A Distributed Control Approach to A Robust Output Regulation Problem for Multi-Agent Linear Systems

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***Abstract*—In this note, the robust output regulation problem of a multi-agent system is considered. An internal model based distributed control scheme is adopted to achieve the objectives of asymptotic tracking and disturbance rejection in an uncertain multi-agent system where both the reference inputs and disturbances are generated by an exosystem.**

***Index Terms*—Distributed control, internal model, multi-agent systems, robust output regulation.**

I. Introduction

Control of a group of subsystems called agents has been an active and important area, and distributed control has become a successful strategy to handle such design issues as stabilization, formation control, and output regulation of multi-agent systems [5], [7], [9], [11], [13]–[15].

In this note, we consider the robust output regulation problem of a linear uncertain system composed of N agents. What makes our problem distinct from the standard linear output regulation problem as can be found in ([2]–[4] and [12]) is that the control design for each agent cannot access all the information of measurement outputs. To be more specific, the N agents are classified into two groups: each agent of the first group can only use its measurement output for feedback control, and each agent of the second group can only use the measurement outputs of itself and its neighbors for feedback control. Moreover, the regulated output of these agents may not be readable from the measurement output. Therefore, the problem cannot be solved by a decentralized control scheme in which each local control law can only takes the measurement output of each agent [3]. Under some standard assumptions plus a mild assumption that the leader node of the connection graph is global reachable, we have managed to solve the above problem by both state feedback and output feedback controls.

The problem is motivated by many practical problems. Such problems include having a group of tanks to follow a leader tank for the scenario where some tanks may not see their leader in the parading team but they can get the position information from those tanks located just before them using vision-based sensors, or having a team of cooperative mobile robots to catch a moving target in an uncertain environment.

In contrast with the similar problem studied in [1] and [15], we allow parameter uncertainties in system matrices by employing the internal model technique. Under mild assumptions such as the global reachability of the leader node (corresponding to the exosystem), we establish that this distributed scheme can lead to the solution of the output regulation problem for uncertain systems by both state feedback and output feedback control.

The rest of the note is organized as follows. Section II introduces some preliminaries and the problem formulation, while Section III proposes distributed output regulation via state and output feedback using an internal model idea. A numerical example is given in Section IV. Finally, some concluding remarks are given in Section V.

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II. Preliminaries and Problem Formulation

Consider an uncertain linear system composed of N interconnected agents as follows:

$$\begin{cases} \dot{v} = \Gamma v, \\ \dot{x}_i = \bar{A}_i x_i + \bar{B}_i u_i + \bar{E}_i v, \\ y_i = \bar{C}_i x_i + F_i v, \\ e_i = \bar{C}_i x_i + F v \end{cases} \quad i = 1, \cdots, N \tag{1}$$

with $x_i \in \mathbb{R}^n$ as the state of ith subsystem, $u_i \in \mathbb{R}^m$ as the control input, $y_i \in \mathbb{R}^p$ as the measurement output, i.e., the output that can be measured for feedback control, $e_i \in \mathbb{R}^p$ as the regulated output, i.e., the output that is to be regulated to the origin, and $v \in \mathbb{R}^q$ represent both the reference input and disturbance and is generated by the exosystem $\dot{v} = \Gamma v$. We assume the matrices Γ, F, and $F_i, i = 1, \cdots, N$, are known, and, on the other hand, the matrices $\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{D}_i, i = 1, \cdots, N$ are uncertain and they admit the following forms:

$$\begin{aligned} \bar{A}_i &= A_i + \delta A_i, \quad \bar{B}_i = B_i + \delta B_i, \\ \bar{C}_i &= C_i + \delta C_i, \quad \bar{E}_i = E_i + \delta E_i, \quad i = 1, \cdots, N \end{aligned} \tag{2}$$

where A_i, B_i, C_i, and E_i $(i = 1, \cdots, N)$ are known and δA_i, δB_i, δC_i, and δE_i describe the perturbation of A_i, B_i, C_i, and E_i $(i = 1, \cdots, N)$ from their nominal values, respectively.

For convenience, define

$$\begin{aligned} w = &(vec(\delta A_1 \cdots \delta A_N) vec(\delta B_1 \cdots \delta B_N) vec(\delta C_1 \cdots \delta C_N) \\ &\quad vec(\delta E_1 \cdots \delta E_N))^T \\ &\in \mathbb{R}^{Nn(n+m+p+q)} \end{aligned} \tag{3}$$

where $vec(Y) = (Y_1 \cdots Y_{l_1})$ with Y_i the ith row of $Y \in \mathbb{R}^{l_1 \times l_2}$. The system with $w = 0$ is called a nominal system.

Some standard assumptions are listed below:

Assumption A1: The real parts of the eigenvalues of Γ are nonnegative.

Assumption A2: (A_i, B_i) is stabilizable for $i = 1, \ldots, N$.

Assumption A3:

$$rank \begin{pmatrix} A_i - \lambda I & B_i \\ C_i & 0 \end{pmatrix} = n + p, \quad \lambda \in \sigma(\Gamma), \quad i = 1, \ldots, N \tag{4}$$

where $\sigma(\Gamma)$ denotes the spectrum of Γ.

Assumption A4: (C_i, A_i) is detectable for $i = 1, \ldots, N$.

Remark 2.1: The robust output regulation problem has been thoroughly studied in, e.g., [2]–[4] and [10]. In particular, under Assumptions A1 to A4, the robust output regulation problem of each subsystem can be solved by a decentralized output feedback controller of the following form (Theorem 1.31 of [10])

$$\begin{cases} u_i = K_{zi} z_i \\ \dot{z}_i = E_{zi} z_i + E_{yi} y_i \end{cases} \quad z_i \in \mathbb{R}^s, \quad i = 1, \dots, N \tag{5}$$

provided that, for all $i = 1, \cdots, N$, the regulated output e_i is readable from the measurement output y_i, i.e., there exists a constant matrix T_i such that $e_i(t) = T_i y_i(t)$ for all $t \geq 0$ [4],[8], (also see Remark 1.29 of [10]).

However, as will soon be seen from our problem formulation, what makes our problem interesting is that even though our problem cannot be solved by the decentralized control law (5) because the readability condition is not satisfied for all subsystems,[1] it is still possible to overcome the "non-readability difficulty" by introducing a virtual regulated output for each subsystem. As a result, we can obtain a so-called distributed control law as will be defined in (8) and (9) to achieve our objectives. We stress that our approach is totally different from the decentralized control scheme in [3] where the output regulation problem for each agent is assumed to be solved by a local control law.

To describe our problem, we need to give a brief introduction to the concept of graph ([6]). A directed graph or digraph $\mathcal{G}$ is a pair of sets $(\mathcal{E}, \mathcal{N})$, where $\mathcal{N} = \{0, 1, 2, \cdots, N\}$ is called a node set and $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is called an edge set. If $(i, j) \in \mathcal{E}$, then node i is said to be the father of node j and node j is said to be the child of node i. All the fathers of node i constitute an in-neighboring set of node i and will be denoted by $\mathcal{N}_i$.

If the digraph $\mathcal{G}$ contains a sequence of edges of the form (i_1, i_2), $(i_2, i_3), \cdots, (i_k, i_{k+1})$, then the set $\{(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_{k+1})\}$ is called a path of $\mathcal{G}$ from i_1 to i_{k+1} and node i_1 is said to be reachable from node i_{k+1}. If $i_{k+1} = i_1$, then the path is called a loop. If a node is reachable from every other node of the digraph, then the node is called globally reachable.

The adjacency matrix of $\mathcal{G}$ is denoted as $A_L = (a_{ij})_{(N+1)\times(N+1)} \in \mathbb{R}^{(N+1)\times(N+1)}$ where $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The diagonal matrix $D_L = diag(\sum_{j=0}^{N} a_{0j}, \dots, \sum_{j=0}^{N} a_{Nj})$ is called the degree matrix of $\mathcal{G}$ and the matrix $L = D_L - A_L$ is called the Laplacian of $\mathcal{G}$.

A digraph $\mathcal{G}$ together with its Laplacian L can be used to describe the information exchange of a multi-agent system. Given the system (1), we can define a digraph $\mathcal{G}$ with $N + 1$ nodes in which node 0 is associated with the exosystem, and the other N nodes are associated with the N subsystems. The edge set $\mathcal{E}$ contains an edge (i, j) iff the subsystem j can use the measurement output y_i for feedback design.

Having associated system (1) with the digraph $\mathcal{G}$, we can further make an assumption on the measurement output as follows:

Assumption A5: For $i = 1, \cdots, N$, $F_i = F$ if $(0, i) \in \mathcal{E}$, and $F_i = 0$ otherwise.

Remark 2.2: Assumption A5 means that, for subsystems that are the children of the leader node (exosystem), the measurement output and the regulated output are the same, and for subsystems that are not the children of the leader node, the measurement output and the regulated output are generally not the same. This assumption reflects the fact that a subsystem can access the state v of the leader iff it is the child of the leader.

Now we are ready to introduce our distributed control law. First, let us define a virtual regulated output e_{iv} for subsystem i as follows:

$$e_{iv} = g_{iv}(y_i, y_j, j \in \mathcal{N}_i) \tag{6}$$

[1]Readability of e_i from y_i implies $F_i = F, i = 1, 2, \cdots, N$.

where $g_{iv} = y_i$ if $(0, i) \in \mathcal{E}$, and otherwise

$$g_{iv} = \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|}(e_i - e_j) = \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|}(y_i - y_j) \tag{7}$$

where $|\mathcal{N}_i|$ is the cardinality of the set $\mathcal{N}_i$ $(i = 1, \dots, N)$.

Clearly, g_{iv} is a linear combination of the measurable outputs of these subsystems associated with the in-neighboring set of node i. Thus we can define the following two classes of distributed control laws:

1) **Dynamic State Feedback**:

$$\begin{cases} u_i = K_{zi} z_i + K_{xi} x_i \\ \dot{z}_i = G_1 z_i + G_2 e_{iv}, \quad z_i \in \mathbb{R}^s \end{cases} \quad i = 1, \dots, N \tag{8}$$

where K_{zi}, K_{xi}, G_1 and G_2 are some constant matrices to be described in Section III.

2) **Dynamic Output Feedback**:

$$\begin{cases} u_i = K_i z_i \\ \dot{z}_i = \bar{G}_{1i} z_i + \bar{G}_{2i} e_{iv}, \quad z_i \in \mathbb{R}^s \end{cases} \quad i = 1, \dots, N \tag{9}$$

where K_i, $\bar{G}_{1i}$ and $\bar{G}_{2i}$ are some constant matrices to be described in Section III.

We now describe the distributed robust output regulation problem as follows:

Problem Statement: Given system (1), find a dynamic feedback control law of the form (8) or (9) such that the nominal closed-loop system matrix is Hurwitz, and, for all initial conditions of the closed-loop system and exosystem, and all sufficiently small parameter perturbation w, the trajectories of the closed-loop system satisfy

$$\lim_{t \to +\infty} e_i(t) = 0, \quad i = 1, \dots, N. \tag{10}$$

Remark 2.3: If we view system (1) as a multi-input multi-output system and assume that the local feedback design of each subsystem can use all the output measurements, then we can achieve the objectives of the above problem straightforwardly using the standard internal model approach as can be found in [2]–[4] and [10]. Unfortunately, our problem formulation only allows the control of each subsystem to use the measurement output of itself and its neighbors. As mentioned in Section I, this formulation is motivated from such problem as stabilization, formation, and output regulation of multi-agent systems [5], [11], [13]. Also, as pointed out in Remark 2.1, the above objectives cannot be achieved using the decentralized control scheme in [3] because the readability condition for each subsystem may not be satisfied.

III. Solvability of the Problem

In order to make our problem more trackable, we will make the following assumption on the digraph.

Assumption A6: The digraph $\mathcal{G}$ contains no loop and node 0 is globally reachable.

Obviously, $(j, i) \notin \mathcal{E}$ once $(i, j) \in \mathcal{E}$ under Assumption A5.

Under this assumption, we can see that

$$\lim_{t \to +\infty} e_i(t) = 0 \quad \Leftrightarrow \quad \lim_{t \to +\infty} e_{iv}(t) = 0, \quad i = 1, \dots, N. \tag{11}$$

Let $z = (z_1^T \cdots z_N^T)^T$, $\tilde{E} = (\bar{E}_1^T \cdots \bar{E}_N^T)^T$, $\tilde{A} = block\ diag(\bar{A}_1 \cdots \bar{A}_N)$, $\tilde{B} = block\ diag(\bar{B}_1 \cdots \bar{B}_N)$, $\tilde{C} = block\ diag(\bar{C}_1 \cdots \bar{C}_N)$, $\tilde{K}_x = block\ diag(K_{x1} \cdots K_{xN})$,

$\tilde{K}_z = block\ diag(K_{z1} \cdots K_{zN})$, $\tilde{K} = block\ diag(K_1 \cdots K_N)$, $\tilde{G}_i = block\ diag(\bar{G}_{i1} \cdots \bar{G}_{iN})$ ($i = 1, 2$), and

$$\tilde{H} = diag\begin{pmatrix} \frac{1}{|\mathcal{N}_1|} & \cdots & \frac{1}{|\mathcal{N}_N|} \end{pmatrix} \tilde{L} + A_0,$$
$$A_0 = diag(a_{10} \quad \cdots \quad a_{N0})$$

where $\tilde{L}$ is the Laplacian associated with a subgraph of $\mathcal{G}$ with node set $\{1, \ldots, N\}$.

Then, from (1) and (8) or (9), the closed-loop system can be put as follows:

$$\begin{cases} \dot{\xi} = A_w \xi + B_w v \\ \dot{v} = \Gamma v \\ e = C_w x + F_w v \end{cases} \quad \xi = \begin{pmatrix} x \\ z \end{pmatrix}, \quad e = \begin{pmatrix} e_{1v} \\ \vdots \\ e_{Nv} \end{pmatrix} \tag{12}$$

with $C_w = (\tilde{H} \otimes I_p)\tilde{C}$, $F_w = (A_0 \mathbf{1}) \otimes F$, and

$$A_w = \begin{pmatrix} \tilde{A} + \tilde{B}\tilde{K}_x & \tilde{B}\tilde{K}_z \\ (\tilde{H} \otimes G_2)\tilde{C} & I_N \otimes G_1 \end{pmatrix},$$
$$B_w = \begin{pmatrix} \tilde{E} \\ (A_0 \mathbf{1})(G_2 \otimes F) \end{pmatrix}$$

in the case of (8), or

$$A_w = \begin{pmatrix} \tilde{A} & \tilde{B}\tilde{K} \\ (\tilde{H} \otimes I_s)\tilde{G}_2\tilde{C} & \tilde{G}_1 \end{pmatrix},$$
$$B_w = \begin{pmatrix} \tilde{E} \\ (A_0 \otimes I_s)\tilde{G}_2(\mathbf{1} \otimes F) \end{pmatrix}$$

in the case of (9), where $\otimes$ denotes the Kronecker product, I_s is an $s \times s$ identity matrix and $\mathbf{1} = (1 \ldots 1)^T$. We use (A_c, B_c) to denote the nominal system value of (A_w, B_w).

Applying Lemma 1.20 of [10] to system (12) immediately gives a solvability condition on the robust output regulation of system (12) as follows.

Lemma 3.1: Under Assumptions A1 and A6, the controller (8) or (9) solves the distributed robust output regulation of system (1) if and only if A_c is Hurwitz, and for each sufficiently small w, there exists a unique matrix X_w that satisfies

$$\begin{cases} X_w \Gamma = A_w X_w + B_w \\ (C_w \quad 0) X_w = F_w. \end{cases} \tag{13}$$

Remark 3.1: The first equation of (13) is a Sylvester equation which always has a unique solution X_w as long as w is such that A_w is stable. If this control law happens to also make X_w satisfy the second equation of (13) regardless of the variations of w, then the control law solves the problem. In what follows, we will employ the so-called internal model based control law to handle our problem, this control law is of particular interest because, as will be pointed out in Remark 3.3, it can always make the solution of the first equation of (13) satisfy the second equation of (13) regardless of the small variations of w. That is how the internal model control law works and that is why the internal model control law is robust with respect to small variation of the plant parameter.

We now introduce the concept of internal model as follows ([2], [10]).

Definition 3.1: A pair of matrices (M_1, M_2) is said to incorporate a p-copy internal model of matrix Γ if

$$M_1 = S\begin{pmatrix} T_1 & T_2 \\ 0 & G_1 \end{pmatrix} S^{-1}, \quad M_2 = S\begin{pmatrix} T_3 \\ G_2 \end{pmatrix} \tag{14}$$

where T_i, $i = 1, 2, 3$, are any constant matrices with appropriate dimensions, S is any nonsingular matrix, and

$$G_1 = \text{block diag } \underbrace{[\beta, \cdots, \beta]}_{p-tuple}, \quad G_2 = \text{block diag } \underbrace{[\sigma, \cdots, \sigma]}_{p-tuple}$$

where β is a square matrix and σ is a column vector such that (β, σ) is controllable and the minimal polynomial of Γ equals the characteristic polynomial of β.

Remark 3.2: In particular, the pair of matrices (G_1, G_2) incorporates a p-copy internal model of matrix Γ since G_1 and G_2 are special cases of M_1 and M_2, respectively.

Remark 3.3: An interesting property of the internal model that can be derived from Lemma 1.27 of [10] is that, if there exists a pair of matrices Z, Y such that

$$Z\Gamma = M_1 Z + M_2 Y \tag{15}$$

then $Y = 0$. It is this property that makes the solution of the first equation of (13) satisfy the second equation of (13) regardless of the small variations of w. The usage of this property will be seen in the proofs of Theorems 3.1 and 3.2.

Theorem 3.1: Under Assumptions A1–A3, and A5–A6, the distributed output regulation of system (1) can be solved by a dynamic state feedback control of the form (8).

Proof: Define (G_1, G_2) as in Definition 3.1 with G_1 satisfying

$$rank\begin{pmatrix} A_i - \lambda I & B_i \\ C_i & 0 \end{pmatrix} = n + p, \quad \forall \lambda \in \sigma(G_1) \tag{16}$$

for $i = 1, \ldots, N$. From Lemma 1.26 of [10], the pairs

$$\left(\begin{pmatrix} A_i & 0 \\ G_2 C_i & G_1 \end{pmatrix}, \quad \begin{pmatrix} B_i \\ 0 \end{pmatrix} \right), \quad i = 1, \ldots, N$$

are stabilizable. Then there exist $(K_{xi}\ K_{zi})$, $i = 1, \ldots, N$ such that

$$\begin{pmatrix} A_i & 0 \\ G_2 C_i & G_1 \end{pmatrix} + \begin{pmatrix} B_i \\ 0 \end{pmatrix} (K_{xi} \quad K_{zi})$$
$$= \begin{pmatrix} A_i + B_i K_{xi} & B_i K_{zi} \\ G_2 C_i & G_1 \end{pmatrix}, \quad i = 1, \ldots, N \tag{17}$$

are Hurwitz.

By Assumption A6, we can always label the subsystems of (1) such that $i < j$ if $(i, j) \in \mathcal{E}$. Let T be such that

$$\begin{bmatrix} x_1 \\ z_1 \\ \vdots \\ x_N \\ z_N \end{bmatrix} = T\xi.$$

Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{z}_1 \\ \vdots \\ \dot{x}_N \\ \dot{z}_N \end{bmatrix} = T A_w T^{-1} \begin{bmatrix} x_1 \\ z_1 \\ \vdots \\ x_N \\ z_N \end{bmatrix} + T B_w v.$$

Under Assumption A6, the Laplacian $\tilde{L}$ is a block lower triangular matrix. Thus A_c, the nominal value of A_w, is also a block lower triangular matrix with the block diagonal entries being given by the matrix (17), and is thus Hurwitz.

Next we will verify (13). Consider the two cases:

i) If $(0,i) \in \mathcal{E}$, then $e_{iv} = e_i$. For each such i, the closed-loop subsystem is composed of the i^{th} controller of (8) and the i^{th} subsystem of (1). Since (17) is Hurwitz, and (G_1, G_2) incorporates a p-copy internal model of matrix Γ, it follows from Lemma 1.27 of [10] that, there exist X_i and Z_i that satisfy

$$\begin{aligned} X_i\Gamma &= (\bar{A}_i + \bar{B}_i K_{xi})X_i + \bar{B}_i K_{zi} Z_i + \bar{E}_i \\ Z_i\Gamma &= G_1 Z_i + G_2(\bar{C}_i X_i + F) \\ \bar{C}_i X_i + F &= 0. \end{aligned} \tag{18}$$

ii) If $(0,i) \notin \mathcal{E}$, then e_{iv} is given by (7). For each such i, the closed-loop subsystem is composed of the i^{th} controller of (8) and the i^{th} subsystem of (1). Again, since (17) is Hurwitz, there exist X_i and Z_i to satisfy

$$\begin{aligned} X_i\Gamma &= (\bar{A}_i + \bar{B}_i K_{xi})X_i + \bar{B}_i K_{zi} Z_i + \bar{E}_i \\ Z_i\Gamma &= G_1 Z_i + G_2\left(\bar{C}_i X_i - \sum_{j\in\mathcal{N}_i} \frac{1}{|\mathcal{N}_i|}\bar{C}_j X_j\right) \end{aligned} \tag{19}$$

for any $\bar{E}_i$ and any X_j with $j \in \mathcal{N}_i$, and any $\bar{F}_i = -(1/|\mathcal{N}_i|)\sum_{j\in\mathcal{N}_i}\bar{C}_j X_j$. Now note that the second equation of (19) is the special case of (15) with $G_1 = M_1$, $G_2 = M_2$, and $Y = \bar{C}_i X_i + \bar{F}_i$. By Remark 3.3

$$\bar{C}_i X_i + \bar{F}_i = 0. \tag{20}$$

Let $X_w = (X_1^T\ X_2^T\ \cdots\ X_N^T\ Z_1^T\ Z_2^T\ \cdots\ Z_N^T)^T$. Then (18), (19) and (20) imply that

$$\begin{aligned} X_w\Gamma &= A_w X_w + B_w \\ 0 &= C_w X_w + F_w. \end{aligned} \tag{21}$$

Thus, by Lemma 3.1, the controller (8) solves the robust output regulation problem. ■

Remark 3.4: Theorem 3.1 and its proof provide a design procedure for feedback control law (8): select (G_1, G_2) as the minimal p-copy internal model of Γ, and then select $(K_{xi}\ K_{zi})$ to make (17) stable.

Next, we present the output feedback result.

Theorem 3.2: Under Assumptions A1–A6, the distributed output regulation of system (1) can be solved by a dynamic output feedback controller of the form (9).

Proof: Define $(K_{xi}\ K_{zi})$, $i = 1,\ldots,N$, (G_1, G_2) as in Theorem 3.1. Let $K_i = (K_{xi}\ K_{zi})$, $i = 1,\ldots,N$. Since (C_i, A_i) is detectable, there exists $\bar{L}_i$ such that $A_i - \bar{L}_i C_i$ is Hurwitz, $i = 1,\ldots,N$. Set

$$\begin{cases} u_i = K_i z_i \\ \dot{z}_i = \begin{pmatrix} A_i + B_i K_{xi} - \bar{L}_i C_i & B_i K_{zi} \\ 0 & G_1 \end{pmatrix} z_i + \begin{pmatrix} \bar{L}_i \\ G_2 \end{pmatrix} e_{iv} \\ \quad = \bar{G}_{i1} z_i + \bar{G}_{i2} e_{iv} \end{cases} \tag{22}$$

with $i = 1,\ldots,N$. Clearly, the pair $(\bar{G}_{i1}, \bar{G}_{i2})$ incorporates a p-copy internal model of Γ for $i = 1,\ldots,N$.

Under Assumption A6, the nominal closed-loop system A_c is a block lower triangular matrix with the block diagonal entries being given by the following matrices

$$\begin{pmatrix} A_i & B_i K_i \\ \bar{G}_{i2} C_i & \bar{G}_{i1} \end{pmatrix} = \begin{pmatrix} A_i & B_i K_{xi} & B_i K_{zi} \\ \bar{L}_i C_i & A_i + B_i K_{xi} - \bar{L}_i C_i & B_i K_{zi} \\ G_2 C_i & 0 & G_1 \end{pmatrix}$$

for $i = 1,\ldots,N$, which are similar to the following one:

$$\begin{pmatrix} A_i + B_i K_{xi} & B_i K_{xi} & B_i K_{zi} \\ 0 & A_i - \bar{L}_i C_i & 0 \\ G_2 C_i & 0 & G_1 \end{pmatrix}, \quad i = 1,\ldots,N \tag{23}$$

and are thus Hurwitz.

Next, similar to the proof of Theorem 3.1, under controller (9), we have the following conclusion.

i) If $(0,i) \in \mathcal{E}$, then there exist X_i and Z_i such that

$$\begin{aligned} X_i\Gamma &= \bar{A}_i X_i + \bar{B}_i K_i Z_i + \bar{E}_i \\ Z_i\Gamma &= G_1 Z_i + G_2(\bar{C}_i X_i + F) \\ \bar{C}_i X_i + F &= 0 \end{aligned} \tag{24}$$

ii) If $(0,i) \notin \mathcal{E}$, then there exist X_i and Z_i such that

$$\begin{aligned} X_i\Gamma &= \bar{A}_i X_i + \bar{B}_i K_i Z_i + \bar{E}_i \\ Z_i\Gamma &= \bar{G}_{i2} Z_i + \bar{G}_{i2}\bar{C}_i X_i - \bar{G}_{i2}\sum_{j\in\mathcal{N}_i}\frac{1}{|\mathcal{N}_i|}\bar{C}_j X_j \end{aligned} \tag{25}$$

for any $\bar{E}_i$ and any $\bar{F}_i = -\sum_{j\in\mathcal{N}_i}(1/|\mathcal{N}_i|)\bar{C}_j X_j$. Now note that the second equation of (25) is the special case of (15) with $\bar{G}_{i1} = M_1$, $\bar{G}_{i2} = M_2$, and $Y = \bar{C}_i X_i + \bar{F}_i$. By Remark 3.3 again, we obtain $\bar{C}_i X_i + \bar{F}_i = 0$.

Take

$$X_w = \begin{pmatrix} X_1^T & X_2^T & \cdots & X_N^T & Z_1^T & Z_2^T & \cdots & Z_N^T \end{pmatrix}^T.$$

It is not hard to see that

$$\begin{aligned} X_w\Gamma &= A_w X_w + B_w \\ 0 &= C_w X_w + F_w. \end{aligned} \tag{26}$$

Thus, the conclusion follows from Lemma 3.1. ■

Remark 3.5: The design procedure feedback control law (9) is summarized as follows: i) select (G_1, G_2) as the minimal p-copy internal model of Γ; ii) take $K_i = (K_{xi}\ K_{zi})$ to stabilize (17); iii) take $\bar{L}_i$ such that $A_i - \bar{L}_i C_i$ is Hurwitz; and iv) construct $\bar{G}_{i1}, \bar{G}_{i2}$ as defined in (22).

Remark 3.6: If the order of the minimal polynomial of Γ is n_m, then $G_1 \in \mathbb{R}^{pn_m\times pn_m}$, $G_2 \in \mathbb{R}^{pn_m\times p}$. As a result, $s = pn_m$ in (8). Moreover, $\bar{G}_{i1} \in \mathbb{R}^{(pn_m+n)\times(pn_m+n)}$, $\bar{G}_{i2} \in \mathbb{R}^{(pn_m+n)\times p}$ [with $\bar{G}_{i1}, \bar{G}_{i2}$ defined in (22)]. Therefore, the dimension of z_i in (9) is $pn_m + n$.

IV. Example

Consider a system of the form (1) with $N = 5$, and

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$E_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad i = 1,\ldots,5$$

and $B_i = (0\ 1)^T$, $C_i = (1\ 0)$, $i = 1,\ldots,5$, $F = (1\ 1\ 1)$. The parameters δ_{ij} $(j = 1, 2)$ are uncertain parameter with their nominal value 0.

The system topology is described by a digraph with $\mathcal{N} = \{0,1,\cdots,5\}$ and the entries of the adjacency matrix are $a_{01} = a_{02} = 1$, $a_{32} = 1$, $a_{41} = 1$, $a_{43} = 1$, $a_{54} = 1$ and all the other entries are zero. Thus, the first two agents can get the exosystem information, and the others cannot.

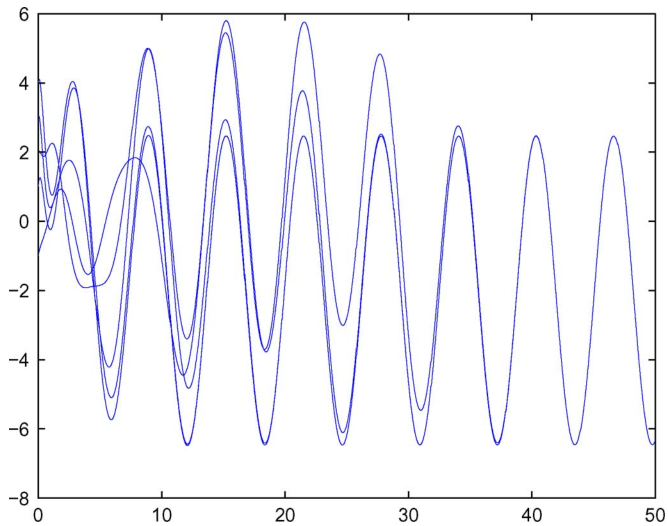

Fig. 1. Positions x_i of the five subsystems.

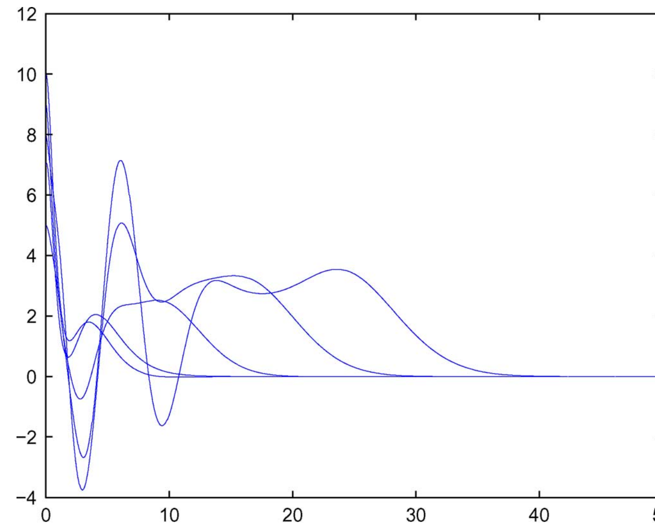

Fig. 2. Regulated outputs e_i of the five subsystems.

As F is not equal to 0, the problem cannot be solved by a decentralized control law. Nevertheless, it can be verified that Assumptions A1 to A6 hold. Thus, it is possible to solve the problem using the distributed control law. For this purpose, let the 1-copy internal model for Γ be

$$G_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Construct the distributed control law as follows:

$$\begin{cases} u_i = k_{1x} x_i + k_{2x} \dot{x}_i + (k_1 \quad k_2 \quad k_3) z_i, \\ \dot{z}_i = G_1 z_i + G_2 e_{iv}, \qquad z_i \in \mathbb{R}^3 \end{cases} \tag{27}$$

where

$$e_{jv} = y_j, \quad j = 1, 2, \quad e_{3v} = y_3 - y_2,$$
$$e_{4v} = y_4 - \frac{1}{2} y_1 - \frac{1}{2} y_3, \quad e_{5v} = y_5 - y_4.$$

Set $k_{1x} = -9$, $k_{2x} = -5$, $k_1 = -1$, $k_2 = 4$, $k_3 = -5$, $K_{xi} = (k_{1x} \; k_{2x})$, $K_{zi} = (k_1 \; k_2 \; k_3)$, $i = 1, \ldots, 5$. Then

$$\bar{A} = \begin{pmatrix} A_i + B_i K_{xi} & B_i K_{zi} \\ G_2 C_i & G_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -9 & -5 & -1 & 4 & -5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}$$

is stable. By Theorem 3.1, the robust output regulation problem is solved (see Figs. 1 and 2).

In the simulations, the initial conditions are randomly selected as follows: $x_1(0) = 1$, $x_2(0) = 4$, $x_3(0) = 3$, $x_4(0) = 2$, $x_5(0) = -1$, $\dot{x}_1(0) = 4$, $\dot{x}_2(0) = 3$, $\dot{x}_3(0) = 2$, $\dot{x}_4(0) = 1$, $\dot{x}_5(0) = 2$, $z_1(0) = (0\;2\;4)^T$, $z_2(0) = (1\;4\;3)^T$, $z_3(0) = (2\;1\;4)^T$, $z_4(0) = (3\;2\;1)^T$, $z_5(0) = (1\;0\;2)^T$, and $w(0) = (4\;1\;1)^T$. Also

$$\delta A_i = \begin{pmatrix} 0 & 0 \\ 0.01 \times i & 0.02 \times i \end{pmatrix}$$

and all other perturbation matrices are taken to be zero.

The numerical results demonstrate the effectiveness of the feedback (27) by showing position x_i and regulated output e_i for $i = 1, \ldots, 5$. ■

V. Conclusion

In this note, we have studied the robust output regulation problem of a multi-agent system by a distributed control scheme. The solvability of the problem has been established and both state and output feedback control laws based on internal model have been constructed.

The extension of the work in this note to nonlinear setting is under consideration.

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