A General Result on the Robust Cooperative Output Regulation for Linear Uncertain Multi-Agent Systems

Youfeng Su, Yiguang Hong, and Jie Huang

***Abstract*—The cooperative output regulation problem for linear uncertain multi-agent systems was studied in [14] via an internal model approach under the assumption that the information graph of the system contains no cycle. In this technical note, we further show that the no-cycle assumption can be removed if all subsystems of the follower have the same nominal dynamics. Moreover, by directly making use of the property of the internal model, we provide a more straightforward proof than the one in [14] in that we don't need to verify the satisfaction of certain matrix equations.**

***Index Terms*—Cooperative control, internal model, linear uncertain systems, multi-agent systems, output regulation.**

I. Introduction

In this technical note, we consider the robust cooperative output regulation of linear uncertain multi-agent systems

$$\begin{aligned}\dot{x}_i &= \bar{A}_i x_i + \bar{B}_i u_i + \bar{E}_i v\\ y_i &= \bar{C}_i x_i, \quad i = 1, \ldots, N\end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^p$ and $u_i \in \mathbb{R}^m$ are the state, measurement output and control input of the ith subsystem, and $v \in \mathbb{R}^q$ is the exogenous signal representing the reference input to be tracked or the disturbance to be rejected and is assumed to be generated by a so-called exosystem as follows:

$$\dot{v} = Sv. \tag{2}$$

The regulated outputs for the subsystems are defined as

$$e_i = y_i - y_0, \quad i = 1, \ldots, N \tag{3}$$

where $y_0 = -Fv$ for some matrix F.

As described in [14], in the context of cooperative output regulation, system (1) and exosystem (2) together are considered as a multi-agent system of $N + 1$ agents with the exosystem as the leader and all N subsystems of system (1) as the followers, and the communication topology of this multi-agent system is described by a digraph[1] $\bar{\mathcal{G}} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$ where $\bar{\mathcal{V}} = \{0, 1, \ldots, N\}$ with the node 0 associated with the exosystem and the other N nodes associated with the N subsystems of system (1), and, for $i = 1, \ldots, N$, $j = 0, 1, \ldots, N$, $i \neq j$, $(j, i) \in \bar{\mathcal{E}}$ if and only if the control u_i of the subsystem i can access the state x_j or the output y_j of subsystem j, and, for $j = 1, \ldots, N$, $(j, 0) \notin \bar{\mathcal{E}}$ since the exosystem does not have a control input. As a result, the subsystems of system (1) can be divided into two groups. The first group consists of those subsystems whose controls can access y_0, while the second group consists of those subsystems whose controls cannot access y_0. Thus, as pointed out in [14], the robust cooperative output regulation problem as described in [14] and will be repeated in the next section in general cannot be solved by (purely) decentralized control. Nevertheless, under the assumption that the information graph of the system contains no cycle, the above problem can be solved using a distributed control law [14]. In this technical note, we will further show that the no-cycle assumption can be removed if all follower subsystems have the same nominal dynamics.

The no-cycle assumption is somehow stringent since it rules out the bidirected graph. This assumption was made in [14] so that the closed-loop system matrix is a block lower triangular matrix. This special structure lends itself to a straightforward stability analysis of the closed-loop system. In contrast, without the no-cycle assumption, the closed-loop system matrix is no longer a block lower triangular matrix and depends on the Laplacian of the information graph. Nevertheless, by applying the simultaneous eigenvalue assignment [12], we have managed to solve the problem without the no-cycle assumption. The establishment of the main result in [14] involves two steps: stability analysis of the closed-loop system and verification of certain matrix equations. In this note, by directly making use of the property of the internal model, we do not need to verify the satisfaction of certain matrix equations, thus significantly simplifying the proof.

It is noted that the cooperative output regulation problem of linear multi-agent system has also been studied by the feedforward control method in [8], [10], [11], [16]. This method is quite different from our internal model based control method, and cannot handle the plant uncertainty. One the other hand, it is interesting to mention two recent papers [9] and [15], which essentially deal with the leaderless consensus/synchronization problem. The difference of these two papers from this technical note will be highlighted in Remark 9 later.

The rest of this technical note is organized as follows: In Section II we introduce a mathematical description of linear robust cooperative output regulation problem. In Section III we present our main result. An example is provided to illustrate our design in Section IV. Finally, we close this note in Section V with some remarks.

II. Preliminaries

Like [14], we allow the entries of all matrices but S and F to be uncertain. We will adopt the following notation. $\bar{A}_i = A_i + \delta_i A$, $\bar{B}_i = B_i + \delta_i B$, $\bar{C}_i = C_i + \delta_i C$, $\bar{E}_i = E_i + \delta_i E$, $i = 1, \ldots, N$, where A_i, B_i, C_i, E_i are the nominal part of these matrices, and $\delta_i A, \delta_i B, \delta_i C, \delta_i E$ are perturbations of these matrices. For convenience, let

$$w = \begin{bmatrix} \mathrm{vec}(\delta_1 A, \ldots, \delta_N A)\\ \mathrm{vec}(\delta_1 B, \ldots, \delta_N B)\\ \mathrm{vec}(\delta_1 C, \ldots, \delta_N C)\\ \mathrm{vec}(\delta_1 E, \ldots, \delta_N E)\end{bmatrix} \in \mathbb{R}^{Nn(n+m+q+p)}.$$

We will consider the class of distributed control laws described as follows:

$$\begin{aligned}u_i &= k_i(z_i, z_j, x_i - x_j, j \in \mathcal{N}_i),\\ \dot{z}_i &= g_i(z_i, y_i - y_j, j \in \mathcal{N}_i), \quad i = 1, \ldots, N\end{aligned} \tag{4}$$

where $\mathcal{N}_i = \{j, (j, i) \in \bar{\mathcal{E}}\}$, and k_i and g_i are linear functions of their arguments. A control law of the form (4) is called a distributed dynamic state feedback control law, and is further called a distributed dynamic

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[1]See Appendix for a summary of digraph.

output feedback control law if the functions k_i are independent of any state variable. We can now state our problem as follows:

Definition 1: Given the plant (1), exosystem (2), and a digraph $\bar{\mathcal{G}}$, find a control law of the form (4) such that:

1) the nominal closed-loop system matrix is Hurwitz;
2) there exists an open neighborhood W of $w = 0$, for any $w \in W$ and any initial condition $x_i(0)$, $z_i(0)$ and $v(0)$, the regulated output $\lim_{t\to\infty} e_i(t) = 0, i = 1, \ldots, N$.

Assumptions needed for solving the above problem are listed below.

Assumption 1: There exist matrices A, B, C such that $A_i = A$, $B_i = B, C_i = C, i = 1, \ldots, N$.

Assumption 2: S has no eigenvalues with negative real parts.

Assumption 3: The pair (A, B) is stabilizable.

Assumption 4: The pair (C, A) is detectable.

Assumption 5: For all $\lambda \in \sigma(S)$, where $\sigma(S)$ is the spectrum of S, rank $\begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix} = n + p$.

Assumption 6: The digraph $\bar{\mathcal{G}}$ contains a directed spanning tree with the node 0 as its root.

Remark 1: Assumption 1 means that all the followers have the same nominal dynamics. Assumptions 2 to 5 are standard ones for guaranteeing the solvability of the linear robust output regulation problem by classical method or the centralized control method in [1]–[3], [7]. However, due to the limited information exchange among the individual agents, the centralized control method is not allowed here. Reference [14] has solved the above problem using a distributed control law of the form (4) under the condition that the digraph $\bar{\mathcal{G}}$ contains no cycle. Assumption 1 is not needed in [14]. It is introduced here in order to remove the assumption that the digraph $\bar{\mathcal{G}}$ contains no cycle.

III. Main Result

Let us first define a subgraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ of $\bar{\mathcal{G}}$ with $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E}$ being obtained from $\bar{\mathcal{E}}$ by removing all edges of $\bar{\mathcal{E}}$ which are incident with the node 0. Let $\mathcal{A} = [a_{ij}]_{i,j=1}^N \in \mathbb{R}^{N\times N}$ be any weighted adjacency matrix of $\mathcal{G}$, and $\mathcal{L}$ be the Laplacian of $\mathcal{G}$ corresponding to $\mathcal{A}$. Also let $\Delta = \text{diag}[a_{10}, \cdots, a_{N0}]$ where, for $i = 1, \ldots, N, a_{i0} > 0$ if $(0, i) \in \bar{\mathcal{E}}$ and $a_{i0} = 0$ if otherwise, and $H = \mathcal{L} + \Delta$. Then

$$H\mathbf{1}_N = \Delta\mathbf{1}_N. \tag{5}$$

where $\mathbf{1}_N$ denotes an N-dimensional column vector with all elements being 1.

Remark 2: It is shown in Lemma 4 of [6] that all of the nonzero eigenvalues of the matrix H have positive real parts. Furthermore, all the eigenvalues of the matrix H have positive real parts if and only if Assumption 6 is satisfied.

In terms of $\mathcal{A}$ and Δ, we can define a virtual regulated output $e_{vi}(t)$ for each follower subsystem i as follows:

$$e_{vi} = \sum_{j\in\mathcal{N}_i} a_{ij}(y_i - y_j), \quad i = 1, \ldots, N. \tag{6}$$

Note that the subsystem i can access the regulated error e_i, i.e., the ith subsystem is of the first group, if and only if $a_{i0} > 0$.

Our control law will make use of the internal model principle in order to handle the plant uncertainty. For this purpose, we will introduce the concept of the minimum p-copy internal model as follows [1], [7].

Definition 2: A pair of matrices (G_1, G_2) is said to incorporate the minimum p-copy internal model of the matrix S if

$$G_1 = \text{blockdiag}\underbrace{[\beta, \ldots, \beta]}_{p-tuple}, \quad G_2 = \text{blockdiag}\underbrace{[\sigma, \ldots, \sigma]}_{p-tuple} \tag{7}$$

where β is a constant square matrix whose characteristic polynomial equals the minimal polynomial of S, and σ is a constant column vector such that (β, σ) is controllable.

Having defined the virtual regulated output $e_{vi}(t)$ and introduced the p-copy internal model, we can describe our *distributed dynamic state feedback control law*

$$\begin{aligned} u_i &= K_x\left(\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j) + a_{i0}x_i\right) + K_z z_i \\ \dot{z}_i &= G_1 z_i + G_2 e_{vi}, \quad i = 1, \ldots, N \end{aligned} \tag{8}$$

where $z_i \in \mathbb{R}^{n_z}$ with n_z to be specified later and K_x, K_z are some constant matrices to be designed, and, respectively, our *distributed dynamic output feedback control law*

$$\begin{aligned} u_i &= K_1\left(\sum_{j\in\mathcal{N}_i} a_{ij}(\xi_i - \xi_j) + a_{i0}\xi_i\right) + K_2 z_i \\ \dot{\xi}_i &= A\xi_i + Bu_i - LC\left(\sum_{j\in\mathcal{N}_i} a_{ij}(\xi_i - \xi_j) + a_{i0}\xi_i\right) + Le_{vi} \\ \dot{z}_i &= G_1 z_i + G_2 e_{vi}, \quad i = 1, \ldots, N \end{aligned} \tag{9}$$

where $\xi_i \in \mathbb{R}^n$ and $z_i \in \mathbb{R}^{n_z}$ with n_z to be specified later, and K_1, K_2, L are some constant matrices to be designed.

Remark 3: The virtual regulated output in (6) is somehow different from that of [14] which is repeated as follows

$$e_{vi} = \begin{cases} e_i, & (0, i) \in \bar{\mathcal{E}}; \\ \sum\limits_{j\in\mathcal{N}_i} \frac{1}{|\mathcal{N}_i|}(y_i - y_j), & (0, i) \notin \bar{\mathcal{E}}. \end{cases} \tag{10}$$

It is noted that the virtual regulated output e_{vi} as defined in (10) has the property that $e_{vi} = 0, i = 1, \ldots, N$ if and only if $e_i = 0, i = 1, \ldots, N$ under Assumption 6 plus the no cycle assumption. This is a key property for the success of the control law in [14]. It will be shown in Lemma 1 that, under Assumption 6 alone, the virtual regulated output e_{vi} as defined in (6) also possesses this key property.

Remark 4: In [14], y_i is defined as $y_i = \bar{C}_i x_i + F_i v$, where $F_i = F$ when $(0, i) \in \bar{\mathcal{E}}$ and $F_i = 0$ if otherwise (See Assumption 5 of [14]). However, this definition may lead to inconsistency of (7) in [14]. The inconsistency of (7) of [14] can be corrected by modifying Assumption 5 of [14] to "$F_i = 0, i = 1, \ldots, N$". Therefore, in this technical note, we have $y_i = \bar{C}_i x_i$ for all $i = 1, \ldots, N$.

Let $x = [x_1^T, \ldots, x_N^T]^T$, $e_v = [e_{v1}^T, \ldots, e_{vN}^T]^T$, $u = [u_1^T, \ldots, u_N^T]^T$, $\tilde{A} = \text{blockdiag}[\bar{A}_1, \ldots, \bar{A}_N]$, $\tilde{B} = \text{blockdiag}[\bar{B}_1, \ldots, \bar{B}_N]$, $\tilde{E} = [\bar{E}_1^T, \ldots, \bar{E}_N^T]^T$, $\tilde{C} = (H \otimes I_p)\text{blockdiag}[\bar{C}_1, \ldots, \bar{C}_N]$, $\tilde{F} = (\Delta\mathbf{1}_N) \otimes F$, where $\otimes$ denotes the Kronecker product of matrices. Also, let $\tilde{G}_1 = I_N \otimes G_1$ and $\tilde{G}_2 = I_N \otimes G_2$. Then we can define an augmented system as follows:

$$\begin{aligned} \dot{x} &= \tilde{A}x + \tilde{B}u + \tilde{E}v \\ \dot{z} &= \tilde{G}_1 z + \tilde{G}_2 e_v \\ \dot{v} &= Sv \\ e_v &= \tilde{C}x + \tilde{F}v. \end{aligned} \tag{11}$$

Remark 5: It is easy to see that Assumptions 3 implies that the pair $(\tilde{A}, \tilde{B})$ is stabilizable. It can also be seen that the pair $(\tilde{G}_1, \tilde{G}_2)$ incorporates the minimum pN-copy internal model of the matrix S. Thus, this system is stabilizable and detectable at $w = 0$ with the state $[x^T, z^T]^T$, input u, and regulated output e_v [3]. Moreover, from Lemma 20 of [7], if any state feedback control law of the form $u = \tilde{K}_x x + \tilde{K}_z z$, or output feedback control law of the form $u = \tilde{K}_1\xi + \tilde{K}_2 z, \dot{\xi} = \tilde{F}_1\xi + \tilde{F}_2 e_v$, stabilizes the nominal plant of the augmented system (11), then the dynamic state feedback control law of the form $u = \tilde{K}_x x + \tilde{K}_z z, \dot{z} = \tilde{G}_1 z + \tilde{G}_2 e_v$, or the dynamic output feedback control law of the form $u = \tilde{K}_1 \xi + \tilde{K}_2 z, \dot{\xi} = \tilde{F}_1 \xi + \tilde{F}_2 e_v, \dot{z} = \tilde{G}_1 z + \tilde{G}_2 e_v$, solves the robust output regulation problem of the following plant:

$$\begin{aligned}\dot{x} &= \tilde{A}x + \tilde{B}u + \tilde{E}v\\ \dot{v} &= Sv\\ e_v &= \tilde{C}x + \tilde{F}v.\end{aligned} \tag{12}$$

Moreover, what makes our approach interesting is that the same control law will also solve the robust cooperative output regulation problem as described in Definition 1 due to the following lemma.

Lemma 1: Under Assumption 6, for $i = 1, \ldots, N, \lim_{t\to\infty} e_i(t) = 0$ if and only if $\lim_{t\to\infty} e_{vi}(t) = 0$.

Proof: It is noted by (5) and (12) that

$$\begin{aligned}e_v &= (H\otimes I_p)\tilde{C}x + \Delta \mathbb{1}_N \otimes Fv = (H\otimes I_p)\tilde{C}x + H\mathbb{1}_N\otimes Fv\\ &= (H\otimes I_p)(\tilde{C}x + \mathbb{1}_N\otimes Fv) = (H\otimes I_p)e.\end{aligned}$$

By Remark 2, H is nonsingular under Assumption 6. Therefore, $\lim_{t\to\infty} e_i(t) = 0$ if and only if $\lim_{t\to\infty} e_{vi}(t) = 0, i = 1, \ldots, N$. □

Lemma 2 (Lemma 1 in [12]): Given any stabilizable pair (A, B), $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, the algebraic Riccati equation $A^T P + PA + I_n - PBB^T P = 0$ admits a positive definite solution. Then for any $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) \geq 1$, $A - \mu BB^T P$ is Hurwitz.

Lemma 3: Let (G_1, G_2) be the minimal p-copy internal model of S as defined in (7). Then, under Assumptions 2, 3, and 5, there exist matrices (K_x, K_z) such that

$$A_c \triangleq \begin{bmatrix} I_N\otimes A + H\otimes BK_x & I_N\otimes (BK_z)\\ H\otimes (G_2C) & I_N\otimes G_1\end{bmatrix}$$

is Hurwitz if and only if Assumption 6 is satisfied.

Proof: (If part:) Denote the eigenvalues of H by λ_i, $i = 1, \ldots, N$. Let T_1 be the nonsingular matrix such that $J_H = T_1 H T_1^{-1}$ is the Jordan form of H. Let $T_2 = \begin{bmatrix} T_1\otimes I_n & 0\\ 0 & T_1\otimes I_n\end{bmatrix}$. Then $\bar{A}_c \triangleq T_2 A_c T_2^{-1} = \begin{bmatrix} I_N\otimes A + J_H\otimes BK_x & I_N\otimes (BK_z)\\ J_H\otimes (G_2C) & I_N\otimes G_1\end{bmatrix}$.

Let $T_3 = [i_1^T, i_{N+1}^T, i_2^T, i_{N+2}^T, \ldots, i_N^T, i_{2N}^T]^T$, where i_k is the k-th row of I_{2N}. Then $\hat{A}_c \triangleq (T_3\otimes I_n)\bar{A}_c(T_3^{-1}\otimes I_n)$ is a lower block triangular matrix whose diagonal blocks are

$$\hat{A}_{ci} \triangleq \begin{bmatrix} A + \lambda_i BK_x & BK_z\\ \lambda_i G_2 C & G_1\end{bmatrix}, \quad i = 1, \ldots, N \tag{13}$$

Under Assumption 6, by Remark 2, for all $i = 1, \ldots, N, \lambda_i$ have positive real part. Let $T_{4i} = \begin{bmatrix} I_n & 0\\ 0 & \lambda_i^{-1} I_n\end{bmatrix}$ and $\tilde{A}_{ci} \triangleq \begin{bmatrix} A + \lambda_i BK_x & \lambda_i BK_z\\ G_2C & G_1\end{bmatrix}$, $i = 1, \ldots, N$. It can be verified that $T_{4i}\hat{A}_{ci}T_{4i}^{-1} = \tilde{A}_{ci}$. Therefore, A_c is Hurwitz if and only if, for all $i = 1, \ldots, N$, $\hat{A}_{ci}$ and hence $\tilde{A}_{ci}$ are.

Let $Y \triangleq \begin{bmatrix} A & 0\\ G_2C & G_1\end{bmatrix}$, $J \triangleq \begin{bmatrix} B\\ 0\end{bmatrix}$. Under Assumptions 2 and 3, by Remark 5, (Y, J) is stabilizable. Therefore, the Riccati Equation

$$Y^T P + PY - PJJ^T P + I = 0 \tag{14}$$

admits a positive definite solution P. Let

$$[K_x, K_z] \triangleq K = -\nu_1^{-1} J^T P \tag{15}$$

where $\nu_1 \in \mathbb{R}$ satisfies

$$0 < \nu_1 \leq \mathrm{Re}(\lambda_i), \quad i = 1, \ldots, N. \tag{16}$$

Therefore, $\tilde{A}_{ci} = Y + \lambda_i JK = Y - \lambda_i \nu_1^{-1} JJ^T P$. Since $\mathrm{Re}(\lambda_i\nu_1^{-1}) = \nu_1^{-1}\mathrm{Re}(\lambda_i) \geq 1$, it follows from Lemma 2 that, for all $i = 1, \ldots, N$, $\tilde{A}_{ci}$ are Hurwitz. Thus, A_c is Hurwitz.

(Only if part:) Suppose the digraph $\bar{\mathcal{G}}$ does not satisfy Assumption 6. Then by Remark 2, H has at least one eigenvalue at the origin. Without loss of generality, we assume that $\lambda_1 = 0$. Then by (13), $\hat{A}_{c1} = \begin{bmatrix} A & BK_z\\ 0 & G_1\end{bmatrix}$. Since the spectrum of G_1 is the same as the spectrum of S, $\hat{A}_{c1}$ and hence A_c cannot be Hurwitz regardless of the choice of $[K_x, K_z]$. The proof is thus completed. □

Remark 6: The sufficient part of Lemma 3 was first presented in Theorem 2 of [5]. The proof here has somehow simplified the one in [5].

We are now ready to present our main result.

Theorem 1: Under Assumptions 1 to 3, and 5, there exist matrices K_x, K_z such that the robust cooperative output regulation problem can be solved by the distributed state feedback control law (8) with (G_1, G_2) incorporating the minimal p-copy internal model of S if and only if Assumption 6 is satisfied.

Proof: Under the state feedback control law (8), the nominal closed-loop system matrix is given by A_c. The proof is completed by noting Lemma 3 and Remark 5. □

Remark 7: Recently, under the assumption that C is of full row rank, a similar problem was studied in [5] by a distributed control law that contains a feedforward term $C^\dagger Fv$, where $C^\dagger$ is the Moore-Penrose inverse of the matrix C. Thus, the control law in [5] is not robust with respect to C, and has to make use of Fv.

Theorem 2: Under Assumptions 1 to 5, there exist matrices K_1, K_2, L such that the robust cooperative output regulation problem can be solved by the distributed output feedback control law (9) with (G_1, G_2) incorporating the minimal p-copy internal model of S if and only if Assumption 6 is satisfied.

Proof: Under the control law (9), the nominal closed-loop system matrix takes the following form:

$$A_c = \begin{bmatrix} I_N\otimes A & H\otimes BK_1 & I_N\otimes BK_2\\ H\otimes LC & I_N\otimes A + H\otimes(BK_1 - LC) & I_N\otimes BK_2\\ H\otimes G_2C & 0 & I_N\otimes G_1\end{bmatrix}$$

Let

$$T = \begin{bmatrix} I_{Nn} & 0 & 0\\ 0 & 0 & I_{Nn}\\ -I_{Nn} & I_{Nn} & 0\end{bmatrix}.$$

Then

$$\begin{aligned}\bar{A}_c &\triangleq TA_cT^{-1}\\ &= \left[\begin{array}{cc|c} I_N\otimes A + H\otimes BK_1 & I_N\otimes BK_2 & H\otimes BK_1\\ H\otimes G_2C & I_N\otimes G_1 & 0\\ \hline 0 & 0 & I_N\otimes A - H\otimes LC\end{array}\right]\end{aligned}$$

Let $K_1 = K_x$, $K_2 = K_z$, where $[K_x, K_z]$ is defined in (15). By Lemma 3, the first diagonal block matrix $\bar{A}_c$ can be made Hurwitz by this choice of $[K_1, K_2]$ if and only if the digraph $\bar{\mathcal{G}}$ satisfies Assumption 6. Thus the only if part has been proved. To complete the proof of if part, it suffices to show that there exists L such that $I_N\otimes A - H\otimes LC$ or equivalently $I_N\otimes A - J_H\otimes LC$ is Hurwitz. Or $A - \lambda_i LC$ is Hurwitz, for $i = 1, \ldots, N$. Since (C, A) is detectable, (A^T, C^T) is stabilizable. Therefore, the Riccati Equation

$$A\tilde{P} + \tilde{P}A^T - \tilde{P}C^T C\tilde{P} + I = 0 \tag{17}$$

admits a positive definite solution $\tilde{P}$. Let

$$L^T = \nu_2^{-1} C\tilde{P} \tag{18}$$

where $\nu_2 \in \mathbb{R}$ satisfies

$$0 < \nu_2 \leq \mathrm{Re}(\lambda_i), \quad i = 1, \ldots, N. \tag{19}$$

Then by Lemma 2, $A^T - \lambda_i C^T L^T$ and hence $A - \lambda_i LC, i = 1, \ldots, N$ are Hurwitz. The proof is then completed by noting Remark 5. □

Remark 8: From Theorems 1 and 2, the nonzero weights a_{ij} can be arbitrary assigned as long as its associated digraph $\bar{\mathcal{G}}$ satisfies Assumption 6. Different choices of the nonzero weights may lead to different convergent rates. However, the transient response of the closed-loop system can be more effectively shaped by the locations of the eigenvalues of the closed-loop system matrix which are determined by the gain matrices K_1, K_2 and L.

Remark 9: It is interesting to mention two recent papers [9] and [15], which are relevant to this technical note. Both of these two papers essentially deal with the leaderless consensus/synchronization problem. The objective of the control is to make all the outputs or states equal and does not care and cannot dictate the asymptotic behavior of the output/state of each individual subsystem. While [9] makes use of the internal model design, [15] employs the feedforward approach. In contrast, our problem requires the output of each subsystem asymptotically track a reference input and reject a disturbance asymptotically by a distributed control law. Here the reference input and disturbance are generated by an exosystem called the leader system. Since our problem includes the leader-following consensus/synchronization problem as a special case, its difference from the above papers is comparable with the difference between the leaderless and leader-following consenuse/synchronization problems. Moreover, our control laws are also quite different from those in [9] and [15] in that ours make the closed-loop system matrix Hurwitz, while in [9] and [15], the closed-loop system is not asymptotically stable.

Remark 10: The output regulation problem of linear multi-agent systems with a type of symmetric switching graph was studied recently in [13] by a so-called canonical internal model. Our design method is different than that of [13] in several aspects. First, the method in [13] only applies to the case where $p = m$. Second, the method in [13] can only lead to an output feedback control law. Third, the dimension of the output feedback control law in [13] is much higher than that in this technical note because the control law in [13] needs to estimate the state of the internal model while the output feedback control law of this note does not.

IV. An Example

Consider a group of 4 robots modeled as double integrator with parameter uncertainties and external disturbance:

$$\begin{aligned} \dot{x}_{1i} &= x_{2i} \\ \dot{x}_{2i} &= \delta_{1i} x_{1i} + \delta_{2i} x_{2i} + u_i + \mu_i v_3 \\ y_i &= x_{1i} \\ e_i &= x_{1i} - (v_1 + v_2), \quad i = 1, 2, 3, 4 \end{aligned} \tag{20}$$

where $\mu_i = i, \delta_{ji} = 0.05 * j * i, j = 1, 2, i = 1, 2, 3, 4$, are parameter uncertainties, and the exogenous signal $v = (v_1, v_2, v_3)^T$ is generated by the exosystem of form (2) with $S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. System (20) is in the form (1) with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $E_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \end{bmatrix}$, $C = [1 \ \ 0]$, $F = [-1 \ \ -1 \ \ 0]$, $\delta_i A = \begin{bmatrix} 0 & 0 \\ \delta_{1i} & \delta_{2i} \end{bmatrix}$. The information exchange among all agents is described by the digraph $\bar{\mathcal{G}}$ as shown in Fig. 1. Let

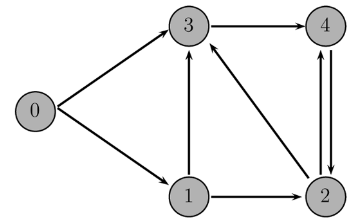

Fig. 1. Network topology $\bar{\mathcal{G}}$.

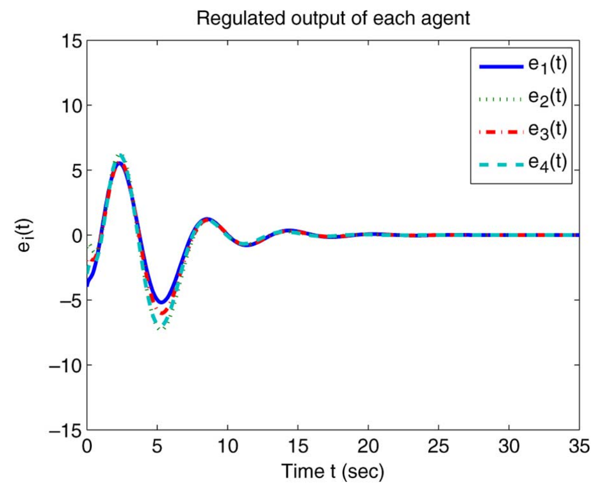

Fig. 2. Tracking performance under the dynamic state feedback control law.

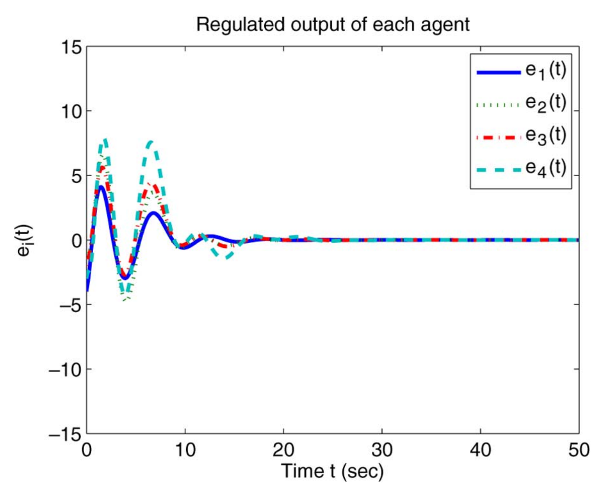

Fig. 3. Tracking performance under the dynamic output feedback control law.

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & -1 & 3 & 0 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

whose eigenvalues are $\{0.7944, 1.0000, 3.1028 + 0.6655\jmath, 3.1028 - 0.6655\jmath\}$, where $\jmath \triangleq \sqrt{-1}$.

It can be verified that Assumptions 2–6 are all satisfied. Thus, it is possible to solve the problem using our distributed control laws (8) and (9), respectively. The design process is given as follows. Let

$$G_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

By solving the Riccati Equation (14), we obtain

$$P = \begin{bmatrix} 10.6869 & 4.2948 & 3.0967 & -0.3431 & 8.7226 \\ 4.2948 & 3.0967 & 1.0000 & -0.6316 & 2.6128 \\ 3.0967 & 1.0000 & 3.6632 & 2.6128 & 4.2948 \\ -0.3431 & -0.6316 & 2.6128 & 5.6194 & 2.9133 \\ 8.7226 & 2.6128 & 4.2948 & 2.9133 & 11.5645 \end{bmatrix}.$$

Let $\nu_1 = 1/3$ which satisfies (16). Then by (15), we have $[K_x, K_z] = -3 * [4.2948\ 3.0967\ 1.0000\ -0.6316\ 2.6128]$. By solving the Riccati Equation (17), we obtain $\tilde{P} = \begin{bmatrix} 1.7321 & 1.0000 \\ 1.0000 & 1.7321 \end{bmatrix}$. Let $\nu_2 = 1/4$ which satisfies (19). Then by (18), we have $L = 4 * \begin{bmatrix} 1.7321 \\ 1.0000 \end{bmatrix}$. Applying controllers (8) and (9) with the initial conditions being $x_{11}(0) = 1$, $x_{21}(0) = 4$, $x_{12}(0) = 4$, $x_{22}(0) = 3$, $x_{13}(0) = 3$, $x_{23}(0) = 2$, $x_{14}(0) = 2$, $x_{24}(0) = 1$, $z_1(0) = [0, 2, 4]^T$, $z_2(0) = [1, 4, 3]^T$, $z_3(0) = [2, 1, 4]^T$, $z_4(0) = [3, 2, 1]^T$, $v(0) = [4, 1, 1]^T$, and $\xi_1(0) = [1, 4]^T$, $\xi_2(0) = [4, 3]^T$, $\xi_3(0) = [3, 2]^T$, $\xi_4(0) = [1, 2]^T$, we obtain the simulation results as shown in Figs. 2 and 3. It can be seen that all the regulated outputs of the subsystems converge to the origin asymptotically.

V. Conclusion

In this technical note, we have removed the no-cycle condition on the solvability of the cooperative output regulation problem studied in [14]. Our future work will focus on the same problem for nonlinear multi-agent systems.

Appendix

We summarize some graph terminologies in [4]. A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge of $\mathcal{E}$ from node i to node j is denoted by (i, j), where the nodes i and j are called the parent node and the child node of each other. If the digraph $\mathcal{G}$ contains a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})$, then the set $\{(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})\}$ is called a path of $\mathcal{G}$ from node i_1 to node i_{k+1}, and node i_{k+1} is said to be reachable from node i_1. If $i_1 = i_{k+1}$, the path is called a cycle. The edge (i, j) is called bidirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. The graph is called bidirected if every edge in $\mathcal{E}$ is bidirected. A digraph $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ is a subgraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if $\mathcal{V}_s \subseteq \mathcal{V}$ and $\mathcal{E}_s \subseteq \mathcal{E} \cap (\mathcal{V}_s \times \mathcal{V}_s)$. A directed tree is a digraph in which every node has exactly one parent except for one node, called the root, which has no parent and from which every other node is reachable. A subgraph $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ of the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called a directed spanning tree of $\mathcal{G}$ if $\mathcal{G}_s$ is a directed tree and $\mathcal{V}_s = \mathcal{V}$, and, in this case, we say the digraph $\mathcal{G}$ contains a directed spanning tree.

A matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is said to be a weighted adjacency matrix of a digraph $\mathcal{G}$ if $a_{ii} = 0$, $a_{ij} > 0 \Leftrightarrow (j, i) \in \mathcal{E}$, and $a_{ij} = a_{ji}$ if (j, i) is a bidirected edge of $\mathcal{E}$. Let $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ be such that $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$ if $i \neq j$. Then $\mathcal{L}$ is called the Laplacian of the graph $\mathcal{G}$ corresponding to $\mathcal{A}$.

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