given structural restrictions. All computations have performed using Matlab LMI Control Toolbox [23].

V. Conclusion

The main contribution of the note is a set of conditions which extend the Inclusion Principle to solve overlapping guaranteed cost control problems using static output controllers for a class of linear norm bounded uncertain state-delayed discrete-time systems. These conditions ensure the expansion-contraction of the closed loop systems and the equality of the guaranteed cost bounds. The controller design is performed for the expanded system using a delay independent LMI, which is adapted to this class of problems. In this context, the LMI is used as a tool that allows a computable control, which is further contracted to be implemented in the original system. Other design tools could also be possible within the same framework. The results are specialized for the overlapping output feedback control design under decentralized information structure constraints. It leads to a block tridiagonal structure of controllers. A numerical illustrative example is supplied.

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Finite-Time Stabilization of Nonlinear Systems With Parametric and Dynamic Uncertainties

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***Abstract*—In this note, non-smooth finite-time stabilization of nonlinear systems with parametric and dynamic uncertainties is investigated. To solve this problem, the input-to-state stability property is used to characterize unmeasured dynamic uncertainties. A constructive partial-state control design is proposed on the basis of involved combined use of Lyapunov, backstepping and input-to-state stability techniques. Under small-gain type local conditions, a solution for the finite-time regulation of a class of uncertain nonlinear systems is obtained.**

***Index Terms*—Dynamic uncertainties, finite-time convergence, input-to-state stability (ISS), nonsmooth feedback, parametric uncertainties.**

I. Introduction

The control synthesis to deal with uncertain nonlinear systems becomes more and more important following various practical demands. Many (robust and/or adaptive) nonlinear control approaches were proposed in the last decade [5], [6], [14], [16], [17]. The design for nonlinear control systems with dynamic uncertainty has also been studied widely, partially because dynamic uncertainty often arises from many different control engineering applications; see [10], [12], [13], [26] and references therein.

Meanwhile, non-smooth control has drawn increasing attention in nonlinear control system design. One of the main benefits of the non-smooth finite-time control strategy is that it can force a control system to reach a desirable target in finite time. This approach was first studied in the literature of optimal control. Despite its potential application to practical problems, the study of finite-time stabilization is quite under-developed, partially because of the lack of effective and constructive

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tools in non-smooth analysis. In recent years, finite-time stabilizing controllers were constructed for some classes of nonlinear systems [2], [3], [7]–[9], [11]. Moreover, finite-time stabilization of discontinuous systems has recently been studied in light of differential inclusions in [19]. These finite-time stabilizing controllers can also yield, in some sense, fast response and high tracking precision as well as disturbance-rejection properties because of their non-smoothness [2].

The purpose of this note is to propose a global robust finite-time control design for a class of nonlinear systems with dynamic uncertainty, leading to an extension of the existing results on (adaptive) finite-time stabilization. The crucial idea behind our study on finite-time stabilization is to use Sontag's concept of input-to-state stability (ISS) [23] and its Lyapunov characterization [25] to characterize unmeasured dynamic uncertainties as in [12]. A novel controller design is presented through a rather involved combined application of Lyapunov, backstepping and ISS techniques; see [23], [24], [16], [14], [18] for related applications.

The rest of the note is organized as follows. In Section II, preliminary knowledge and the problem formulation are given. Section III handles the construction of robust finite-time controllers in the presence of parametric and dynamic uncertainties, while Section IV provides the analysis for finite-time convergence. Finally, concluding remarks are proposed in Section V.

II. Preliminaries

We will consider the finite-time stabilization for uncertain nonlinear systems of the strict-feedback form

$$\begin{cases} \dot{z} = \psi(z, \xi_1) \\ \dot{\xi}_1 = \xi_2 + \phi_1(\xi_1, z, t) \\ \dot{\xi}_2 = \xi_3 + \phi_2(\xi_1, \xi_2, z, t) \\ \vdots \\ \dot{\xi}_n = u + \phi_n(\xi_1, \dots, \xi_n, z, t) \end{cases} \tag{1}$$

where $u \in R$ is the control input; $\xi := (\xi_1, \dots, \xi_n)^T \in R^n$ is the measured portion of the state; $z \in R^{n_0}$ is the remaining unmeasured state, which we refer to as [12]. For each $i = 1, 2, \dots, n$, ϕ_i is an unknown and Lipschitz continuous function, and ψ is piecewise continuous with respect to z and Lipschitz continuous with respect to ξ_1. For simplicity, we assume $\psi(0,0) = 0$. Nonlinear systems transformable into (1) have been studied extensively over the last 15 years from both theoretical and practical viewpoints; see, e.g., [15]–[17].

For convenience, for any given ξ_1, we rewrite $\psi(z, \xi_1) = \psi_{\xi_1}(z)$; that is,

$$\dot{z} = \psi(z, \xi_1) = \psi_{\xi_1}(z) \tag{2}$$

where $\psi_{\xi_1}(z)$ is a piecewise continuous vector field. The set of all the discontinuous points of $\psi_{\xi_1}(z)$, denoted by ∂G, is of measure zero, and let $F_{\xi_1}(z)$ be the smallest convex closed set containing all the limit values of $\psi_{\xi_1}(\hat{z})$ for all $\hat{z} \notin \partial G$ with $\hat{z} \to z$. A solution $z(t)$ of (2) is called a solution of the differential inclusion $\dot{z} \in F_{\xi_1}(z)$ (referring to [4], [19]). If $V_0(z)$ is C^1 with respect to z, then for convenience, in what follows, we denote

$$\dot{V}_0|_{(2)} := \sup_{\bar{z} \in F_{\xi_1}(z)} \left\{ \frac{\partial V_0}{\partial z} \bar{z} \right\}. \tag{3}$$

If $\psi_{\xi_1}(z)$ is continuous, then $F_{\xi_1}(z) = \{f_{\xi_1}(z)\}$, and $\dot{V}_0$ can be understood in the conventional sense.

Throughout the note, the following assumptions are made regarding system (1):

(A1) For each $i = 1, \dots, n$, there are C^1 known functions κ_1, κ_{i2} vanishing at the origin such that,

$$|\phi_i(\xi_1, \dots, \xi_i, z, t)| \le \kappa_1(\|z\|) + p\kappa_{i2}(\|\xi_1, \dots, \xi_i\|) \tag{4}$$

(for all $\xi_1, \dots, \xi_i, z, t$) where $p > 0$ is an unknown constant. In other words, the functions $\phi_i (i = 1, \dots, n)$ are allowed to be dependent on unknown parameters, as in [12], [13].

(A2) System (2), that is, $\dot{z} = \psi(z, \xi_1)$, has an ISS-Lyapunov function $V_0(z)$ (see [23], [25] for the definition and implications of an input-to-state stable (ISS) system). Namely, V_0 is a positive-definite and proper function that satisfies

$$\dot{V}_0|_{(2)} := \sup_{\bar{z} \in F_{\xi_1}(z)} \frac{\partial V_0}{\partial z} \bar{z} \le -c_0 V_0^{\alpha_0} + \gamma_0(|\xi_1|) \tag{5}$$

where c_0, α_0 are positive constants with $\alpha_0 < 1$, and γ_0 is a class-$\mathcal{K}_\infty$ function. Moreover, $\overline{\pi}$ and $\underline{\pi}$ are class-$\mathcal{K}_\infty$ functions such that

$$\underline{\pi}(\|z\|) \le V_0(z) \le \overline{\pi}(\|z\|).$$

(A3) It is assumed that $\limsup_{s \to 0^+}(\gamma_0(s)/s^2) < +\infty$ and $\limsup_{s \to 0^+}(\kappa_1(s)^2/\underline{\pi}(s)) < +\infty$.

Note that the inequalities in (A3) are the well-known (small-gain type) conditions for small signals; see [12], [13] for more details.

Remark 1: Obviously, condition (5) implies that the z-subsystem is globally finite time stable when $\xi_1 = 0$. The converse Lyapunov-like conditions for finite-time stable systems are discussed under the assumption that the settling time T is continuous at the origin in [3]. However, converse Lyapunov conditions may not be extendible directly to ISS. In fact, a 0-input finite-time stable system, $\dot{z} = \psi(z, 0)$, may not lead to (5) with $\xi_1 \neq 0$. Here we give an example for illustrating ISS inequality (5). Consider

$$\dot{z} = -\mathrm{sgn}(z) - z^3 + \xi_1^2$$

where $sgn(\cdot)$ is the sign function. Take $V_0 = \frac{z^2}{2}$ (with $\underline{\pi}(s) = \overline{\pi}(s) = \frac{s^2}{2}$). As it can be directly checked,

$$\dot{V}_0 = -|z| - z^4 + z\xi_1^2 \le -\frac{1}{\sqrt{2}} V_0^{\frac{1}{2}} + \gamma_0(\xi_1)$$

with $\gamma_0(\xi_1) = \xi_1^4/2$ and $\alpha_0 = 1/2$.

Let $0 < \nu < 1$ be a real number in the form of $1 - \frac{p_0}{q_0}$, where $0 < p_0 < q_0$ are two odd integers. Moreover, when $n > 1$, take $\nu > 0$ so that

$$\nu < \frac{1 - \alpha_0}{1 + (n-3)\alpha_0}, \qquad r_1 = 1 > \dots > r_i := r_{i-1} - \nu > 0 \tag{6}$$

and

$$\beta_0 = 1 - \nu, \qquad (\beta_{i-1} + 1)r_i = (\beta_{i-2} + 1)r_{i-1} > 0 \tag{7}$$

for $i = 2, \dots, n$. Note that (6) can be satisfied as ν is sufficiently small. It is easy to see that $1 - \nu \le r_i\beta_{i-1} < r_{i+1}\beta_i$ and $\beta_{i-1} < \beta_i$ $(i = 1, \dots, n-1)$.

The next lemma is useful in the following finite-time convergence analysis.

Lemma 1: Define

$$\hat{\kappa}_1(s) = \sum_{i=1}^{n} \kappa_1(s)^{\frac{2-2\nu}{2-2\nu-r_i\beta_{i-1}}} + \sum_{i=2}^{n} \kappa_1(s)^{\frac{2-2\nu}{2-2\nu-r_i}}. \quad (8)$$

Then we have

$$\limsup_{s\to 0+} \frac{\hat{\kappa}_1(s)}{\underline{\pi}(s)^{\alpha_0}} < +\infty. \quad (9)$$

Proof: Note that $\kappa_1(s)$ is C^1, $\frac{2-2\nu}{2-2\nu-\beta_0} = 2$, and

$$\frac{2-2\nu}{2-2\nu-r_i\beta_{i-1}} > 2, \qquad \frac{2-2\nu}{2-2\nu-r_i} > 2\alpha_0, \qquad i = 2,\ldots,n,$$

according to (6). Then, (9) follows straightforwardly from (A3).

The control objective is to find a continuous dynamic partial-state feedback law of the form

$$u = u(\xi,\hat{\theta}) \quad \dot{\hat{\theta}} = \varphi(\xi,\hat{\theta}) \quad (10)$$

such that the state $x = (z^T, \xi^T)^T$ of system (1) is globally finite-time convergent to the origin. Namely, for any initial conditions $x(t_0) = x_0, \hat{\theta}(t_0) = \theta_0$, there is a *finite* settling time $T \geq t_0$ such that the solution $x(t; t_0, x_0, \theta_0)$ of system (1) under control law (10) satisfies

$$\lim_{t\to T} x(t; t_0, x_0, \theta_0) = 0 \quad x(t; t_0, x_0, \theta_0) = 0 \qquad \forall t > T.$$

Note that $\hat{\theta}(t)$ needs not converge to θ.

For the need of the analysis later, we introduce some useful inequalities [1].

Lemma 2: (Jensen's inequality):

$$\left(\sum_{i=1}^{n} x_i^{a_2}\right)^{1/a_2} \leq \left(\sum_{i=1}^{n} x_i^{a_1}\right)^{1/a_1}, \qquad 0 < a_1 < a_2 \quad (11)$$

with $x_i \geq 0, 1 \leq i \leq n$.

Lemma 3: (Young's inequality):

$$ab \leq \frac{a^{1+c}}{1+c} + \frac{c\, b^{1+\frac{1}{c}}}{1+c}, \quad a \geq 0, b \geq 0, c > 0. \quad (12)$$

III. Control Design

This section is to develop a constructive procedure for robust finite time stabilization by partial-state feedback. In what follows, we denote $\theta = \max\{1, p^{\frac{2-2\nu}{r_n}}\}$, which is unknown since p is unknown. Denote $\hat{\theta}(t)$ as a time-varying function to estimate the unknown constant θ, and $\tilde{\theta} = \theta - \hat{\theta}$.

Define

$$\begin{cases} v_0 := 0 \\ w_j := \xi_j^{\beta_{j-1}} - v_{j-1}(\xi_1,\ldots,\xi_{j-1},\hat{\theta})^{\beta_{j-1}} \\ v_j := -w_j^{\frac{r_j-\nu}{r_j\beta_{j-1}}} \Phi_j(\xi_1,\ldots,\xi_j,\hat{\theta}), \qquad 1 \leq j \leq n \end{cases} \quad (13)$$

where, for each $1 \leq j \leq n$, Φ_j is a C^1 positive function to be determined later.

Remark 2: It is not hard to show that $|\xi_j - v_{j-1}| \leq 2|w_j|^{\frac{1}{\beta_{j-1}}}$ for $\beta_{j-1} \geq 1 (j \geq 2)$.

Set

$$W_j := \int_{v_{j-1}}^{\xi_j} \left[s^{\beta_{j-1}} - v_{j-1}^{\beta_{j-1}}\right] \; ds, \qquad 1 \leq j \leq n \quad (14)$$

and

$$\eta_j := -\sum_{l=1}^{j} \frac{\partial W_l}{\partial \hat{\theta}}, \qquad 1 \leq j \leq n. \quad (15)$$

It is easy to see that $\eta_1 = 0$ and $w_j = (\partial W_j / \partial \xi_j), 1 \leq j \leq n$.

Furthermore, we define

$$V_j = \sum_{i=1}^{j} W_i, \qquad j = 1,\ldots,n; \quad (16)$$

$$Q_j(x) = (|w_1|^2 + |w_2|^{2-2\nu} + \cdots + |w_j|^{\frac{2-2\nu}{r_j\beta_{j-1}}})^{\frac{1}{2-2\nu}} \quad (17)$$

for $j = 1,\ldots,n$.

Then, a constructive design procedure can be developed, which is quite complicated since it combines the scheme given in [8], [9] with that given in [13]. Here, due to space limitation, some details are omitted and the ideas for the detailed backstepping-like procedure can be found in [8], [13], or the references therein. In fact, this type of Lyapunov function candidates (14) is originally used in [20]–[22].

Step 1: Consider

$$\dot{\xi}_1 = \xi_2 + \phi_1(\xi_1, z, t). \quad (18)$$

Take $V_1 = W_1 = (\xi_1^{2-\nu}/2-\nu)$. By (A2) and $\theta \geq 1$

$$\begin{aligned} \xi_1^{1-\nu}\phi_1 &\leq |\xi_1^{1-\nu}|\kappa_1(\|z\|) + |\xi_1^{1-\nu}|p\kappa_{12}(|\xi_1|) \\ &\leq \frac{\xi_1^{2-2\nu}}{4} + \kappa_1(\|z\|)^2 + p\xi_1^{2-2\nu}\left|\xi_1^{\nu}\int_0^1 \kappa_{12}'(\lambda|\xi_1|)d\lambda\right| \\ &\leq \theta\xi_1^{2-2\nu}\hat{b}_1(\xi_1) + \kappa_1(\|z\|)^2 \end{aligned}$$

where $\hat{b}_1(\xi_1)$ is a C^1 nonnegative function dominating $\frac{1}{4} + |\xi_1^{\nu}\int_0^1 \kappa_{12}'(\lambda|\xi_1|)d\lambda|$. Take $v_1(\xi_1,\hat{\theta}) = -\xi_1^{1-\nu}\Phi_1(\xi_1,\hat{\theta})$ with $\Phi_1(\xi_1)$ a C^1 positive function that dominates the following function:

$$\Phi_1^0 = L_1 + \frac{\gamma_0(|\xi_1|)}{|\xi_1|^{2-2\nu}}[\mu(\xi_1)+1] + \hat{\theta}\hat{b}_1(\xi_1) \quad (19)$$

where μ is a C^1 function. Φ_1^0 is continuous due to $\nu > 0$ and (A3).

Consider the derivative of V_1, that is

$$\begin{aligned}\dot{V}_1|_{(18)} &= \xi_1^{1-\nu}(\xi_2+\phi_1)\\ &\le \xi_1^{1-\nu}(\xi_2 - v_1 + v_1 + \xi_1^{1-\nu}\theta\hat{b}_1) + \kappa_1^2\\ &\le -\gamma_0(|\xi_1|)[\mu(\xi_1)+1] - L_1\xi_1^{2-2\nu}\\ &\quad + |\xi_1^{1-\nu}||\xi_2 - v_1| + \kappa_1(\|z\|)^2\\ &\quad + (\tilde{\theta}+\eta_1)\varphi_1(\xi_1) - \eta_1\dot{\hat{\theta}},\end{aligned}$$

where $\eta_1 = 0$ defined in (15), $\varphi_1 = \xi_1^{2-2\nu}\hat{b}_1 \ge 0$ and $\dot{\hat{\theta}}$ will be determined later.

After Step $j-1(2 \le j \le n)$: For system

$$\begin{cases}\dot{\xi}_1 = \xi_2 + \phi_1(\xi_1, z, t)\\ \vdots\\ \dot{\xi}_{j-1} = \xi_j + \phi_{j-1}(\xi_1,\ldots,\xi_{j-1}, z, t)\end{cases} \tag{20}$$

we assume that, with $V_{j-1} = \sum_{i=1}^{j-1} W_i$, we have

$$\begin{aligned}\dot{V}_{j-1}|_{(20)} &\le -\gamma_0(|\xi_1|)[\mu(\xi_1)+1] - L_{j-1}Q_{j-1}^{2-2\nu}\\ &\quad + |w_{j-1}||\xi_j - v_{j-1}| + (\tilde{\theta}+\eta_{j-1})\varphi_{j-1}(\xi,\hat{\theta}) - \eta_{j-1}\dot{\hat{\theta}}\\ &\quad + \sum_{i=1}^{j-1}\kappa_1^{\frac{2-2\nu}{2-2\nu-r_i\beta_{i-1}}} + \sum_{i=2}^{j-1}\kappa_1^{\frac{2-2\nu}{2-2\nu-r_i}}, \qquad L_{j-1}>0\end{aligned} \tag{21}$$

where η_{j-1} was defined in (15) and the function $\varphi_{j-1} \ge 0$ is C^1. The construction of $v_{j-1}, w_{j-1}, \varphi_{j-1}$, and V_{j-1}, based on $v_{j-2}, w_{j-2}, \varphi_{j-2}$, and V_{j-2}, will be introduced in detail in Step j. We give the assumptions which will be verified later in Step j.

i) For $1 \le i \le j-1 (j \le n)$, $v_i^{\beta_i}$ with v_i defined in (13) is C^1, and therefore, W_{i+1}, defined in (14), is C^1.

ii) There are C^1 functions $\rho_{i,l}(\xi,\hat{\theta}) \ge 0$ and $\hat{v}_i(\xi,\hat{\theta}) \ge 0$ (for $1 \le i \le l \le j-1$) such that $|(\partial v_l^{\beta_l}/\partial\xi_i)| \le Q_l^{r_{l+1}\beta_l - r_i}\rho_{i,l}$ and $|(\partial v_i^{\beta_i}/\partial\hat{\theta})| \le \hat{v}_i$.

When $j=2$, Assumptions i) and ii) can be easily proved using (12).

Step j: Consider system

$$\begin{cases}\dot{\xi}_1 = \xi_2 + \phi_1(\xi_1, z, t)\\ \vdots\\ \dot{\xi}_{j-1} = \xi_j + \phi_{j-1}(\xi_1,\ldots,\xi_{j-1}, z, t)\\ \dot{\xi}_j = \xi_{j+1} + \phi_j(\xi_1,\ldots,\xi_j, z, t).\end{cases} \tag{22}$$

Clearly, W_j is nonnegative and even positive when $\xi_j \ne v_{j-1}$, and W_j is C^1 owing to Assumption i) in Step $j-1$. Then, with (16), $V_j(\xi)$ is C^1 and positive definite with respect to $\xi_1,\ldots,\xi_j$. Consider

$$\begin{aligned}\dot{V}_j|_{(22)} &\le -\gamma_0(|\xi_1|)[\mu(\xi_1)+1] - L_{j-1}Q_{j-1}^{2-2\nu}\\ &\quad + |w_{j-1}||\xi_j - v_{j-1}|\\ &\quad + (\tilde{\theta}+\eta_{j-1})\varphi_{j-1} - \eta_{j-1}\dot{\hat{\theta}} + \sum_{i=1}^{j}\frac{\partial W_j}{\partial\xi_i}(\xi_{i+1}+\phi_i)\\ &\quad + \frac{\partial W_j}{\partial\hat{\theta}}\dot{\hat{\theta}} + \sum_{i=1}^{j-1}\kappa_1^{\frac{2-2\nu}{2-2\nu-r_i\beta_{i-1}}} + \sum_{i=2}^{j-1}\kappa_1^{\frac{2-2\nu}{2-2\nu-r_i}}.\end{aligned} \tag{23}$$

Then, we analyze each term on the right hand side of inequality (23).

i) With (12) and Remark 2, we first obtain

$$\begin{aligned}|w_{j-1}||\xi_j - v_{j-1}| &\le 2|w_{j-1}|\,|w_j|^{\frac{1}{\beta_{j-1}}}\\ &\le \frac{L_{j-1}}{4n}Q_{j-1}^{2-2\nu} + l_j|w_j|^{\frac{2-2\nu}{r_j\beta_{j-1}}}\end{aligned} \tag{24}$$

where l_j is a positive constant depending on ν and L_{j-1}.

ii) By Assumption ii) in Step $j-1$ and Remark 2, we have

$$\begin{aligned}\left|\frac{\partial W_j}{\partial\xi_i}\right| &= |v_{j-1} - \xi_j|\left|\frac{\partial v_{j-1}^{\beta_{j-1}}}{\partial\xi_i}\right|\\ &\le 2|w_j|^{\frac{1}{\beta_{j-1}}}Q_{j-1}^{r_j\beta_{j-1}-r_i}\rho_{i,j-1}\\ &\qquad\qquad i \le j-1\end{aligned}$$

where $\rho_{i,j-1}$ are C^1 nonnegative functions for $i \le j-1$. Moreover, a repeated application of Young's inequality (12) along with Remark 2 gives

$$\begin{aligned}&|\xi_{i+1}+\phi_i|\\ &\le |\xi_{i+1} - v_i| + |v_i| + |\phi_i|\\ &\le 2|w_{i+1}|^{\frac{1}{\beta_i}} + |w_i|^{\frac{r_i-\nu}{r_i\beta_{i-1}}}\Phi_i + |\phi_i|\\ &\le (2+\Phi_i)Q_{i+1}^{r_i-\nu} + \kappa_1 + p\kappa_{i2}\end{aligned}$$

where, for simplicity, we dropped the arguments. Then, due to (12) and $(2-2\nu/r_j) \le (2-2\nu/r_n)$, we have

$$\begin{aligned}&\left|\sum_{i=1}^{j-1}\frac{\partial W_j}{\partial\xi_i}(\xi_{i+1}+\phi_i)\right|\\ &\le 2|w_j|^{\frac{1}{\beta_{j-1}}}[Q_j^{r_j\beta_{j-1}-\nu}\sum_{i=1}^{j-1}\rho_{i,j-1}(2+\Phi_i)\\ &\quad + \sum_{i=1}^{j-1}Q_j^{r_j\beta_{j-1}-r_i}\rho_{i,j-1}(\kappa_1 + p\kappa_{i2})]\\ &\le \frac{L_{j-1}}{4n}Q_j^{2-2\nu} + |w_j|^{\frac{2-2\nu}{r_j\beta_{j-1}}}(\tilde{\psi}_j + \theta\tilde{b}_j)\\ &\quad + \kappa_1^{\frac{2-2\nu}{2-2\nu-r_j}}\end{aligned} \tag{25}$$

where $\tilde{\psi}_j(\xi,\hat{\theta})$ and $\tilde{b}_j(\xi,\hat{\theta})$ are C^1 nonnegative functions with $\tilde{b}_j(0,\hat{\theta}) = 0$.

Similar to the analysis given for (25), with the help of (12), we have

$$\begin{aligned}&\left|\frac{\partial W_j}{\partial\xi_j}(\xi_{j+1}+\phi_j)\right|\\ &\le |w_j\xi_{j+1}| + |w_j|(\kappa_1 + p\kappa_{j2})\\ &\le |w_j||\xi_{j+1} - v_j| + w_jv_j + \frac{L_{j-1}}{4n}Q_j^{2-2\nu}\\ &\quad + |w_j|^{\frac{2-2\nu}{r_j\beta_{j-1}}}[\tilde{\psi}_j^0(\xi,\hat{\theta}) + \theta\tilde{b}_j^0(\xi,\hat{\theta})]\\ &\quad + \kappa_1^{\frac{2-2\nu}{2-2\nu-r_j\beta_{j-1}}}\end{aligned}$$

where $\tilde{\psi}_j^0, \tilde{b}_j^0$ are C^1 nonnegative functions with $\tilde{b}_j^0(0,\hat{\theta}) = 0$, and $w_jv_j \ge 0$ with v_j to be determined.

Thus

$$\left|\sum_{i=1}^{j}\frac{\partial W_j}{\partial \xi_i}(\xi_{i+1}+\phi_i)\right| \le |w_j||\xi_{j+1}-v_j| + \frac{L_{j-1}}{2n}Q_j^{2-2\nu} + w_j v_j + |w_j|^{\frac{2-2\nu}{r_j\beta_{j-1}}}(\psi_j+\theta\hat{b}_j) + \kappa_1^{\frac{2-2\nu}{2-2\nu-r_j}} + \kappa_1^{\frac{2-2\nu}{2-2\nu-r_j\beta_{j-1}}} \tag{26}$$

where $\psi_j = \tilde{\psi}_j + \tilde{\psi}_j^0$ and $\hat{b}_j = \tilde{b}_j + \tilde{b}_j^0$.

iii) Similar to the analysis given in ii), for the other terms of (23), by Assumption ii) in Step $j-1$, (12) and (15), we can obtain (referring to [8, Lemma 5])

$$(\tilde{\theta}+\eta_{j-1})\varphi_{j-1} - \eta_{j-1}\dot{\hat{\theta}} + \frac{\partial W_j}{\partial \hat{\theta}}\dot{\hat{\theta}} = (\tilde{\theta}+\eta_{j-1})\varphi_{j-1} - \eta_j\dot{\hat{\theta}} \le \frac{L_{j-1}}{4n}Q_j^{2-2\nu} + |w_j|^{\frac{2-2\nu}{r_j\beta_{j-1}}}(\hat{\phi}_j - \tilde{\theta}\hat{b}_j) + (\tilde{\theta}+\eta_j)\varphi_j - \eta_j\dot{\hat{\theta}} \tag{27}$$

where $\varphi_j = \varphi_{j-1} + |w_j|^{(2-2\nu/r_j\beta_{j-1})}\hat{b}_j$.

Construct

$$v_j = -w_j^{\frac{r_j-\nu}{r_j\beta_{j-1}}}\Phi_j \tag{28}$$

where $\Phi_j = (n-1/n)L_{j-1} + l_j + \psi_j + \hat{\phi}_j + (1+\hat{\theta}^2)\hat{b}_j, (j \ge 2)$ is C^1 and positive. Then

$$w_j v_j + |w_j|^{(2-2\nu/r_j\beta_{j-1})}(l_j+\psi_j+\hat{\phi}_j+\hat{\theta}\hat{b}_j) \le -L_j|w_j|^{(2-2\nu/r_j\beta_{j-1})}$$

for $L_j = \frac{n-1}{n}L_{j-1}$. Therefore, based on (24) and (26)–(28), we have

$$\dot{V}_j|_{(22)} \le -\gamma_0(|\xi_1|)[\mu(\xi_1)+1] - L_jQ_j^{2-2\nu} + |w_j||\xi_{j+1}-v_j| + \sum_{i=1}^{j}\kappa_1^{\frac{2-2\nu}{2-2\nu-r_i\beta_{i-1}}} + \sum_{i=2}^{j}\kappa_1^{\frac{2-2\nu}{2-2\nu-r_i}} + (\tilde{\theta}+\eta_j)\varphi_j - \eta_j\dot{\hat{\theta}}. \tag{29}$$

Moreover, because Φ_j and w_j are C^1, $v_j^{\beta_j}$ is C^1 by $(r_j-\nu)\beta_j \ge r_j\beta_{j-1}$. Therefore, Assumption i) given in Step $j-1$ is still valid in Step j. In addition, Assumption ii) can also be verified for Step j by induction with the help of (12).

Step n: Take V_n defined in 16. According to the previous inductive arguments, the control law can be constructed as

$$u = v_n = -w_n^{\frac{r_n-\nu}{r_n\beta_{n-1}}}\Phi_n \tag{30}$$

Furthermore, according to $(\xi_i - v_{i-1})w_i \ge 0$ and Remark 2, we have

$$V_n = \sum_{j=1}^{n}\int_{v_{j-1}}^{\xi_j}\left[s^{\beta_{j-1}} - v_{j-1}^{\beta_{j-1}}\right]ds \le \sum_{j=1}^{n}2|w_j|^{\frac{2-\nu}{r_j\beta_{j-1}}}.$$

With (11), we have $V_n^{(2-2\nu/2-\nu)} \le 2Q_n^{2-2\nu}$. Therefore, it holds

$$\dot{V}_n|_{(1)} \le -L_nQ_n^{2-2\nu} - \gamma_0(|\xi_1|)[\mu(\xi_1)+1] + \sum_{i=1}^{n}\kappa_1^{\frac{2-2\nu}{2-2\nu-r_i\beta_{i-1}}} + \sum_{i=2}^{n}\kappa_1^{\frac{2-2\nu}{2-2\nu-r_i}} + (\tilde{\theta}+\eta_n)\varphi_n - \eta_n\dot{\hat{\theta}} \le -\frac{L_n}{2}V_n^{\frac{2-2\nu}{2-\nu}} - \gamma_0(\mu+1) + \hat{\kappa}_1 + (\tilde{\theta}+\eta_n)\varphi_n - \eta_n\dot{\hat{\theta}} \tag{31}$$

with $\hat{\kappa}_1(\|z\|)$ given in (8).

IV. Finite-Time Convergence Analysis

In the previous section, we constructed a feedback control law via partial-state variables ξ. Now, we consider the finite-time convergence of the closed-loop trajectories under the proposed control law (30) along with an update law

$$\dot{\hat{\theta}} = -\dot{\tilde{\theta}} = \varphi_n. \tag{32}$$

Clearly, we have the following.

Lemma 4: The trajectories $z(t), \xi(t)$ and $\hat{\theta}(t)$ are bounded for the closed-loop system (1) with (30) and (32).

Proof: Consider the following (positive–definite) Lyapunov function:

$$V_c = V_n(\xi) + V_*(z) + \frac{1}{2}\tilde{\theta}^2 \quad V_*(z) = \int_0^{V_0(z)}\rho(s)ds$$

where $\rho : R_+ \to R_+$ is a continuous nondecreasing function with $\rho(s) > 0$ for every $s \ge 0$.

Therefore, according to (31)

$$\dot{V}_c \le -\frac{L_n}{2}V_n^{\frac{2-2\nu}{2-\nu}} - \gamma_0(|\xi_1|)[\mu(\xi_1)+1] + \hat{\kappa}_1(\|z\|) - \rho(V_0(z))V_0(z)^{\alpha_0} + \rho(V_0(z))\gamma_0(|\xi_1|).$$

According to (9), we can find a desired function ρ and then a C^1 function μ (see [12] and [13] for the details) such that

$$\begin{cases}(1-\epsilon)\rho(\underline{\pi}(\|z\|))V_0^{\alpha_0} \ge 2\hat{\kappa}_1(\|z\|) \\ \mu(\xi_1)+1 \ge \rho\left(\left(\frac{\gamma_0(|\xi_1|)}{\epsilon}\right)^{\frac{1}{\alpha_0}}\right), \qquad 0<\epsilon<1.\end{cases}$$

Therefore, we have

$$\dot{V}_c \le -\frac{L_n}{2}V_n^{\frac{2-2\nu}{2-\nu}} - \frac{1}{2}\rho(V_0(z))V_0(z)^{\alpha_0} \le 0 \tag{33}$$

which implies the boundedness of z, ξ and $\hat{\theta}$. △

We are in a position to state our main result.

Theorem 1: The solutions of system (1) in closed-loop with feedback law (30) are bounded. In particular, each trajectory $(z(t), \xi(t))$ is finite-time convergent to the origin $0 \in R^{n+n_0}$.

Proof: Consider $\bar{V}_c = V_n(\xi) + V_*(z)$, which is positive definite with respect to (z, ξ). According to (33), we have

$$\dot{\bar{V}}_c \leq -\frac{L_n}{2} V_n^{\frac{2-2\nu}{2-\nu}} - \frac{1}{2}\rho(V_0(z))V_0(z)^{\alpha_0} + \tilde{\theta}\phi_n. \tag{34}$$

At first, we consider local finite-time convergence in a small neighborhood around $(z, \xi) = (0, 0)$.

It is not hard to see that, locally around $z = 0$

$$\lim_{\|z\| \to 0} \frac{V_0(z)^{\alpha_0}}{\rho(V_0)V_0(z)^{\alpha_0}} = \lim_{\|z\| \to 0} \frac{1}{\rho(V_0(z))} = \frac{1}{\rho(0)} < \infty$$

which implies

$$V_0(z)^{\alpha_0} = O(\rho(V_0)V_0(z)^{\alpha_0}). \tag{35}$$

On the other hand, by definition of $V_*(z)$

$$\lim_{\|z\| \to 0} \frac{V_*(z)}{V_0(z)} = \lim_{\|z\| \to 0} \rho(V_0(z)) < \infty$$

which implies $V_*(z) = O(V_0)$. Thus

$$V_*(z)^{\alpha_0} = O(V_0(z)^{\alpha_0}) = O(\rho(V_0)V_0(z)^{\alpha_0}) \tag{36}$$

when $\|z\|$ is near 0. Therefore, when $\|z\|$ is sufficiently small, there is a constant c_* such that $V_*^{\alpha_0} \leq \frac{1}{c_*}\rho(V_0)V_0^{\alpha_0}$ or, equivalently $-\rho(V_0)V_0^{\alpha_0} \leq -c_* V_*^{\alpha_0}$.

For convenience, denote $\alpha_1 := \frac{2-2\nu}{2-\nu} < 1$. Then, based on (34), (35), and (36), we have, locally around $(\xi, z) = (0, 0)$,

$$\dot{\bar{V}}_c \leq -\frac{L_n}{2} V_n^{\alpha_1} - \frac{c_*}{2} V_*^{\alpha_0} + \tilde{\theta}\varphi_n.$$

Note that

$$\varphi_n(\xi, \hat{\theta}) = \varphi_1 + \sum_{j=2}^{n} |w_j|^{\frac{2-2\nu}{r_j \beta_{j-1}}} \hat{b}_j \leq Q^{2-2\nu}\varphi_0(\xi, \hat{\theta})$$

where $\varphi_0 = \sum_{j=1}^{n} \hat{b}_j$ is continuous with $\varphi_0(0, \hat{\theta}) = 0$ because $\hat{b}_j(0, \hat{\theta}) = 0 (j = 1, \ldots, n)$ and $\hat{\theta}$ is bounded. In other words, $\varphi_n = o(V_n^{\alpha_1})$ with respect to ξ. Therefore, in a small neighborhood of $(z, \xi) = (0, 0), \varphi_n = V_n(\xi, \hat{\theta})^{\alpha_1}\tilde{V}(\xi, \hat{\theta})$ with $\tilde{V}(0, \hat{\theta}) = 0$, and then we have

$$\dot{\bar{V}}_c \leq -\frac{L_n}{4} V_n^{\alpha_1} - \frac{c_*}{2} V_*^{\alpha_0} - \frac{L_n}{4} V_n^{\alpha_1}(1 - \tilde{V}).$$

In fact, due to the boundedness of $\hat{\theta}$ (from Lemma 4) and $\tilde{V}(0, \hat{\theta}) = o(1)$ with respect to (z, ξ) around $(z, \xi) = (0, 0)$, we can prove that there is a constant $0 < \varrho \leq 1$ such that $\tilde{V}(x, \hat{\theta}) \leq 1$ if $\bar{V}_c \leq \varrho$. Note that when $\bar{V}_c \leq 1$, there is $c_0 > 0$ such that $\frac{L_n}{4} V_n^{\alpha_1} + \frac{c_*}{2} V_*^{\alpha_0} \geq c_0 (V_n + V_*)^{\alpha}$ with $\alpha := \max\{\alpha_1, \alpha_0\}$ and $c_0 := \min\{\frac{L_n}{4}, \frac{c_*}{2}\}$, by inequality (11). Thus, in a neighborhood $\Omega = \{(z, \xi, \hat{\theta}) : \bar{V}_c \leq \varrho\}$, we have

$$\dot{\bar{V}}_c \leq -c_0 \bar{V}_c^{\alpha} \tag{37}$$

which implies $\bar{V} \to 0$ in finite time or, equivalently, the local finite time convergence in Ω (noting that (37) guarantees that Ω is an invariant set).

Next, we consider the global finite-time convergence. The finite-time convergence in Ω has been proved. Now, we study the situation outside Ω. It is not hard to see that there is $\bar{c}_0$ such that $(L_n/2)V_n^{\alpha_1} + (c_*/2)V_*^{\alpha_0} \geq \bar{c}_0 \varrho^{\alpha}$. Note that when the initial condition $(z(0), \xi(0), \hat{\theta}(0)) \notin \Omega$, we have $\bar{V}_c \geq \varrho$. Therefore, based on (33) and the aforementioned discussion

$$\begin{aligned} V_c(0) \geq V_c(0) - V_c(\tau) &= \int_0^{\tau} -\dot{V}_c(s)ds \\ &= \int_0^{\tau} (-\dot{\bar{V}}_c + \tilde{\theta}\varphi_n)ds \\ &\geq \int_0^{\tau} (\frac{L_n}{2} V_n^{\alpha_1} + \frac{c_*}{2} V_*^{\alpha_0})ds \geq c_0 \, \varrho^{\alpha}\tau. \end{aligned} \tag{38}$$

If $(z(t), \xi(t), \hat{\theta}(t))$ do not reach Ω in finite time, (38) will lead to a contradiction. Therefore, $(z(t), \xi(t), \hat{\theta}(t))$ will enter Ω in finite time. Thus, the conclusion follows. △

Remark 3: If there is no dynamic uncertainty, then the feedback law becomes full-state, which is consistent with the adaptive finite-time control given in [8]. On the other hand, if we take $\nu = 0$, then following the proposed design procedure, we will obtain an asymptotically stabilizing partial-state control, which is consistent with the result given in [13].

The settling time analysis is often helpful for predicting when the target is reached. In practice, although we do not have the exact knowledge about θ (or p), we usually know its range. Without loss of generality, we assume that $0 < \theta \leq \Theta$ for some constant $\Theta > 0$.

From (33), $\tilde{\sigma}(t)^2 \leq 2V_c(t) \leq 2V_c(0)$ and $\dot{V}_c(t) \leq 0$, which implies that, for any given initial conditions $x(0), \hat{\theta}(0)$, we have $|\tilde{\theta}(t)| \leq \sqrt{2V_c(0)} \leq \sqrt{2\bar{V}_c(0) + (|\hat{\theta}(0)| + \Theta)^2}$, only depending on the initial condition $(x(0), \hat{\theta}(0))$ and S. Therefore

$$V_c(0) \leq \bar{V}_c(0) + \frac{1}{2}\tilde{\theta}(0)^2 \leq 2\bar{V}_c(0) + \frac{1}{2}(|\hat{\theta}(0)| + \Theta)^2.$$

Based on the proof of Theorem 1 [especially, (37) and (38)], the settling time can be estimated as follows:

$$T \leq \begin{cases} \frac{(\bar{V}_c(z(0), \xi(0), \hat{\theta}(0)))^{1-\max\{\alpha_1, \alpha_0\}}}{c_0(1-\max\{\alpha_1, \alpha_0\})}, & \text{if } (z, \xi, \hat{\theta}) \in \Omega \\ \frac{\varrho^{1-\max\{\alpha_1, \alpha_0\}}}{c_0(1-\max\{\alpha_1, \alpha_0\})} \\ \quad + \frac{4\bar{V}_c(z(0), \xi(0), \hat{\theta}(0)) + (|\hat{\theta}(0)| + \Theta)^2}{c_0 \varrho^{\max\{\alpha_1, \alpha_0\}}}, & \text{otherwise} \end{cases}$$

V. Conclusion

In this note, the finite-time stabilization problem for a class of nonlinear systems with parametric and dynamic uncertainties is investigated. Due to unmeasured zero dynamics, a (dynamic) partial-state control strategy is used for solving the finite-time regulation problem. Under the assumptions related to ISS, the proposed partial-state feedback controller renders the system state variables finite-time convergent. It is under current investigation to propose a robustification tool for a larger class of systems with possibly nonvanishing disturbances and more general dynamic uncertainties. Our findings along this direction will be reported elsewhere.

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On a New Method for H_2-Based Decomposition

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***Abstract*—This note proposes a simple linear matrix inequality (LMI) system for the design of static precompensators to reduce the interactions of a multivariable system. The approach is based on minimizing the H_2 norm of a modified system and the LMIs are derived from the well known technique of pseudodiagonalisation. The approach is applied to two complex real-life benchmark problems with high levels of interaction. It is shown that its performance is significantly better than previously proposed LMI optimization techniques for designing static precompensators.**

***Index Terms*—Decentralized control, H_2 optimization, large-scale systems, linear matrix inequality (LMI), pseudodiagonalisation.**

I. Introduction

Despite much recent effort in developing advanced techniques for the design of multivariable controllers, it is fair to say the serious difficulties which arise in their application have halted their widespread use. These relate to the complexity of design procedure and the complexity of the resulting controller. In addition, many of these techniques are effectively only applicable to low dimensional multivariable systems because the methodologies breakdown on a large-scale platform. Under these circumstances, it is not surprising that for larger multivariable systems, decentralized control remains the quintessential methodology.

Decentralized control design techniques can be categorized into those which are based on the extension of single-input–single-output (SISO) techniques to multiple-input–multiple-output (MIMO) system (such as sequential loop closing techniques [1]) and those which design the decentralized controller as a single transfer function [2]. While from the application point of view decentralized control is more practical, the added structural constraints means that from a design and analysis perspective it is less tractable. These issues range from difficulties in the parameterization of all stabilizing decentralized controllers [3], [4], to the fact that often their use is limited to so called open-loop "weakly coupled" systems [5].

Unfortunately, many systems cannot satisfy the weakly-coupled requirement and therefore do not yield to decentralized control. A convenient remedy in these cases is to employ a precompensator K such if $G(s)$ is the original system, then the open-loop gain $Q(s) = G(s)K$ satisfies the weakly coupled requirements. $Q(s)$ may then be treated as a new system and any number of available closed-loop decentralized methods (e.g., PI) be used to obtain the overall desired responses. The implications of this are that the design of K is independent of the decentralized controller, and also that K is the solution to an open-loop problem whereas the decentralized controller is designed in closed-loop. In order to ensure that the structure of the final overall resulting controller is as simple as possible, K is typically required to be constant. A tradeoff of this structural constraint is that the performance is in general substantially less than those achieved by decoupling controllers which are not static, and whose order is often at least the same as the system's. Indeed, previously proposed methods provide state–space formulations for both the H_∞ [6], [7], and H_2 [7] based decomposition problems using dynamic controllers. In particular, the powerful method

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