes along with the arc weights in the proposed algorithm.

Index Terms—Heterogeneous stepsizes, joint connection, multi-agent systems, Nash equilibrium, weight-unbalanced graphs.

I. INTRODUCTION

IN recent years, distributed control and optimization of multi-agent systems have drawn much research attention due to their broad applications in various fields of science, engineering, computer science, and social science. Various tasks including consensus, localization, and convex optimization can be accomplished cooperatively for a group of autonomous

agents via distributed algorithm design and local information exchange [8], [9], [14], [15], [20]–[22], [37].

Distributed optimization has been widely investigated for agents to achieve a global optimization objective by cooperating with each other [14], [15], [20]–[22]. Furthermore, distributed optimization algorithms in the presence of adversaries have gained rapidly growing interest [2], [3], [23], [30]–[32]. For instance, a non-model based approach was proposed for seeking a Nash equilibrium of noncooperative games in [30], while distributed methods to compute Nash equilibria based on extreme-seeking technique were developed in [31]. A distributed continuous-time set-valued dynamical system solution to seek a Nash equilibrium of zero-sum games was first designed for undirected graphs and then for weight-balanced directed graphs in [23]. It is worthwhile to mention that, in the special case of additively separable objective functions, the considered distributed Nash equilibrium computation problem is equivalent to the well-known distributed optimization problem: multiple agents cooperatively minimize a sum of their own convex objective functions [11], [12], [14]–[19], [24], [29].

One main approach to distributed optimization is based on subgradient algorithms with each node computing a subgradient of its own objective function. Distributed subgradient-based algorithms with constant and time-varying stepsizes, respectively, were proposed in [14] and [15] with detailed convergence analysis. A distributed iterative algorithm that avoids choosing a diminishing stepsize was proposed in [29]. Both deterministic and randomized versions of distributed projection-based protocols were studied in [20]–[22].

In existing works on distributed optimization, most of the results were obtained for switching weight-balanced graphs because there usually exists a common Lyapunov function to facilitate the convergence analysis in this case [14], [15], [18], [23], [24]. Sometimes, the weight-balance condition is hard to preserve in the case when the graph is time-varying and with communication delays [38], and it may be quite restrictive and difficult to verify in a distributed setting. However, in the case of weight-unbalanced graphs, there may not exist a common (quadratic) Lyapunov function or it may be very hard to construct one even for simple consensus problems [10], and hence, the convergence analysis of distributed problems become extremely difficult. Recently, many efforts have been made to handle the weight unbalance problem, though very few results have been obtained on distributed optimization. For instance, the effect of the Perron vector of the adjacency matrix on the optimal convergence of distributed subgradient and dual averaging algorithms were investigated for a fixed

Manuscript received August 26, 2014; revised October 7, 2015; accepted November 23, 2015. Date of publication December 3, 2015; date of current version September 23, 2016. This work was supported by the NNSF of China under Grants 71401163 and 61333001, by the Hong Kong Scholars Program under Grant XJ2015049, by the Beijing Natural Science Foundation under Grant 4152057, by the National Research Foundation of Singapore under grant NRF-CRP8-2011-03, by the Knut and Alice Wallenberg Foundation, and by the Swedish Research Council. Recommended by Associate Editor D. Bauso.

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Digital Object Identifier 10.1109/TAC.2015.2504962

weight-unbalanced graph in [39] and [40]. Some methods were developed for the unbalanced graph case such as the reweighting technique [39] (for a fixed graph with a known Perron vector) and the subgradient-push methods [41], [42] (where each node has to know its out-degree all the time). To our knowledge, there are no theoretical results on distributed Nash equilibrium computation for switching weight-unbalanced graphs.

In this paper, we consider the distributed zero-sum game Nash equilibrium computation problem proposed in [23], where a multi-agent network consisting of two subnetworks, with one minimizing the objective function and the other maximizing it. The agents play a zero-sum game. The agents in two different subnetworks play antagonistic roles against each other, while the agents in the same subnetwork cooperate. The objective of the network is to achieve a Nash equilibrium via distributed computation based on local communications under time-varying connectivity. The considered Nash equilibrium computation problem is motivated by power allocation problems [23] and saddle point searching problems arising from Lagrangian dual optimization problems [13], [18], [25]–[28]. The contribution of this paper can be summarized as follows:

- We propose a subgradient-based distributed algorithm to compute a saddle-point Nash equilibrium under time-varying graphs, and show that our algorithm with homogeneous stepsizes can achieve a Nash equilibrium under uniformly jointly strongly connected (UJSC) weight-balanced digraphs.
- We further consider the weight-unbalanced case, though most existing results on distributed optimization were obtained for weight-balanced graphs, and show that distributed homogeneous-stepsize algorithms may fail in the unbalanced case, even for the special case of identical subnetworks.
- We propose a heterogeneous stepsize rule and study how to cooperatively find a Nash equilibrium in general weight-unbalanced cases. We find that, for UJSC time-varying digraphs, there always exist (heterogeneous) stepsizes to make the network achieve a Nash equilibrium. Then we construct an adaptive algorithm to update the stepsizes to achieve a Nash equilibrium in two standard cases: one with a common left eigenvector associated with eigenvalue one of adjacency matrices and the other with periodically switching graphs.

The paper is organized as follows. Section II gives some preliminary knowledge, while Section III formulates the distributed Nash equilibrium computation problem and proposes a novel algorithm. Section IV provides the main results followed by Section V that contains all the proofs of the results. Then Section VI provides numerical simulations for illustration. Finally, Section VII gives some concluding remarks.

Notations: $|\cdot|$ denotes the Euclidean norm, $\langle \cdot, \cdot \rangle$ the Euclidean inner product and \otimes the Kronecker product. $\mathbf{B}(z, \varepsilon)$ is a ball with z the center and $\varepsilon > 0$ the radius, $\mathcal{S}_n^+ = \{\mu | \mu_i > 0, \sum_{i=1}^n \mu_i = 1\}$ is the set of all n -dimensional positive stochastic vectors. z' denotes the transpose of vector z , A_{ij} the i -th

row and j -th column entry of matrix A and $\text{diag}\{c_1, \dots, c_n\}$ the diagonal matrix with diagonal elements c_1, \dots, c_n . $\mathbf{1} = (1, \dots, 1)'$ is the vector of all ones with appropriate dimension.

II. PRELIMINARIES

In this section, we give preliminaries on graph theory [4], convex analysis [5], and Nash equilibrium.

A. Graph Theory

A digraph (directed graph) $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ consists of a node set $\bar{\mathcal{V}} = \{1, \dots, \bar{n}\}$ and an arc set $\bar{\mathcal{E}} \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$. Associated with graph $\bar{\mathcal{G}}$, there is a (weighted) adjacency matrix $\bar{A} = (\bar{a}_{ij}) \in \mathbb{R}^{\bar{n} \times \bar{n}}$ with nonnegative adjacency elements \bar{a}_{ij} , which are positive if and only if $(j, i) \in \bar{\mathcal{E}}$. Node j is a neighbor of node i if $(j, i) \in \bar{\mathcal{E}}$. Assume $(i, i) \in \bar{\mathcal{E}}$ for $i = 1, \dots, \bar{n}$. A path in $\bar{\mathcal{G}}$ from i_1 to i_p is an alternating sequence $i_1 e_1 i_2 e_2 \dots i_{p-1} e_{p-1} i_p$ of nodes $i_r, 1 \leq r \leq p$ and arcs $e_r = (i_r, i_{r+1}) \in \bar{\mathcal{E}}, 1 \leq r \leq p-1$. $\bar{\mathcal{G}}$ is said to be bipartite if $\bar{\mathcal{V}}$ can be partitioned into two disjoint parts $\bar{\mathcal{V}}_1$ and $\bar{\mathcal{V}}_2$ such that $\bar{\mathcal{E}} \subseteq \bigcup_{\ell=1}^2 (\bar{\mathcal{V}}_\ell \times \bar{\mathcal{V}}_{3-\ell})$.

Consider a multi-agent network Ξ consisting of two subnetworks Ξ_1 and Ξ_2 with respective n_1 and n_2 agents. Ξ is described by a digraph, denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which contains self-loops, i.e., $(i, i) \in \mathcal{E}$ for each i . Here \mathcal{G} can be partitioned into three digraphs: $\mathcal{G}_\ell = (\mathcal{V}_\ell, \mathcal{E}_\ell)$ with $\mathcal{V}_\ell = \{\omega_1^\ell, \dots, \omega_{n_\ell}^\ell\}$, $\ell = 1, 2$, and a bipartite graph $\mathcal{G}_{\bowtie} = (\mathcal{V}, \mathcal{E}_{\bowtie})$, where $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_{\bowtie}$. In other words, Ξ_1 and Ξ_2 are described by the two digraphs, \mathcal{G}_1 and \mathcal{G}_2 , respectively, and the interconnection between Ξ_1 and Ξ_2 is described by \mathcal{G}_{\bowtie} . Here \mathcal{G}_{\bowtie} is called bipartite without isolated nodes if, for any $i \in \mathcal{V}_\ell$, there is at least one node $j \in \mathcal{V}_{3-\ell}$ such that $(j, i) \in \mathcal{E}$ for $\ell = 1, 2$. Let A_ℓ denote the adjacency matrix of \mathcal{G}_ℓ , $\ell = 1, 2$. Digraph \mathcal{G}_ℓ is strongly connected if there is a path in \mathcal{G}_ℓ from i to j for any pair node $i, j \in \mathcal{V}_\ell$. A node is called a root node if there is at least a path from this node to any other node. In the sequel, we still write $i \in \mathcal{V}_\ell$ instead of $\omega_i^\ell \in \mathcal{V}_\ell$, $\ell = 1, 2$ for simplicity if there is no confusion.

Let $A_\ell = (a_{ij})_{i, j \in \mathcal{V}_\ell} \in \mathbb{R}^{n_\ell \times n_\ell}$ be the adjacency matrix of \mathcal{G}_ℓ . Graph \mathcal{G}_ℓ is weight-balanced if $\sum_{j \in \mathcal{V}_\ell} a_{ij} = \sum_{j \in \mathcal{V}_\ell} a_{ji}$ for $i \in \mathcal{V}_\ell$; and weight-unbalanced otherwise.

A vector is said to be stochastic if all its components are nonnegative and the sum of its components is one. A matrix is a stochastic matrix if each of its row vectors is stochastic. A stochastic vector is positive if all its components are positive.

Let $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ be a stochastic matrix. Define $\mathcal{G}_B = (\{1, \dots, n\}, \mathcal{E}_B)$ as the graph associated with B , where $(j, i) \in \mathcal{E}_B$ if and only if $b_{ij} > 0$ (its adjacency matrix is B). According to Perron-Frobenius theorem [1], there is a unique positive stochastic left eigenvector of B associated with eigenvalue one if \mathcal{G}_B is strongly connected. We call this eigenvector the Perron vector of B .

B. Convex Analysis

A set $K \subseteq \mathbb{R}^m$ is convex if $\lambda z_1 + (1 - \lambda)z_2 \in K$ for any $z_1, z_2 \in K$ and $0 < \lambda < 1$. A point z is an interior point of K if $\mathbf{B}(z, \varepsilon) \subseteq K$ for some $\varepsilon > 0$. For a closed convex set K in \mathbb{R}^m , we can associate with any $z \in \mathbb{R}^m$ a unique element $P_K(z) \in K$ satisfying $|z - P_K(z)| = \inf_{y \in K} |z - y|$, where P_K is the

projection operator onto K . The following property for the convex projection operator P_K holds by Lemma 1 (b) in [15]

$$|P_K(y) - z| \leq |y - z| \text{ for any } y \in \mathbb{R}^m \text{ and any } z \in K. \quad (1)$$

A function $\varphi(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$ is (strictly) convex if $\varphi(\lambda z_1 + (1-\lambda)z_2) \leq \lambda\varphi(z_1) + (1-\lambda)\varphi(z_2)$ for any $z_1 \neq z_2 \in \mathbb{R}^m$ and $0 < \lambda < 1$. A function φ is (strictly) concave if $-\varphi$ is (strictly) convex. A convex function $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous.

For a convex function φ , $v(\hat{z}) \in \mathbb{R}^m$ is a subgradient of φ at point \hat{z} if $\varphi(z) \geq \varphi(\hat{z}) + \langle z - \hat{z}, v(\hat{z}) \rangle$, $\forall z \in \mathbb{R}^m$. For a concave function φ , $v(\hat{z}) \in \mathbb{R}^m$ is a subgradient of φ at \hat{z} if $\varphi(z) \leq \varphi(\hat{z}) + \langle z - \hat{z}, v(\hat{z}) \rangle$, $\forall z \in \mathbb{R}^m$. The set of all subgradients of (convex or concave) function φ at \hat{z} is denoted by $\partial\varphi(\hat{z})$, which is called the subdifferential of φ at \hat{z} .

C. Saddle Point and Nash Equilibrium

A function $\phi(\cdot, \cdot) : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}$ is (strictly) convex-concave if it is (strictly) convex in first argument and (strictly) concave in second one. Given a point (\hat{x}, \hat{y}) , we denote by $\partial_x \phi(\hat{x}, \hat{y})$ the subdifferential of convex function $\phi(\cdot, \hat{y})$ at \hat{x} and $\partial_y \phi(\hat{x}, \hat{y})$ the subdifferential of concave function $\phi(\hat{x}, \cdot)$ at \hat{y} .

A pair $(x^*, y^*) \in X \times Y$ is a saddle point of ϕ on $X \times Y$ if

$$\phi(x^*, y) \leq \phi(x^*, y^*) \leq \phi(x, y^*), \quad \forall x \in X, y \in Y.$$

The next lemma presents a necessary and sufficient condition to characterize the saddle points (see Proposition 2.6.1 in [33]).

Lemma 2.1: Let $X \subseteq \mathbb{R}^{m_1}$, $Y \subseteq \mathbb{R}^{m_2}$ be two closed convex sets. Then a pair (x^*, y^*) is a saddle point of ϕ on $X \times Y$ if and only if

$$\sup_{y \in Y} \inf_{x \in X} \phi(x, y) = \inf_{x \in X} \sup_{y \in Y} \phi(x, y) = \phi(x^*, y^*)$$

and x^* is an optimal solution of optimization problem

$$\text{minimize } \sup_{y \in Y} \phi(x, y) \quad \text{subject to } x \in X \quad (2)$$

while y^* is an optimal solution of optimization problem

$$\text{maximize } \inf_{x \in X} \phi(x, y) \quad \text{subject to } y \in Y. \quad (3)$$

From Lemma 2.1, we find that all saddle points of ϕ on $X \times Y$ yield the same value. The next lemma can be obtained from Lemma 2.1.

Lemma 2.2: If (x_1^*, y_1^*) and (x_2^*, y_2^*) are two saddle points of ϕ on $X \times Y$, then (x_1^*, y_2^*) and (x_2^*, y_1^*) are also saddle points of ϕ on $X \times Y$.

Remark 2.1: Denote by \bar{Z} the set of all saddle points of function ϕ on $X \times Y$, \bar{X} and \bar{Y} the optimal solution sets of optimization problems (2) and (3), respectively. Then from Lemma 2.1 it is not hard to find that if \bar{Z} is nonempty, then \bar{X} , \bar{Y} are nonempty, convex, and $\bar{Z} = \bar{X} \times \bar{Y}$. Moreover, if X and Y are convex, compact and ϕ is convex-concave, then \bar{Z} is nonempty (see Proposition 2.6.9 in [33]).

The saddle point computation can be related to a zero-sum game. In fact, a (strategic) game is described as a triple $(\mathcal{I}, \mathcal{W}, \mathcal{U})$, where \mathcal{I} is the set of all players; $\mathcal{W} = \mathcal{W}_1 \times \dots \times$

\mathcal{W}_n , n is the number of players, \mathcal{W}_i is the set of actions available to player i ; $\mathcal{U} = (u_1, \dots, u_n)$, $u_i : \mathcal{W} \rightarrow \mathbb{R}$ is the payoff function of player i . The game is said to be zero-sum if $\sum_{i=1}^n u_i(w_i, w_{-i}) = 0$, where w_{-i} denotes the actions of all players other than i . A profile action $w^* = (w_1^*, \dots, w_n^*)$ is said to be a Nash equilibrium if $u_i(w_i^*, w_{-i}^*) \geq u_i(w_i, w_{-i}^*)$ for each $i \in \mathcal{I}$ and $w_i \in \mathcal{W}_i$. The Nash equilibria set of a two-person zero-sum game ($n = 2, u_1 + u_2 = 0$) is exactly the saddle point set of payoff function u_2 .

III. DISTRIBUTED NASH EQUILIBRIUM COMPUTATION

In this section, we introduce a distributed Nash equilibrium computation problem and then propose a subgradient-based algorithm as a solution.

Consider a network Ξ consisting of two subnetworks Ξ_1 and Ξ_2 . Agent i in Ξ_1 is associated with a convex-concave objective function $f_i(x, y) : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}$, and agent i in Ξ_2 is associated with a convex-concave objective function $g_i(x, y) : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}$. Each agent only knows its own objective function. The two subnetworks have a common sum objective function with closed convex constraint sets $X \subseteq \mathbb{R}^{m_1}$, $Y \subseteq \mathbb{R}^{m_2}$

$$U(x, y) = \sum_{i=1}^{n_1} f_i(x, y) = \sum_{i=1}^{n_2} g_i(x, y), \quad x \in X, y \in Y.$$

Then the network is engaged in a (generalized) zero-sum game $(\{\Xi_1, \Xi_2\}, X \times Y, u)$, where Ξ_1 and Ξ_2 are viewed as two players, their respective payoff functions are $u_{\Xi_1} = -\sum_{i=1}^{n_1} f_i$ and $u_{\Xi_2} = \sum_{i=1}^{n_2} g_i$. The objective of Ξ_1 and Ξ_2 is to achieve a Nash equilibrium of the zero-sum game.

Remark 3.1: Despite that the contribution of this paper is mainly theoretical, the considered model appears also in applications. Here we illustrate that by discussing two practical examples in the literature. In the first example, from [23] note that for multiple Gaussian communication channels with budget constrained signal power and noise levels, the capacity of each channel is concave in signal power and convex in noise level. Suppose there are two subnetworks, one of which is more critical than the other. The critical subnetwork aims to maximize its capacity by raising its transmission power while the other aims to reduce the interference to other channels by minimizing its transmission power (and thus the capacity). The objective of the two subnetworks is then to find the Nash equilibrium of the sum of all channels' capacities, see Remark 3.1 in [23] for more details. For the second example, recall that many practical problems (for example, distributed estimation, resource allocation, optimal flow control) can be formulated as distributed convex constrained optimization problems, in which the associated Lagrangian function can be expressed as a sum of individual Lagrangian functions, which are convex in the optimization variable and linear (hence concave) in the Lagrangian multiplier. Under Salter's condition, the optimal solutions can be found by computing the saddle-points of the convex-concave Lagrangian function, or equivalently, the Nash equilibrium of the corresponding zero-sum game, see [18] for further discussions.

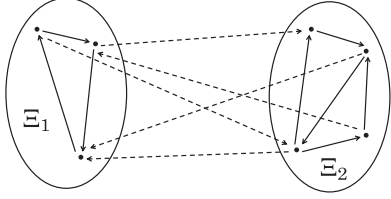


Fig. 1. The zero-sum game communication graph.

We next provide a basic assumption.

A1 (Existence of Saddle Points) For each stochastic vector μ , $\sum_{i=1}^{n_1} \mu_i f_i$ has at least one saddle point over $X \times Y$.

Clearly, **A1** holds if X and Y are bounded (see Proposition 2.6.9 in [33] for other conditions guaranteeing the existence of saddle points). However, in this paper we do not require X and Y to be bounded. Let

$$Z^* = X^* \times Y^* \subseteq X \times Y$$

denote the set of all saddle points of U on $X \times Y$. Notice that $X^* \times Y^*$ is also the set of Nash equilibria of the generalized zero-sum game.

Denote the state of node $i \in \mathcal{V}_1$ as $x_i(k) \in \mathbb{R}^{m_1}$ and the state of node $i \in \mathcal{V}_2$ as $y_i(k) \in \mathbb{R}^{m_2}$ at time $k = 0, 1, \dots$

Definition 3.1: The network Ξ is said to achieve a Nash equilibrium if, for any initial condition $x_i(0) \in \mathbb{R}^{m_1}$, $i \in \mathcal{V}_1$ and $y_i(0) \in \mathbb{R}^{m_2}$, $i \in \mathcal{V}_2$, there are $x^* \in X^*$ and $y^* \in Y^*$ such that

$$\lim_{k \rightarrow \infty} x_i(k) = x^*, i \in \mathcal{V}_1, \quad \lim_{k \rightarrow \infty} y_i(k) = y^*, i \in \mathcal{V}_2.$$

The interconnection in the network Ξ is time-varying and modeled as three digraph sequences

$$\mathcal{G}_1 = \{\mathcal{G}_1(k)\}, \mathcal{G}_2 = \{\mathcal{G}_2(k)\}, \mathcal{G}_{\bowtie} = \{\mathcal{G}_{\bowtie}(k)\}$$

where $\mathcal{G}_1(k) = (\mathcal{V}_1, \mathcal{E}_1(k))$ and $\mathcal{G}_2(k) = (\mathcal{V}_2, \mathcal{E}_2(k))$ are the graphs to describe subnetworks Ξ_1 and Ξ_2 , respectively, and $\mathcal{G}_{\bowtie}(k) = (\mathcal{V}, \mathcal{E}_{\bowtie}(k))$ is the bipartite graph to describe the interconnection between Ξ_1 and Ξ_2 at time $k \geq 0$ (see Fig. 1). For $k_2 > k_1 \geq 0$, denote by $\mathcal{G}_{\bowtie}([k_1, k_2])$ the union graph with node set \mathcal{V} and arc set $\bigcup_{s=k_1}^{k_2-1} \mathcal{E}_{\bowtie}(s)$, and $\mathcal{G}_\ell([k_1, k_2])$ the union graph with node set \mathcal{V}_ℓ and arc set $\bigcup_{s=k_1}^{k_2-1} \mathcal{E}_\ell(s)$ for $\ell = 1, 2$. The following assumption on connectivity is made.

A2 (Connectivity) (i) The graph sequence \mathcal{G}_{\bowtie} is uniformly jointly bipartite; namely, there is an integer $T_{\bowtie} > 0$ such that $\mathcal{G}_{\bowtie}([k, k + T_{\bowtie}])$ is bipartite without isolated nodes for $k \geq 0$.

(ii) For $\ell = 1, 2$, the graph sequence \mathcal{G}_ℓ is uniformly jointly strongly connected (UJSC); namely, there is an integer $T_\ell > 0$ such that $\mathcal{G}_\ell([k, k + T_\ell])$ is strongly connected for $k \geq 0$.

Remark 3.2: The agents in Ξ_ℓ connect directly with those in $\Xi_{3-\ell}$ for all the time in [23], while the agents in two subnetworks are connected at least once in each interval of length T_{\bowtie} according to **A2** (i). In fact, it may be practically hard for the agents of different subnetworks to maintain communications all the time. Moreover, even if each agent in Ξ_ℓ can receive the information from $\Xi_{3-\ell}$, agents may just send or receive once during a period of length T_{\bowtie} to save energy or communication cost.

To handle the distributed Nash equilibrium computation problem, we propose a subgradient-based algorithm, called *Distributed Nash Equilibrium Computation Algorithm*

$$\begin{cases} x_i(k+1) = P_X(\hat{x}_i(k) - \alpha_{i,k} q_{1i}(k)), \\ q_{1i}(k) \in \partial_x f_i(\hat{x}_i(k), \check{x}_i(k)), i \in \mathcal{V}_1 \\ y_i(k+1) = P_Y(\hat{y}_i(k) + \beta_{i,k} q_{2i}(k)), \\ q_{2i}(k) \in \partial_y g_i(\check{y}_i(k), \hat{y}_i(k)), i \in \mathcal{V}_2 \end{cases} \quad (4)$$

with

$$\begin{aligned} \hat{x}_i(k) &= \sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k) x_j(k), \quad \check{x}_i(k) = \sum_{j \in \mathcal{N}_i^2(\check{k}_i)} a_{ij}(\check{k}_i) y_j(\check{k}_i) \\ \hat{y}_i(k) &= \sum_{j \in \mathcal{N}_i^2(k)} a_{ij}(k) y_j(k), \quad \check{y}_i(k) = \sum_{j \in \mathcal{N}_i^1(\check{k}_i)} a_{ij}(\check{k}_i) x_j(\check{k}_i) \end{aligned}$$

where $\alpha_{i,k} > 0$, $\beta_{i,k} > 0$ are the stepsizes at time k , $a_{ij}(k)$ is the time-varying weight of arc (j, i) , $\mathcal{N}_i^\ell(k)$ is the set of neighbors in \mathcal{V}_ℓ of node i at time k , and

$$\check{k}_i = \max \{s | s \leq k, \mathcal{N}_i^{3-\ell}(s) \neq \emptyset\} \leq k \quad (5)$$

which is the last time before k when node $i \in \mathcal{V}_\ell$ has at least one neighbor in $\mathcal{V}_{3-\ell}$.

Remark 3.3: When all objective functions f_i, g_i are additively separable, i.e., $f_i(x, y) = f_i^1(x) + f_i^2(y)$, $g_i(x, y) = g_i^1(x) + g_i^2(y)$, the considered distributed Nash equilibrium computation problem is equivalent to two separated distributed optimization problems with respective objective functions $\sum_{i=1}^{n_1} f_i^1(x)$, $\sum_{i=1}^{n_2} g_i^2(y)$ and constraint sets X, Y . In this case, the set of Nash equilibria is given by

$$X^* \times Y^* = \arg \min_X \sum_{i=1}^{n_1} f_i^1 \times \arg \max_Y \sum_{i=1}^{n_2} g_i^2.$$

Since $\partial_x f_i(x, y) = \partial_x f_i^1(x)$ and $\partial_y g_i(x, y) = \partial_y g_i^2(y)$, algorithm (4) becomes in this case the well-known distributed subgradient algorithms [14], [15].

Remark 3.4: To deal with *weight-unbalanced* graphs, some methods, the rescaling technique [34] and the push-sum protocols [35], [36], [38] have been proposed for average consensus problems; reweighting the objectives [39] and the subgradient-push protocols [41], [42] for distributed optimization problems. Different from these methods, in this paper we propose a distributed algorithm to handle weight-unbalanced graphs when the stepsizes taken by agents are not necessarily the same.

Remark 3.5: Different from the extreme-seeking techniques used in [30] and [31], our method uses the subgradient to compute the Nash equilibrium.

The next assumption was also used in [14], [15], [18], and [21].

A3 (Weight Rule) (i) There is $0 < \eta < 1$ such that $a_{ij}(k) \geq \eta$ for all i, k and $j \in \mathcal{N}_i^1(k) \cup \mathcal{N}_i^2(k)$;

(ii) $\sum_{j \in \mathcal{N}_i^\ell(k)} a_{ij}(k) = 1$ for all k and $i \in \mathcal{V}_\ell, \ell = 1, 2$;

(iii) $\sum_{j \in \mathcal{N}_i^{3-\ell}(\check{k}_i)} a_{ij}(\check{k}_i) = 1$ for $i \in \mathcal{V}_\ell, \ell = 1, 2$.

Conditions (ii) and (iii) in **A3** state that the information from an agent's neighbors is used through a weighted average. The next assumption is about subgradients of objective functions.

A4 (Boundedness of Subgradients) There is $L > 0$ such that, for each i, j

$$|q| \leq L, \quad \forall q \in \partial_x f_i(x, y) \cup \partial_y g_j(x, y), \quad \forall x \in X, y \in Y.$$

Obviously, **A4** holds if X and Y are bounded. A similar bounded assumption has been widely used in distributed optimization [12]–[15].

Note that the stepsize in our algorithm (4) is *heterogenous*, i.e., the stepsizes may be different for different agents, in order to deal with general unbalanced cases. One challenging problem is how to select the stepsizes $\{\alpha_{i,k}\}$ and $\{\beta_{i,k}\}$. The *homogenous* stepsize case is to set $\alpha_{i,k} = \beta_{j,k} = \gamma_k$ for $i \in \mathcal{V}_1, j \in \mathcal{V}_2$ and all k , where $\{\gamma_k\}$ is given as follows.

A5 $\{\gamma_k\}$ is non-increasing, $\sum_{k=0}^{\infty} \gamma_k = \infty$ and $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$.

Conditions $\sum_{k=0}^{\infty} \gamma_k = \infty$ and $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$ in **A5** are well-known in homogeneous stepsize selection for distributed subgradient algorithms for distributed optimization problems with weight-balanced graphs, e.g., [15], [16], [18].

Remark 3.6: While weight-balanced graphs are considered in [14], [15], [18], [23] and [24], we consider general (weight-unbalanced) digraphs, and provide a heterogeneous stepsize design method for the desired Nash equilibrium convergence.

IV. MAIN RESULTS

In this section, we start with homogeneous stepsizes to achieve a Nash equilibrium for weight-balanced graphs (in Section IV-A). Then we focus on a special weight-unbalanced case to show how a homogeneous-stepsize algorithm may fail to achieve our aim (in Section IV-B). Finally, we show that the heterogeneity of stepsizes can help us achieve a Nash equilibrium in some weight-unbalanced graph cases (in Section IV-C).

A. Weight-Balanced Graphs

Here we consider algorithm (4) with homogeneous stepsizes $\alpha_{i,k} = \beta_{i,k} = \gamma_k$ for weight-balanced digraphs. The following result, in fact, provides two sufficient conditions to achieve a Nash equilibrium under switching weight-balanced digraphs.

Theorem 4.1: Suppose **A1–A5** hold and digraph $\mathcal{G}_\ell(k)$ is weight-balanced for $k \geq 0$ and $\ell = 1, 2$. Then the multi-agent network Ξ achieves a Nash equilibrium by algorithm (4) with the homogeneous stepsizes $\{\gamma_k\}$ if either of the following two conditions holds:

- (i) U is strictly convex-concave;
- (ii) $X^* \times Y^*$ contains an interior point.

The proof can be found in Section V-B.

Remark 4.1: The authors in [23] developed a continuous-time dynamical system to solve the Nash equilibrium computation problem for fixed weight-balanced digraphs, and showed that the network converges to a Nash equilibrium for a strictly convex-concave differentiable sum objective function. Different from [23], here we allow time-varying communication structures and a non-smooth objective function U . The same result may also hold for the continuous-time solution in [23] under our problem setup, but the analysis would probably be much more involved.

B. Homogenous Stepsizes vs. Unbalanced Graphs

In the preceding subsection, we showed that a Nash equilibrium can be achieved with homogeneous stepsizes when the graphs of two subnetworks are weight-balanced. Here we demonstrate that the homogenous stepsize algorithm may fail to guarantee the Nash equilibrium convergence for general weight-unbalanced digraphs unless certain conditions about the objective function hold.

Consider a special case, called the completely identical sub-network case, i.e., Ξ_1 and Ξ_2 are completely identical

$$n_1 = n_2, \quad f_i = g_i, \quad i = 1, \dots, n_1; \quad A_1(k) = A_2(k) \\ \mathcal{G}_{\Xi}(k) = \{(\omega_i^\ell, \omega_i^{3-\ell}), \ell = 1, 2, i = 1, \dots, n_1\}, \quad k \geq 0.$$

In this case, agents $\omega_i^\ell, \omega_i^{3-\ell}$ have the same objective function, neighbor set and can communicate with each other at all times. Each pair of agents $\omega_i^\ell, \omega_i^{3-\ell}$ can be viewed as one agent labeled as “ i ”. Then algorithm (4) with homogeneous stepsizes $\{\gamma_k\}$ reduces to the following form:

$$\begin{cases} x_i(k+1) = P_X(\sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k)x_j(k) - \gamma_k q_{1i}(k)) \\ y_i(k+1) = P_Y(\sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k)y_j(k) + \gamma_k q_{2i}(k)) \end{cases} \quad (6)$$

for $i = 1, \dots, n_1$, where $q_{1i}(k) \in \partial_x f_i(\hat{x}_i(k), y_i(k))$, $q_{2i}(k) \in \partial_y f_i(x_i(k), \hat{y}_i(k))$.

Remark 4.2: Similar distributed saddle point computation algorithms have been proposed in the literature, for example, the distributed saddle point computation for the Lagrange function of constrained optimization problems in [18]. In fact, algorithm (6) can be used to solve the following distributed saddle-point computation problem: consider a network Ξ_1 consisting of n_1 agents with node set $\mathcal{V}_1 = \{1, \dots, n_1\}$, its objective is to seek a saddle point of the sum objective function $\sum_{i=1}^{n_1} f_i(x, y)$ in a distributed way, where f_i can only be known by agent i . In (6), (x_i, y_i) is the state of node “ i .” Moreover, algorithm (6) can be viewed as a distributed version of the following centralized algorithm:

$$\begin{cases} x(k+1) = P_X(x(k) - \gamma q_1(k)), \quad q_1(k) \in \partial_x U(x(k), y(k)) \\ y(k+1) = P_Y(y(k) + \gamma q_2(k)), \quad q_2(k) \in \partial_y U(x(k), y(k)) \end{cases}$$

which was proposed in [13] to solve the approximate saddle point problem with a constant stepsize.

We first show that, algorithm (4) with homogeneous stepsizes [or equivalently (6)] cannot seek the desired Nash equilibrium though it is convergent, even for *fixed* weight-unbalanced graphs.

Theorem 4.2: Suppose **A1**, **A3–A5** hold, and $f_i, i = 1, \dots, n_1$ are strictly convex-concave and the graph is fixed with $\mathcal{G}_1(0)$ strongly connected. Then, with (6), all the agents converge to the unique saddle point, denoted as (\bar{x}, \bar{y}) , of an objective function $\sum_{i=1}^{n_1} \mu_i f_i$ on $X \times Y$, where $\mu = (\mu_1, \dots, \mu_{n_1})'$ is the Perron vector of the adjacency matrix $A_1(0)$ of graph $\mathcal{G}_1(0)$.

The proof is almost the same as that of Theorem 4.1, by replacing $\sum_{i=1}^{n_1} |x_i(k) - \bar{x}|^2, \sum_{i=1}^{n_2} |y_i(k) - \bar{y}|^2$ and $U(x, y)$ with $\sum_{i=1}^{n_1} \mu_i |x_i(k) - \bar{x}|^2, \sum_{i=1}^{n_1} \mu_i |y_i(k) - \bar{y}|^2$ and $\sum_{i=1}^{n_1} \mu_i f_i(x, y)$, respectively. Therefore, the proof is omitted.

Although it is hard to achieve the desired Nash equilibrium with the homogeneous-stepsize algorithm in general, we can still achieve it in some cases. Here we can give a necessary and sufficient condition to achieve a Nash equilibrium for any UJSC switching digraph sequence.

Theorem 4.3: Suppose **A1**, **A3–A5** hold and $f_i, i = 1, \dots, n_1$ are strictly convex-concave. Then the multi-agent network Ξ achieves a Nash equilibrium by algorithm (6) for any UJSC switching digraph sequence \mathcal{G}_1 if and only if $f_i, i = 1, \dots, n_1$ have the same saddle point on $X \times Y$.

The proof can be found in Section V-C.

Remark 4.3: The strict convexity-concavity of f_i implies that the saddle point of f_i is unique. From the proof we can find that the necessity of Theorem 4.3 does not require that each objective function f_i is strictly convex-concave, but the strict convexity-concavity of the sum objective function $\sum_{i=1}^{n_1} f_i$ suffices.

C. Weight-Unbalanced Graphs

The results in the preceding subsections showed that the homogenous-stepsize algorithm may not make a weight-unbalanced network achieve its Nash equilibrium. Here we first show the existence of a heterogeneous-stepsize design to make the (possibly weight-unbalanced) network achieve a Nash equilibrium.

Theorem 4.4: Suppose **A1**, **A3**, **A4** hold and U is strictly convex-concave. Then for any time-varying communication graphs $\mathcal{G}_\ell, \ell = 1, 2$ and $\mathcal{G}_{\triangleright\triangleleft}$ that satisfy **A2**, there always exist stepsize sequences $\{\alpha_{i,k}\}$ and $\{\beta_{i,k}\}$ such that the multi-agent network Ξ achieves a Nash equilibrium by algorithm (4).

The proof is in Section V-D. In fact, it suffices to design stepsizes $\alpha_{i,k}$ and $\beta_{i,k}$ as follows:

$$\alpha_{i,k} = \frac{1}{\hat{\alpha}_k^i} \gamma_k, \quad \beta_{i,k} = \frac{1}{\hat{\beta}_k^i} \gamma_k \quad (7)$$

where $(\alpha_k^1, \dots, \alpha_k^{n_1})' = \phi^1(k+1)$, $(\beta_k^1, \dots, \beta_k^{n_2})' = \phi^2(k+1)$, $\phi^\ell(k+1)$ is the vector for which $\lim_{r \rightarrow \infty} \Phi^\ell(r, k+1) =: \mathbf{1}(\phi^\ell(k+1))'$, $\Phi^\ell(r, k+1) := A_\ell(r)A_\ell(r-1) \cdots A_\ell(k+1)$, $\ell = 1, 2$, $\{\gamma_k\}$ satisfies the following conditions:

$$\begin{aligned} \lim_{k \rightarrow \infty} \gamma_k \sum_{s=0}^{k-1} \gamma_s = 0, \quad \{\gamma_k\} \text{ is non-increasing} \\ \sum_{k=0}^{\infty} \gamma_k = \infty, \quad \sum_{k=0}^{\infty} \gamma_k^2 < \infty. \end{aligned} \quad (8)$$

Remark 4.4: The stepsize design in Theorem 4.4 is motivated by the following two ideas. On one hand, agents need to eliminate the imbalance caused by the weight-unbalanced graphs, which is done by $\{1/\alpha_k^i\}, \{1/\beta_k^i\}$, while on the other hand, agents also need to achieve a consensus within each subnetwork and cooperative optimization, which is done by $\{\gamma_k\}$, as in the balanced graph case.

Remark 4.5: Condition (8) can be satisfied by letting $\gamma_k = c/(k+b)^{1/2+\epsilon}$ for $k \geq 0$, $c > 0$, $b > 0$, $0 < \epsilon \leq 1/2$. Moreover, from the proof of Theorem 4.4 we find that, if the sets X and Y are bounded, the system states are naturally bounded, and then (8) can be relaxed as **A5**.

Clearly, the above choice of stepsizes at time k depend on the adjacency matrix sequences $\{A_1(s)\}_{s \geq k+1}$ and $\{A_2(s)\}_{s \geq k+1}$, which is not so practical. Therefore, we will consider how to design adaptive algorithms to update the stepsize sequences $\{\alpha_{i,k}\}$ and $\{\beta_{i,k}\}$ such that the Nash equilibrium can be achieved, where the (heterogeneous) stepsizes at time k just depend on the local information that agents can obtain before time k .

Take

$$\alpha_{i,k} = \frac{1}{\hat{\alpha}_k^i} \gamma_k, \quad \beta_{i,k} = \frac{1}{\hat{\beta}_k^i} \gamma_k \quad (9)$$

where $\{\gamma_k\}$ satisfies (8). The only difference between stepsize selection rule (9) and (7) is that α_k^i and β_k^i are replaced with $\hat{\alpha}_k^i$ and $\hat{\beta}_k^i$, respectively. We consider how to design distributed adaptive algorithms for $\hat{\alpha}^i$ and $\hat{\beta}^i$ such that

$$\begin{aligned} \hat{\alpha}_k^i &= \hat{\alpha}^i(a_{ij}(s), j \in \mathcal{N}_i^1(s), s \leq k) \\ \hat{\beta}_k^i &= \hat{\beta}^i(a_{ij}(s), j \in \mathcal{N}_i^2(s), s \leq k) \end{aligned} \quad (10)$$

$$\lim_{k \rightarrow \infty} (\hat{\alpha}_k^i - \alpha_k^i) = 0, \quad \lim_{k \rightarrow \infty} (\hat{\beta}_k^i - \beta_k^i) = 0. \quad (11)$$

Note that $(\alpha_k^1, \dots, \alpha_k^{n_1})'$ and $(\beta_k^1, \dots, \beta_k^{n_2})'$ are the Perron vectors of the two limits $\lim_{r \rightarrow \infty} \Phi^1(r, k+1)$ and $\lim_{r \rightarrow \infty} \Phi^2(r, k+1)$, respectively.

The next theorem shows that, in two standard cases, we can design distributed adaptive algorithms satisfying (10) and (11) to ensure that Ξ achieves a Nash equilibrium. How to design them is given in the proof.

Theorem 4.5: Consider algorithm (4) with stepsize selection rule (9). Suppose **A1–A4** hold, U is strictly convex-concave. For the following two cases, with the adaptive distributed algorithms satisfying (10) and (11), network Ξ achieves a Nash equilibrium.

- (i) For $\ell = 1, 2$, the adjacency matrices $A_\ell(k), k \geq 0$ have a common left eigenvector with eigenvalue one;
- (ii) For $\ell = 1, 2$, the adjacency matrices $A_\ell(k), k \geq 0$ are switching periodically, i.e., there exist positive integers p^ℓ and two finite sets of stochastic matrices $A_\ell^0, \dots, A_\ell^{p^\ell-1}$ such that $A_\ell(rp^\ell + s) = A_\ell^s$ for $r \geq 0$ and $s = 0, \dots, p^\ell - 1$.

The proof is given in Section V-E.

Remark 4.6: Regarding case (i), note that for a *fixed* graph, the adjacency matrices obviously have a common left eigenvector. Moreover, periodic switching can be interpreted as a simple scheduling strategy. At each time agents may choose some neighbors to communicate with in a periodic order.

Remark 4.7: In the case of a fixed unbalanced graph, the optimization can also be solved by either reweighting the objectives [39], or by the subgradient-push protocols [41], [42], where the Perron vector of the adjacency matrix is required to be known in advance or each agent is required to know its out-degree. These requirements may be quite restrictive in a distributed setting. Theorem 4.5 shows that, in the fixed graph case, agents can adaptively learn the Perron vector by the adaptive learning scheme and then achieve the desired convergence without knowing the Perron vector and their individual out-degrees.

When the adjacency matrices $A_\ell(k)$ have a common left eigenvector, the designed distributed adaptive learning strategy (43) can guarantee that the differences between $\hat{\alpha}_k^i = \alpha_k^i(k)$, $\hat{\beta}_k^i = \beta_k^i(k)$ and the “true stepsizes” $\phi_k^1(k+1)$, $\phi_k^2(k+1)$ asymptotically tend to zero. The converse is also true for some cases. In fact, if the time-varying adjacency matrices are switching within finite matrices and $\lim_{k \rightarrow \infty} (\alpha_k^i(k) - \phi_k^1(k+1)) = 0$, $\lim_{k \rightarrow \infty} (\beta_k^i(k) - \phi_k^2(k+1)) = 0$, then we can show that the finite adjacency matrices certainly have a common left eigenvector.

Moreover, when the adjacency matrices have no common left eigenvector, the adaptive learning strategy (43) generally cannot make $\hat{\alpha}_k^i$, $\hat{\beta}_k^i$ asymptotically learn the true stepsizes and then cannot achieve a Nash equilibrium. For instance, consider the special distributed saddle-point computation algorithm (6) with strictly convex-concave objective functions f_i . Let $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_{n_1})'$, $\hat{\alpha} = (\hat{\alpha}_1, \dots, \hat{\alpha}_{n_1})'$ be two different positive stochastic vectors. Suppose $A_1(0) = \mathbf{1}\bar{\alpha}'$ and $A_1(k) = \mathbf{1}\hat{\alpha}'$ for $k \geq 1$. In this case, $\alpha_k^i(k) = \bar{\alpha}_i$, $\phi_k^1(k+1) = \hat{\alpha}_i$ for all $k \geq 0$ and then (11) is not true. According to Theorem 4.2, the learning strategy (43) can make $(x_i(k), y_i(k))$ converge to the (unique) saddle point of the function $\sum_{i=1}^{n_1} (\hat{\alpha}_i/\bar{\alpha}_i) f_i(x, y)$ on $X \times Y$, which is not necessarily the saddle point of $\sum_{i=1}^{n_1} f_i(x, y)$ on $X \times Y$.

V. PROOFS

In this section, we first introduce some useful lemmas and then present the proofs of the theorems in last section.

A. Supporting Lemmas

First of all, we introduce two lemmas. The first lemma is the deterministic version of Lemma 11 on page 50 in [6], while the second one is Lemma 7 in [15].

Lemma 5.1: Let $\{a_k\}$, $\{b_k\}$, and $\{c_k\}$ be non-negative sequences with $\sum_{k=0}^{\infty} b_k < \infty$. If $a_{k+1} \leq a_k + b_k - c_k$ holds for any k , then $\lim_{k \rightarrow \infty} a_k$ is a finite number.

Lemma 5.2: Let $0 < \lambda < 1$ and $\{a_k\}$ be a positive sequence. If $\lim_{k \rightarrow \infty} a_k = 0$, then $\lim_{k \rightarrow \infty} \sum_{r=0}^k \lambda^{k-r} a_r = 0$. Moreover, if $\sum_{k=0}^{\infty} a_k < \infty$, then $\sum_{k=0}^{\infty} \sum_{r=0}^k \lambda^{k-r} a_r < \infty$.

Next, we show some useful lemmas.

Lemma 5.3: For any $\mu \in \mathcal{S}_n^+$, there is a stochastic matrix $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ such that \mathcal{G}_B is strongly connected and $\mu'B = \mu'$.

Proof: Take $\mu = (\mu_1, \dots, \mu_n)' \in \mathcal{S}_n^+$. Without loss of generality, we assume $\mu_1 = \min_{1 \leq i \leq n} \mu_i$ (otherwise we can rearrange the index of agents). Let B be a stochastic matrix such that the graph \mathcal{G}_B associated with B is a directed cycle: $1e_n n \dots 2e_1 1$ with $e_r = (r+1, r)$, $1 \leq r \leq n-1$ and $e_n = (1, n)$. Clearly, \mathcal{G}_B is strongly connected. Then all nonzero entries of B are $\{b_{ii}, b_{i(i+1)}, 1 \leq i \leq n-1, b_{nn}, b_{n1}\}$ and $\mu'B = \mu'$ can be rewritten as $b_{11}\mu_1 + (1 - b_{nn})\mu_n = \mu_1$, $(1 - b_{rr})\mu_r + b_{(r+1)(r+1)}\mu_{r+1} = \mu_{r+1}$, $1 \leq r \leq n-1$. Equivalently

$$\begin{cases} (1 - b_{22})\mu_2 = (1 - b_{11})\mu_1 \\ (1 - b_{33})\mu_3 = (1 - b_{11})\mu_1 \\ \vdots \\ (1 - b_{nn})\mu_n = (1 - b_{11})\mu_1 \end{cases} \quad (12)$$

Let $b_{11} = b_{11}^*$ with $0 < b_{11}^* < 1$. Clearly, there is a solution to (12): $b_{11} = b_{11}^*$, $0 < b_{rr} = 1 - (1 - b_{11}^*)\mu_1/\mu_r < 1$, $2 \leq r \leq n$. Then the conclusion follows. \square

The following lemma is about stochastic matrices, which can be found from Lemma 3 in [7].

Lemma 5.4: Let $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ be a stochastic matrix and $\bar{h}(\mu) = \max_{1 \leq i, j \leq n} |\mu_i - \mu_j|$, $\mu = (\mu_1, \dots, \mu_n)' \in \mathbb{R}^n$. Then $\bar{h}(B\mu) \leq \mu(B)\bar{h}(\mu)$, where $\mu(B) = 1 - \min_{j_1, j_2} \sum_{i=1}^n \min\{b_{j_1 i}, b_{j_2 i}\}$, is called “the ergodicity coefficient” of B .

We next give a lemma about the transition matrix sequence $\Phi^\ell(k, s) = A_\ell(k)A_\ell(k-1) \dots A_\ell(s)$, $k \geq s$, $\ell = 1, 2$, where (i), (ii), and (iv) are taken from Lemma 4 in [14], while (iii) can be obtained from Lemma 2 in [14].

Lemma 5.5: Suppose **A2** (ii) and **A3** (i), (ii) hold. Then for $\ell = 1, 2$, we have

- (i) The limit $\lim_{k \rightarrow \infty} \Phi^\ell(k, s)$ exists for each s ;
- (ii) There is a positive stochastic vector $\phi^\ell(s) = (\phi_1^\ell(s), \dots, \phi_{n_\ell}^\ell(s))'$ such that $\lim_{k \rightarrow \infty} \Phi^\ell(k, s) = \mathbf{1}(\phi^\ell(s))'$;
- (iii) For every $i = 1, \dots, n_\ell$ and s , $\phi_i^\ell(s) \geq \eta^{(n_\ell-1)T_\ell}$;
- (iv) For every i , the entries $\Phi^\ell(k, s)_{ij}$, $j = 1, \dots, n_\ell$ converge to the same limit $\phi_j^\ell(s)$ at a geometric rate, i.e., for every $i = 1, \dots, n_\ell$ and all $s \geq 0$

$$|\Phi^\ell(k, s)_{ij} - \phi_j^\ell(s)| \leq C_\ell \rho_\ell^{k-s}$$

for all $k \geq s$ and $j = 1, \dots, n_\ell$, where $C_\ell = 2(1 + \eta^{-M_\ell})/(1 - \eta^{M_\ell})$, $\rho_\ell = (1 - \eta^{M_\ell})^{1/M_\ell}$, and $M_\ell = (n_\ell - 1)T_\ell$.

The following lemma shows a relation between the left eigenvectors of stochastic matrices and the Perron vector of the limit of their product matrix.

Lemma 5.6: Let $\{B(k)\}$ be a sequence of stochastic matrices. Suppose $B(k)$, $k \geq 0$ have a common left eigenvector μ corresponding to eigenvalue one and the associated graph sequence $\{\mathcal{G}_{B(k)}\}$ is UJSC. Then, for each s

$$\lim_{k \rightarrow \infty} B(k) \dots B(s) = \frac{\mathbf{1}\mu'}{\mu'\mathbf{1}}$$

Proof: Since μ is the common left eigenvector of $B(r)$, $r \geq s$ associated with eigenvalue one, $\mu' \lim_{k \rightarrow \infty} B(k) \dots B(s) = \lim_{k \rightarrow \infty} \mu' B(k) \dots B(s) = \mu'$. In addition, by Lemma 5.5, for each s , the limit $\lim_{k \rightarrow \infty} B(k) \dots B(s) =: \mathbf{1}\phi'(s)$ exists. Therefore, $\mu' = \mu'(\mathbf{1}\phi'(s)) = (\mu'\mathbf{1})\phi'(s)$, which implies $(\mu'\mathbf{1})\phi(s) = \mu$. The conclusion follows. \square

Basically, the two dynamics of algorithm (4) are in the same form. Let us check the first one

$$x_i(k+1) = P_X(\hat{x}_i(k) - \alpha_{i,k} q_{1i}(k)), \\ q_{1i}(k) \in \partial_x f_i(\hat{x}_i(k), \check{x}_i(k)), \quad i \in \mathcal{V}_1. \quad (13)$$

By treating the term containing y_j ($j \in \mathcal{V}_2$) as “disturbance,” we can transform (13) to a simplified model in the following form with disturbance ϵ_i :

$$x_i(k+1) = \sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k)x_j(k) + \epsilon_i(k), \quad i \in \mathcal{V}_1 \quad (14)$$

where $\epsilon_i(k) = P_X(\hat{x}_i(k) + w_i(k)) - \hat{x}_i(k)$. It follows from $x_j(k) \in X$, the convexity of X and **A3** (ii) that $\hat{x}_i(k) = \sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k)x_j(k) \in X$. Then from (1), $|\epsilon_i(k)| \leq |w_i(k)|$.

The next lemma is about a limit for the two subnetworks. Denote

$$\bar{\alpha}_k = \max_{1 \leq i \leq n_1} \alpha_{i,k}, \quad \bar{\beta}_k = \max_{1 \leq i \leq n_2} \beta_{i,k}.$$

Lemma 5.7: Consider algorithm (4) with **A3** (ii) and **A4**. If $\lim_{k \rightarrow \infty} \bar{\alpha}_k \sum_{s=0}^{k-1} \bar{\alpha}_s = \lim_{k \rightarrow \infty} \bar{\beta}_k \sum_{s=0}^{k-1} \bar{\beta}_s = 0$, then for any x, y , $\lim_{k \rightarrow \infty} \bar{\alpha}_k \max_{1 \leq i \leq n_1} |x_i(k) - x| = \lim_{k \rightarrow \infty} \bar{\beta}_k \max_{1 \leq i \leq n_2} |y_i(k) - y| = 0$.

Proof: We will only show $\lim_{k \rightarrow \infty} \bar{\alpha}_k \max_{1 \leq i \leq n_1} |x_i(k) - x| = 0$ since the other one about $\bar{\beta}_k$ can be proved similarly. At first, it follows from $\lim_{k \rightarrow \infty} \bar{\alpha}_k \sum_{s=0}^{k-1} \bar{\alpha}_s = 0$ that $\lim_{k \rightarrow \infty} \bar{\alpha}_k = 0$. From **A4** we have $|\epsilon_i(k)| \leq \bar{\alpha}_k L$. Then from (14) and **A3** (ii) we obtain

$$\max_{1 \leq i \leq n_1} |x_i(k+1) - x| \leq \max_{1 \leq i \leq n_1} |x_i(k) - x| + \bar{\alpha}_k L, \quad \forall k.$$

Therefore, $\max_{1 \leq i \leq n_1} |x_i(k) - x| \leq \max_{1 \leq i \leq n_1} |x_i(0) - x| + L \sum_{s=0}^{k-1} \bar{\alpha}_s$ and then, for each k

$$\bar{\alpha}_k \max_{1 \leq i \leq n_1} |x_i(k) - x| \leq \bar{\alpha}_k \max_{1 \leq i \leq n_1} |x_i(0) - x| + \bar{\alpha}_k \sum_{s=0}^{k-1} \bar{\alpha}_s L.$$

Taking the limit over both sides of the preceding inequality yields the conclusion. \square

We assume without loss of generality that $m_1 = 1$ in the sequel of this subsection for notational simplicity. Denote $x(k) = (x_1(k), \dots, x_{n_1}(k))'$, $\epsilon(k) = (\epsilon_1(k), \dots, \epsilon_{n_1}(k))'$. Then system (14) can be written in a compact form

$$x(k+1) = A_1(k)x(k) + \epsilon(k), \quad k \geq 0.$$

Recall transition matrix

$$\Phi^\ell(k, s) = A_\ell(k)A_\ell(k-1) \cdots A_\ell(s), \quad k \geq s, \quad \ell = 1, 2.$$

Therefore, for each k

$$x(k+1) = \Phi^1(k, s)x(s) + \sum_{r=s}^{k-1} \Phi^1(k, r+1)\epsilon(r) + \epsilon(k). \quad (15)$$

At the end of this section, we present three lemmas for (4) (or (14) and the other one for y). The first lemma gives an estimation for $h_1(k) = \max_{1 \leq i, j \leq n_1} |x_i(k) - x_j(k)|$ and $h_2(k) = \max_{1 \leq i, j \leq n_2} |y_i(k) - y_j(k)|$ over a bounded interval.

Lemma 5.8: Suppose **A2** (ii), **A3** and **A4** hold. Then for $\ell = 1, 2$ and any $t \geq 1, 0 \leq q \leq T^\ell - 1$

$$h_\ell(tT^\ell + q) \leq (1 - \eta^{T^\ell})h_\ell((t-1)T^\ell + q) + 2L \sum_{r=(t-1)T^\ell + q}^{tT^\ell + q-1} \lambda_r^\ell \quad (16)$$

where $\lambda_r^1 = \bar{\alpha}_r$, $\lambda_r^2 = \bar{\beta}_r$, $T^\ell = (n_\ell(n_\ell - 2) + 1)T_\ell$ for a constant T_ℓ given in **A2** and L as the upper bound on the subgradients of objective functions in **A4**.

Proof: Here we only show the case of $\ell = 1$ since the other one can be proven in the same way. Consider $n_1(n_1 - 2) + 1$ time intervals $[0, T_1 - 1], [T_1, 2T_1 - 1], \dots, [n_1(n_1 - 2)T_1, (n_1(n_1 - 2) + 1)T_1 - 1]$. By the definition of UJSC graph, $\mathcal{G}_1([tT_1, (t+1)T_1 - 1])$ contains a root node for $0 \leq t \leq n_1(n_1 - 2)$. Clearly, the set of the $n_1(n_1 - 2) + 1$ root nodes contains at least one node, say i_0 , at least $n_1 - 1$ times. Assume without loss of generality that i_0 is a root node of $\mathcal{G}_1([tT_1, (t+1)T_1 - 1])$, $t = t_0, \dots, t_{n_1-2}$.

Take $j_0 \neq i_0$ from \mathcal{V}_1 . It is not hard to show that there exist a node set $\{j_1, \dots, j_q\}$ and time set $\{k_0, \dots, k_q\}$, $q \leq n_1 - 2$ such that $(j_{r+1}, j_r) \in \mathcal{E}_1(k_{q-r})$, $0 \leq r \leq q - 1$ and $(i_0, j_q) \in \mathcal{E}_1(k_0)$, where $k_0 < \dots < k_{q-1} < k_q$ and all k_r belong to different intervals $[t_r T_1, (t_r + 1)T_1 - 1]$, $0 \leq r \leq n_1 - 2$.

Noticing that the diagonal elements of all adjacency matrices are positive, and moreover, for matrices $D^1, D^2 \in \mathbb{R}^{n_1 \times n_1}$ with nonnegative entries

$$(D^1)_{r_0 r_1} > 0, (D^2)_{r_1 r_2} > 0 \implies (D^1 D^2)_{r_0 r_2} > 0$$

so we have $\Phi^1(T^1 - 1, 0)_{j_0 i_0} > 0$. Because j_0 is taken from \mathcal{V}_1 freely, $\Phi^1(T^1 - 1, 0)_{j i_0} > 0$ for $j \in \mathcal{V}_1$. As a result, $\Phi^1(T^1 - 1, 0)_{j i_0} \geq \eta^{T^1}$ for $j \in \mathcal{V}_1$ with **A3** (i) and so $\mu(\Phi^1(T^1 - 1, 0)) \leq 1 - \eta^{T^1}$ by the definition of ergodicity coefficient given in Lemma 5.4. According to (15), the inequality $h(\mu + \nu) \leq h(\mu) + 2 \max_i \nu_i$, Lemma 5.4 and **A4**

$$\begin{aligned} h_1(T^1) &\leq h_1(\Phi^1(T^1 - 1, 0)x(0)) + 2L \sum_{r=0}^{T^1-1} \bar{\alpha}_r \\ &\leq \mu(\Phi^1(T^1 - 1, 0)) h_1(0) + 2L \sum_{r=0}^{T^1-1} \bar{\alpha}_r \\ &\leq (1 - \eta^{T^1})h_1(0) + 2L \sum_{r=0}^{T^1-1} \bar{\alpha}_r \end{aligned}$$

which shows (16) for $\ell = 1, t = 1, q = 0$. Analogously, we can show (16) for $\ell = 1, 2$ and $t \geq 1, 0 \leq q \leq T^\ell - 1$. \square

Lemma 5.9: Suppose **A2** (ii), **A3** and **A4** hold.

- (i) If $\sum_{k=0}^{\infty} \bar{\alpha}_k^2 < \infty$ and $\sum_{k=0}^{\infty} \bar{\beta}_k^2 < \infty$, then $\sum_{k=0}^{\infty} \bar{\alpha}_k h_1(k) < \infty$, $\sum_{k=0}^{\infty} \bar{\beta}_k h_2(k) < \infty$;
- (ii) If for each i , $\lim_{k \rightarrow \infty} \alpha_{i,k} = 0$ and $\lim_{k \rightarrow \infty} \beta_{i,k} = 0$, then the subnetworks Ξ_1 and Ξ_2 achieve a consensus, respectively, i.e., $\lim_{k \rightarrow \infty} h_1(k) = 0$, $\lim_{k \rightarrow \infty} h_2(k) = 0$.

Note that (i) is an extension of Lemma 8 (b) in [15] dealing with weight-balanced graph sequence to general graph sequence (possibly weight-unbalanced), while (ii) is about the consensus within the subnetworks, and will be frequently used in the sequel. This lemma can be shown by Lemma 5.8 and similar arguments to the proof of Lemma 8 in [15], and hence, the proof is omitted here.

The following provides the error estimation between agents' states and their average.

Lemma 5.10: Suppose **A2–A4** hold, and $\{\bar{\alpha}(k)\}, \{\bar{\beta}(k)\}$ are non-increasing with $\sum_{k=0}^{\infty} \bar{\alpha}_k^2 < \infty$, $\sum_{k=0}^{\infty} \bar{\beta}_k^2 < \infty$. Then for each $i \in \mathcal{V}_1$ and $j \in \mathcal{V}_2$, $\sum_{k=0}^{\infty} \bar{\beta}_k |\dot{x}_i(k) - \bar{y}(k)| < \infty$, $\sum_{k=0}^{\infty} \bar{\alpha}_k |\dot{y}_j(k) - \bar{x}(k)| < \infty$, where $\bar{x}(k) = (1/n_1) \sum_{i=1}^{n_1} x_i(k) \in X$, $\bar{y}(k) = (1/n_2) \sum_{i=1}^{n_2} y_i(k) \in Y$.

Proof: We only need to show the first conclusion since the second one can be obtained in the same way. At first, from **A3** (iii) and $|y_j(\check{k}_i) - \bar{y}(\check{k}_i)| \leq h_2(\check{k}_i)$ we have

$$\sum_{k=0}^{\infty} \bar{\beta}_k |\check{x}_i(k) - \bar{y}(\check{k}_i)| \leq \sum_{k=0}^{\infty} \bar{\beta}_k h_2(\check{k}_i). \quad (17)$$

Let $\{s_{ir}, r \geq 0\}$ be the set of all moments when $\mathcal{N}_i^2(s_{ir}) \neq \emptyset$. Recalling the definition of \check{k}_i in (5), $\check{k}_i = s_{ir}$ when $s_{ir} \leq k < s_{i(r+1)}$. Since $\{\bar{\beta}_k\}$ is non-increasing and $\sum_{k=0}^{\infty} \bar{\beta}_k h_2(k) < \infty$ (by Lemma 5.9), we have

$$\begin{aligned} \sum_{k=0}^{\infty} \bar{\beta}_k h_2(\check{k}_i) &\leq \sum_{k=0}^{\infty} \bar{\beta}_{\check{k}_i} h_2(\check{k}_i) \\ &= \sum_{r=0}^{\infty} \bar{\beta}_{s_{ir}} |s_{i(r+1)} - s_{ir}| h_2(s_{ir}) \\ &\leq T_{\boxtimes} \sum_{r=0}^{\infty} \bar{\beta}_{s_{ir}} h_2(s_{ir}) \leq T_{\boxtimes} \sum_{k=0}^{\infty} \bar{\beta}_k h_2(k) < \infty \end{aligned}$$

where T_{\boxtimes} is the constant in **A2** (i). Thus, the preceding inequality and (17) imply $\sum_{k=0}^{\infty} \bar{\beta}_k |\check{x}_i(k) - \bar{y}(\check{k}_i)| < \infty$.

Since $y_i(k) \in Y$ for all i and Y is convex, $\bar{y}(k) \in Y$. Then, from the non-expansiveness property of the convex projection operator

$$\begin{aligned} &|\bar{y}(k+1) - \bar{y}(k)| \\ &= \left| \frac{\sum_{i=1}^{n_2} (P_Y(\hat{y}_i(k) + \beta_{i,k} q_{2i}(k)) - P_Y(\bar{y}(k)))}{n_2} \right| \\ &\leq \frac{1}{n_2} \sum_{i=1}^{n_2} |\hat{y}_i(k) + \beta_{i,k} q_{2i}(k) - \bar{y}(k)| \\ &\leq h_2(k) + \bar{\beta}_k L. \end{aligned} \quad (18)$$

Based on (18), the non-increasingness of $\{\bar{\beta}_k\}$ and $\check{k}_i \geq k - T_{\boxtimes} + 1$, we also have

$$\begin{aligned} &\sum_{k=0}^{\infty} \bar{\beta}_k |\bar{y}(\check{k}_i) - \bar{y}(k)| \\ &\leq \sum_{k=0}^{\infty} \bar{\beta}_k \sum_{r=\check{k}_i}^{k-1} |\bar{y}(r) - \bar{y}(r+1)| \\ &\leq \sum_{k=0}^{\infty} \bar{\beta}_k \sum_{r=\check{k}_i}^{k-1} (h_2(r) + \bar{\beta}_r L) \\ &\leq \sum_{k=0}^{\infty} \bar{\beta}_k \sum_{r=k-T_{\boxtimes}+1}^{k-1} (h_2(r) + \bar{\beta}_r L) \\ &\leq \sum_{k=0}^{\infty} \bar{\beta}_k \sum_{r=k-T_{\boxtimes}+1}^{k-1} h_2(r) + \frac{(T_{\boxtimes}-1)L}{2} \sum_{k=0}^{\infty} \bar{\beta}_k^2 \\ &\quad + \frac{L}{2} \sum_{k=0}^{\infty} \sum_{r=k-T_{\boxtimes}+1}^{k-1} \bar{\beta}_r^2 \\ &\leq (T_{\boxtimes}-1) \sum_{k=0}^{\infty} \bar{\beta}_k h_2(k) + \frac{(T_{\boxtimes}-1)L}{2} \sum_{k=0}^{\infty} \bar{\beta}_k^2 \\ &\quad + \frac{(T_{\boxtimes}-1)L}{2} \sum_{k=0}^{\infty} \bar{\beta}_k^2 < \infty \end{aligned}$$

where $h_2(r) = \bar{\beta}_r = 0$, $r < 0$. Since $|\check{x}_i(k) - \bar{y}(k)| \leq |\check{x}_i(k) - \bar{y}(\check{k}_i)| + |\bar{y}(\check{k}_i) - \bar{y}(k)|$, the first conclusion follows. \square

Remark 5.1: From the proof we find that Lemma 5.10 still holds when the non-increasing condition of $\{\bar{\alpha}_k\}$ and $\{\bar{\beta}_k\}$ is replaced by that there are an integer $T^* > 0$ and $c^* > 0$ such that $\bar{\alpha}_{k+T^*} \leq c^* \bar{\alpha}_k$ and $\bar{\beta}_{k+T^*} \leq c^* \bar{\beta}_k$ for all k .

B. Proof of Theorem 4.1

We complete the proof by the following two steps.

Step 1: We first show that the states of (4) are bounded. Take $(x, y) \in X \times Y$. By (4) and (1)

$$\begin{aligned} |x_i(k+1) - x|^2 &\leq |\hat{x}_i(k) - \gamma_k q_{1i}(k) - x|^2 = |\hat{x}_i(k) - x|^2 \\ &\quad + 2\gamma_k \langle \hat{x}_i(k) - x, -q_{1i}(k) \rangle + \gamma_k^2 |q_{1i}(k)|^2. \end{aligned} \quad (19)$$

It is easy to see that $|\cdot|^2$ is a convex function from the convexity of $|\cdot|$ and the convexity of scalar function $h(c) = c^2$. From this and **A3** (ii), $|\hat{x}_i(k) - x|^2 \leq \sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k) |x_j(k) - x|^2$. Moreover, since $q_{1i}(k)$ is a subgradient of $f_i(\cdot, \check{x}_i(k))$ at $\hat{x}_i(k)$, $\langle x - \hat{x}_i(k), q_{1i}(k) \rangle \leq f_i(x, \check{x}_i(k)) - f_i(\hat{x}_i(k), \check{x}_i(k))$. Thus, based on (19) and **A4**,

$$\begin{aligned} |x_i(k+1) - x|^2 &\leq \sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k) |x_j(k) - x|^2 + L^2 \gamma_k^2 \\ &\quad + 2\gamma_k (f_i(x, \check{x}_i(k)) - f_i(\hat{x}_i(k), \check{x}_i(k))). \end{aligned} \quad (20)$$

Again employing **A4**, $|f_i(x, y_1) - f_i(x, y_2)| \leq L|y_1 - y_2|$, $|f_i(x_1, y) - f_i(x_2, y)| \leq L|x_1 - x_2|$, $\forall x, x_1, x_2 \in X, y, y_1, y_2 \in Y$. This imply

$$|f_i(x, \check{x}_i(k)) - f_i(x, \bar{y}(k))| \leq L|\check{x}_i(k) - \bar{y}(k)| \quad (21)$$

$$\begin{aligned} &|f_i(\hat{x}_i(k), \check{x}_i(k)) - f_i(\bar{x}(k), \bar{y}(k))| \\ &\leq L(|\hat{x}_i(k) - \bar{x}(k)| + |\check{x}_i(k) - \bar{y}(k)|) \\ &\leq L(h_1(k) + |\check{x}_i(k) - \bar{y}(k)|). \end{aligned} \quad (22)$$

Hence, by (20)–(22)

$$\begin{aligned} |x_i(k+1) - x|^2 &\leq \sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k) |x_j(k) - x|^2 \\ &\quad + 2\gamma_k (f_i(x, \bar{y}(k)) - f_i(\bar{x}(k), \bar{y}(k))) + L^2 \gamma_k^2 + 2L\gamma_k e_{i1}(k) \end{aligned} \quad (23)$$

where $e_{i1}(k) = h_1(k) + 2|\check{x}_i(k) - \bar{y}(k)|$.

It follows from the weight balance of $\mathcal{G}_1(k)$ and **A3** (ii) that $\sum_{i \in \mathcal{V}_1} a_{ij}(k) = 1$ for all $j \in \mathcal{V}_1$. Then, from (23), we have

$$\begin{aligned} \sum_{i=1}^{n_1} |x_i(k+1) - x|^2 &\leq \sum_{i=1}^{n_1} |x_i(k) - x|^2 + 2\gamma_k (U(x, \bar{y}(k)) \\ &\quad - U(\bar{x}(k), \bar{y}(k))) + n_1 L^2 \gamma_k^2 + 2L\gamma_k \sum_{i=1}^{n_1} e_{i1}(k). \end{aligned} \quad (24)$$

Analogously

$$\begin{aligned} \sum_{i=1}^{n_2} |y_i(k+1) - y|^2 &\leq \sum_{i=1}^{n_2} |y_i(k) - y|^2 + 2\gamma_k (U(\bar{x}(k), \bar{y}(k)) \\ &\quad - U(\bar{x}(k), y)) + n_2 L^2 \gamma_k^2 + 2L\gamma_k \sum_{i=1}^{n_2} e_{i2}(k) \end{aligned} \quad (25)$$

where $e_{i2}(k) = h_2(k) + 2|\tilde{y}_i(k) - \bar{x}(k)|$. Let $(x, y) = (x^*, y^*) \in X^* \times Y^*$, which is nonempty by **A1**. Denote $\xi(k, x^*, y^*) = \sum_{i=1}^{n_1} |x_i(k) - x^*|^2 + \sum_{i=1}^{n_2} |y_i(k) - y^*|^2$. Then adding (24) and (25) together leads to

$$\xi(k+1, x^*, y^*) \leq \xi(k, x^*, y^*) - 2\gamma_k \Upsilon(k) + (n_1 + n_2)L^2\gamma_k^2 + 2L\gamma_k \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} e_{i\ell}(k) \quad (26)$$

where

$$\begin{aligned} \Upsilon(k) &= U(\bar{x}(k), y^*) - U(x^*, \bar{y}(k)) \\ &= U(x^*, y^*) - U(x^*, \bar{y}(k)) + U(\bar{x}(k), y^*) - U(x^*, y^*) \\ &\geq 0 \end{aligned} \quad (27)$$

following from $U(x^*, y^*) - U(x^*, \bar{y}(k)) \geq 0$, $U(\bar{x}(k), y^*) - U(x^*, y^*) \geq 0$ for $k \geq 0$ since (x^*, y^*) is a saddle point of U on $X \times Y$. Moreover, by $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$ and Lemmas 5.9, 5.10

$$\sum_{k=0}^{\infty} \gamma_k \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} e_{i\ell}(k) < \infty. \quad (28)$$

Therefore, by virtue of $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$ again, (28), (26) and Lemma 5.1, $\lim_{k \rightarrow \infty} \xi(k, x^*, y^*)$ is a finite number, denoted as $\xi(x^*, y^*)$. Thus, the conclusion follows. \square

Step 2: We next show that the limit points of all agents satisfy certain objective function equations, and then prove the Nash equilibrium convergence under either of the two conditions: (i) and (ii).

As shows in *Step 1*, $(x_i(k), y_i(k))$, $k \geq 0$ are bounded. Moreover, it also follows from (26) that

$$\begin{aligned} 2 \sum_{r=0}^k \gamma_r \Upsilon(r) &\leq \xi(0, x^*, y^*) + (n_1 + n_2)L^2 \sum_{r=0}^k \gamma_r^2 \\ &\quad + 2L \sum_{r=0}^k \gamma_r \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} e_{i\ell}(r) \end{aligned}$$

and then by $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$ and (28) we have

$$0 \leq \sum_{k=0}^{\infty} \gamma_k \Upsilon(k) < \infty. \quad (29)$$

The stepsize condition $\sum_{k=0}^{\infty} \gamma_k = \infty$ and (29) imply $\liminf_{k \rightarrow \infty} \Upsilon(k) = 0$. As a result, there is a subsequence $\{k_r\}$ such that $\lim_{r \rightarrow \infty} U(x^*, \bar{y}(k_r)) = U(x^*, y^*)$ and $\lim_{r \rightarrow \infty} U(\bar{x}(k_r), y^*) = U(x^*, y^*)$. Let (\tilde{x}, \tilde{y}) be any limit pair of $\{(\bar{x}(k_r), \bar{y}(k_r))\}$ (noting that the finite limit pairs exist by the boundedness of system states). Because $U(x^*, \cdot)$, $U(\cdot, y^*)$ are continuous and the Nash equilibrium point (x^*, y^*) is taken from $X^* \times Y^*$ freely, the limit pair (\tilde{x}, \tilde{y}) must satisfy that for any $(x^*, y^*) \in X^* \times Y^*$

$$U(x^*, \tilde{y}) = U(\tilde{x}, y^*) = U(x^*, y^*). \quad (30)$$

We complete the proof by discussing the proposed two sufficient conditions: (i) and (ii).

- (i). For the strictly convex-concave function U , we claim that $X^* \times Y^*$ is a single-point set. If it contains two different points (x_1^*, y_1^*) and (x_2^*, y_2^*) (without loss of generality, assume $x_1^* \neq x_2^*$), it also contains point

(x_2^*, y_1^*) by Lemma 2.2. Thus, $U(x_1^*, y_1^*) \leq U(x, y_1^*)$ and $U(x_2^*, y_1^*) \leq U(x, y_1^*)$ for any $x \in X$, which yields a contradiction since $U(\cdot, y_1^*)$ is strictly convex and then the minimizer of $U(\cdot, y_1^*)$ is unique. Thus, $X^* \times Y^*$ contains only one single-point (denoted as (x^*, y^*)).

Then $\tilde{x} = x^*$, $\tilde{y} = y^*$ by (30). Consequently, each limit pair of $\{(\bar{x}(k_r), \bar{y}(k_r))\}$ is (x^*, y^*) , i.e., $\lim_{r \rightarrow \infty} \bar{x}(k_r) = x^*$ and $\lim_{r \rightarrow \infty} \bar{y}(k_r) = y^*$. By Lemma 5.9, $\lim_{r \rightarrow \infty} x_i(k_r) = x^*$, $i \in \mathcal{V}_1$ and $\lim_{r \rightarrow \infty} y_i(k_r) = y^*$, $i \in \mathcal{V}_2$. Moreover, $\lim_{k \rightarrow \infty} \xi(k, x^*, y^*) = \xi(x^*, y^*)$ as given in *Step 1*, so $\xi(x^*, y^*) = \lim_{r \rightarrow \infty} \xi(k_r, x^*, y^*) = 0$, which in return implies $\lim_{k \rightarrow \infty} x_i(k) = x^*$, $i \in \mathcal{V}_1$ and $\lim_{k \rightarrow \infty} y_i(k) = y^*$, $i \in \mathcal{V}_2$.

- (ii). In *Step 1*, we proved $\lim_{k \rightarrow \infty} \xi(k, x^*, y^*) = \xi(x^*, y^*)$ for any $(x^*, y^*) \in X^* \times Y^*$. We check the existence of the two limits $\lim_{k \rightarrow \infty} \bar{x}(k)$ and $\lim_{k \rightarrow \infty} \bar{y}(k)$. Let (x^+, y^+) be an interior point of $X^* \times Y^*$ for which $\mathbf{B}(x^+, \varepsilon) \subseteq X^*$ and $\mathbf{B}(y^+, \varepsilon) \subseteq Y^*$ for some $\varepsilon > 0$. Clearly, any two limit pairs (\hat{x}_1, \hat{y}_1) , (\hat{x}_2, \hat{y}_2) of $\{(\bar{x}(k), \bar{y}(k))\}$ must satisfy $n_1|\hat{x}_1 - x^+|^2 + n_2|\hat{y}_1 - y^+|^2 = n_1|\hat{x}_2 - x^+|^2 + n_2|\hat{y}_2 - y^+|^2$, $\forall x \in \mathbf{B}(x^+, \varepsilon)$, $y \in \mathbf{B}(y^+, \varepsilon)$. Take $y = y^+$. Then for any $x \in \mathbf{B}(x^+, \varepsilon)$

$$n_1|\hat{x}_1 - x|^2 = n_1|\hat{x}_2 - x|^2 + n_2(|\hat{y}_2 - y^+|^2 - |\hat{y}_1 - y^+|^2). \quad (31)$$

Taking the gradient with respect to x on both sides of (31) yields $2n_1(x - \hat{x}_1) = 2n_1(x - \hat{x}_2)$, namely, $\hat{x}_1 = \hat{x}_2$. Similarly, we can show $\hat{y}_1 = \hat{y}_2$. Thus, the limits, $\lim_{k \rightarrow \infty} \bar{x}(k) = \hat{x} \in X$ and $\lim_{k \rightarrow \infty} \bar{y}(k) = \hat{y} \in Y$, exist. Based on Lemma 5.9 (ii), $\lim_{k \rightarrow \infty} x_i(k) = \hat{x}$, $i \in \mathcal{V}_1$ and $\lim_{k \rightarrow \infty} y_i(k) = \hat{y}$, $i \in \mathcal{V}_2$.

We claim that $(\hat{x}, \hat{y}) \in X^* \times Y^*$. First it follows from (24) that, for any $x \in X$, $\sum_{k=0}^{\infty} \gamma_k \sum_{i=1}^{n_1} (U(\bar{x}(k), \bar{y}(k)) - U(x, \bar{y}(k))) < \infty$. Moreover, recalling $\sum_{k=0}^{\infty} \gamma_k = \infty$, we obtain

$$\liminf_{k \rightarrow \infty} (U(\bar{x}(k), \bar{y}(k)) - U(x, \bar{y}(k))) \leq 0. \quad (32)$$

Then $U(\hat{x}, \hat{y}) - U(x, \hat{y}) \leq 0$ for all $x \in X$ due to $\lim_{k \rightarrow \infty} \bar{x}(k) = \hat{x}$, $\lim_{k \rightarrow \infty} \bar{y}(k) = \hat{y}$, the continuity of U , and (32). Similarly, we can show $U(\hat{x}, \hat{y}) - U(\hat{x}, y) \leq 0$ for all $y \in Y$. Thus, (\hat{x}, \hat{y}) is a saddle point of U on $X \times Y$, which implies $(\hat{x}, \hat{y}) \in X^* \times Y^*$.

Thus, the proof is completed. \square

C. Proof of Theorem 4.3

(Necessity) Let (x^*, y^*) be the unique saddle point of strictly convex-concave function U on $X \times Y$. Take $\mu = (\mu_1, \dots, \mu_{n_1})' \in \mathcal{S}_{n_1}^+$. By Lemma 5.3 again, there is a stochastic matrix A_1 such that $\mu' A_1 = \mu'$ and \mathcal{G}_{A_1} is strongly connected. Let $\mathcal{G}_1 = \{\mathcal{G}_1(k)\}$ be the graph sequence of algorithm (4) with $\mathcal{G}_1(k) = \mathcal{G}_{A_1}$ for $k \geq 0$ and A_1 being the adjacency matrix of $\mathcal{G}_1(k)$. Clearly, \mathcal{G}_1 is UJSC. On one hand, by Proposition 4.2, all agents converge to the unique saddle point of $\sum_{i=1}^{n_1} \mu_i f_i$ on $X \times Y$. On the other hand, the necessity condition states that $\lim_{k \rightarrow \infty} x_i(k) = x^*$ and $\lim_{k \rightarrow \infty} y_i(k) = y^*$ for $i = 1, \dots, n_1$. Therefore, (x^*, y^*) is a saddle point of $\sum_{i=1}^{n_1} \mu_i f_i$ on $X \times Y$.

Because μ is taken from $\mathcal{S}_{n_1}^+$ freely, we have that, for any $\mu \in \mathcal{S}_{n_1}^+$, $x \in X$, $y \in Y$

$$\sum_{i=1}^{n_1} \mu_i f_i(x^*, y) \leq \sum_{i=1}^{n_1} \mu_i f_i(x^*, y^*) \leq \sum_{i=1}^{n_1} \mu_i f_i(x, y^*). \quad (33)$$

We next show by contradiction that, given any $i = 1, \dots, n_1$, $f_i(x^*, y^*) \leq f_i(x, y^*)$ for all $x \in X$. Hence suppose there are i_0 and $\hat{x} \in X$ such that $f_{i_0}(x^*, y^*) > f_{i_0}(\hat{x}, y^*)$. Let $\mu_i, i \neq i_0$ be sufficiently small such that $|\sum_{i \neq i_0} \mu_i f_i(x^*, y^*)| < (\mu_{i_0}/2)(f_{i_0}(x^*, y^*) - f_{i_0}(\hat{x}, y^*))$ and $|\sum_{i \neq i_0} \mu_i f_i(\hat{x}, y^*)| < (\mu_{i_0}/2)(f_{i_0}(x^*, y^*) - f_{i_0}(\hat{x}, y^*))$. Consequently, $\sum_{i=1}^{n_1} \mu_i f_i(x^*, y^*) > (\mu_{i_0}/2)(f_{i_0}(x^*, y^*) + f_{i_0}(\hat{x}, y^*)) > \sum_{i=1}^{n_1} \mu_i f_i(\hat{x}, y^*)$, which contradicts the second inequality of (33). Thus, $f_i(x^*, y^*) \leq f_i(x, y^*)$ for all $x \in X$. Analogously, we can show from the first inequality of (33) that for each i , $f_i(x^*, y) \leq f_i(x^*, y^*)$ for all $y \in Y$. Thus, we obtain that $f_i(x^*, y) \leq f_i(x^*, y^*) \leq f_i(x, y^*)$, $\forall x \in X, y \in Y$, or equivalently, (x^*, y^*) is the saddle point of f_i on $X \times Y$.

(Sufficiency) Let (x^*, y^*) be the unique saddle point of $f_i, i = 1, \dots, n_1$ on $X \times Y$. Similar to (23), we have

$$\begin{aligned} |y_i(k+1) - y^*|^2 &\leq \sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k) |y_j(k) - y^*|^2 \\ &+ 2\gamma_k (f_i(\bar{x}(k), \bar{y}(k)) - f_i(\bar{x}(k), y^*)) + L^2\gamma_k^2 + 2L\gamma_k u_2(k) \end{aligned} \quad (34)$$

where $u_2(k) = 2h_1(k) + h_2(k)$. Merging (23) and (34) gives

$$\begin{aligned} \zeta(k+1) &\leq \zeta(k) + 2\gamma_k \max_{1 \leq i \leq n_1} (f_i(x^*, \bar{y}(k)) - f_i(\bar{x}(k), y^*)) \\ &+ 2L^2\gamma_k^2 + 2L\gamma_k (u_1(k) + u_2(k)) \\ &= \zeta(k) + 2\gamma_k \max_{1 \leq i \leq n_1} (f_i(x^*, \bar{y}(k)) - f_i(x^*, y^*) \\ &\quad + f_i(x^*, y^*) - f_i(\bar{x}(k), y^*)) + 2L^2\gamma_k^2 \\ &+ 6L\gamma_k (h_1(k) + h_2(k)) \end{aligned} \quad (35)$$

where $\zeta(k) = \max_{1 \leq i \leq n_1} (|x_i(k) - x^*|^2 + |y_i(k) - y^*|^2)$, $u_1(k) = h_1(k) + 2h_2(k)$. Since $f_i(x^*, \bar{y}(k)) - f_i(x^*, y^*) \leq 0$ and $f_i(x^*, y^*) - f_i(\bar{x}(k), y^*) \leq 0$ for all i, k , the second term in (35) is non-positive. By Lemma 5.1

$$\lim_{k \rightarrow \infty} \zeta(k) = \zeta^* \geq 0 \quad (36)$$

for a finite number ζ^* , which implies that $(x_i(k), y_i(k)), k \geq 0$ are bounded.

Denote $\wp(k) = \min_{1 \leq i \leq n_1} (f_i(x^*, y^*) - f_i(x^*, \bar{y}(k)) + f_i(\bar{x}(k), y^*) - f_i(x^*, y^*))$. From (35), we also have

$$\begin{aligned} 0 &\leq 2 \sum_{l=0}^k \gamma_l \wp(l) \leq \zeta(0) - \zeta(k+1) + 2L^2 \sum_{l=0}^k \gamma_l^2 \\ &+ 6L \sum_{l=0}^k \gamma_l (h_1(l) + h_2(l)), \quad k \geq 0 \end{aligned}$$

and hence $0 \leq \sum_{k=0}^{\infty} \gamma_k \wp(k) < \infty$. The stepsize condition $\sum_{k=0}^{\infty} \gamma_k = \infty$ implies that there is a subsequence $\{k_r\}$ such that

$$\lim_{r \rightarrow \infty} \wp(k_r) = 0.$$

We assume without loss of generality that $\lim_{r \rightarrow \infty} \bar{x}(k_r) = \hat{x}$, $\lim_{r \rightarrow \infty} \bar{y}(k_r) = \hat{y}$ for some \hat{x}, \hat{y} (otherwise we can find a subsequence of $\{k_r\}$ recalling the boundedness of system states). Due to the finite number of agents and the continuity of f_i s, there exists i_0 such that $f_{i_0}(x^*, y^*) = f_{i_0}(x^*, \hat{y})$ and $f_{i_0}(\hat{x}, y^*) = f_{i_0}(x^*, y^*)$. It follows from the strict convexity-concavity of f_{i_0} that $\hat{x} = x^*, \hat{y} = y^*$.

Since the consensus is achieved within two subnetworks, $\lim_{r \rightarrow \infty} x_i(k_r) = x^*$ and $\lim_{r \rightarrow \infty} y_i(k_r) = y^*$, which leads to $\zeta^* = 0$ based on (36). Thus, the conclusion follows. \square

D. Proof of Theorem 4.4

We design the stepsizes $\alpha_{i,k}$ and $\beta_{i,k}$ as that given before Remark 4.4. First by Lemma 5.5 (i) and (ii), the limit $\lim_{r \rightarrow \infty} \Phi^\ell(r, k) = \mathbf{1}(\phi^\ell(k))'$ exists for each k . Let (x^*, y^*) be the unique Nash equilibrium. From (23) we have

$$\begin{aligned} |x_i(k+1) - x^*|^2 &\leq \sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k) |x_j(k) - x^*|^2 \\ &+ 2\alpha_{i,k} (f_i(x^*, \bar{y}(k)) - f_i(\bar{x}(k), \bar{y}(k))) \\ &+ L^2\alpha_{i,k}^2 + 2L\alpha_{i,k} e_{i1}(k). \end{aligned} \quad (37)$$

Analogously

$$\begin{aligned} |y_i(k+1) - y^*|^2 &\leq \sum_{j \in \mathcal{N}_i^2(k)} a_{ij}(k) |y_j(k) - y^*|^2 \\ &+ 2\beta_{i,k} (g_i(\bar{x}(k), \bar{y}(k)) - g_i(\bar{x}(k), y^*)) \\ &+ L^2\beta_{i,k}^2 + 2L\beta_{i,k} e_{i2}(k). \end{aligned} \quad (38)$$

Denote

$$\begin{aligned} \Lambda_k^1 &= \text{diag} \left\{ \frac{1}{\alpha_k^1}, \dots, \frac{1}{\alpha_k^{n_1}} \right\}, \quad \Lambda_k^2 = \text{diag} \left\{ \frac{1}{\beta_k^1}, \dots, \frac{1}{\beta_k^{n_2}} \right\}; \\ \psi^\ell(k) &= (\psi_1^\ell(k), \dots, \psi_{n_\ell}^\ell(k))', \quad \ell = 1, 2 \\ \psi_i^1(k) &= |x_i(k) - x^*|^2, \quad \psi_i^2(k) = |y_i(k) - y^*|^2; \\ \vartheta^\ell(k) &= (\vartheta_1^\ell(k), \dots, \vartheta_{n_\ell}^\ell(k))' \\ \vartheta_i^1(k) &= f_i(\bar{x}(k), \bar{y}(k)) - f_i(x^*, \bar{y}(k)) \\ \vartheta_i^2(k) &= g_i(\bar{x}(k), y^*) - g_i(\bar{x}(k), \bar{y}(k)); \\ e_\ell(k) &= (e_{1\ell}(k), \dots, e_{n_\ell\ell}(k))'. \end{aligned}$$

Then it follows from (37) and (38) that

$$\psi^\ell(k+1) \leq A_\ell(k) \psi^\ell(k) - 2\gamma_k \Lambda_k^\ell \vartheta^\ell(k) + \delta_*^2 L^2 \gamma_k^2 \mathbf{1} + 2\delta_* L \gamma_k e_\ell(k)$$

where $\delta_* = \sup_{i,k} \{1/\alpha_k^i, 1/\beta_k^i\}$. By Lemma 5.5 (iii), $\alpha_k^i \geq \eta^{(n_1-1)T_1}$, $\beta_k^i \geq \eta^{(n_2-1)T_2}$, $\forall i, k$ and then δ_* is a finite number. Therefore

$$\begin{aligned} \psi^\ell(k+1) &\leq \Phi^\ell(k, r) \psi^\ell(r) - 2 \sum_{s=r}^{k-1} \gamma_s \Phi^\ell(k, s+1) \Lambda_s^\ell \vartheta^\ell(s) \\ &+ \delta_*^2 L^2 \sum_{s=r}^k \gamma_s^2 \mathbf{1} + 2\delta_* L \sum_{s=r}^{k-1} \gamma_s \Phi^\ell(k, s+1) e_\ell(s) \\ &- 2\gamma_k \Lambda_k^\ell \vartheta^\ell(k) + 2\delta_* L \gamma_k e_\ell(k). \end{aligned} \quad (39)$$

Then (39) can be written as

$$\begin{aligned}
 & \psi^\ell(k+1) \\
 & \leq \Phi^\ell(k, r)\psi^\ell(r) - 2 \sum_{s=r}^{k-1} \gamma_s \mathbf{1} (\phi^\ell(s+1))' \Lambda_s^\ell \vartheta^\ell(s) \\
 & \quad + \delta_*^2 L^2 \sum_{s=r}^k \gamma_s^2 \mathbf{1} + 2\delta_* L \sum_{s=r}^{k-1} \gamma_s \mathbf{1} (\phi^\ell(s+1))' e_\ell(s) \\
 & \quad + 2 \sum_{s=r}^{k-1} \gamma_s \left(\mathbf{1} (\phi^\ell(s+1))' - \Phi^\ell(k, s+1) \right) \Lambda_s^\ell \vartheta^\ell(s) \\
 & \quad - 2\gamma_k \Lambda_k^\ell \vartheta^\ell(k) + 2\delta_* L \gamma_k e_\ell(k) \\
 & \quad + 2\delta_* L \sum_{s=r}^{k-1} \gamma_s \left(\Phi^\ell(k, s+1) - \mathbf{1} (\phi^\ell(s+1))' \right) e_\ell(s). \quad (40)
 \end{aligned}$$

The subsequent proof is given as follows. First, we show that the designed stepsizes (7) can eliminate the imbalance caused by the weight-unbalanced graphs [see the second term in (40)], and then we prove that all the terms from the third one to the last one in (40) is summable based on the geometric rate convergence of transition matrices. Finally, we show the desired convergence based on inequality (40), as (26) for the weight-balance case in Theorem 4.1.

Clearly, $\mathbf{1}(\phi^\ell(s+1))' \Lambda_s^\ell = \mathbf{1}'$, $\ell = 1, 2$. From Lemma 5.5 (iv) we also have that $|\Phi^\ell(k, s)_{ij} - \phi_j^\ell(s)| \leq C\rho^{k-s}$ for $\ell = 1, 2$, every $i = 1, \dots, n_\ell$, $s \geq 0$, $k \geq s$, and $j = 1, \dots, n_\ell$, where $C = \max\{C_1, C_2\}$, $0 < \rho = \max\{\rho_1, \rho_2\} < 1$. Moreover, by **A4**, $|\vartheta_i^1(s)| = |f_i(\bar{x}(s), \bar{y}(s)) - f_i(x^*, \bar{y}(s))| \leq L|\bar{x}(s) - x^*|$ for $i \in \mathcal{V}_1$, and $|\vartheta_i^2(s)| = |f_i(\bar{x}(s), y^*) - f_i(\bar{x}(s), \bar{y}(s))| \leq L|\bar{y}(s) - y^*|$ for $i \in \mathcal{V}_2$. Based on these observations, multiplying $(1/n_\ell)\mathbf{1}'$ on the both sides of (40) and taking the sum over $\ell = 1, 2$ yield

$$\begin{aligned}
 \sum_{\ell=1}^2 \frac{1}{n_\ell} \mathbf{1}' \psi^\ell(k+1) & \leq \sum_{\ell=1}^2 \frac{1}{n_\ell} \mathbf{1}' \Phi^\ell(k, r) \psi^\ell(r) \\
 & \quad - 2 \sum_{s=r}^{k-1} \gamma_s \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} \vartheta_i^\ell(s) + 2\delta_*^2 L^2 \sum_{s=r}^k \gamma_s^2 \\
 & \quad + 2\delta_* L \sum_{s=r}^{k-1} \gamma_s \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} e_{i\ell}(s) \\
 & \quad + 2CL\delta_*(n_1+n_2) \sum_{s=r}^{k-1} \rho^{k-s-1} \gamma_s \varsigma(s) \\
 & \quad + 2L\delta_* \gamma_k \varsigma(k) + 2\delta_* L \gamma_k \sum_{\ell=1}^2 \frac{1}{n_\ell} \sum_{i=1}^{n_\ell} e_{i\ell}(k) \\
 & \quad + 2CL\delta_* \sum_{s=r}^{k-1} \gamma_s \rho^{k-s-1} \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} e_{i\ell}(s) \quad (41)
 \end{aligned}$$

$$\begin{aligned}
 & =: \sum_{\ell=1}^2 \frac{1}{n_\ell} \mathbf{1}' \Phi^\ell(k, r) \psi^\ell(r) \\
 & \quad - 2 \sum_{s=r}^{k-1} \gamma_s \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} \vartheta_i^\ell(s) + \varrho(k, r) \quad (42)
 \end{aligned}$$

where $\varsigma(s) = \max\{|x_i(s) - x^*|, i \in \mathcal{V}_1, |y_j(s) - y^*|, j \in \mathcal{V}_2\}$, $\varrho(k, r)$ is the sum of all terms from the third one to the last one in (41).

We next show $\lim_{r \rightarrow \infty} \sup_{k \geq r} \varrho(k, r) = 0$. First by Lemmas 5.9, 5.10, and Remark 5.1, $\sum_{s=r}^{\infty} \gamma_s \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} e_{i\ell}(s) < \infty$ and hence $\lim_{k \rightarrow \infty} \gamma_k \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} e_{i\ell}(k) = 0$. It follows from $0 < \rho < 1$ that for each k

$$\sum_{s=r}^{k-1} \gamma_s \rho^{k-s-1} \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} e_{i\ell}(s) \leq \sum_{s=r}^{\infty} \gamma_s \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} e_{i\ell}(s) < \infty.$$

Moreover, by Lemma 5.7, $\lim_{r \rightarrow \infty} \gamma_r \varsigma(r) = 0$, which implies $\lim_{r \rightarrow \infty} \sup_{k \geq r+1} \sum_{s=r}^{k-1} \rho^{k-s-1} \gamma_s \varsigma(s) = 0$ along with $\sum_{s=r}^{k-1} \rho^{k-s-1} \gamma_s \varsigma(s) \leq (1/(1-\rho)) \sup_{s \geq r} \gamma_s \varsigma(s)$. From the preceding zero limit results, we have $\lim_{r \rightarrow \infty} \sup_{k \geq r} \varrho(k, r) = 0$. Then from (42) $\sum_{s=r}^{\infty} \gamma_s \sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} \vartheta_i^\ell(s) < \infty$. Clearly, from (27) $\sum_{\ell=1}^2 \sum_{i=1}^{n_\ell} \vartheta_i^\ell(s) = \Upsilon(s) \geq 0$. By the similar procedures to the proof of Theorem 4.1, we can show that there is a subsequence $\{k_l\}$ such that $\lim_{l \rightarrow \infty} \bar{x}(k_l) = x^*$, $\lim_{l \rightarrow \infty} \bar{y}(k_l) = y^*$.

Now let us show $\lim_{k \rightarrow \infty} \sum_{\ell=1}^2 (1/n_\ell) \mathbf{1}' \psi^\ell(k) = 0$. First it follows from $\lim_{r \rightarrow \infty} \sup_{k \geq r} \varrho(k, r) = 0$ that, for any $\varepsilon > 0$, there is a sufficiently large l_0 such that when $l \geq l_0$, $\sup_{k \geq k_l} \varrho(k, k_l) \leq \varepsilon$. Moreover, since the consensus is achieved within the two subnetworks, l_0 can be selected sufficiently large such that $|x_i(k_{l_0}) - x^*| \leq \varepsilon$ and $|y_i(k_{l_0}) - y^*| \leq \varepsilon$ for each i . With (42), we have that, for each $k \geq k_l$

$$\begin{aligned}
 \sum_{\ell=1}^2 \frac{1}{n_\ell} \mathbf{1}' \psi^\ell(k+1) & \leq \sum_{\ell=1}^2 \frac{1}{n_\ell} \mathbf{1}' \Phi^\ell(k, k_l) \psi^\ell(k_l) \\
 & \quad + \sup_{k \geq k_l} \varrho(k, k_l) \leq 2\varepsilon^2 + \varepsilon
 \end{aligned}$$

which implies $\lim_{k \rightarrow \infty} \sum_{\ell=1}^2 (1/n_\ell) \mathbf{1}' \psi^\ell(k) = 0$. Therefore, $\lim_{k \rightarrow \infty} x_i(k) = x^*$, $i \in \mathcal{V}_1$ and $\lim_{k \rightarrow \infty} y_i(k) = y^*$, $i \in \mathcal{V}_2$. Thus, the proof is completed. \square

E. Proof of Theorem 4.5

(i). In this case we design a dynamics for auxiliary states $\alpha^i = (\alpha_{n_1}^i, \dots, \alpha_{n_1}^i)' \in \mathbb{R}^{n_1}$ for $i \in \mathcal{V}_1$ and $\beta^i = (\beta_{n_2}^i, \dots, \beta_{n_2}^i)' \in \mathbb{R}^{n_2}$ for $i \in \mathcal{V}_2$ to estimate the respective desired stepsizes

$$\begin{cases} \alpha^i(k+1) = \sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k) \alpha^j(k), & k \geq 0 \\ \beta^i(k+1) = \sum_{j \in \mathcal{N}_i^2(k)} a_{ij}(k) \beta^j(k), & k \geq 0 \end{cases} \quad (43)$$

with the initial value $\alpha_i^i(0) = 1$, $\alpha_j^i(0) = 0$, $\forall j \neq i$; $\beta_i^i(0) = 1$, $\beta_j^i(0) = 0$, $\forall j \neq i$.

Then for each i and k , let $\hat{\alpha}_k^i = \alpha_i^i(k)$, $\hat{\beta}_k^i = \beta_i^i(k)$. Clearly, (10) holds.

First by **A3** (i) and algorithm (43), $\alpha_i^i(k) \geq \eta^k > 0$ and $\beta_i^i(k) \geq \eta^k > 0$ for each k , which guarantees that the stepsize selection rule (9) is well-defined. Let $\phi^\ell = (\phi_1^\ell, \dots, \phi_{n_\ell}^\ell)'$ be the common left eigenvector of $A_\ell(r)$, $r \geq 0$ associated with eigenvalue one, where $\sum_{i=1}^{n_\ell} \phi_i^\ell = 1$. According to Lemma 5.6, $\lim_{r \rightarrow \infty} \Phi^\ell(r, k) = \lim_{r \rightarrow \infty} A_\ell(r) \cdots A_\ell(k) = \mathbf{1}(\phi^\ell)'$ for each k . As a result, $\alpha_k^i = \phi_i^1$, $i = 1, \dots, n_1$; $\beta_k^i = \phi_i^2$, $i = 1, \dots, n_2$ for all k .

Let $\theta(k) = ((\alpha^1(k))', \dots, (\alpha^{n_1}(k))')'$. From (43) we have

$$\theta(k+1) = (A_1(k) \otimes I_{n_1}) \theta(k)$$

and then $\lim_{k \rightarrow \infty} \theta(k) = \lim_{k \rightarrow \infty} (\Phi^1(k, 0) \otimes I_{n_1}) \theta(0) = (\mathbf{1}(\phi^1)' \otimes I_{n_1}) \theta(0) = \mathbf{1} \otimes \phi^1$. Therefore, $\lim_{k \rightarrow \infty} \alpha_i^i(k) = \phi_i^1$ for $i \in \mathcal{V}_1$. Similarly, $\lim_{k \rightarrow \infty} \beta_i^i(k) = \phi_i^2$ for $i \in \mathcal{V}_2$. Since $\alpha_k^i = \phi_i^1$ and $\beta_k^i = \phi_i^2$ for all k , (11) holds. Moreover, the above convergence is achieved with a geometric rate by Lemma 5.5. Without loss of generality, suppose $|\alpha_i^i(k) - \phi_i^1| \leq \bar{C} \bar{\rho}^k$ and $|\beta_i^i(k) - \phi_i^2| \leq \bar{C} \bar{\rho}^k$ for some $\bar{C} > 0$, $0 < \bar{\rho} < 1$, and all i, k .

The only difference between the models in Theorem 4.4 and the current one is that the terms α_k^i and β_k^i (equal to ϕ_i^1 and ϕ_i^2 in case (i), respectively) in stepsize selection rule (7) are replaced with $\hat{\alpha}_k^i$ and $\hat{\beta}_k^i$ (equal to $\alpha_i^i(k)$ and $\beta_i^i(k)$, respectively) in stepsize selection rule (9). We can find that all lemmas involved in the proof of Theorem 4.4 still hold under the new stepsize selection rule (9). Moreover, all the analysis is almost the same as that in Theorem 4.4 except that the new stepsize selection rule will yield an error term (denoted as $\varpi^\ell(k, r)$) on the right-hand side of (39). In fact

$$\varpi^\ell(k, r) = 2 \sum_{s=r}^{k-1} \gamma_s \Phi^\ell(k, s+1) \varpi_s^\ell \vartheta^\ell(s) + 2\gamma_k \varpi_k^\ell \vartheta^\ell(k)$$

where $\varpi_s^1 = \text{diag}\{1/\phi_1^1 - 1/\alpha_1^1(s), \dots, 1/\phi_{n_1}^1 - 1/\alpha_{n_1}^1(s)\}$, $\varpi_s^2 = \text{diag}\{1/\phi_1^2 - 1/\beta_1^2(s), \dots, 1/\phi_{n_2}^2 - 1/\beta_{n_2}^2(s)\}$. Moreover, since $\lim_{s \rightarrow \infty} \alpha_i^i(s) = \phi_i^1$, $\alpha_i^i(s) \geq \phi_i^1/2 \geq \eta^{(n_1-1)T_1}/2$

$$\left| \frac{1}{\alpha_i^i(s)} - \frac{1}{\phi_i^1} \right| = \left| \frac{\alpha_i^i(s) - \phi_i^1}{\alpha_i^i(s) \phi_i^1} \right| \leq \frac{2|\alpha_i^i(s) - \phi_i^1|}{(\eta^{(n_1-1)T_1})^2} \leq \frac{2\bar{C}\bar{\rho}^s}{\eta^{2(n_1-1)T_1}}$$

for a sufficiently large s . Analogously, $|(1/\beta_i^i(s)) - (1/\phi_i^2)| \leq (2\bar{C}\bar{\rho}^s/\eta^{2(n_2-1)T_2})$. Then for a sufficiently large r and any $k \geq r+1$

$$\begin{aligned} & \left| \sum_{\ell=1}^2 \frac{1}{n_\ell} \mathbf{1}' \varpi^\ell(k, r) \right| \\ & \leq 4\bar{C}L\varepsilon_1 \sum_{s=r}^{k-1} \gamma_s \bar{\rho}^s \max_{i,j} \{|x_i(s) - x^*|, |y_j(s) - y^*|\} \\ & \leq 4\bar{C}L\varepsilon_1 \varepsilon_2 \sum_{s=r}^{k-1} \bar{\rho}^s \leq 4\bar{C}L\varepsilon_1 \varepsilon_2 \bar{\rho}^r / (1 - \bar{\rho}) \end{aligned} \quad (44)$$

where $\varepsilon_1 = \max\{1/\eta^{2(n_1-1)T_1}, 1/\eta^{2(n_2-1)T_2}\}$, $\varepsilon_2 = \sup_s \{\gamma_s \max_{i,j} \{|x_i(s) - x^*|, |y_j(s) - y^*|\}\} < \infty$ due to $\lim_{s \rightarrow \infty} \gamma_s \max_{i,j} \{|x_i(s) - x^*|, |y_j(s) - y^*|\} = 0$ by Lemma 5.7. From the proof of Theorem 4.4, we can find that the relation (44) makes all the arguments hold and then a Nash equilibrium is achieved for case (i).

(ii). Here we design a dynamics for the auxiliary states $\alpha^{(\nu)i} = (\alpha_1^{(\nu)i}, \dots, \alpha_{n_1}^{(\nu)i})'$, $\nu = 0, \dots, p^1 - 1$ for $i \in \mathcal{V}_1$ and $\beta^{(\nu)i} = (\beta_1^{(\nu)i}, \dots, \beta_{n_2}^{(\nu)i})'$, $\nu = 0, \dots, p^2 - 1$ for $i \in \mathcal{V}_2$ to estimate the respective desired stepsizes:

$$\begin{cases} \alpha^{(\nu)i}(s+1) = \sum_{j \in \mathcal{N}_1^1(s)} a_{ij}(s) \alpha^{(\nu)j}(s) \\ \beta^{(\nu)i}(s+1) = \sum_{j \in \mathcal{N}_2^2(s)} a_{ij}(s) \beta^{(\nu)j}(s), \end{cases} \quad s \geq \nu + 1 \quad (45)$$

with the initial value $\alpha_i^{(\nu)i}(\nu+1) = 1$, $\alpha_j^{(\nu)i}(\nu+1) = 0$, $j \neq i$; $\beta_i^{(\nu)i}(\nu+1) = 1$, $\beta_j^{(\nu)i}(\nu+1) = 0$, $j \neq i$.

Then, for each $r \geq 0$, let $\hat{\alpha}_{rp^1+\nu}^i = \alpha_i^{(\nu)i}(rp^1 + \nu)$ for $i \in \mathcal{V}_1$, $\nu = 0, \dots, p^1 - 1$; let $\hat{\beta}_{rp^2+\nu}^i = \beta_i^{(\nu)i}(rp^2 + \nu)$ for $i \in \mathcal{V}_2$, $\nu = 0, \dots, p^2 - 1$.

Note that **A2** implies that the union graphs $\bigcup_{s=0}^{p^\ell-1} \mathcal{G}_{A_s^\ell}$, $\ell = 1, 2$ are strongly connected. Let $\phi^{\ell(0)}$ be the Perron vector of $\lim_{r \rightarrow \infty} \Phi^\ell(rp^\ell - 1, 0)$, i.e., $\lim_{r \rightarrow \infty} \Phi^\ell(rp^\ell - 1, 0) = \lim_{r \rightarrow \infty} (A_\ell^{p^\ell-1} \dots A_\ell^0)^r = \mathbf{1}(\phi^{\ell(0)})'$. Then for $\nu = 1, \dots, p^\ell - 1$

$$\begin{aligned} & \lim_{r \rightarrow \infty} \Phi^\ell(rp^\ell + \nu - 1, \nu) \\ & = \lim_{r \rightarrow \infty} \left(A_\ell^{\nu-1} \dots A_\ell^0 A_\ell^{p^\ell-1} \dots A_\ell^{\nu+1} A_\ell^\nu \right)^r \\ & = \lim_{r \rightarrow \infty} \left(A_\ell^{p^\ell-1} \dots A_\ell^0 \right)^r A_\ell^{p^\ell-1} \dots A_\ell^{\nu+1} A_\ell^\nu \\ & = \mathbf{1}(\phi^{\ell(0)})' A_\ell^{p^\ell-1} \dots A_\ell^{\nu+1} A_\ell^\nu = \mathbf{1}(\phi^{\ell(\nu)})'. \end{aligned} \quad (46)$$

Consequently, for each $r \geq 0$, $\alpha_{rp^1+\nu}^i = \phi_i^{1(\nu+1)}$, $\nu = 0, 1, \dots, p^1 - 2$, $\alpha_{rp^1+p^1-1}^i = \phi_i^{1(0)}$. Moreover, from (45) and (46) we obtain that for $\nu = 0, 1, \dots, p^1 - 1$

$$\begin{aligned} & \lim_{r \rightarrow \infty} \theta^\nu(r) \\ & = \left(\lim_{r \rightarrow \infty} \Phi^1(r, \nu+1) \otimes I_{n_1} \right) \theta^\nu(\nu+1) \\ & = \left(\lim_{r \rightarrow \infty} \Phi^1(r, 0) A_\ell^{p^1-1} \dots A_\ell^{\nu+1} \otimes I_{n_1} \right) \theta^\nu(\nu+1) \\ & = (\mathbf{1}(\phi^{1(\nu+1)})' \otimes I_{n_1}) \theta^\nu(\nu+1) \end{aligned}$$

where $\theta^\nu = ((\alpha^{(\nu)1})', \dots, (\alpha^{(\nu)n_1})')'$, $\phi^{1(p^1)} = \phi^{1(0)}$. Then $\lim_{r \rightarrow \infty} \alpha_i^{(\nu)i}(r) = \phi_i^{1(\nu+1)}$ for $i \in \mathcal{V}_1$. Hence

$$\lim_{r \rightarrow \infty} (\hat{\alpha}_{rp^1+\nu}^i - \alpha_{rp^1+\nu}^i) = 0, \quad \nu = 0, \dots, p^1 - 1.$$

Analogously, we have $\lim_{r \rightarrow \infty} (\hat{\beta}_{rp^2+\nu}^i - \beta_{rp^2+\nu}^i) = 0$, $\nu = 0, \dots, p^2 - 1$. Moreover, the above convergence is achieved with a geometric rate. Similar to the proof of case (i), we can prove case (ii). Thus, the conclusion follows. \square

VI. NUMERICAL EXAMPLES

In this section, we provide examples to illustrate the obtained results in both the balanced and unbalanced graph cases.

Consider a network of five agents, where $n_1 = 3$, $n_2 = 2$, $m_1 = m_2 = 1$, $X = Y = [-5, 5]$, $f_1(x, y) = x^2 - (20 - x^2)(y - 1)^2$, $f_2(x, y) = |x - 1| - |y|$, $f_3(x, y) = (x - 1)^4 - 2y^2$ and $g_1(x, y) = (x - 1)^4 - |y| - (5/4)y^2 - (1/2)(20 - x^2)(y - 1)^2$, $g_2(x, y) = x^2 + |x - 1| - (3/4)y^2 - (1/2)(20 - x^2)(y - 1)^2$. Notice that $\sum_{i=1}^3 f_i = \sum_{i=1}^2 g_i$ and all objective functions are strictly convex-concave on $[-5, 5] \times [-5, 5]$. The unique saddle point of the sum objective function $g_1 + g_2$ on $[-5, 5] \times [-5, 5]$ is (0.6102, 0.8844).

Take initial conditions $x_1(0) = 2$, $x_2(0) = -0.5$, $x_3(0) = -1.5$ and $y_1(0) = 1$, $y_2(0) = 0.5$. When $\hat{x}_2(k) = 1$, we take $q_{12}(k) = 1 \in \partial_x f_2(1, \hat{x}_2(k)) = [-1, 1]$; when $\hat{y}_1(k) = 0$, we take $q_{21}(k) = -1 + (20 - \hat{y}_1^2(k)) \in \partial_y g_1(\hat{y}_1(k), 0) = \{r + (20 - \hat{y}_1^2(k)) | -1 \leq r \leq 1\}$. Let $\gamma_k = 1/(k + 50)$, $k \geq 0$, which satisfies **A5**.

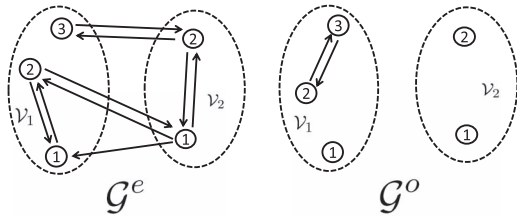
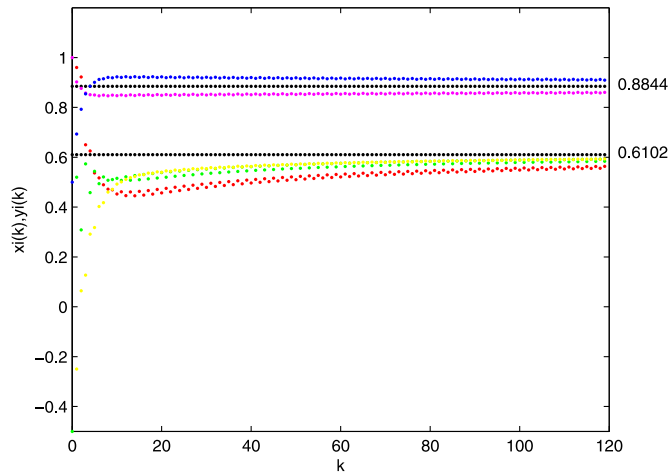


Fig. 2. Two possible communication graphs in Example 6.1.


 Fig. 3. The Nash equilibrium is achieved (i.e., $x_i \rightarrow x^*$ and $y_i \rightarrow y^*$) for the time-varying weight-balanced digraphs with homogeneous stepsizes.

We discuss three examples. The first example is given for verifying the convergence of the proposed algorithm with homogeneous stepsizes in the case of weight-balanced graphs, while the second one is for the convergence with the stepsizes provided in the existence theorem in the case of weight-unbalanced graphs. The third example demonstrates the efficiency of the proposed adaptive learning strategy for periodical switching unbalanced graphs.

Example 6.1: The communication graph is switching periodically over the two graphs $\mathcal{G}^e, \mathcal{G}^o$ given in Fig. 2, where $\mathcal{G}(2k) = \mathcal{G}^e, \mathcal{G}(2k+1) = \mathcal{G}^o, k \geq 0$. Denote by \mathcal{G}_1^e and \mathcal{G}_2^e the two subgraphs of \mathcal{G}^e describing the communications within the two subnetworks. Similarly, the subgraphs of \mathcal{G}^o are denoted as \mathcal{G}_1^o and \mathcal{G}_2^o . Here the adjacency matrices of $\mathcal{G}_1^e, \mathcal{G}_2^e$ and \mathcal{G}_1^o are as follows:

$$A_1(2k) = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.4 & 0.6 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2(2k) = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$$

$$A_1(2k+1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.7 & 0.3 \\ 0 & 0.3 & 0.7 \end{pmatrix}.$$

Clearly, with the above adjacency matrices, the three digraphs $\mathcal{G}_1^e, \mathcal{G}_2^e$ and \mathcal{G}_1^o are weight-balanced. Let the stepsize be $\alpha_{i,k} = \beta_{j,k} = \gamma_k$ for all i, j and $k \geq 0$. Fig. 3 shows that the agents converge to the unique Nash equilibrium $(x^*, y^*) = (0.6102, 0.8844)$.

Example 6.2: Consider the same switching graphs given in Example 6.1 except that a new arc $(2, 3)$ is added in \mathcal{G}_1^e . The new

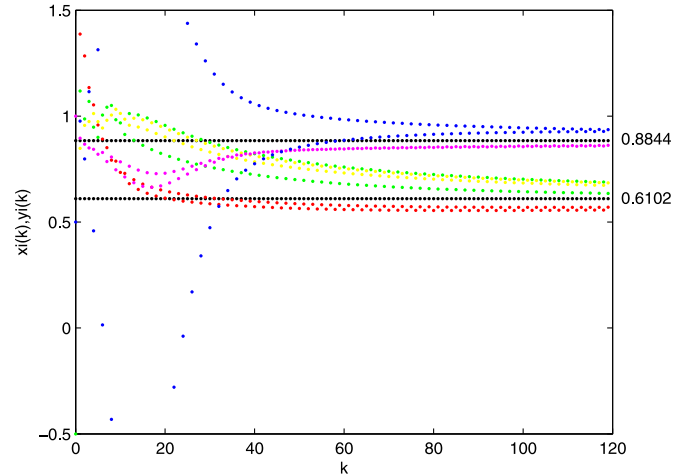


Fig. 4. The Nash equilibrium is achieved for weight-unbalanced digraphs with heterogeneous stepsizes.

graph is still denoted as \mathcal{G}_1^e for simplicity. Here the adjacency matrices of the three digraphs $\mathcal{G}_1^e, \mathcal{G}_2^e$, and \mathcal{G}_1^o are given by

$$A_1(2k) = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.7 & 0.3 & 0 \\ 0 & 0.6 & 0.4 \end{pmatrix}, A_2(2k) = \begin{pmatrix} 0.9 & 0.1 \\ 0.8 & 0.2 \end{pmatrix}$$

$$A_1(2k+1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.3 & 0.7 \\ 0 & 0.4 & 0.6 \end{pmatrix}.$$

In this case, $\mathcal{G}_1^e, \mathcal{G}_2^e$ and \mathcal{G}_1^o are weight-unbalanced with $(\alpha_{2k}^1, \alpha_{2k}^2, \alpha_{2k}^3) = (0.5336, 0.1525, 0.3139), (\alpha_{2k+1}^1, \alpha_{2k+1}^2, \alpha_{2k+1}^3) = (0.5336, 0.3408, 0.1256)$ and $(\beta_k^1, \beta_k^2) = (0.8889, 0.1111), \forall k \geq 0$. We design the heterogeneous stepsizes as follows: $\alpha_{i,2k} = (1/\alpha_1^i)\gamma_{2k}, \alpha_{i,2k+1} = (1/\alpha_0^i)\gamma_{2k+1}, i = 1, 2, 3; \beta_{i,k} = (1/\beta_0^i)\gamma_k, i = 1, 2$. Fig. 4 shows that the agents converge to the unique Nash equilibrium with those heterogeneous stepsizes.

Example 6.3: Here we verify the result obtained in Theorem 4.5 (ii). Consider Example 6.2, where $p^1 = p^2 = 2$. Design adaptive stepsize algorithms: for $\nu = 0, 1$

$$\theta^\nu(r) = (A_1(r) \cdots A_1(\nu+1) \otimes I_3) \theta^\nu(\nu+1), r \geq \nu+1$$

where $\theta^\nu(r) = ((\alpha^{(\nu)1}(r))', (\alpha^{(\nu)2}(r))', (\alpha^{(\nu)3}(r))')'$, $\theta^\nu(\nu+1) = (1, 0, 0, 0, 1, 0, 0, 0, 1)'$; for $\nu = 0, 1$

$$\vartheta^\nu(r) = (A_2(r) \cdots A_2(\nu+1) \otimes I_2) \vartheta^\nu(\nu+1), r \geq \nu+1$$

where $\vartheta^\nu(r) = ((\beta^{(\nu)1}(r))', (\beta^{(\nu)2}(r))')'$, $\vartheta^\nu(\nu+1) = (1, 0, 0, 1)'$.

Let $\hat{\alpha}_{2k}^i = \alpha_i^{(0)i}(2k), \hat{\alpha}_{2k+1}^i = \alpha_i^{(1)i}(2k+1), \hat{\beta}_{2k}^i = \beta_i^{(0)i}(2k), \hat{\beta}_{2k+1}^i = \beta_i^{(1)i}(2k+1)$ and

$$\alpha_{i,2k} = \frac{1}{\hat{\alpha}_{2k}^i} \gamma_{2k}, \quad \alpha_{i,2k+1} = \frac{1}{\hat{\alpha}_{2k+1}^i} \gamma_{2k+1}, \quad i = 1, 2, 3$$

$$\beta_{i,2k} = \frac{1}{\hat{\beta}_{2k}^i} \gamma_{2k}, \quad \beta_{i,2k+1} = \frac{1}{\hat{\beta}_{2k+1}^i} \gamma_{2k+1}, \quad i = 1, 2.$$

Fig. 5 shows that the agents converge to the unique Nash equilibrium under the above designed adaptive stepsizes.

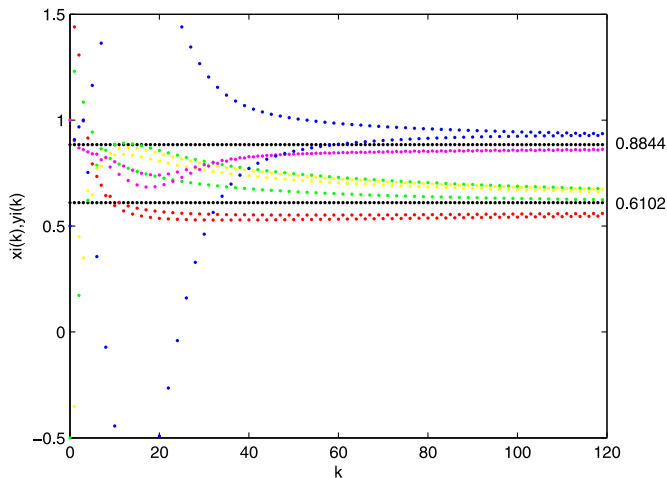


Fig. 5. The Nash equilibrium is achieved for weight-unbalanced digraphs by adaptive stepsizes.

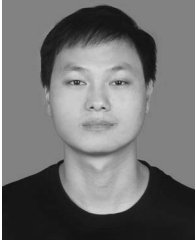
VII. CONCLUSION

A subgradient-based distributed algorithm was proposed to solve a Nash equilibrium computation problem as a zero-sum game with switching communication graphs. Sufficient conditions were provided to achieve a Nash equilibrium for switching weight-balanced digraphs by an algorithm with homogenous stepsizes. In the case of weight-unbalanced graphs, it was demonstrated that the algorithm with homogeneous stepsizes might fail to reach a Nash equilibrium. Then the existence of heterogeneous stepsizes to achieve a Nash equilibrium was established. Furthermore, adaptive algorithms were designed to update the heterogeneous stepsizes for the Nash equilibrium computation in two special cases.

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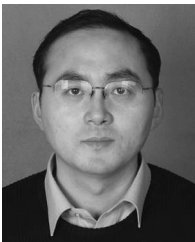
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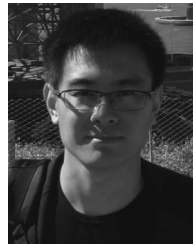


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