from the regime marked by 0 of R_5 (which is very small), say, as $x_1 = 0.2$ and $x_2 = 0.02$. Thus, the stabilizing PI controller is given by

$$C(z) = \frac{0.2z + 0.02}{z - 1.3} \tag{16}$$

and the characteristic polynomial of the controlled system is obtained as

$$\delta(z) = 76z^7 - 52z^6 + 16.28z^5 - 8.56z^4 - 36.12z^3 - 11.78z^2 + 34.38z - 17.6 \tag{17}$$

which has seven eigenvalues, with the modulars: 0.5794, 0.8400, 0.9905, and 0.9964. They are indeed all less than 1, satisfying the requirement.

IV. Conclusion

In this note, from the geometrical view of point, a new method is developed for the design of stabilization of discrete systems using first-order controllers. The method combines analytical, numerical and graphical approaches to generate an effective procedure, which can be used to exactly determine the complete set of parameter values of x_1, x_2, and x_3 for first-order controllers. Explicit algebraic conditions are given for seven critical cases. An example is presented to show the exact critical boundaries, which was not obtained in the existing literature.

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Stabilization of Uncertain Chained Form Systems Within Finite Settling Time

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***Abstract*—This note considers finite time stabilization of uncertain chained form systems. The objective is to design a nonsmooth state feedback law such that the controlled chained form system is both Lyapunov stable and finite-time convergent within any given settling time. We propose a novel switching control strategy with help of homogeneity, time-rescaling, and Lyapunov-based method. Also, the simulation results show the effectiveness of the proposed control design approach.**

***Index Terms*—Chained form systems, finite time convergence, Lyapunov stability, time-rescaling, uncertainty.**

I. Introduction

The last several decades have witnessed the increasing interest in the control of nonholonomic systems (see, for example, [3], [4], and [8]–[12]). In the literature on nonholonomic control systems, smooth time-varying or nonsmooth state-feedback laws can be applied to achieve asymptotic stabilization. In recent years, important steps in feedback design have been made for the exponential regulation of chained form systems [10], [11], [1], [8].

On the other hand, nonsmooth finite time control, which makes the controlled system reach the target in a finite time, provides fast response and high tracking precision, and moreover, shows disturbance-rejection properties because of its nonsmoothness [2]. This provides the motivation for us to apply the finite time control technique to achieve the control target for nonholonomic systems. To date, finite-time stabilizing control has been studied and several explicitly-constructed continuous (but nonsmooth) finite-time controllers for some nonlinear systems (with single input) have been obtained in the references such as [2], [6], and [7].

The purpose of this note is to study finite-time stabilization for a class of uncertain chained form systems. We propose a novel control design procedure to construct a switching nonlinear control scheme that solves the problem of finite time convergence and Lyapunov stability for these nonholonomic systems, where a time-rescaling technique is employed in stabilizing these controlled systems within any given settling time.

II. Problem Formulation

Consider a class of uncertain chained form system in the following form:

$$\begin{cases} \dot{x}_0 = q_0 u_0 \\ \dot{x}_1 = q_1 x_2 u_0 \\ \vdots \\ \dot{x}_{n-1} = q_{n-1} x_n u_0 \\ \dot{x}_n = q_n u + \psi_n(x) \end{cases} \tag{1}$$

Manuscript received November 22, 2004; revised March 8, 2005. Recommended by Associate Editor L. Xie.

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The research is supported by the NNSF of China under Grants 60425307, 10472129, 60404008, and 60221301.

Digital Object Identifier 10.1109/TAC.2005.854620

where $x=(x_1,\ldots,x_n)^T \in R^n$; $q_i>0, i=0,\ldots,n$ are uncertain parameters but they are located in known intervals (i.e., $0<q_i\in[q_i^{\min},q_i^{\max}], i=0,\ldots,n$); u_0, u are control inputs; and ψ_n is an uncertain function satisfying

$$|\psi_n(x)|\le M\sum_{i=1}^{n}|x_i|, \qquad M>0. \tag{2}$$

Before the analysis of (1), we first introduce the concept of finite time stability.

Definition 1: Consider a system

$$\dot{x}=f(x) \quad f(0)=0, \qquad x\in R^n \tag{3}$$

where $f: U_0\to R^n$ is continuous with respect to x on an open neighborhood U_0 of the origin $x=0$. The equilibrium $x=0$ of the system is finite-time stable if it is Lyapunov stable and finite-time convergent in a neighborhood $U\subseteq U_0$ of the origin. By "finite-time convergence," we mean: If, for any initial condition $x(0)\in U$, there is a settling time $T>0$, such that every solution $x(t)$ with $x(0)$ as its initial condition of (3) is defined with $x(t)\in U/\{0\}$ for $t\in[0,T)$ and satisfies $\lim_{t\to T}x(t)=0$ and $x(t)=0$ for any $t\ge T$.

The following result can be found in [2].

Lemma 1: Consider the nonlinear system described in (3). Suppose there is a C^1 function $V(x)$ defined in a neighborhood $\hat{U}\subset R^n$ of the origin, real numbers $c>0$ and $0<a<1$, such that $V(x)$ is positive definite on $\hat{U}$ and $\dot{V}(x)+cV^a(x)$ is negative semidefinite (along the trajectory) on $\hat{U}$. Then, the origin of (3) is finite-time stable with $T\le(V(x(0))^{1-a})/(c(1-a))$ for initial condition $x(0)$ in some open neighborhood $U\subset\hat{U}$ of the origin.

This note aims to find controllers for (1) with any initial condition $(x_0(0),x(0))$

$$u_0=u_0(x_0,x) \quad u=u(x_0,x) \tag{4}$$

such that the closed-loop system (1) with (4) is finite time stable within a given settling time T.

In the following analysis, the concept of homogeneity is used, though the system under consideration is not necessarily homogeneous.

Definition 2: $r_1,\ldots,r_n$ are positive real numbers. A function $V(x_1,x_2,\ldots,x_n)$ is homogeneous of degree $\sigma>0$ with respect to $(r_1,\ldots,r_n)$ if $V(\epsilon^{r_1}x_1,\epsilon^{r_2}x_2,\ldots,\epsilon^{r_n}x_n)=\epsilon^{\sigma}V(x_1,x_2,\ldots,x_n)$.

Remark 1: Homogeneity facilitates some analysis because it sometimes can limit the discussion to a compact set (for example, a unit sphere defined as $S^j=\{y\in R^j:\|(y_1,\ldots,y_j)\|=1\}$ in R^j). To be specific, consider a homogeneous function $\tilde{V}(x_1,\ldots,x_j)$ of degree σ with respect to $(r_1,\ldots,r_j)$. If $\tilde{V}(y)<0$ holds for any $y\in S^j$, then $\tilde{V}(x)<0$ for any $0\ne x\in R^j$, because there are $\epsilon>0$ and $y\in S^j$ such that $x_i=\epsilon^{r_i}y_i, i=1,\ldots,j$ and, therefore, $\tilde{V}(x)=\epsilon^{\sigma}\tilde{V}(y)$.

The terms with fractional powers are widely used in the sequel. It is known that, for any $b\in R, n_i>0, i=1,2,3$ with n_1,n_3 odd integers and n_2 even integers

$$|b|^{\frac{n_3}{n_1}}\operatorname{sgn}(b)=b^{\frac{n_3}{n_1}} \quad |b|^{\frac{n_2}{n_1}}=b^{\frac{n_2}{n_1}} \tag{5}$$

where $\operatorname{sgn}(\cdot)$ is the sign function. In fact, taking notice of (5) simplifies the expression in the following analysis.

III. Finite-Time Stabilization

In this section, we give a constructive procedure for the finite-time stabilizing control of system (1) within any given settling time T. As usual, we first discuss the problem in a special case when $x_0(0)\ne 0$, and then we extend our result to the case for any $x_0(0)$.

A. Analysis When $x_0(0)\ne 0$

For x_0-subsystem, we take the following control law:

$$u_0=-K_0x_0^{\alpha} \quad 0<\alpha=\frac{a_1}{a_2}<1 \tag{6}$$

where K_0 is a positive design parameter and $a_i, i=1,2$ are positive odd numbers. Take its Lyapunov function $V_0=x_0^2$ and then we have $\dot{V}_0+2q_0K_0V_0^{(1+\alpha)/(2)}=0$. By Lemma 1, the x_0-subsystem is finite time stable and the settling time T_s is

$$T_s=\frac{V(x_0(0))^{\frac{1-\alpha}{2}}}{q_0K_0(1-\alpha)}=\frac{|x_0(0)|^{1-\alpha}}{q_0K_0(1-\alpha)}\le\frac{|x_0(0)|^{1-\alpha}}{q_0^{\min}K_0(1-\alpha)}\triangleq T_0. \tag{7}$$

It is easy to see that when $t<T_s, x_0(t)\ne 0$ and therefore, $x_0(t)\in R$ does not change its sign. To secure finite time convergence within T for any $q_0\in[q_0^{\min},q_0^{\max}]$, we need to keep $T_0\le T$, or equivalently, $(|x_0(0)|^{1-\alpha})/(q_0^{\min}T(1-\alpha))\le K_0$. If we take $T_*=(T_0q_0^{\min})/(2q_0^{\max})$, then we obtain $|x_0(0)|\ge|x_0(t)|\ge(|x_0(0)|)/(2^{1/(1-\alpha)})$ during $[0,T_*]$ without changing the sign of $x_0(t)$. Hence, when $t\in[0,T_*]$

$$\hat{\gamma}\triangleq K_0|x_0(0)|^{\alpha}\ge|u_0|\ge\frac{K_0|x_0(0)|^{\alpha}}{2^{\frac{\alpha}{1-\alpha}}}\triangleq\gamma>0. \tag{8}$$

Then, we only need to stabilize the time-varying x-subsystem

$$\begin{cases}\dot{x}_1=q_1x_2u_0\\ \vdots\\ \dot{x}_{n-1}=q_{n-1}x_nu_0,\\ \dot{x}_n=q_nu+\psi_n(x)\end{cases} \tag{9}$$

within the given settling time T_*. This can be carried out as follows. We first find finite-time stabilizing feedback for (9) with assuming that (8) holds for any $t\ge 0$ (see Theorem 1). Then, we easily find that it is sufficient to assume (8) for $t\in[0,T_x]$ with T_x denoted as its actual settling time. After that, we reconstruct the controller via time-rescaling to make $T_x\le T_*$ and, therefore, remove this assumption because (8) naturally holds for $t\in[0,T_*]$ (see Theorem 2).

To solve the finite time stabilization problem with (8) holding for any $t\ge 0$, here we can extend the design idea given in [6] to this uncertain time-varying system (9).

Theorem 1: Let $r_i>0, \beta_{i-1}>0, i=1,\ldots,n$ and $k=p/q-1<0$ (with $q>0$ and $p>0$ odd integers) be real numbers satisfying

$$r_1=1,\ldots,r_i=r_{i-1}+k, \qquad r_i>-k>0 \qquad i=1,\ldots,n \tag{10}$$

and

$$\beta_0=r_2 \quad (\beta_i+1)r_{i+1}=(\beta_{i-1}+1)r_i>0, \qquad i=1,\ldots,n-1. \tag{11}$$

Then, finite-time stabilizing control law of (9), with (8) holding for any $t\ge 0$, can be constructed in the form of

$$u=v_n=-\gamma l_nw_n^{\frac{r_n+k}{r_n\beta_{n-1}}}-\frac{M}{q_n^{\min}}\left(\sum_{i=1}^{n}|x_i|\right)\operatorname{sgn}(w_n) \tag{12}$$

where w_n is defined by (13), as shown at the bottom of the next page. $\operatorname{sgn}(\cdot)$ denotes the sign function, and $l_i>0, i=1,\ldots,n$ are suitable constants.

Proof: Define

$$W_j(x) \triangleq \int_{v_{j-1}}^{x_j} w_j(x_1,\ldots,x_{j-1},s)\,ds = \frac{x_j^{\beta_{j-1}+1} + \beta_{j-1}|v_{j-1}|^{\beta_{j-1}+1}}{\beta_{j-1}+1} - x_j(v_{j-1})^{\beta_{j-1}} \quad (14)$$

which is nonnegative and even positive when $x_j \neq v_{j-1}(x_1,\ldots,x_{j-1})$. It is easy to see that W_j is C^1 if $v_{j-1}^{\beta_{j-1}}$ is C^1. Since $\mathrm{sgn}(u_0)$ in $v_j, j=1,\ldots n-1$ is either 1 or -1 and keeps unchanged due to (8), it does not influence the C^1 smoothness of $v_j^{\beta_j}, j=1,\ldots n-1$ with respect to x.

Then, the proof can be given step-by-step.

Step 1: Take a homogeneous function $V_1(x) = (x_1^{1+r_2})/(1+r_2)$ for the first equation $\dot{x}_1 = q_1 x_2 u_0$ and, therefore, for any given $l_1 > 0, \dot{V}_1 = x_1^{r_2} q_1 x_2 u_0 \leq -(\gamma/2) q_1^{\min} l_1 w_1^{(1+k+r_2)/(r_1\beta_0)} + q_1 u_0 w_1(x_2 - v_1)$. Obviously, $v_1^{\beta_1}$ is C^1 since $\beta_1 = 1/r_2 = 1/(1+k)$ by (11).

After Step $j-1$: For system

$$\begin{cases} \dot{x}_1 = q_1 x_2 u_0 \\ \cdots \\ \dot{x}_{j-1} = q_{j-1} x_j u_0 \end{cases} \quad (15)$$

we already have that $v_i^{\beta_i}$ is C^1 of homogeneity degree $(r_i+k)\beta_i > 0$ for $i=1,\ldots,j-1, V_{j-1} = \sum_{i=1}^{j-1} W_i$ is a homogeneous function of degree $1+r_2$ with respect to $(r_1,\ldots,r_{j-1})$ and is C^1 positive definite with respect to $x_1,\ldots,x_{j-1}$, with its derivative as

$$\dot{V}_{j-1}|_{(15)} \leq -\sum_{i=1}^{j-1} \frac{\gamma}{2} q_i^{\min} l_i w_i^{\frac{1+k+r_2}{r_i\beta_{i-1}}} + q_{j-1} u_0 w_{j-1}(x_j - v_{j-1}) \quad (16)$$

for some suitable constants $l_i > 0, i=1,\ldots,j-1$.

Step $j(<n)$: Consider system

$$\begin{cases} \dot{x}_1 = q_1 x_2 u_0 \\ \cdots \\ \dot{x}_{j-1} = q_{j-1} x_j u_0 \\ \dot{x}_j = q_j x_{j+1} u_0. \end{cases} \quad (17)$$

Note that the function $V_j(x) = W_j(x) + V_{j-1}(x) = \sum_{i=1}^{j} W_i$ is positive definite with respect to $x_1,\ldots,x_j$ in R^j, and it is also C^1 and homogeneous of degree $1+r_2$ with respect to $(r_1,\ldots,r_j)$ because W_j and V_{j-1} are so.

Take v_j as given in (13), where l_j will be determined later. $u_0 w_j v_j \leq 0$ implies $q_j u_0 w_j v_j \leq -\gamma q_j^{\min} w_j v_j$. Then, we have

$$\begin{aligned} \dot{V}_j|_{(17)} &\leq V_j^0(x) + q_j u_0 w_j(x_{j+1} - v_j) + q_j u_0 w_j v_j \\ &\leq \tilde{V}_j + q_j u_0 w_j(x_{j+1} - v_j) \end{aligned} \quad (18)$$

with

$$V_j^0(x) \triangleq \hat{\gamma} \sum_{i=1}^{j-1} \left| \frac{\partial W_j}{\partial x_i} q_i^{\max} x_{i+1} \right| - \sum_{i=1}^{j-1} \frac{\gamma}{2} q_i^{\min} l_i w_i^{\frac{1+k+r_2}{r_i\beta_{i-1}}} + \hat{\gamma} q_{j-1}^{\max} |w_{j-1}(x_j - v_{j-1})|$$

and $\tilde{V}_j \triangleq V_j^0 - \gamma q_j^{\min} l_j w_j^{(1+k+r_2)/(r_j\beta_{j-1})}$, where only $l_j > 0$ should be determined to make $\tilde{V}_j$ negative definite. Clearly, both V_j^0 and $\tilde{V}_j$ are homogeneous (for any constant l_j) of degree $1+k+r_2 > 0$ with respect to $(r_1,\ldots,r_j)$.

It is obvious that, when $w_j = 0$ (that is, $x_j = v_{j-1}$), we have $W_j = 0$ according to (14), which implies

$$\tilde{V}_j = V_j^0 = -\sum_{i=1}^{j-1} \frac{\gamma}{2} q_i^{\min} l_i w_i^{\frac{1+k+r_2}{r_i\beta_{i-1}}} < 0 \quad \text{if } (x_1,\ldots,x_j) \neq 0. \quad (19)$$

S^j is the unit sphere of R^j as defined in Remark 1. Define $S_+^j = \{x \in S^j : V_j^0(x) \geq 0\}$ and $S_-^j = \{x \in S^j : V_j^0(x) < 0\}$. S_-^j is not empty due to (19). If S_+^j is empty, then $\tilde{V}_j$ is negative definite. Here, suppose S_+^j is nonempty. Then, since S_+^j is closed and compact, we can take

$$M_1 \triangleq \max_{x\in S_+^j} V_j^0(x) \geq 0 \quad \text{and} \quad M_2 \triangleq \min_{x \in S_+^j} \frac{\gamma}{2} q_j^{\min} w_j(x)^{\frac{1+k+r_2}{r_j\beta_{j-1}}} \geq 0.$$

$M_2 > 0$ because $M_2 = 0$ implies $w_j = 0$ [noting that $\{x \in S^j : w_j = 0\} \subset S_-^j$ from (19)]. Hence, if we take $l_j > M_1/M_2 \geq 0$, we have, along with (19)

$$\tilde{V}_j(x) \leq -\sum_{i=1}^{j} \frac{\gamma}{2} q_i^{\min} l_i w_i^{\frac{1+k+r_2}{r_i\beta_{i-1}}} < 0 \qquad \forall x \in S^j. \quad (20)$$

Recalling Remark 1, (20) holds for any $x \neq 0$ with the selected l_j. Thus, we have

$$\begin{aligned} \dot{V}_j &\leq \tilde{V}_j + q_j u_0 w_j(x_{j+1} - v_j) \\ &\leq -\sum_{i=1}^{j} \frac{\gamma}{2} q_i^{\min} l_i w_i^{\frac{1+k+r_2}{r_i\beta_{i-1}}} + q_j u_0 w_j(x_{j+1} - v_j) \end{aligned}$$

which is consistent with (16).

In addition, $v_j^{\beta_j} = -l_j^{\beta_j}[x_j^{\beta_{j-1}} - v_{j-1}^{\beta_{j-1}}]^{(r_{j+1}\beta_j)/(r_j\beta_{j-1})}$ is C^1 because $v_{j-1}^{\beta_{j-1}}$ is C^1 and $\beta_j(r_j+k) = \beta_j r_{j+1} > \beta_{j-1} r_j$ from (11) and $r_j > r_{j+1} > 0$.

Up to Step n: By induction, we have $V_n = \sum_{i=1}^n W_i$ and then

$$\begin{aligned} \dot{V}_n \leq & \sum_{i=1}^{n-1} \frac{\partial W_n}{\partial x_i} q_i x_{i+1} u_0 + w_n(q_n u + \psi_n(x)) \\ & - \sum_{i=1}^{n-1} \frac{\gamma}{2} q_i^{\min} l_i w_i^{\frac{1+k+r_2}{r_i\beta_{i-1}}} + q_{n-1} u_0 w_{n-1}(x_n - v_{n-1}). \end{aligned}$$

Take u of form (12). Still using the analysis given in Step j, we obtain $l_n > 0$ such that

$$\dot{V}_n \leq \tilde{V}_n \triangleq V_n^0 - \gamma q_n^{\min} l_n w_n^{\frac{1+k+r_2}{r_n\beta_{n-1}}} \leq -\frac{\gamma}{2} \sum_{i=1}^{n} q_i^{\min} l_i w_i^{\frac{1+k+r_2}{r_i\beta_{i-1}}}. \quad (21)$$

Because $(x_i - v_{i-1}) w_i(x_1,\ldots,x_i) \geq 0$, it is not hard to get that

$$V_n^{\frac{1+k+r_2}{1+r_2}} = \left(\sum_{i=1}^{n} \int_{u_{i-1}}^{x_i} w_i(x_1,\ldots,x_{i-1},s)\,ds \right)^{\frac{1+k+r_2}{1+r_2}} \leq \left(2 \sum_{i=1}^{n} w_i^{\frac{1+r_2}{r_i\beta_{i-1}}} \right)^{\frac{1+k+r_2}{1+r_2}}. \quad (22)$$

$$\begin{cases} w_1 \triangleq x_1^{1+k} \\ w_i(x_1,\ldots,x_i) \triangleq x_i^{\beta_{i-1}} - v_{i-1}^{\beta_{i-1}} \quad v_{i-1} = -l_{i-1} w_{i-1}^{\frac{r_{i-1}+k}{r_{i-1}\beta_{i-2}}} \mathrm{sgn}(u_0), \qquad i = 2,\ldots,n \end{cases} \quad (13)$$

Then, putting (22) back to (21) gives

$$\dot{V}_n \leq -\frac{\gamma l}{2^{\frac{2+k+2r_2}{1+r_2}}} V_n^{\frac{1+k+r_2}{1+r_2}} \quad l = \min\{l_i, i = 1, \ldots, n\}. \tag{23}$$

By Lemma 1 and $k < 0$, system (9) under control law (12) is finite time stable with its settling time denoted as T_x

$$T_x \leq -\frac{2^{\frac{2+k+2r_2}{1+r_2}}(1+r_2)}{\gamma l k} V_n(x(0))^{\frac{-k}{1+r_2}}. \tag{24}$$

△

Remark 2: From the proof of Theorem 1, we can find that the assumption that (8) holds for any t can be replaced by that (8) holds for $t \in [0, T_x]$ with T_x as its settling time, because we need not care about (8) after $x = 0$ when $t \geq T_x$.

Then, we render the system finite time stable within any given settling time via time-rescaling, and remove the assumption in Theorem 1 (i.e., (8) for any $t \in [0, T_x]$) by making $T_x \leq T_*$.

Theorem 2: If $x_0(0) \neq 0$, system (1) is finite-time stable within any given settling time T under the feedback law in the form of

$$u = u_* = K^n v_n\left(x_1, \frac{x_2}{K}, \ldots, \frac{x_n}{K^{n-1}}\right) \tag{25}$$

with v_n defined in (12) and suitable $K \geq 1$.

Proof: First of all, it is easy to see that we can choose suitable K_0 such that the system converges within $T_0 \leq T$ for the system $\dot{x}_0 = q_0 u_0$. Then, we should construct a controller $u(x)$ for system (9) to make its settling time $T_x \leq T_* = (T_0 q_0^{\min})/(2q_0^{\max})$.

If T_x given in (24) satisfies $T_x \leq T_*$, then we can take $u_* = v_n$ or, equivalently, in the form of (25) with $K = 1$.

If $T_x > T_*$, we will employ a *time-rescaling* technique to reconstruct a finite-time controller to make the closed-loop system with a "modified" settling time $T_x^K \leq T_*$.

Take $\bar{t} = Kt$ and $\bar{x}_i = K^{1-i} x_i$ with $K \geq 1$, then we get equations from (9)

$$\begin{cases} \frac{d\bar{x}_1}{d\bar{t}} = q_1 \bar{x}_2 u_0 \\ \vdots \\ \frac{d\bar{x}_{n-1}}{d\bar{t}} = q_2 \bar{x}_n u_0 \\ \frac{d\bar{x}_n}{d\bar{t}} = q_n \frac{u}{K^n} + \frac{\psi_n(x)}{K^n} \triangleq q_n \bar{u} + \bar{\psi}_n(\bar{x}) \end{cases} \tag{26}$$

where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T$. Note that $|\bar{\psi}_n| \leq M \sum_{i=1}^n |x_i|/K^n \leq M \sum_{i=1}^n |\bar{x}_i|$ still satisfying (2). Therefore, (26) is still in the form of (9), and then control $\bar{u} = v_n(\bar{x})$ given in (12) with the same $l_i, i = 1, \ldots, n$, can finite-time stabilize (26). Here, we only need that (8) should hold when $\bar{t} \in [0, \bar{T}_x]$, or equivalently $t \in [0, \bar{T}_x/K]$, as discussed in Remark 2, where $\bar{T}_x(\bar{x})$ is its settling time in the time scale $\bar{t}$

$$\bar{T}_x \leq -\frac{2^{\frac{2+k+2r_2}{1+r_2}}(1+r_2)}{\gamma l k} V_n(\bar{x}(0))^{\frac{-k}{1+r_2}}. \tag{27}$$

Next, we only need to prove that there is a suitable K such that the "modified" settling time $T_x^K = \bar{T}_x/K \leq T_*$ of the system with (25) in time scale t, or in other words, $\bar{t} \in [0, \bar{T}_x]$ (or equivalently, $t \in [0, T_x^K]$) implies $t \in [0, T_*]$. If so, (8) for any $t \in [0, T_x^K]$ is not an assumption any longer since (8) is certainly true for any $t \in [0, T_*]$ and $T_x^K \leq T_*$.

Note that, because of the continuity of V_n, for any fixed $x(0)$ and $K \geq 1$, we will have

$$\sup_{K \geq 1} V_n\left(x_1(0), \frac{x_2(0)}{K}, \ldots, \frac{x_n(0)}{K^{n-1}}\right)$$

and, therefore, $\bar{T}_x$ in (27) is bounded. Hence, $\lim_{K \to \infty} T_x^K = 0$. Then, with a suitable constant $K \geq 1$, we can make the settling time $T_x^K \leq T_*$. Thus, the conclusion follows. △

B. Control Scheme

In the last subsection, we gave the controller expressions (6) and (12) for u_0 and u of system (9) in the case of $x_0(0) \neq 0$. Now, we consider finite-time control laws for any $x_0(0)$.

Theorem 3: Let $\Gamma = \{(0, x) : \|x\| \neq 0\}$ and let $\beta > 0$ be a suitably selected real constant. Then, the following feedback law globally finite-time stabilizes the system (9) within any given finite time T.

i) When $(x_0(0), x(0)) = (0, x(0)) \in \Gamma$

$$u_0(t) = \begin{cases} \beta, & \text{if } t < t_s(\|x(0)\|) \\ -K_0 x_0^\alpha, & \text{if } t \geq t_s(\|x(0)\|) \end{cases} \tag{28}$$

$$u(t) = \begin{cases} 0, & \text{if } t < t_s(\|x(0)\|) \\ u_*(x, u_0), & \text{if } t \geq t_s(\|x(0)\|) \end{cases} \tag{29}$$

where α is given in (6), $t_s = \min\{(T/4), \|x(0)\|\}$

$$K_0 \geq \frac{4(\beta t_s)^{1-\alpha}}{3q_0^{\min} T(1-\alpha)} \quad T_0 = \frac{(\beta t_s)^{1-\alpha}}{q_0^{\min} K_0(1-\alpha)} \leq \frac{3T}{4} \tag{30}$$

and $u = u_*$ is taken in the form of (25) to stabilize x-subsystem within finite time $T_* = (q_0^{\min} T_0)/(2q_0^{\max})$.

ii) When $(x_0(0), x(0)) = (0, 0)$

$$\begin{cases} u_0 = 0 \\ u = 0 \end{cases}. \tag{31}$$

iii) When $(x_0(0), x(0)) \notin \Gamma \cup \{(0,0)\}$, u_0 is given in the form of (6) with $K_0 \geq (x_0(0)^{1-\alpha})/(q_0^{\min} T(1-\alpha))$ and

$$u = u_*(x, u_0) \tag{32}$$

in the form of (25) to stabilize x-subsystem in finite time $T_* = (q_0^{\min} T_0)/(2q_0^{\max})$ with T_0 given in (7).

Proof: If $(x_0(0), x(0)) = (0,0)$ and $(x_0(0), x(0)) \notin \Gamma \cup \{(0,0)\}$, Theorem 3 is a direct consequence of Theorem 2 with the controller given in (25). Therefore, we only need to consider the case when $(x_0(0), x(0)) \in \Gamma$.

When $t \in [0, t_s]$, we have

$$|x_0(t)| = \beta t, \qquad t_s \geq t \geq 0 \tag{33}$$

and

$$\|x(t)\| \leq \gamma_0 \|x(0)\| \qquad \forall t \in [0, t_s] \tag{34}$$

for some constant $\gamma_0 > 0$ because u_0 is a constant and the x-subsystem becomes linear. Then, in light of Theorem 2, it is not hard to get the finite-time convergence of the closed-loop system within $T_0 \leq 3T/4$ by taking α as in (6) and K_0 as in (30).

The remaining task is to consider the Lyapunov stability. When $t \in [0, t_s], |x_0(t)| = \beta t \leq \beta t_s \leq \beta \|x(0)\|$ according to (33); when $t \geq t_s$, we take $V_0(x_0) = x_0^2$ with $\dot{V}_0 \leq 0$, which implies that V_0 is decreasing with t when $t \geq t_s$ and, therefore, $|x_0(t)| \leq \beta \|(x_0(0), x(0))\|, \forall t \geq 0$. Moreover, with (34) and the Lyapunov stability of x-subsystem under $u = u_*$ when $t \geq t_s$ (from Theorem 2), it is quite obvious to see that there is a constant $\gamma_* > 0$ such that

$$\|x(t)\| \leq \gamma_* \|x(t_s)\| \leq \gamma_* \gamma_0 \|x(0)\|. \tag{35}$$

Inequality (35) along with inequality $|x_0(t)| \leq \beta \|(x_0(0), x(0))\|$ secures the Lyapunov stability of the closed-loop system. △

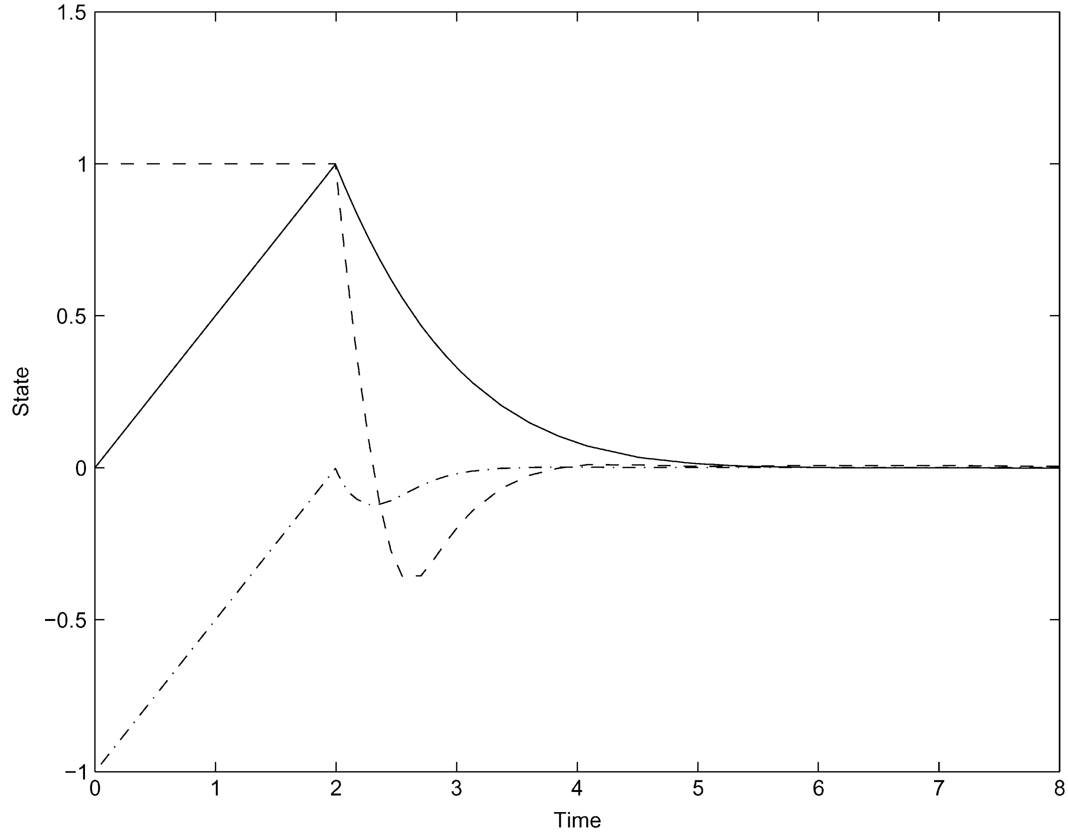

Fig. 1. Trajectories of x_0 (solid line), x_1 (dot line), and x_2 (dash line) of system (37).

Remark 3: Different from most existing results about finite time control, we have to finite-time stabilize the two interconnected subsystems of the considered nonholonomic system. To do this, we finite-time stabilize one subsystem within given settling time at first, and then, with this control input kept at a nonzero value, we make the other subsystem finite time convergent.

Remark 4: Taking $u = 0$ for $t \in [0, t_s]$ in (26) is not necessary. In fact, many choices can be given (but the selected controllers should not yield finte time escape). Here, we take $u = 0$, partially to simplify the proof of Lyapunov stability of the closed-loop system.

IV. EXAMPLE

A mobile robot with nonholonomic constraints has often been used as a benchmark example in the literature [5], [8]. In [5], the parking problem for a mobile robot with parametric uncertainties is given as follows:

$$\begin{cases} \dot{x} = p_1 u \cos\psi \\ \dot{y} = p_1 u \sin\psi \\ \dot{\psi} = p_2 u_0 \end{cases} \tag{36}$$

where (x, y) denotes the position of the center of mass of the robot, ψ is the heading angle of the robot, v is the forward velocity, ω is the angular velocity of the robot, $p_1 > 0$ and $p_2 > 0$ are uncertain positive parameters determined by the radius of the rear wheels and the distance between them. Using the following change of coordinates and feedback [8]:

$$\begin{aligned} x_0 &= \psi \\ x_1 &= x\sin\psi - y\cos\psi \\ x_2 &= x\cos\psi + y\sin\psi \end{aligned}$$

system (36) was transformed into the following system, of form (1):

$$\begin{cases} \dot{x}_0 = p_2 u_0 \\ \dot{x}_1 = p_2 x_2 u_0 \\ \dot{x}_2 = p_1 u - p_2 x_1 u_0. \end{cases} \tag{37}$$

As in [12], $p_1^{\min} = p_2^{\min} = 1, p_1^{\max} = p_2^{\max} = 2$, which are known for us in contructing control laws to make (36) finite-time stable, and parameters $p_1 = p_2 = 1.5$ are unknown. Then, based on Theorem 3, the following controllers can be obtained for a given settling time $T = 8$.

With $x_0(0) = 0, (x_1(0), x_2(0)) = (-1, 1) \neq (0, 0)$, the controllers are shown in (38) and (39) at the bottom of the page.

The simulation results in Fig. 1 show that the effectiveness of the controller.

$$\begin{cases} u_0(t) = 1/2 \\ u(t) = 0 \end{cases} \quad \text{if } t < 2 \tag{38}$$

$$\begin{cases} u_0(t) = -x_0^{\frac{9}{11}} \\ u(t) = -4(x_2^{\frac{11}{9}} - 3.5x_1)^{9/11} - 4|x_1|\operatorname{sgn}(x_2^{\frac{11}{9}} - 3.5x_1) - \frac{3x_1 u_0}{2} \end{cases} \quad \text{if } t \geq 2 \tag{39}$$

V. Conclusion

In this note, the problem of finite time stabilization is considered for a class of uncertain chained systems. Using homogeneity, time-rescaling and Lyapunov function techniques, a finite-time stabilizing feedback law is designed in order to guarantee both Lyapunov stability and finite time convergence in any given settling time for the closed-loop system. The numerical results in a wheeled mobile robot demonstrate the effectiveness of the proposed control design.

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Is Normalization Necessary for Stable Model Reference Adaptive Control?

Nikita Barabanov, Romeo Ortega, and Alessandro Astolfi

***Abstract*—One of the longest standing open questions in adaptive control concerns the correctness of the stability claim of the un-normalized model reference scheme proposed by Monopoli in 1974. Although provably correct solutions to the problem now abound, in particular, it is well known that adding a normalization to Monopoli's original scheme ensures global convergence, it is interesting to know whether this technique-driven modification is really necessary or only required to complete the stability proof in the absence of more elaborate arguments. In this note, we construct a counterexample that provides a definite-unfortunately, negative-answer to the claim. Instrumental for the establishment of this result is a technical lemma that shows that, under some conditions on the regressor that may appear in Monopoli's scheme, the parameter error freezes as the adaptation gain goes to infinity. On the lighter side, we also prove that Monopoli's scheme is semiglobally stable, underscoring the relevance of this important contribution.**

***Index Terms*—Adaptive control, model reference control, nonlinear control, stability of adaptive systems.**

I. Introduction

Model reference adaptive control (MRAC) is unquestionably the most widely studied problem in the adaptive literature that has a very long history going back to the 1950s and extending to the present time.[1] The earliest attempts to solve the MRAC problem followed the classical path of designing an observer, that had to be made adaptive because of the unknown plant parameters, and then feeding back the observed state [10]. A first major breakthrough, essentially due to [3] and [13], was the introduction of the so-called direct control parameterization which revealed that the estimation of the plant state could be obviated and only a "good" estimation of the controller parameters was needed to achieve the asymptotic model matching objective.

A second fundamental development, also reported in [13], was the derivation of a suitable error signal, called the augmented error, that can be used to identify the controller parameters with a quality that, as time evolves, is nondegrading. (More precisely, the norm of the parameter estimation error is a nonincreasing function.) Motivated by this important property Monopoli also presented some arguments intended to establish global convergence of his scheme, that in the sequel we will call M-MRAC. A flaw in the proof of M-MRAC was indicated in [7]. As pointed out in that paper, the authors do not provide a counterexample to the claim of stability but only question the correctness of the proof, and the problem of deciding whether or not M-MRAC is globally convergent remained unsolved for 30 years (see [16].) The purpose of this note is to give a definite *negative* answer to the question. For, we construct a *bona-fide* analytical counterexample proving

Manuscript received December 7, 2004; revised April 25, 2005. Recommended by Associate Editor S. S. Ge.

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Digital Object Identifier 10.1109/TAC.2005.854625

[1]The interested reader is referred to [14] for a vivid description of the history as well as to the existing textbooks [21], [17], [12], [9] for further information on MRAC.