respectively. It can be seen in Fig. 2 that when $G \neq B^{+}$ the control acts in the opposite direction, i.e., it's effect is *counter effective*.

For comparison purposes, we have in Fig. 3 a plot of $\|z\|_{\mathcal{L}_2}/\|w\|_{\mathcal{L}_2}$ for each controller. When G is selected improperly, the value is *increased* due to the amplification of ϕ_{u}. When G is selected properly, the value is, after a short transient, lower than the one obtained by $\mathcal{H}_\infty$ alone, even though the discontinuous component was included in the penalty variable (i.e., $z = Cx + D(u_0 + u_1)$).

V. Conclusion

In this note, we studied the effects that the projection matrix has on the resulting (equivalent) perturbation. It was shown that in the presence of unmatched disturbances the projection matrix of an ISM controller should be selected carefully, for the resulting controller could amplify them. Two propositions provide a way for selecting the projection matrix correctly. The proposed parameters ensure that the effect of the unmatched disturbance will *not be amplified* by the discontinuous control. It is also shown that the discontinuous control can not attenuate the unmatched disturbances.

The analysis is aimed at combining ISMC with other robust techniques. $\mathcal{H}_\infty$ control was selected as a specific case, but other techniques could be used as well. Simulation results support the analysis developed.

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Adaptive Finite-Time Control of Nonlinear Systems With Parametric Uncertainty

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***Abstract*—In this note, global finite time stabilization is investigated for a class of nonlinear systems in p normal form with parametric uncertainties. To achieve finite-time stabilization, a constructive control design approach is proposed by following backstepping methodology, and an adaptive finite-time control law is obtained in the form of continuous time-invariant feedback.**

***Index Terms*—Adaptive control, finite-time stability, nonsmooth feedback, settling time.**

I. Introduction

Nonsmooth finite-time control can make the controlled systems to reach their targets in a finite time. It was first studied in the literature of optimal control. In recent years, finite-time stabilization controllers have been investigated for a number of class of systems [2], [4], [5], [7]–[9]. In particular, [5] has constructed finite-time stabilizing laws for the nonlinear systems in p normal form without parametric uncertainties. Indeed, asymptotic/exponential stabilizations are sufficient for many engineering applications. For very demanding applications, finite-time stabilization offers an effective alternative, which yields, in some sense, fast response, high tracking precision, and disturbance-rejection properties because of their nonsmoothness [2], [8]. The studies of finite-time control can show us how to increase the precision in a given settling time or make the system convergent fast to the target within arbitrary given precision.

Uncertainties do exist in any real world systems. Adaptive control is one of the effective ways to deal with control systems with parametric uncertainty. Although it is not easy to propose adaptive control strategies for general nonlinear systems, a great deal of efforts have been made in this area and some well-known adaptive design methods are proposed for nonlinear systems with uncertain parameters (referring

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to [6] and [10]). Adaptive finite-time control design is more complicated. In the most studies of adaptive (asymptotic) control, Barbalat's lemma is widely used, but it may not be applicable to the analysis of finite-time convergence. Using terminal sliding mode control, [11] obtained adaptive finite-time convergence for a class of nonlinear systems with its uncertainty satisfying the matching condition, but the discontinuous controller contained some singularities. On the other hand, the finite-time convergence for nonlinear systems without parametric uncertainties can usually be derived from some inequalities in the form of $\dot{V} \leq -cV^{\alpha}$ with $c > 0, 0 < \alpha < 1$ (e.g., in [5], and [7]). However, this idea cannot be employed directly to check the finite-time convergence for the systems with parametric uncertainties.

In this note, a continuous adaptive finite-time controller is constructed for a class of nonlinear systems. This backstepping-like procedure for adaptive finite-time control is proposed in light of the results on both finite-time stabilization (e.g., [4] and [5]) and adaptive (asymptotic) control (e.g., [10]). The nonsmoothness in our proposed control law not only results probably from the (maybe underactuated) systems under consideration, but also comes inherently from finite-time convergence.

II. Preliminaries

To study finite-time stability, some basic concepts can be introduced following the conventions in the literature.

Definition 1: Consider a system

$$\dot{x} = f(x,t), \qquad f(0,t) = 0, \quad x \in U_0 \subset R^n \tag{1}$$

where $f : U_0 \times R^+ \to R^n$ is continuous with respect to x on an open neighborhood U_0 of the origin $x = 0$. The equilibrium $x = 0$ of the system is (locally) finite-time stable if it is Lyapunov stable and finite-time convergent in a neighborhood $U \subseteq U_0$ of the origin. By "finite-time convergence," we mean: If, for any initial condition $x(t_0) = x_0 \in U$ at any given initial time t_0, there is a settling time $T > 0$, such that every solution $x(t;t_0,x_0)$ of system (1) is defined with $x(t;t_0,x_0) \in U/\{0\}$ for $t \in [t_0, T)$ and

$$\lim_{t \to T} x(t;t_0,x_0) = 0 \quad x(t;t_0,x_0) = 0 \qquad \forall t > T.$$

When $U = R^n$, the origin is a globally finite-time stable equilibrium.

Lemma 1: Suppose that, for system (1), there are a C^1 positive–definite function $V(x,t)$ (defined on $\hat{U} \times R^+$, where $\hat{U} \subset U_0 \subset R^n$ is a neighborhood of the origin), real numbers $c > 0$ and $0 < \alpha < 1$, such that $\dot{V}(x,t) + cV^{\alpha}(x,t)$ is negative semidefinite (along the trajectory) on $\hat{U}$. Then $V(x,t)$ is locally finite-time convergent, or equivalently, becomes 0 locally in finite time, with its settling time $T \leq (V(x(t_0),t_0)^{1-\alpha}/c(1-\alpha))$ for a given initial condition $x(t_0)$ in a neighborhood of the origin in $\hat{U}$.

Proof: The proof is straightforward by following that in [2], and thus omitted. △

Definition 2: Consider the following nonlinear system:

$$\dot{x} = f(x,\theta) + g(x)u \qquad x \in R^n, \qquad u \in R^m \tag{2}$$

where f and g are smooth with $f(0,\theta) = 0$ and $g(0) \neq 0$, and θ is an uncertain parameter vector. The problem of global adaptive finite-time stabilization is to find a continuous control law

$$\begin{cases} u = v(x,\hat{\sigma}), & v(0,\hat{\sigma}) = 0 \\ \dot{\hat{\sigma}} = \phi(x,\hat{\sigma}), & \phi(0,\hat{\sigma}) = 0 \end{cases} \tag{3}$$

such that the trajectory $(x(t),\hat{\sigma}(t))$ of system (2) under control law (3) is bounded, and moreover, for any initial condition $(x(t_0),\hat{\sigma}(t_0))$, $x(t)$ converges to 0 in finite time. In other words, for any initial condition $(x(t_0),\hat{\sigma}(t_0))$, there exists $T > t_0$ such that $x(t) = 0$ for any $t > T$.

Here, we consider the adaptive finite-time control for smooth nonlinear systems of the form (called p normal form in [3])

$$\begin{cases} \dot{x}_1 = x_2^{m_1} + \sum_{i=0}^{m_1-1} x_2^i f_{1,i}(x_1,\theta) \\ \cdots \\ \dot{x}_j = x_{j+1}^{m_j} + \sum_{i=0}^{m_j-1} x_{j+1}^i f_{j,i}(x_1,\ldots,x_j,\theta) \\ \cdots \\ \dot{x}_{n-1} = x_n^{m_{n-1}} + \sum_{i=0}^{m_{n-1}-1} x_n^i f_{n-1,i}(x_1,\ldots,x_{n-1},\theta) \\ \dot{x}_n = f_n(x,\theta) + u \end{cases} \tag{4}$$

where $\theta \in R^N$ (for some integer $N > 0$) is a vector of uncertain parameters, m_i, $i = 1,\ldots,n-1$ are odd positive integers, and

$$f_j(x,\theta) = \sum_{i=0}^{m_j-1} x_{j+1}^i f_{j,i}(x_1,\ldots,x_j,\theta), \qquad j = 1,\ldots,n-1$$

are smooth with $f_{j,i}(0,\theta) = 0, i = 0,\ldots,m_j-1, j = 1,2,\ldots,n-1$ and $f_n(0,\theta) = 0$.

III. Adaptive Finite-Time Control

In this section, we propose a backstepping-like procedure to construct adaptive finite-time controllers of system (4).

To do this, we need to select some constants that will be used in the control design. Take $m_0 = 1$ and $p_0 < q_0$ with $p_0 > 0$ and $q_0 > 0$ two odd integers such that

$$\nu = \frac{p_0}{q_0} - 1 < 0$$
$$r_1 = 1 > \cdots > r_i = \frac{r_{i-1}+\nu}{m_{i-1}} > \cdots > r_n = \frac{r_{n-1}+\nu}{m_{n-1}} > 0. \tag{5}$$

As in [4], we can select

$$\beta_0 = r_2 \quad (\beta_i m_i + 1)r_{i+1} = (\beta_{i-1}m_{i-1}+1)r_i > 0, \qquad i = 1,\ldots,n-1. \tag{6}$$

Moreover, to simplify the following analysis, we define the following functions. Take

$$w_1 = x_1^{r_2} \quad w_j = x_j^{m_{j-1}\beta_{j-1}} - v_{j-1}^{\beta_{j-1}}, \qquad 2 \leq j \leq n \tag{7}$$

with

$$v_j(x_1,\ldots,x_j,\hat{\sigma}) = -2w_j^{(r_j+\nu)/r_j m_{j-1}\beta_{j-1}} \Phi_j(x,\hat{\sigma}), 1 \leq j \leq n \tag{8}$$

where $\hat{\sigma}(t)$ and a C^1 positive function Φ_j $(1 \leq j \leq n)$ will be determined in the following recursive design procedure. Note that, v_j and w_j are computable once Φ_j $(1 \leq j \leq n)$ are fixed. Set

$$W_j(x) = \int_{v_{j-1}^{1/m_{j-1}}}^{x_j} \left[s^{m_{j-1}\beta_{j-1}} - v_{j-1}^{\beta_{j-1}} \right] ds, \qquad 1 \leq j \leq n. \tag{9}$$

It is easy to see that W_j is nonnegative and even positive when $x_j^{m_{j-1}} \neq v_{j-1}(x_1,\ldots,x_{j-1},\hat{\sigma})$[4], [5]. Furthermore, take

$$V_j^*(x,\hat{\sigma}) = \sum_{i=1}^{j} W_i(x,\hat{\sigma}) = W_j(x,\hat{\sigma}) + V_{j-1}^*(x,\hat{\sigma}), \qquad j = 1,\ldots,n. \tag{10}$$

In addition, we define a group of functions

$$Q_j(w) = (|w_1|^{(1+\nu+r_2)/r_2} + |w_2|^{1+\nu+r_2} + \cdots + |w_j|^{(1+\nu+r_2)/r_j\beta_{j-1}m_{j-1}})^{1/(1+\nu+r_2)} \tag{11}$$

for $j=1,\ldots,n$. It is easy to see that $Q_l \le Q_i$ if $l \le i$.

Lemma 2: For system (4), there is a smooth positive function $b_j(x_1,\ldots,x_j)$ such that

$$|f_j| \le \frac{1}{2}|x_{j+1}|^{m_j} + \sum_{l=1}^{j}|x_l| b_j(x_1,\ldots,x_j)\sigma \tag{12}$$

where $\sigma \ge 1$ is an uncertain constant depending on uncertain θ [10].

In what follows, we consider the adaptive finite-time control design for system (4).

Step 1: Consider system

$$\dot{x}_1 = x_2^{m_1} + f_1(x_1,x_2,\theta) \tag{13}$$

with $f_1 = \sum_{i=0}^{m_1-1} x_2^i f_{1,i}(x_1,\theta)$. From Lemma 2, we have $|f_1| \le |x_2|^{m_1}/2 + |x_1| b_1(x_1)\sigma$.

Take $v_1 = -2w_1^{m_1}\Phi_1(x,\hat{\sigma}) = -2x_1^{1/\beta_1}\Phi_1(x,\hat{\sigma})$ with

$$\Phi_1 = \left[\frac{2+\nu}{2} + \frac{-\nu}{2}(1+\hat{\sigma}^2)^{1/-\nu}x_1^2\right] b_1(x_1) + l_1$$

where $l_1 > 0$ is a constant. It is easy to see that Φ_1 is positive and C^1.

Take $L_1 = l_1$. Take a function $V_1 = V_1^* + \tilde{\sigma}^2/2$, where $\hat{\sigma}(t)$ is a time-varying function to estimate σ, $\tilde{\sigma}(t) = \sigma - \hat{\sigma}(t)$, and V_1^* defined in (10). Obviously, $\dot{\tilde{\sigma}} = -\dot{\hat{\sigma}}$ because σ is a constant.

Note that $-\hat{\sigma}|x_1^{1+r_2}|b_1 + \sigma|x_1^{1+r_2}|b_1 = \tilde{\sigma}|x_1^{1+r_2}|b_1$, we have

$$\begin{aligned} \dot{V}_1|_{(13)} &\le x_1^{r_2}x_2^{m_1} + \left|\frac{x_1^{r_2}x_2^{m_1}}{2}\right| + |x_1|^{1+r_2}b_1\sigma + \tilde{\sigma}\dot{\tilde{\sigma}} \\ &\le -L_1 x_1^{1+\nu+r_2} + \frac{3}{2}|w_1||x_2^{m_1} - v_1| \\ &\quad + (\tilde{\sigma}+\eta_1)[\phi_1(x_1) - \dot{\hat{\sigma}}] \end{aligned} \tag{14}$$

where $\eta_1 = 0$, $\phi_1 = |x_1|^{1+r_2}b_1$ is a C^1 function, and $\dot{\hat{\sigma}}(t)$ will be given later.

It is obvious that $v_1^{\beta_1}$ is C^1, and there are smooth nonnegative functions $\rho_{1,1}$ and $\hat{v}_1$ such that

$$\left|\frac{\partial v_1^{\beta_1}}{\partial x_1}\right| \le \rho_{1,1}(x_1,\hat{\sigma}) \quad \left|\frac{\partial v_1^{\beta_1}}{\partial \hat{\sigma}}\right| \le \hat{v}_1(x_1,\hat{\sigma}). \tag{15}$$

After Step $j-1$ ($j \ge 2$): For system

$$\begin{cases} \dot{x}_1 = x_2^{m_1} + \sum\limits_{i=0}^{m_1-1} x_2^i f_{1,i}(x_1,\theta) \\ \cdots \\ \dot{x}_{j-2} = x_{j-1}^{m_{j-2}} + \sum\limits_{i=0}^{m_{j-2}-1} x_{j-1}^i f_{j-2,i}(x_1,\ldots,x_{j-1},\theta) \\ \dot{x}_{j-1} = x_j^{m_{j-1}} + f_{j-1}(x,\theta) \end{cases} \tag{16}$$

we assume that, by taking $V_{j-1} = V_{j-1}^* + \tilde{\sigma}^2/2$ with V_{j-1}^* defined in (10)

$$\begin{aligned} &\dot{V}_{j-1}(x,\tilde{\sigma})|_{(16)} \\ &\quad \le -L_{j-1}Q_{j-1}(w_1,w_2,\ldots,w_{j-1})^{1+\nu+r_2} \\ &\qquad + \frac{3}{2}|w_{j-1}||x_j^{m_{j-1}} - v_{j-1}| \\ &\qquad + [\tilde{\sigma}+\eta_{j-1}(x_1,\ldots,x_{j-1},\hat{\sigma})][\phi_{j-1}(x_1,\ldots,x_{j-1},\hat{\sigma}) - \dot{\hat{\sigma}}] \end{aligned} \tag{17}$$

where $L_{j-1} > 0$ is a constant, and functions ϕ_{j-1} and η_{j-1} are continuous (their recursive definitions will be introduced in Step j).

For clarity, we make the following assumptions.

A1) For $1 \le i \le j-1$ ($j \le n$), $\Phi_i(x,\hat{\sigma})$ is positive and C^1 with respect to x and $\hat{\sigma}$. $v_i^{\beta_i}$ with v_i defined in (8) is C^1 and, therefore, W_{i+1}, defined in (9), is C^1.

A2) There is a C^1 nonnegative function $\rho_{i,l}$, for any given $1 \le i \le l \le j-1$, such that

$$\left|\frac{\partial v_l^{\beta_l}}{\partial x_i}\right| \le Q_l^{(r_l+\nu)\beta_l - r_i}(x)\rho_{i,l}(x,\hat{\sigma}). \tag{18}$$

Moreover, there is a C^1 nonnegative function $\hat{v}_i$ for $1 \le i \le j-1$ such that

$$\left|\frac{\partial v_i^{\beta_i}}{\partial \hat{\sigma}}\right| \le \hat{v}_i(x,\hat{\sigma}). \tag{19}$$

Step j: It is time to consider system

$$\begin{cases} \dot{x}_1 = x_2^{m_1} + \sum\limits_{i=0}^{m_1-1} x_2^i f_{1,i}(x_1,\theta) \\ \cdots \\ \dot{x}_{j-1} = x_j^{m_{j-1}} + \sum\limits_{i=0}^{m_{j-1}-1} x_j^i f_{j-1,i}(x_1,\ldots,x_{j-1},\theta) \\ \dot{x}_j = x_{j+1}^{m_j} + f_j(x,\theta) \end{cases} \tag{20}$$

with $f_j = \sum_{i=0}^{m_j-1} x_{j+1}^i f_{j,i}$ ($1 \le j < n$).

W_j is C^1 owing to Assumption **A1)** in Step $j-1$. Then, we can construct C^1 and positive–definite function as follows:

$$V_j = V_{j-1} + W_j = V_j^* + \frac{\tilde{\sigma}^2}{2} \tag{21}$$

with V_j^* defined in (10).

Let us consider the derivative of V_j:

$$\begin{aligned} \dot{V}_j|_{(20)} \le &-L_{j-1}Q_{j-1}^{1+\nu+r_2} + \frac{3}{2}|w_{j-1}||x_j^{m_{j-1}} - v_{j-1}| \\ &+ (\tilde{\sigma}+\eta_{j-1})\phi_{j-1} + \eta_{j-1}\dot{\tilde{\sigma}} + w_j(x_{j+1}^{m_j} + f_j) \\ &+ \sum_{i=1}^{j-1}\frac{\partial W_j}{\partial x_i}(x_{i+1}^{m_i} + f_i) + \frac{\partial W_j}{\partial \hat{\sigma}}\dot{\hat{\sigma}} + \tilde{\sigma}\dot{\tilde{\sigma}}. \end{aligned} \tag{22}$$

By Young's inequality [1], we first have

$$\begin{aligned} &\frac{3}{2}|w_{j-1}||x_j^{m_{j-1}} - v_{j-1}| \\ &\le \frac{3}{2^{1/\beta_{j-1}}}|w_{j-1}||w_j|^{1/\beta_{j-1}} \\ &\le \frac{L_{j-1}}{4n}Q_{j-1}^{1+\nu+r_2} + \bar{l}_j|w_j|^{(1+\nu+r_2)/r_j\beta_{j-1}m_{j-1}} \end{aligned} \tag{23}$$

where $\bar{l}_j > 0$ is a constant depending on $\nu, m_1,\ldots,m_{j-1}$.

Based on Young's inequality [1], the following three lemmas are given to analyze the other terms on the right hand side of inequality (22). Due to the space limitations, their proofs are omitted here.

Lemma 3: There is a C^1 nonnegative function $\tilde{b}_j$ with $\tilde{b}_j(0,\hat{\sigma}) = 0$ satisfying

$$\begin{aligned} w_j\left(x_{j+1}^{m_j} + f_j\right) \le &\frac{3}{2}|w_j||x_{j+1}^{m_j} - v_j| + \frac{w_j v_j}{2} \\ &+ \frac{L_{j-1}}{4n}Q_j^{1+\nu+r_2} + |w_j|^{(1+\nu+r_2)/r_j\beta_{j-1}m_{j-1}}\sigma\tilde{b}_j. \end{aligned} \tag{24}$$

Lemma 4: For $\forall j \geq 2$, there exist smooth nonnegative functions $\tilde{\psi}_j$ and $\tilde{\gamma}_j$ such that

$$\left|\sum_{i=1}^{j-1} \frac{\partial W_j}{\partial x_i}\left(x_{i+1}^{m_i} + f_i\right)\right| \leq \frac{L_{j-1}}{4n} Q_j^{1+\nu+r_2} + |w_j|^{(1+\nu+r_2)/r_j\beta_{j-1}m_{j-1}}\left[\tilde{\psi}_j(x,\hat{\sigma}) + \sigma\tilde{\gamma}_j(x,\hat{\sigma})\right] \quad (25)$$

where $\tilde{\gamma}_j(0,\hat{\sigma}) = 0$.

Lemma 5: For $\forall j \geq 2$, there exists a C^1 positive function $\tilde{\phi}_j$ satisfying

$$\begin{aligned}(\tilde{\sigma} + \eta_{j-1})\phi_{j-1} - \left(\eta_{j-1} - \frac{\partial W_j}{\partial \hat{\sigma}}\right)\dot{\hat{\sigma}} &\leq \frac{L_{j-1}}{4n} Q_{j-1}^{1+\nu+r_2} + |w_j|^{(1+\nu+r_2)/r_j\beta_{j-1}m_{j-1}}\tilde{\phi}_j \\ &\quad - |w_j|^{(1+\nu+r_2)/r_j\beta_{j-1}m_{j-1}}(\tilde{b}_j + \tilde{\gamma}_j)\tilde{\sigma} \\ &\quad + (\tilde{\sigma} + \eta_j)\phi_j - \eta_j\dot{\hat{\sigma}} \end{aligned} \quad (26)$$

where

$$\begin{aligned}\eta_j &= \eta_{j-1} - \frac{\partial W_j}{\partial \hat{\sigma}} \\ \phi_j &= \phi_{j-1} + |w_j|^{(1+\nu+r_2)/r_j\beta_{j-1}m_{j-1}}(\tilde{\gamma}_j + \tilde{b}_j). \end{aligned} \quad (27)$$

Based on the previous analysis, take

$$v_j = -2w_j^{(r_j+\nu)/r_j m_{j-1}\beta_{j-1}}\Phi_j, \quad (28)$$

where, for $j \geq 2$

$$\Phi_j(x,\hat{\sigma}) = l_j + \frac{L_{j-1}}{2n} + \bar{l}_j + \tilde{\psi}_j(x,\hat{\sigma}) + \tilde{\phi}_j(x,\hat{\sigma}) + (1+\hat{\sigma}^2)^{1/2}[\tilde{\gamma}_j(x,\hat{\sigma}) + \tilde{b}_j(x,\hat{\sigma})]$$

with $l_j > 0$ a constant, $\tilde{b}_j(0,\hat{\sigma}) = 0$, and $\tilde{\gamma}_j(0,\hat{\sigma}) = 0$. Therefore

$$\begin{aligned}&\frac{w_j v_j}{2} + |w_j|^{(1+\nu+r_2)/r_j\beta_{j-1}m_{j-1}} \\ &\quad \times \left(\frac{L_{j-1}}{2n} + \bar{l}_j + \tilde{\psi}_j + \tilde{\phi}_j + \hat{\sigma}(\tilde{b}_j + \tilde{\gamma}_j)\right) \\ &\leq -l_j|w_j|^{(1+\nu+r_2)/r_j\beta_{j-1}m_{j-1}}.\end{aligned}$$

Then, combining (23)–(26) leads to

$$\dot{V}_j(x,\hat{\sigma})|_{(20)} \leq -L_j Q_j^{1+\nu+r_2} + \frac{3}{2}|w_j||x_{j+1}^{m_j} - v_j| + (\tilde{\sigma} + \eta_j)(\phi_j - \dot{\hat{\sigma}}) \quad (29)$$

where $L_j = \min\{(n-1/n)L_{j-1}, l_j\} > 0$. Note that (29) is consistent with (17).

Moreover, Φ_j is C^1 because $\tilde{\psi}_j$, $\tilde{\phi}_j$, $\tilde{\gamma}_j$, $\tilde{b}_j$ are so. Therefore, $v_j^{\beta_j}$ is C^1 because w_j is C^1 and $(r_j + \nu)\beta_j \geq r_j m_{j-1}\beta_{j-1}$. Therefore, Assumption **A1)** given in Step $j-1$ is still valid in Step j. Moreover, by induction, we can also verify Assumption **A2** for Step j. The detailed analysis is omitted because of space limitation.

Up to Step n: Take $V_n = V_{n-1} + W_n = \sum_{i=1}^n W_i + (1/2)\tilde{\sigma}^2$, which is positive definite. Then, the adaptive control law can be constructed as

$$\begin{cases} u = v_n = -2w_n^{(r_n+\nu)/r_n m_{n-1}\beta_{n-1}}\Phi_n \\ -\dot{\tilde{\sigma}} = \dot{\hat{\sigma}} = \phi_n(x,\hat{\sigma}). \end{cases} \quad (30)$$

Then, for the closed-loop system, that is

$$\begin{cases} \dot{x}_1 = x_2^{m_1} + \sum_{i=0}^{m_1-1} x_2^i f_{1,i}(x_1,\theta) \\ \cdots \\ \dot{x}_{n-1} = x_n^{m_{n-1}} + \sum_{i=0}^{m_{n-1}-1} x_n^i f_{n-1,i}(x_1,\ldots,x_{n-1},\theta) \\ \dot{x}_n = f_n(x,\theta) + v_n(x,\hat{\sigma}(t)) \\ \dot{\hat{\sigma}} = \phi_n(x,\hat{\sigma}) \end{cases} \quad (31)$$

we have

$$\dot{V}_n|_{(31)} \leq -L_n Q_n(w_1,\ldots,w_n)^{1+\nu+r_2}, \quad (32)$$

where Q_n is positive definite with respect to $w_1,\ldots,w_n$ and $L_n > 0$ is a constant.

Theorem 1: System (4), under the control law (30), is globally adaptive finite-time stable in the sense of Definition 2.

Proof (Outline): Clearly, from (32), $\dot{V}_n \leq 0$, and therefore, x and $\tilde{\sigma}$ (or, equivalently, $\hat{\sigma}$) are bounded. Moreover, $\hat{\sigma}(t)$ is nonnegative if $\hat{\sigma}(0) \geq 0$ because ϕ_n is nonnegative. Therefore, without loss of generality, we assume $\hat{\sigma} \in [0,C]$, where C is a constant depending on initial values $x(0)$ and $\hat{\sigma}(0)$.

Take a Lyapunov function $V_n^*(x,\hat{\sigma}) = \sum_{i=1}^n W_i$, which is positive definite with respect to $x_1,\ldots,x_j$, for any fixed $\hat{\sigma}$. Because $(x_i - v_{i-1})w_i(x_1,\ldots,x_i) \geq 0$ and $|x_i - v_{i-1}| \leq 2|w_i|^{1/\beta_{i-1}}$, we have

$$\begin{aligned}V_n^* &= \sum_{j=1}^n \int_{v_{j-1}^{1/m_{j-1}}}^{x_j}\left[s^{m_{j-1}\beta_{j-1}} - v_{j-1}(x_1,\ldots,x_{j-1})^{\beta_{j-1}}\right]ds \\ &\leq \sum_{j=1}^n 2|w_j|^{(r_2+1)/r_j m_{j-1}\beta_{j-1}}.\end{aligned}$$

Then, we have

$$(V_n^*)^{(1+\nu+r_2)/(1+r_2)} \leq 2Q_n^{1+\nu+r_2}. \quad (33)$$

Note that $\phi_1(x) = Q_1^{1+\nu+r_2}x_1^{-\nu}b_1(x)$ and

$$|w_j|^{(1+\nu+r_2)/r_j\beta_{j-1}m_{j-1}}(\tilde{\gamma}_j + \tilde{b}_j) \leq Q_j^{1+\nu+r_2}(\tilde{\gamma}_j + \tilde{b}_j).$$

Therefore, recalling ϕ_n defined in (27), we have

$$\begin{aligned}\phi_n(x,\hat{\sigma}) &= \phi_1(x) + \sum_{j=2}^n |w_j|^{(1+\nu+r_2)/r_j\beta_{j-1}m_{j-1}}(\tilde{\gamma}_j + \tilde{b}_j) \\ &\leq Q_n^{1+\nu+r_2}\phi_0(x,\hat{\sigma}) \\ \text{where} \quad \phi_0 &= x_1^{-\nu}b_1(x) + \sum_{j=2}^n(\tilde{\gamma}_j + \tilde{b}_j)\end{aligned}$$

which is continuous with $\phi_0(0,\hat{\sigma}) = 0$ because $\tilde{\gamma}_j(0,\hat{\sigma}) = \tilde{b}_j(0,\hat{\sigma}) = 0$ for $j = 2,\ldots,n$.

By (32) and (33)

$$\begin{aligned}&\dot{V}_n^* \\ &\leq -L_n Q_n^{1+\nu+r_2} + \tilde{\sigma}\phi_n \\ &\leq -\frac{L_n}{2}Q_n^{1+\nu+r_2} - \frac{L_n}{2}Q_n^{1+\nu+r_2}\left(1 - \frac{2(C+\sigma)}{L_n}\phi_0\right) \\ &\leq -\frac{L_n}{4}(V_n^*)^{(1+\nu+r_2)/(1+r_2)} - \frac{L_n}{2}Q_n^{1+\nu+r_2}\left(1 - \frac{2(C+\sigma)}{L_n}\phi_0\right).\end{aligned} \quad (34)$$

It is not hard to prove that, with taking a continuous function $\tilde{V}(x,\hat{\sigma}) = (2(C+\sigma)/L_n)\phi_0$, which satisfies $\tilde{V}(0,\hat{\sigma}) = 0$ for any given $\hat{\sigma} \in [0,C]$, we can get a constant $\rho > 0$ such that for any $x \in \Omega = \{(x,\hat{\sigma}) : V_n^*(x,\hat{\sigma}(t)) \leq \rho\}$, $\tilde{V} < 1$ and, therefore, $Q_n^{1+\nu+r_2}(1-\tilde{V}) \geq 0$ because $Q_n \geq 0$ for all $(x,\hat{\sigma})$. Therefore, once $(x,\hat{\sigma}) \in \Omega$, it will be always in Ω.

There are two cases for finite-time convergence analysis.

If the initial condition $(x(0),\hat{\sigma}(0)) \in \Omega$, it is not hard to see that, if $(x,\hat{\sigma}) \in \Omega$, we have

$$\dot{V}_n^* \leq -\frac{L_n}{4}(V_n^*)^{(1+\nu+r_2)/(1+r_2)}$$

and then V_n^* is (locally) finite-time convergent because $0 < 1 + \nu + r_2 < 1 + r_2$ by Lemma 1. Since $V_n^* = \sum_{i=1}^n W_i = 0$ if and only if $x = 0$, x becomes 0 within T

$$T \leq \frac{4(1+r_2)(V_n^*(x(0), \hat{\sigma}(0)))^{-\nu/1+r_2}}{-\nu L_n}$$

where $\nu < 0$ and $\hat{\sigma}(0)$ is known.

If the initial condition $(x(0), \hat{\sigma}(0))$ is not in Ω, then we first estimate its maximum reaching time T_2 to Ω. Before the state enters Ω, we have $V_n^* > \rho$, and then

$$\begin{aligned} V_n(x(0), \hat{\sigma}(0)) &\geq V_n(x(0), \hat{\sigma}(0)) - V_n(x(\tau), \hat{\sigma}(\tau)) \\ &\geq \int_0^\tau L_n Q_n(x(s))^{1+\nu+r_2} ds \\ &\geq \int_0^\tau \frac{L_n}{2} (V_n^*)^{(1+\nu+r_2)/(1+r_2)} ds \\ &\geq \frac{L_n}{2} \rho^{(1+\nu+r_2)/(1+r_2)} \tau. \end{aligned}$$

Therefore, $(x, \hat{\sigma})$ will enter Ω within T_2:

$$T_2 \leq \frac{2V_n^*(x(0), \hat{\sigma}(0)) + \tilde{\sigma}(0)^2}{L_n \rho^{(1+\nu+r_2)/(1+r_2)}}.$$

After the reaching time T_2, the state will be in Ω. It will take

$$T_1 \leq \frac{4(1+r_2)\rho^{-\nu/1+r_2}}{-\nu L_n}$$

to arrive at the origin. Thus, in this case, x becomes 0 within $T \leq T_1 + T_2$.

Therefore, for the closed-loop system (4), $x = 0$ is globally finite-time convergent. $\triangle$

In this note, a constructive procedure for adaptive finite-time control of system (4) is given. In fact, for system (4) without any parametric uncertainty, the design procedure is consistent with the one in [5], where finite-time stabilizing controllers were constructed.

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Circuit Allocation in All Optical Networks With Average Packet Delay Cost Criterion

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***Abstract*—An optimal circuit allocation problem for all-optical circuit-switched backbone networks with average packet delay cost criterion is considered. Multiple classes of traffic flows arrive at the network edge routers, where they can be queued until an end-to-end optical circuit can be allocated. Assuming fluid traffic and circuit allocation of fixed periods, a lower bound on the optimal average packet delay is derived and the cost of two allocation policies are evaluated exactly. The cost of both policies are demonstrated for a variety of deterministic and random networks and are compared with the lower bound.**

***Index Terms*—Circuit switching, golden ratio, Markov decision processes, optical networks, packet delay, TDM.**

I. INTRODUCTION

Circuit switching is a classical network architecture used for real time applications such as telephony, which is also considered attractive for all-optical backbone networks (AON) [3], [14], [15]. Circuit switching in general, provides guaranteed bandwidth and low delay. Optical circuit switching also avoids the electronics associated with high-speed queuing and scheduling hardware at the core routers. Understanding the potential merit of circuit switching for optical networks is becoming of utmost important in light of hybrid switching methods [7], [18] comprising of optical burst switching (OBS) [2] and optical circuit switching (OCS) [6], [8].

AON comprises buffered edge routers at the ingress/egress network nodes and bufferless core routers inside the network. Edge routers multiplex/demultiplex end user traffic flows, e.g., SONET [1], and due to bufferless core routers apply a reservation protocol before transmitting buffered data into the network.

With circuit switching, a two-way reservation protocol is used, by which circuits are allocated for a period of time allowing lossless traffic flows from a subset of sources to their corresponding destinations. Using wavelength division multiplexing (WDM), an optical circuit comprises switching resources and a sequence of wavelengths along a multiple-hop route conforming the wavelength conversion rules [17]. An optical circuit corresponds to a unidirectional lightpath between a source and destination pair of edge routers.

A fundamental control problem in a circuit-switched AON is to find an adaptive policy based on the buffer occupancies for allocating circuits (lightpaths) so as to minimize the average packet delay. Note that with circuit switching, queuing delay is the main concern since by proper edge buffer sizing, packet loss diminishes.

This note derives a lower bound to the optimal long-run average packet delay and evaluates the performance of two heuristic policies. Previous studies have analyzed circuit-switched networks with respect to blocking probability, or equivalently carried traffic. The study in [10] concerns with routing data or voice in a classical circuit-switched

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