Distributed Output Regulation of Nonlinear Multi-Agent Systems via Host Internal Model

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Abstract—**This note presents a host internal model (h-IM) approach to distributed output regulation of reaching nonlinear leader-following consensus. Focusing on a basic nonlinear scenario of identical (or homogeneous) agents of a normal form, it is shown that the problem is solvable as long as its interaction digraph contains a certain directed spanning tree. A constructive Lyapunov protocol is proposed by incorporating a single h-IM. The result is also demonstrated by FitzHugh-Nagumo (FHN) and Lorenz type nonlinear dynamic networks.**

Index Terms—**Consensus, distributed control, multi-agent systems, output regulation.**

I. Introduction

Distributed output regulation has been studied for multi-agent systems in the past few years, coming up with various internal model based approaches (see, e.g. [3], [5], [8], [13], [17]–[19], [21] and references therein), effectively developed by applying modern linear or nonlinear output regulation theory (refer to [7], [10], [20] for an overview). In the present stage and for nonlinear multi-agent systems, it has become an active field with an increasing research interest, e.g., [17] studies nonlinear agents having unity relative degree with an undirected interaction graph and [3] studies an output regulation problem with a loop-free directed graph. Recently, some efforts have been made to construct a networked internal model for a directed chain of multiple agents by allowing signal transmission among internal models of the follower agents (e.g., [23]).

In view of the aforementioned results, potentials of the internal model communication have not been systematically studied for various nonlinear scenarios. Also in the category of distributed feedback control, it is well known that signal transmissions among controllers are basically valuable. From this viewpoint and respecting the indispensable role of internal model in output regulation, it is natural to explore benefits or advantages of internal model communications in multi-agent systems control. This observation therefore provides impetus for investigations of setting up suitable internal model networks to characterize useful channels for communications among local internal models in distributed output regulation.

In this note, we shall start by addressing an h-IM approach to a distributed output regulation problem of nonlinear identical (follower) agents with a general directed interaction graph. What makes the problem challenging is two-fold. First, we need to construct an appropriate h-IM and furthermore derive an augmented dynamics consisting of the agents and the internal model. Second, for a general directed interaction graph, we deal with a stabilization problem of the augmented dynamics by limited measurement output-feedback control with distributed output regulation as the chief control goal. Our main result is summarized as follows. It is shown that the distributed output regulation problem is solvable as long as its interaction digraph contains a directed spanning tree (with the leader node as the root). Moreover, we show that a single h-IM suffices and selection of the host agent can be arbitrary. The h-IM can transmit its produced useful information to all the other agents along with a star-type internal model network for the purpose of reaching agent coordination. Therefore, the proposed distributed control strategy discovers useful internal model communications in a manner of sharing, leading to a notable reduction of the number or redundancy of internal models in resolving distributed output regulation.

The note is organized as follows. In Section II, the concerned model and control problem are formulated. In Section III, the h-IM approach is addressed and the main result is presented. Then in Section IV, the method is applied to solve two illustrative problems of the well-known FHN and Lorenz type dynamic networks. Finally in Section V, the note is closed with some concluding remarks.

II. Formulation

Consider a group of nonlinear agents, globally transformable into a strict-feedback normal form (see, e.g. [10])

$$i \in \mathcal{O}: \quad \dot{z}_i = f(z_i, y_i, w), \quad \dot{y}_i = g(z_i, y_i, w) + bu_i \tag{1}$$

where $\mathcal{O} := \{1, 2, \ldots, \mathbf{n}\}$ is the follower node index set, $(z_i, y_i) \in \mathbb{R}^n \times \mathbb{R}$ is the agent state, $y_i \in \mathbb{R}$ is the output, $u_i \in \mathbb{R}$ is the local control input, $w \in \mathbb{W}$ is an uncertain parameter vector in a compact set $\mathbb{W} \subset \mathbb{R}^{n_w}$, and $b = b(w)$ is the uncertain high frequency gain. In company with (1), the leader (with node index 0) is described by

$$\dot{v} = Sv, \quad y_0 = q_r(v, w) \tag{2}$$

where $v \in \mathbb{R}^{n_v}$ is the state and $y_0 \in \mathbb{R}$ is given as the reference output. To make our problem well posed as in [3], [22], we assume that all the eigenvalues of S are distinct lying on the imaginary axis with its initial condition $v(0)$ starting from a compact set $\mathbb{V} \subset \mathbb{R}^{n_v}$. Both $\mathbb{V}$ and $\mathbb{W}$ are fixed with known boundaries. Obviously, there is a compact set $\mathbb{V}'$ such that for any $v(0) \in \mathbb{V}$, its response $v(t, v(0)) \in \mathbb{V}'$ for all $t \geq 0$. Denote[1] $\mu(t) := (v(t), w)^\top$ and $\mathbb{D} := \mathbb{V}' \times \mathbb{W}$ for notational convenience. It is also assumed that all functions f, g, q_r, b are smooth in their arguments and $f(0, 0, w) = 0$, $g(0, 0, w) = 0$, $q_r(0, w) = 0$, $b(w) > 0$ for all $w \in \mathbb{W}$.

Remark 2.1: The nonlinear agent (1) has unity relative degree, which can model a broad class of practical nonlinear systems such as two well-known types of nonlinear dynamics discussed later in Section IV. The z_i-system can be viewed as the agent inverse dynamics

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[1]Throughout this technical note, if no confusion, by $(v(t), w)^\top$, it means the column vector $[v(t)^\top, w^\top]^\top$.

or dynamic uncertainty (see [11]). The leader (2), as a reference source, can be viewed as an exosystem to generate the fundamental sinusoid/step signals.

Graph Notation: To formulate the concerned distributed feedback control problem, we shall summarize some usual graph notations (see, e.g. [4]) to describe the local measurement signals for individual agents of (1). Denote the directed interaction graph or interaction digraph, associated with the leader and controlled follower nodes, by a triplet $\mathcal{G} := \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ where $\mathcal{V} := \{0, 1, 2, \cdots, \mathbf{n}\}$ is the node set, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set (with self-loops excluded), and $\mathcal{A} = [a_{ij}]_{i,j=0,1,\cdots,\mathbf{n}}$ is the weighted adjacency matrix, explained as follows. An edge of $\mathcal{G}$ is denoted by an ordered pair of nodes $(j, i) \in \mathcal{E}$ with j being indicated as a neighbor of i. A directed path of $\mathcal{G}$ is an ordered sequence of distinct nodes in $\mathcal{V}$ such that any consecutive nodes in the sequence correspond to an edge of the digraph. A node j is said to be connected to another node i if there is a directed path from j to i. $\mathcal{G}$ is said to contain a directed spanning tree if there is at least one node, called the root, being connected to every other node. The matrix $\mathcal{A}$ is an essentially nonnegative matrix and $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$, $i, j \in \mathcal{V}$. The Laplacian associated with $\mathcal{G}$ is an induced matrix $\mathcal{L} = [l_{ij}]_{i,j=0,1,\cdots,\mathbf{n}}$ with $l_{ii} = \sum_{j=0}^{\mathbf{n}} a_{ij}$ and $l_{ij} = -a_{ij}, i \neq j$.

For (1) and (2), we can write $e \in \mathbb{R}^{\mathbf{n}}$ as the output to be regulated with each entry given by $e_i = y_i - y_0$, $i \in \mathcal{O}$. Regarding distributed output regulation for multi-agent systems to realize convergence of every output y_i to the reference y_0, e is generally unavailable to every agent. In practical situations, the measurement output e_{mi} of each agent is given by

$$i \in \mathcal{O}: \quad e_{mi} = \sum_{j \in \mathcal{V}} a_{ij}(y_i - y_j) \tag{3}$$

which can be understood as reasonable output interactions with its neighbors (refer to [16] for a sound introduction). In other words, we are only allowed to use this local information e_{mi} to treat the output regulation of the ith controlled agent. Denote that $e_m := (e_{m1}, \cdots, e_{m\mathbf{n}})^\top$.

Next, to carry out the study in the next section, we list some mild assumptions for the systems (1) and (2).

- **H1** The interaction digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ contains a directed spanning tree with the leader node as the root.

Remark 2.2: **H1** leads to a useful property. First the Laplacian $\mathcal{L}$ of $\mathcal{G}$ can be written as

$$\mathcal{L} = \left[\begin{array}{c|c} 0 & 0 \\ \hline -\mathbf{a}_0 & H \end{array}\right], \quad \mathbf{a}_0 = (a_{10}, \cdots, a_{\mathbf{n}0})^\top$$

for a square matrix $H \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}}$. Then under **H1**, by [6, Lemma 1.4], all the eigenvalues of H have positive real parts. As a result, the relationship between the quantities e_m and e can be determined by $e_m = He$ and H is obviously nonsingular.

Further by [1, Theorem 2.3, page 134], there exists a diagonal matrix $R = \mathrm{diag}(r_1, \cdots, r_{\mathbf{n}})$ with positive entries such that $RH + H^\top R$ is positive definite. In fact, **H1** is necessary for the leader-following consensus with fixed interactions. Many existing results, e.g. [3], [17], [19], [23], have been given under certain conditions that $\mathcal{G}$ is a loop-free digraph or its induced digraph $\mathcal{G}_0$ associated with the follower node set $\mathcal{O}$ is undirected (or bidirected). Both the above are special cases of **H1**. Notice that H is positive definite if $\mathcal{G}_0$ is undirected.

- **H2** There is a smooth function $\mathbf{z} : \mathbb{R}^{n_v} \times \mathbb{W} \to \mathbb{R}^n$ with $\mathbf{z}(0, w) = 0$, $\forall w \in \mathbb{W}$ such that

$$\frac{\partial \mathbf{z}(v, w)}{\partial v} \cdot Sv = f\left(\mathbf{z}(v, w), q_r(v, w), w\right), \ \forall (v, w) \in \mathbb{D}. \tag{4}$$

Remark 2.3: As a consequence of **H2**, for systems (1) and (2), the associated regulator equations (REs) (see [7], [10]) can be solved with the steady-state input functions satisfying

$$\mathbf{u}_1(v, w) = \cdots = \mathbf{u}_{\mathbf{n}}(v, w) = \mathbf{u}(v, w) \tag{5}$$

where

$$\mathbf{u}(v, w) = b^{-1} \frac{\partial q_r(v, w)}{\partial v} Sv - b^{-1} g\left(\mathbf{z}(v, w), q_r(v, w), w\right).$$

Also, under **H2** and for the inverse dynamics of (1), by the coordinate transformation $\bar{z}_i = z_i - \mathbf{z}(v, w)$, we obtain

$$i \in \mathcal{O}: \quad \dot{\bar{z}}_i = \bar{f}(\bar{z}_i, e_i, \mu) \tag{6}$$

where $\bar{f}(\bar{z}_i, e_i, \mu)$ is to denote $f(\bar{z}_i + \mathbf{z}(v, w), e_i + q_r(v, w), w) - f(\mathbf{z}(v, w), q_r(v, w), w)$.

- **H3** The function $\mathbf{u}(v, w)$ is polynomial in v with coefficients depending on w.
- **H4** There exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned} &\underline{\alpha}\left(\|\bar{z}_i\|\right) \leq V(\bar{z}_i) \leq \bar{\alpha}\left(\|\bar{z}_i\|\right) \\ &\dot{V}|_{(6)} \leq -\|\bar{z}_i\|^2 + \gamma(e_i)e_i^2, \quad \forall \mu \in \mathbb{D} \end{aligned} \tag{7}$$

  for some smooth functions[2] $\underline{\alpha}(\cdot), \bar{\alpha}(\cdot) \in \mathcal{K}_\infty$ and $\gamma(e_i) \geq 1$.

Remark 2.4: **H2** is used to guarantee certain solvability of the associated REs, which is generally a necessary condition in output regulation (see [7], [10]). Conditions like **H3** and **H4** are frequently used in nonlinear system control such as [7], [11], [22]. It is worth pointing out that **H3** is verifiable if all the functions in (1) and (2) together with the solution satisfying **H2** are polynomials in their arguments. **H4** will suffice the output-feedback design in this note that implies a minimum phase or output-feedback passivity condition for each follower; see [12, pages 517 & 606]; cf. [3, Assumption 3] and [13, Assumption 1]. In addition, **H3** and **H4** can be relaxed in several directions; e.g. one may further consider **H3** by certain relaxed conditions in [2], [9].

The distributed output regulation problem undertaken in this note is formulated as follows.

Problem 1: For the group of agents (1) and (2), if possible, find an index $\jmath \in \mathcal{O}$ and a smooth regulator of the form

$$\dot{\eta}_\jmath = \rho(\eta_\jmath, u_\jmath), \quad u_i = \varrho_i(\psi_\jmath, e_{mi}), \quad i = 1, \cdots, \mathbf{n} \tag{8}$$

where $\psi_\jmath := \Psi\eta_\jmath \in \mathbb{R}$ with a design parameter vector Ψ of an appropriate dimension such that, for each $(v(0), w) \in \mathbb{V} \times \mathbb{W}$ and each initial condition $(z_i(0), y_i(0), \eta_\jmath(0))$ in their respective entire spaces

(i) the trajectory of the closed-loop system exists for all $t \geq 0$ and is bounded over $t \in [0, +\infty)$;
(ii) the regulated output $e(t)$ satisfies $\lim_{t\to+\infty} e(t) = 0$.

In Problem 1, $\jmath \in \mathcal{O}$ is called the host agent index, indicating an h-IM of dynamics $\dot{\eta}_\jmath = \rho(\eta_\jmath, u_\jmath)$. Regarding Problem 1, there are two natural questions: (i) Does the choice of a single host agent suffice the problem? (ii) Which agent can be selected as the unique host? These questions motivate the investigation in the following section. For a better understanding of the resultant closed-loop system structure, a particular example is illustrated in Fig. 1.

[2] $\mathcal{K}_\infty$ is the usual set of continuous, unbounded, and strictly increasing functions $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$.

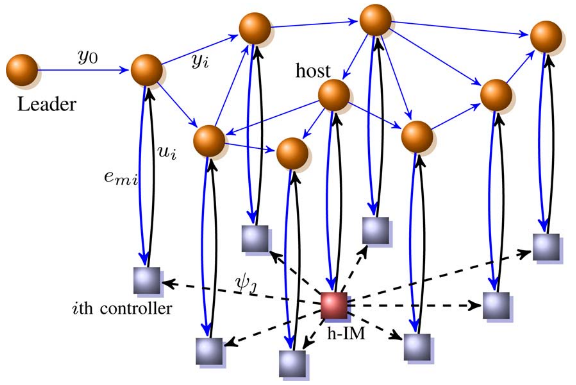

Fig. 1. Structure of closed-loop system containing a single h-IM.

III. Main Result

This section is devoted to the construction of a distributed regulator for Problem 1 to solve the leader-following consensus by employing a single h-IM.

A. Host Internal Model

The (5) describes the zero-error constraint input (see [7, page 83]) that is a necessary steady-state input information to achieve output regulation of the individual nodes. Regarding output regulation of each controlled agent, we will show that the host agent can be selected arbitrarily and the single h-IM can transmit its useful information to every other agents.

In the following, let us first consider the construction of an h-IM. Since $\mathbf{u}(v, w)$ is polynomial in v, its minimal zeroing polynomial (see [7, page 170]) can be given by

$$p(s) = s^{n_s} - c_1 - c_2 s - \cdots - c_{n_s} s^{n_s - 1} \tag{9}$$

determined by an integer $n_s > 0$ and a set of real numbers $\{c_1, \cdots, c_{n_s}\}$, independent of w. Consequently, define $\tau(v, w) = (\tau_1(v, w), \cdots, \tau_{n_s}(v, w))^\top$ and

$$\Phi = \left[\begin{array}{c|c} 0 & I_{n_s - 1} \\ \hline c_1 & c_2, \cdots, c_{n_s} \end{array}\right]_{n_s \times n_s}, \quad \Psi = [1 \quad 0 \quad \cdots \quad 0]_{1 \times n_s}$$

where $\tau_1(v, w) := \mathbf{u}(v, w)$ and along (2), $\tau_j(v, w) := d^{j-1}\mathbf{u}(v, w)/dt^{j-1}$, $j = 2, \cdots, n_s$. It can be verified that

$$\begin{cases} \dfrac{\partial \tau(v, w)}{\partial v} \cdot Sv = \Phi \tau(v, w) \\ \qquad \mathbf{u}_\jmath(v, w) = \Psi \tau(v, w), \ \forall \jmath \in \mathcal{O}, \ \forall (v, w) \in \mathbb{D} \end{cases} \tag{10}$$

which is called a steady-state (input $u_\jmath$) generator (see [7, page 161]). Further recalling (5), the system (10) can generate the common input signal $\mathbf{u}(v, w)$ with the output function $\Psi \tau(v, w)$.

Next, in accordance with the notion of internal models characterized by [7, Definition 6.6], it is straightforward to derive a linear internal model as dynamic compensator for any agent by the following lemma. The interested reader may consult [7, Chapter 6] for more details as well as general arguments.

Lemma 3.1: For any $\jmath \in \mathcal{O}$, there is an internal model of the following form:

$$\dot{\eta}_\jmath = M \eta_\jmath + N u_\jmath \tag{11}$$

with output $u_\jmath$, where (M, N) is a controllable matrix pair with M being Hurwitz.

To briefly show Lemma 3.1, we can also put the generator (10) into the following form:

$$\begin{cases} \dfrac{\partial \tau(v, w)}{\partial v} \cdot Sv = M \tau(v, w) + N \mathbf{u}_\jmath(v, w) \\ \qquad \mathbf{u}_\jmath(v, w) = \Psi \tau(v, w) \end{cases} \tag{12}$$

where $M = \Phi - D_\sigma L \Psi$, $N = D_\sigma L$, and $D_\sigma = \mathrm{diag}(\sigma, \cdots, \sigma^{n_s})$ for a vector $L \in \mathbb{R}^{n_s}$ and a large real number $\sigma \geq 1$ such that M is Hurwitz (see [12]). Then the generator (12) consequently leads to (11). It is noted that the Hurwitz property for matrix M is useful in the stabilization design. This together with **H4** ensures a property of (18) shown later in the next subsection.

Remark 3.1: As we have pointed out, in most existing results, e.g., [3], [17], [19], every local controller contains a local-processed (or local-input driven) internal model of relatively separated. In many situations, it would be a waste to repeat the internal model dynamics in a full or partial manner for all agents. This observation has basically motivated us to address this issue especially for coordination of nonlinear identical agents. In fact, each local internal model may share the same dynamics. Broadly speaking, it is possible to only construct internal models for some selected agents (called hosts) and then transmit the already produced internal model output information to the other agents. The information-receiving agents may have no need of introducing the local-processed internal model any more. Thus far there is no systematic study on a general level for internal model networks in distributed output regulation of multi-agent systems. Nevertheless, with thanks to certain relationships between the agent zero-error constraint inputs like (5), we are able to reduce the repeated internal model dynamics for some followers.

Here, for agents (1) and (2) under **H1** to **H3**, it is valid to select any agent (with index $\jmath$) as the host agent and construct the h-IM because of certain stabilizability of the resultant augmented dynamics, cf. a general nonlinear output regulation framework established in [9]. In the next subsection, it will be shown that the solution of Problem 1 can be finally derived by a distributed stabilization design.

B. Distributed Stabilization of Augmented Network

The augmented dynamics can be obtained as a combination of (1) and (11). In the spirit of [9], some suitable coordinate and input transformations are needed to be done. Then we may convert Problem 1 into an equilibrium stabilization problem for a translated augmented dynamics.

Indeed we can derive such an augmented dynamics which has a stabilizable equilibrium in the sense of leading to a solution to the original problem. To this end, letting

$$\bar{\eta}_\jmath = \eta_\jmath - \tau - b^{-1} N e_\jmath, \quad \bar{u}_i = u_i - \psi_\jmath, \quad i = 1, \cdots, \mathbf{n}$$

yields an interconnected nonlinear system

$$\begin{cases} \dot{\bar{\eta}}_\jmath = M \bar{\eta}_\jmath + \phi(\bar{z}_\jmath, e_\jmath, \mu) \\ \dot{\bar{z}}_i = \bar{f}(\bar{z}_i, e_i, \mu) \\ \dot{e}_i = b \bar{u}_i + \bar{g}(\bar{\eta}_\jmath, \bar{z}_i, e, \mu), \quad i = 1, \cdots, \mathbf{n} \end{cases} \tag{13}$$

where $\bar{z}_i$-system is as in Remark 2.3 and

$$\begin{aligned} &\phi(\bar{z}_\jmath, e_\jmath, \mu) \\ &\quad = b^{-1} M N e_\jmath - b^{-1} N g\left(\bar{z}_\jmath + \mathbf{z}(v, w), e_\jmath + q_r(v, w)\right) \\ &\qquad + b^{-1} N g\left(\mathbf{z}(v, w), q_r(v, w), \mu\right) \\ &\bar{g}(\bar{\eta}_\jmath, \bar{z}_i, e, \mu) \\ &\quad = b \Psi(\bar{\eta}_\jmath + b^{-1} N e_\jmath) + g\left(\bar{z}_i + \mathbf{z}(v, w), e_i + q_r(v, w)\right) \\ &\qquad - g\left(\mathbf{z}(v, w), q_r(v, w), \mu\right). \end{aligned}$$

The system (13) is the so-called translated augmented dynamics whose equilibrium is $(\bar{\eta}_\jmath, \bar{z}, e) = (0,0,0)$ with $\bar{z} := (\bar{z}_1, \cdots, \bar{z}_{\mathbf{n}})^\top$ because $\phi(0,0,\mu) = 0$, $\bar{f}(0,0,\mu) = 0$, $\bar{g}(0,0,0,\mu) = 0$ for all $\mu \in \mathbb{D}$. Consequently, we can formulate a stabilization problem for (13).

Problem 2: For (13) with local measurement output (3), the objective is to find a distributed controller

$$\bar{u}_i = -\kappa_i(e_{mi}), \quad i = 1, \cdots, \mathbf{n} \tag{14}$$

such that the equilibrium $(\bar{\eta}_\jmath, \bar{z}, e) = (0,0,0)$ can be made globally uniformly asymptotically stabilizable.

Remark 3.2: By a general property of [7, Corollary 7.4], we acknowledge that the solution to Problem 2, if there is, can lead to a solution to Problem 1 with the regulator (8) specified by

$$\dot{\eta}_\jmath = M\eta_\jmath + Nu_\jmath, \quad u_i = -\kappa_i(e_{mi}) + \psi_\jmath, \quad i = 1, \cdots, \mathbf{n}. \tag{15}$$

At this point, we have converted the distributed output regulation problem into a distributed stabilization one whose solvability can be in fact ensured by the connectivity of $\mathcal{G}$ specified by **H1**.

Remark 3.3: A key technical barrier to resolving Problem 2 is construction of some manageable Lyapunov-like function especially for $e = (e_1, \cdots, e_{\mathbf{n}})^\top$ dynamics of (13), which could facilitate the stabilizer construction of (14). Recall that, in the case of undirected interaction graphs, the matrix H defined in Remark 2.2 is symmetric and positive definite, which leads to a quadratic function

$$V_m(e) = e^\top He \tag{16}$$

serving the synthesis of intermediate stabilization control in multi-agent coordination (consult [17] for an elegant treatment). However, when the interaction graph is a general digraph of **H1**, the corresponding matrix H is generally not symmetric and not positive definite as well. The quadratic function (16) does not suit the stabilization design any more. Due to this reason, we have to seek a non-quadratic one to cope with this problem. Indeed, based on an analysis of the system interaction structure, we can choose a candidate

$$V_m(e_m) = \sum_{i \in \mathcal{O}} V_{mi}(e_{mi}), \quad V_{mi}(e_{mi}) = \int_0^{e_{mi}} r_i \kappa_i(s) ds \tag{17}$$

for e dynamics of (13), where r_i is specified by Remark 2.2, and $\kappa_i(\cdot)$ is a function of the form $\kappa_i(s) = s\kappa_i^\star(s)$ with $\kappa_i^\star(s) \geq 1$ being a smooth even function, i.e., $\kappa_i^\star(s) = \kappa_i^\star(-s)$ for all $s \in \mathbb{R}$. The method of constructing (17) can be understood as a modified variable gradient method; see, e.g. [12, page 120]. Moreover, it can be shown that $V_m(e_m)$ is positive definite and radially unbounded. This storage function is crucial in the proof of the forthcoming main theorem.

Now we are in a position to state the main result of this note.

Theorem 3.1: For the group of agents (1) and (2) under assumptions **H1** to **H4**, Problem 1 can be solved by a distributed regulator of the form (15).

Proof: For convenience, denote that $\zeta_\jmath := (\bar{\eta}_\jmath, \bar{z}_\jmath)^\top$, $\zeta_i := \bar{z}_i$, $i \neq \jmath$. We only need to solve Problem 2 because of Remark 3.2. First consider (13) under **H4**. For $\zeta_\jmath$ dynamics, by using [22, Lemma 3.1], there exists a smooth function $\bar{V}_\jmath(\zeta_\jmath)$ such that

$$\begin{aligned} &\underline{\alpha}_\jmath(\|\zeta_\jmath\|) \leq \bar{V}_\jmath(\zeta_\jmath) \leq \bar{\alpha}_\jmath(\|\zeta_\jmath\|) \\ &\dot{\bar{V}}_\jmath(\zeta_\jmath)|_{(13)} \leq -\|\zeta_\jmath\|^2 + \gamma_\jmath(e_\jmath)e_\jmath^2, \ \forall \mu \in \mathbb{D} \end{aligned} \tag{18}$$

for some smooth functions $\underline{\alpha}_\jmath(\cdot), \bar{\alpha}_\jmath(\cdot) \in \mathcal{K}_\infty$ and $\gamma_\jmath(e_\jmath) \geq 1$. Also, for $i \neq \jmath$, we have a smooth function $V(\zeta_i)$ satisfying (7) for some smooth functions $\underline{\alpha}(\cdot), \bar{\alpha}(\cdot) \in \mathcal{K}_\infty$ and a smooth function $\gamma(e_i) \geq 1$. Thus, in view of (18) and (7) and by [22, Remark 3.1], for any $i \in \mathcal{O}$ and for any smooth function $\Delta_i(\zeta_i) > 0$, there exists a smooth function $\breve{V}_i(\zeta_i)$ such that

$$\begin{aligned} &\underline{\alpha}_i'(\|\zeta_i\|) \leq \breve{V}_i(\zeta_i) \leq \bar{\alpha}_i'(\|\zeta_i\|) \\ &\dot{\breve{V}}_i(\zeta_i)|_{(13)} \leq -\Delta_i(\zeta_i)\|\zeta_i\|^2 + \gamma_i(e_i)e_i^2, \ \forall \mu \in \mathbb{D} \end{aligned} \tag{19}$$

for some smooth functions $\underline{\alpha}_i'(\cdot), \bar{\alpha}_i'(\cdot) \in \mathcal{K}_\infty$ and $\gamma_i(e_i) \geq 1$. In (19), we state that there are smooth functions $\varphi_{1i}(s)$, $i \in \mathcal{O}$ written by $\varphi_{1i}(s) = s\varphi_{1i}^\star(s)$, $\varphi_{1i}^\star(s) \geq 1$ such that

$$\sum_{i \in \mathcal{O}} \varphi_{1i}^2(e_{mi}) \geq \sum_{i \in \mathcal{O}} \gamma_i(e_i)e_i^2, \ \forall e. \tag{20}$$

Then, we can define

$$V_\zeta(\zeta) = \sum_{i \in \mathcal{O}} \breve{V}_i(\zeta_i), \ \zeta := (\zeta_1, \cdots, \zeta_{\mathbf{n}})^\top \tag{21}$$

which satisfies

$$\dot{V}_\zeta|_{(13)} \leq -\sum_{i \in \mathcal{O}} \Delta_i(\zeta_i)\|\zeta_i\|^2 + \sum_{i \in \mathcal{O}} \varphi_{1i}^2(e_{mi}). \tag{22}$$

Next, consider the Lyapunov function candidate defined by (17) in Remark 3.3 which manifests the following:

$$\dot{V}_{mi}|_{(13)} = r_i\kappa_i(e_{mi}) \sum_{j \in \mathcal{V}} a_{ij}(\dot{e}_i - \dot{e}_j), \ e_0 \equiv 0. \tag{23}$$

The above (23) leads to the following:

$$\dot{V}_{mi}|_{(13)+(14)} = -\Xi_i + r_i\kappa_i \sum_{j \in \mathcal{V}} a_{ij}(\Upsilon_{\jmath i} - \Upsilon_{\jmath j}) \tag{24}$$

where

$$\begin{aligned} &\kappa_i := \kappa_i(e_{mi}), \ \Xi_i := br_i\kappa_i \sum_{j \in \mathcal{V}} a_{ij}(\kappa_i - \kappa_j) \\ &\Upsilon_{\jmath i} := \bar{g}(\bar{\eta}_\jmath, \bar{z}_i, e, \mu), \ \kappa_0 \equiv 0, \ \Upsilon_{\jmath 0} \equiv 0. \end{aligned} \tag{25}$$

In (24), by completing the squares

$$\begin{aligned} &r_i\kappa_i \sum_{j \in \mathcal{V}} a_{ij}(\Upsilon_{\jmath i} - \Upsilon_{\jmath j}) \leq \frac{\epsilon}{2}\kappa_i^2 + \frac{1}{2\epsilon}\Upsilon_i^2 \\ &\Upsilon_i := r_i \sum_{j \in \mathcal{V}} a_{ij}(\Upsilon_{\jmath i} - \Upsilon_{\jmath j}) \end{aligned} \tag{26}$$

for any real number $\epsilon > 0$ to be further specified by (33) later, we have

$$\dot{V}_{mi}|_{(13)+(14)} \leq -\Xi_i + \frac{\epsilon}{2}\kappa_i^2 + \frac{1}{2\epsilon}\Upsilon_i^2. \tag{27}$$

In (27), for the function Ξ_i of (25), it can be shown that

$$\begin{aligned} \sum_{i \in \mathcal{O}} \Xi_i &= b\kappa^\top RH\kappa, \ \kappa := (\kappa_1, \cdots, \kappa_{\mathbf{n}})^\top \\ &= \frac{b}{2}\kappa^\top(RH + H^\top R)\kappa. \end{aligned}$$

This together with the following inequality[3]

$$\frac{b}{2}(RH + H^\top R) \geq \lambda_0 I, \ \forall w \in \mathbb{W} \tag{28}$$

(by Remark 2.2) for a real number $\lambda_0 > 0$ means the following:

$$\sum_{i \in \mathcal{O}} \Xi_i \geq \lambda_0\|\kappa\|^2. \tag{29}$$

[3]In (28), "$\geq$" means the positive semi-definiteness property.

For the last term in the right-hand side of (27), it can be shown (by using [7, Lemma 7.8] and completing the squares again) that, for each $i \in \mathcal{O}$, there are smooth functions $\Delta'_i(\zeta_i) \geq 1$ and $\varphi_{2i}(s)$ written by $\varphi_{2i}(s) = s\varphi^{\star}_{2i}(s)$, $\varphi^{\star}_{2i}(s) \geq 1$ such that, for all (ζ, e_m)

$$\sum_{i\in\mathcal{O}} \frac{1}{2\epsilon}\Upsilon_i^2 \leq \frac{1}{2\epsilon}\sum_{i\in\mathcal{O}}\left\{\varphi^2_{2i}(e_{mi}) + \Delta'_i(\zeta_i)\|\zeta_i\|^2\right\}. \tag{30}$$

Thus, substituting (29) and (30) into (27) yields

$$\begin{aligned}\dot{V}_m|_{(13)+(14)} \leq &-\left(\lambda_0 - \frac{\epsilon}{2}\right)\sum_{i\in\mathcal{O}}\kappa_i^2 \\ &+\frac{1}{2\epsilon}\sum_{i\in\mathcal{O}}\left\{\varphi^2_{2i}(e_{mi}) + \Delta'_i(\zeta_i)\|\zeta_i\|^2\right\}.\end{aligned} \tag{31}$$

Finally, using (21) and (17), define

$$U(\zeta, e_m) = V_\zeta(\zeta) + V_m(e_m)$$

which satisfies, for some smooth functions $\underline{\alpha}''_i(\cdot), \bar{\alpha}''_i(\cdot) \in \mathcal{K}_\infty$

$$\underline{\alpha}''_i\left(\|\zeta, e_m\|\right) \leq U(\zeta, e_m) \leq \bar{\alpha}''_i\left(\|\zeta, e_m\|\right),\ \forall(\zeta, e_m)$$

and also satisfies, by (31) and (22)

$$\begin{aligned}\dot{U}|_{(13)+(14)} \leq &-\left(\lambda_0 - \frac{\epsilon}{2}\right)\sum_{i\in\mathcal{O}}\kappa_i^2 + \sum_{i\in\mathcal{O}}\varphi^2_{1i}(e_{mi}) \\ &+\frac{1}{2\epsilon}\sum_{i\in\mathcal{O}}\left\{\varphi^2_{2i}(e_{mi}) + \left(\Delta'_i - 2\epsilon\Delta_i\right)\|\zeta_i\|^2\right\}.\end{aligned} \tag{32}$$

Consequently, in (32), it is possible to choose ϵ in (26), $\Delta_i \geq 1$ in (19), and the design function κ_i in (14) according to the criteria

$$\begin{aligned}&0 < \epsilon < \lambda_0,\ \Delta_i(\zeta_i) \geq \frac{\Delta'_i(\zeta_i)}{2\epsilon} \\ &\kappa_i^2(e_{mi}) \geq \frac{4}{\lambda_0}\left(\varphi^2_{1i}(e_{mi}) + \frac{1}{2\epsilon}\varphi^2_{2i}(e_{mi})\right),\ \forall(\zeta, e_m)\end{aligned} \tag{33}$$

which means

$$\dot{U}|_{(13)+(14)} \leq -\frac{\lambda_0}{4}\sum_{i\in\mathcal{O}}\kappa_i^2(e_{mi}).$$

Further by LaSalle-Yoshizawa theorem and the facts that $e_m = He$ and H is nonsingular as in Remark 2.2, the proof is complete. □

By Theorem 3.1, for the systems (1) and (2) after verification of all the conditions **H1** to **H4**, we can construct a distributed controller of the form (15) by a two-step algorithm: the first step is to calculate (9) and consequently design (11) by Lemma 3.1 and the second is to calculate (33) in accordance with the proof of Theorem 3.1.

Remark 3.4: We have demonstrated the validity and effectiveness of the extreme case of a single h-IM network, that may be viewed as a star-type network. For a comparison study of the proposed h-IM approach to the other extreme case, we shall discuss the approach of constructing one-to-one internal models for the controlled agents to solving the consensus problem. In this situation, consider a controller of the following form:

$$i \in \mathcal{O}: \quad \dot{\eta}_i = \mathbf{f}(\eta_i, u_i), \quad u_i = \mathbf{k}_i(\eta_i, e_{mi}) \tag{34}$$

by incorporating local-processed internal models

$$i \in \mathcal{O}: \quad \dot{\eta}_i = M\eta_i + Nu_i.$$

Regarding this scheme, in a similar manner as it has been done in the proof of Theorem 3.1, it can be show that, for the network (1) and (2) under **H1** to **H4**, the problem can still be solved by a controller of the form (34).

It can be seen that by (34) and in contrast to (8), there is no communication between controllers and particularly among internal models with emphasis. Clearly, by contrast, controllers of the non-host agents (with indices $i \neq \jmath$) without employing their own internal model dynamics are much simpler than that of the form (34). As a result, especially when the nodes amount $\mathbf{n} \gg 1$, employing controller (8) instead of (34) manifests a striking reduction of the controller order and computing burden, though a static and one-dimensional signal $\psi_\jmath$ is transmitted at an additional communication cost. Finally, it is worth noting that the internal model network (e.g., the bottom layer in Fig. 1) can be validly pre-assigned to admit certain physical requirements. From this viewpoint, it indicates a notable design freedom for distributed output regulation. In other words, besides the extreme situations of (8) and (34), one may appoint multiple host agents to do the task.

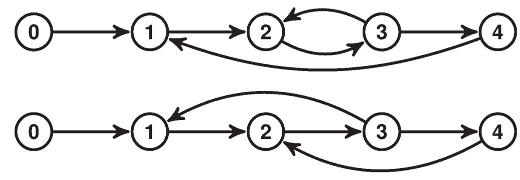

Fig. 2. Interaction graphs for Example 4.1 (upper) and Example 4.2 (lower).

IV. Examples

For illustrations, we present two examples on consensus control of FHN and Lorenz networks. Both will be solved by Theorem 3.1 with interaction graphs depicted in Fig. 2 for $\mathbf{n} = 4$. All their weights are supposed to be unity. The leader (2) is supposed to be a harmonic oscillator

$$\dot{v}_1 = \omega v_2, \quad \dot{v}_2 = -\omega v_1,\ \omega > 0 \tag{35}$$

with $\|v(0)\| \leq 1$ and $y_0 = v_1$ being the reference output. Thus it remains to verify conditions **H2** to **H4** for both examples.

In what follows, the agent (36) is an FHN model adopted from [15]. One may refer to [15] for relevant synchronization studies of certain diffusively coupled FHN networks. The model (37) is a controlled Lorenz system; see, e.g., [14], [22] for some results on Lorenz system synchronization.

Example 4.1: (FHN network) Consider the agents (1) specified by the following equations:

$$\begin{cases}\dot{z}_i = w_1(y_i - w_2 z_i) \\ \dot{y}_i = y_i(y_i - w_3)(1 - y_i) - z_i + bu_i,\ i = 1, \cdots, 4\end{cases} \tag{36}$$

where w_1, w_2, w_3, b belong to the closed interval [0.5,1.5].

By solving the associated REs, we have

$$\begin{aligned}\mathbf{z}(v, w) &= \frac{w_1^2 w_2}{\omega^2 + w_1^2 w_2^2}v_1 - \frac{\omega w_1}{\omega^2 + w_1^2 w_2^2}v_2 \\ \mathbf{u}(v, w) &= b^{-1}\left(v_2 - v_1(v_1 - w_3)(1 - v_1) + \mathbf{z}(v, w)\right)\end{aligned}$$

which verifies both **H2** and **H3**. Since $w_1 w_2 > 0$, **H4** can also be easily verified. For this example, we derive (10) with $n_s = 6$, $c_1 = -36\omega^6$, $c_3 = -49\omega^4$, $c_5 = -14\omega^2$, and $c_2 = c_4 = c_6 = 0$.

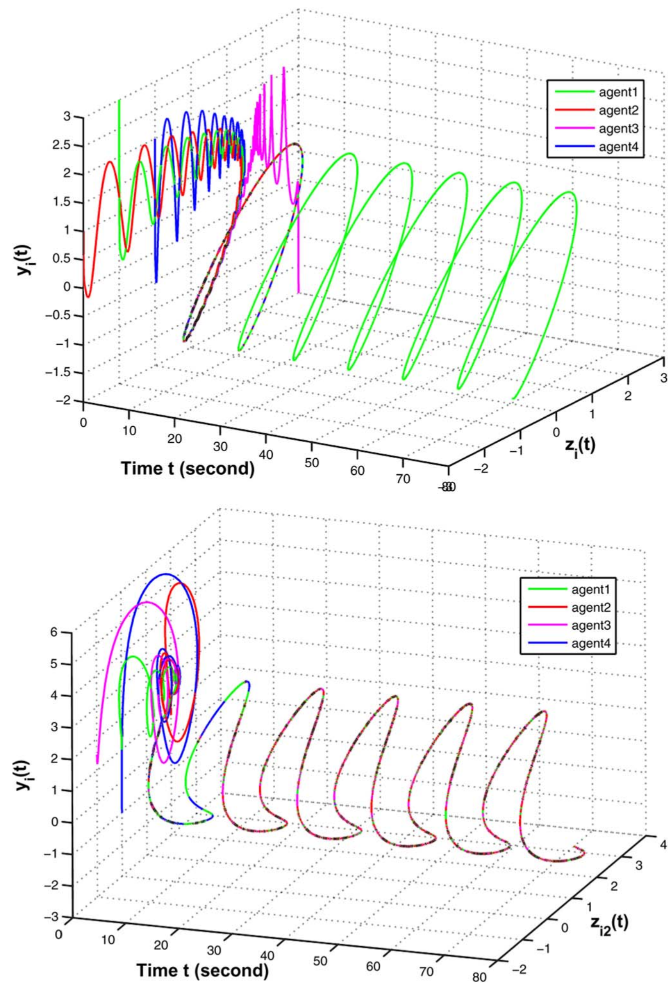

Fig. 3. Consensus of FHN (upper) and Lorenz (lower) agents.

Example 4.2: (Lorenz network) Consider (1) specified by the Lorenz type agents

$$\begin{cases} \dot{z}_{i1} = w_1(y_i - z_{i1}) \\ \dot{z}_{i2} = z_{i1}y_i - w_2 z_{i2} \\ \dot{y}_i = w_3 z_{i1} - y_i - z_{i1}z_{i2} + bu_i,\ i = 1, \cdots, 4 \end{cases} \tag{37}$$

where $9 \le w_1 \le 10$, $25 \le w_2 \le 30$, $2 \le w_3 \le 3$, and $0.5 \le b \le 1.5$. As it has been shown in [22], all conditions of **H2** to **H4** in Theorem 3.1 are verifiable for this example and in particular, (10) can be determined by $n_s = 4$, $c_1 = -9\omega^4$, $c_3 = -10\omega^2$, and $c_2 = c_4 = 0$.

It is concluded that all the conditions in Theorem 3.1 are verifiable for both the above examples. To confirm the effectiveness of the proposed consensus algorithm, the simulation is done. One result is shown in Fig. 3 where for the FHN one in three-axis (z_i, y_i, t), $\omega = \pi/6$, $\sigma = 5$, $\jmath = 2$, $L = (4, 20, 41, 44, 26, 8)^\top$, $\kappa_i = 80(e_{mi}^2 + 6)e_{mi}$, $(w_1, w_2, w_3, b) = (1, 1, 1, 1)^\top$, leader initial condition (1,1), and follower initial condition (-2, 3; -3, 1; 3, -2;-1, 2); for the Lorenz one in three-axis (z_{i2}, y_i, t), $\omega = \pi/7$, $\sigma = 3$, $\jmath = 3$, $L = (4, 12, 13, 6)^\top$, $\kappa_i = 80(e_{mi}^4 + 4)e_{mi}$, $(w_1, w_2, w_3, b) = (10, 28, 8/3, 1)$, leader initial condition (2,1), and follower initial condition (-2, 0, 1; 0, 2, 1; 1, -1, 1.5; 1, 0, -1); all the other controller initial conditions are zero. In the simulation, we have chosen κ_i to be the same and increased to ensure the numerical performance. At the end, it is noticed that for the above Example 4.1 (or 4.2), if we employ (34), the controller order is 24 (or 16) that is certainly larger than 6 (or 4) of the controller (15).

V. Conclusion

An h-IM approach has been proposed to handle distributed output regulation of nonlinear identical agents. For the general interaction digraphs, we have presented a two-step consensus algorithm. The first step is to introduce a single h-IM and the second is to deal with a distributed stabilization problem. Particularly in the stabilization step, by utilizing a suitable non-quadratic storage function, we are able to accomplish the control goal. A couple of examples have been given to illustrate the proposed design. The outcome of future research is hoped on relevant studies for heterogeneous nonlinear multi-agent systems.

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