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On an Output Feedback Finite-Time Stabilization Problem

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***Abstract*—This note studies the problem of finite-time output feedback stabilization for the double integrator system. A class of output feedback controllers that can achieve global finite-time stability for the double integrator system are constructed based on a "finite-time separation principle." Furthermore, it is shown that the closed-loop system resulting from our control law can maintain its local finite-time stability regardless of some nonlinear perturbations. Thus, our control law actually applies to a large class of nonlinear second order systems.**

***Index Terms*—Finite-time observer, finite-time stabilization, nonlinear systems, output feedback.**

I. Introduction

The problem of state feedback finite-time stabilization of a dynamic system has been studied by quite a few people from different perspectives [1], [4], [6], [12], [14]. A most recent result was given by Bhat and Bernstein in [4], where they studied finite-time stabilization of a double integrator system by continuous, unbounded or bounded, state feedback control laws. In addition, the design of finite-time observers was discussed from the probabilistic variational considerations [9].

In this note, we will study the finite-time stabilization of the double integrator system by continuous output feedback control laws. We give a class of nonsmooth finite-time observer for the double integrator system. In conjunction with the state feedback finite-time stabilization control law, our finite-time observer naturally leads to an output feedback finite-time stabilization control law. Using homogeneous techniques as can be found in [3], [7], [13], we also show that our control law for the double integrator system results in a closed-loop system with certain robustness property with respect to a class of nonlinear perturbations. This robust property actually extends the applicability of our control law to a large class of second order nonlinear systems.

The rest of this note is organized as follows. In Section II, we formulate the problem and review some preliminary results. Our major result is presented in Section III, where the output feedback finite-time stabilizing control laws for the double integrator systems are given. In Section IV, we further show that the zero solution of the closed-loop system resulting from our control law given in Section III is still locally finite-time stable in the presence of a class of nonlinear perturbations. Finally, we close this note in Section V with some remarks.

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II. Problem Formulation and Preliminaries

Let us begin by introducing some terminology and results given in [4]. Consider the system

$$\dot{x} = f(x), \qquad f(0) = 0, \qquad x \in R^n, \qquad x(0) = x_0 \tag{1}$$

where $f\colon D \to R^n$ is continuous on an open neighborhood D of the origin $x = 0$. The zero solution of (1) is finite-time convergent if there is an open neighborhood $U \subseteq D$ of the origin and a function $T_x\colon U\backslash\{0\} \to (0, \infty)$, such that $\forall x_0 \in U$, the solution $s_t(0, x_0)$ of system (1) with x_0 as the initial condition is defined and $s_t(0, x_0) \in U\backslash\{0\}$ for $t \in [0, T_x(x_0)]$, and $\lim_{t\to T_x(x_0)} s_t(0, x_0) = 0$. Here $T_x(x_0)$ is called *the settling time* (with respect to initial state x_0). The zero solution of (1) is finite-time stable if it is Lyapunov stable and finite-time convergent. When $U = D = R^n$, then the zero solution is said to be globally finite-time stable.

So far most research on the construction of time-invariant finite-time-stabilizing state feedback has been done for the so-called double integrator system which is described in the following stats space from

$$\begin{cases} \dot{x}_1 = x_2, & x_1(0) = x_{10} \\ \dot{x}_2 = u, & x_2(0) = x_{20} \\ y = x_1. \end{cases} \tag{2}$$

The problem was initially explored in the context of optimal control [1]. A discontinuous finite-time stabilizing controller for double integrators was given in [14]. In [12], a constructive way to design finite-time stabilizing feedback was reported. Further result was given in [6] where a class of continuous time-invariant state feedback controllers was constructed for double integrators as follows:

$$u = -|x_1|^{\alpha_1}\mathrm{sgn}(x_1) - |x_2|^{\alpha_2}\mathrm{sgn}(x_2) \tag{3}$$

where $0 < \alpha_2 < 1$ and $\alpha_1 > \alpha_2/(2-\alpha_2)$. Recently, Bhat and Bernstein have presented two other classes of continuous state-feedback stabilization control laws as follows:

$$u = -|x_2|^{\alpha}\mathrm{sgn}(x_2) - |\phi_\alpha|^{\alpha/(2-\alpha)}\mathrm{sgn}(\phi_\alpha) \tag{4}$$

$$u = -sat(|x_2|^{\alpha}\mathrm{sgn}(x_2)) - sat(|\phi_\alpha|^{\alpha/(2-\alpha)}\mathrm{sgn}(\phi_\alpha)) \tag{5}$$

where $\phi_\alpha = x_1 + (1/(2-\alpha))|x_2|^{2-\alpha}\mathrm{sgn}(x_2)$, $0 < \alpha < 1$, and $\mathrm{sgn}(\cdot)$ and $sat(\cdot)$ are the sign function and the saturation function, respectively. It is noted that (5) is not only continuous but also bounded.

One of the major objectives of this paper is to design output feedback control law of the form

$$u = \mu(\zeta), \qquad \dot{\zeta} = \beta(\zeta, y), \qquad \zeta \in R^2 \tag{6}$$

such that the zero solution of the closed loop systems consisting of (2) and (6) is finite-time stable where $\mu(\cdot)$ and $\beta(\cdot, \cdot)$ are continuous functions satisfying $\mu(0) = 0$ and $\beta(0, 0) = 0$. To this end, we will start from the following result given in [4] and [6].

Lemma 1: Consider the nonlinear system described in (1). Suppose there is a C^1 (continuously differentiable) function $V(x)$ defined in a neighborhood $\hat{U} \subset R^n$ of the origin, and there are real numbers $c > 0$ and $0 < \alpha < 1$, such that $V(x)$ is positive definite on $\hat{U}$ and $\dot{V}(x) + cV^{\alpha}(x)$ is negative semidefinite on $\hat{U}$. Then the zero solution of system (1) is finite-time stable. The settling time, depending on initial state $x(0) = x_0$, is given by

$$T_x(x_0) \le \frac{V(x_0)^{1-\alpha}}{c(1-\alpha)}$$

for all x_0 in some open neighborhood of the origin. If $\hat{U} = R^n$ and $V(x)$ is also radially unbounded [i.e., $V(x) \to +\infty$ as $\|x\| \to +\infty$], the zero solution of system (1) is globally finite-time stable.

Our approach will be based on the properties of homogeneous systems. Therefore, we will also introduce some concepts about homogeneous systems which can be found, for example, in [3], [7], and [13].

A scalar function $V(x)$ is homogeneous of degree $\sigma \in R$ with dilation $(r_1, \cdots, r_n)$, $r_i > 0$, $i = 1, \ldots, n$, if for all $\epsilon > 0$

$$V(\epsilon^{r_1} x_1, \ldots, \epsilon^{r_n} x_n) = \epsilon^{\sigma} V(x). \tag{7}$$

A vector field $f(x) = [f_1(x), \cdots, f_n(x)]^T$ is homogeneous of degree $k \in R$ with dilation $r = (r_1, \ldots, r_n)$ if, for all $\epsilon > 0$,

$$f_i(\epsilon^{r_1} x_1, \ldots, \epsilon^{r_n} x_n) = \epsilon^{k + r_i} f_i(x), \qquad i = 1, \ldots, n. \tag{8}$$

We call (1) a homogeneous system if $f(x)$ is a homogeneous vector field. The stability of the homogeneous systems has been extensively studied in [7], [13]. The following lemma is adapted from [13, Th. 2].

Lemma 2: Suppose that system (1) is homogeneous of degree k with dilation $(r_1, \ldots, r_n)$, f is continuous and $x = 0$ is its asymptotically stable equilibrium. Then, for any positive integer j and any real number $\sigma_0 > j \cdot \max\{r_1, \ldots, r_n\}$, there is a C^j homogeneous function V of degree σ_0 with the same dilation $(r_1, \ldots, r_n)$ such that V is positive definite, radially unbounded, and $\dot{V}(x)|_{(1)} < 0$ for all $x \neq 0$.

III. Global Finite-Time Output Feedback

In this section, we will construct a class of output feedback finite-time stabilizing control law by considering the following output feedback control laws:

$$u = \mu(\zeta_1, \zeta_2) \tag{9}$$

$$\begin{cases} \dot{\zeta}_1 = \zeta_2 - k_1|\zeta_1 - y|^{\sigma_1}\operatorname{sgn}(\zeta_1 - y) \\ \dot{\zeta}_2 = u - k_2|\zeta_1 - y|^{\sigma_2}\operatorname{sgn}(\zeta_1 - y) \end{cases} \tag{10}$$

where $\sigma_1 > 0$, $\sigma_2 > 0$, $k_1 > 0$, $k_2 > 0$, $\zeta_1(0) = \zeta_{10}$, and $\zeta_2(0) = \zeta_{20}$. Clearly, when $\sigma_1 = \sigma_2 = 1$, (10) is nothing but the classic Luenberger observer [10]. Thus, (10) can be viewed as a generalization of the linear observer.

Let $e_1 = \zeta_1 - x_1$, $e_2 = \zeta_2 - x_2$, and $e = (e_1, e_2)^T$, then the error dynamics generated by systems (2) and (10) is given by

$$\begin{cases} \dot{e}_1 = e_2 - k_1|e_1|^{\sigma_1}\operatorname{sgn}(e_1) \\ \dot{e}_2 = -k_2|e_1|^{\sigma_2}\operatorname{sgn}(e_1) \end{cases} \tag{11}$$

with $e_1(0) = \zeta_{10} - x_{10}$ and $e_2(0) = \zeta_{20} - x_{20}$.

With coordinates (x_1, x_2, e_1, e_2), the closed-loop system is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \mu(x_1 + e_1, x_2 + e_2) \\ \dot{e}_1 = e_2 - k_1|e_1|^{\sigma_1}\operatorname{sgn}(e_1) \\ \dot{e}_2 = -k_2|e_1|^{\sigma_2}\operatorname{sgn}(e_1). \end{cases} \tag{12}$$

Remark 1: If the zero solution of the error dynamics (11) is globally finite-time stable with settling time $T_e(e_0)$, then

$$\mu(\zeta_1, \zeta_2) = \mu(x_1, x_2), \qquad t \ge T_e(e_0). \tag{13}$$

Thus, if $u = \mu(x_1, x_2)$ is a state-feedback finite-time stabilizing control law for (2) with settling time $T_x(x_0)$, and the zero solution of the closed-loop system is globally asymptotically stable, then the zero solution of the closed-loop system is also finite-time stable. Thus, in the following, we will show that, by properly selecting the parameters of the observer (10), the zero solution of the error system (11) is finite-time stable. Furthermore, there exist finite-time state-feedback controllers $\mu(x)$ such that the zero solution of the closed-loop system (12) is globally asymptotically stable.

Proposition 1: The zero solution of the error dynamics (11) is globally finite-time stable for any $k_i > 0$, $i = 1, 2$, and for all

$$0 < \sigma_2 < 1, \qquad 2\sigma_1 - \sigma_2 = 1. \tag{14}$$

Proof: Note that the error system (11) is a homogeneous system of degree $k = (\sigma_2/\sigma_1) - 1$ with dilation $(1/\sigma_1, 1)$, and condition (14) implies $k < 0$. By [3, Th. 2], if the degree of a homogeneous system is negative, the zero solution of this system is finite-time stable if and only if the zero solution of this system is asymptotically stable. Therefore, we only need to show that the zero solution of the homogeneous system (11) is asymptotically stable.

To this end, consider a Lyapunov function candidate for system (11):

$$V_*(e) = k_2|e_1|^{\sigma_2 + 1} + \frac{1 + \sigma_2}{2} e_2^2$$

which is positive definite, and radially unbounded since $k_2 > 0$ and $\sigma_2 > 0$. The derivative of V_* along the trajectories of (11) is

$$\dot{V}_*(e) = -k_1 k_2 (1 + \sigma_2)|e_1|^{\sigma_1 + \sigma_2}$$

which is negative semidefinite since $k_1 > 0$. Now note that $\dot{V}_*(e) \equiv 0$ implies $e_1 \equiv 0$ which, in turn, implies $e_2 \equiv 0$ using (11). By LaSalle's invariant set theorem, the zero solution of the homogeneous system (11) is asymptotically stable.

Proposition 2: Consider the closed-loop system (12) where $k_i > 0$, $i = 1, 2$, and $0 < \sigma_2 < 1$, $2\sigma_1 - \sigma_2 = 1$. Assume $\mu(x_1, x_2)$ globally finite-time stabilizes the double integrator system and satisfies

$$\mu(\epsilon^{1/\sigma_1} x_1, \epsilon x_2) = \epsilon^{\sigma_2/\sigma_1} \mu(x). \tag{15}$$

Then, the zero solution of the closed-loop system (12) is globally asymptotically stable.

Proof: Under our assumption, the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \mu(x_1, x_2) \tag{16}$$

is homogeneous of degree $k = (\sigma_2/\sigma_1) - 1 < 0$ with dilation $(r_1, r_2) = (1/\sigma_1, 1)$, and the zero solution of (16) is globally finite-time stable. By Proposition 1, the error system is also homogeneous of degree $k = (\sigma_2/\sigma_1) - 1 < 0$ with dilation $(r_1, r_2) = (1/\sigma_1, 1)$, and its zero solution globally finite-time stable. By Lemma 2, there are radially unbounded positive definite C^1 functions $V_1(x)$ and $V_2(e)$ both of which are homogeneous of degree $\sigma_0 > 1$ with dilation (r_1, r_2) such that $\dot{V}_1(x) \leq -c_1 V_1^{(\sigma_0+k)/\sigma_0}$ for some $c_1 > 0$, and $\dot{V}_2(e) \leq -c_2 V_2^{(\sigma_0+k)/\sigma_0}(e)$ for some $c_2 > 0$. Define a candidate Lyapunov function $V^K(x, e) = V_1(x) + KV_2(e)$ where $K > 0$ for the closed-loop system (12). Clearly, $V^K(x, e)$ is homogeneous function of degree σ_0 with dilation (r_1, r_2, r_1, r_2), and is positive definite and radially unbounded. We will show that, for sufficiently large K, the derivative of $V^K(x, e)$ along all the trajectories of the closed-loop system is negative definite. To this end, note that

$$\dot{V}^K(x, e) = \dot{V}_1(x) + K\dot{V}_2(e) \leq -h_1(x) - Kh_2(e) - h_3(x, e) \tag{17}$$

where

$$h_1(x) = c_1 V_1(x)^{(\sigma_0+k)/\sigma_0}, \qquad h_2(e) = c_2 V_2(e)^{(\sigma_0+k)/\sigma_0}$$
$$h_3(x, e) = -\frac{\partial V_1(x)}{\partial x_2}\,[\mu(x+e) - \mu(x)].$$

Clearly, $h_1(x)$, $h_2(e)$ and $h_3(x, e)$ are continuous functions for any $x \in R^2$ and $e \in R^2$, h_1 and h_2 are positive definite, and $h_3(x, 0) = 0$. It suffices to show $-\dot{V}^K(x, e)$ is positive definite when K is sufficiently large. In fact, it can be verified that $\dot{V}^K(x, e)$ is also a homogeneous function of degree $\sigma_0 + k$ with dilation (r_1, r_2, r_1, r_2). Next, define a positive-definite function $\Gamma(x, e) = (|x_1|^{m/r_1} + |x_2|^{m/r_2} + |e_1|^{m/r_1} + |e_2|^{m/r_2})^{1/m}$ with $m \geq \max\{r_1, r_2\}$. Then, $\Gamma(x, e)$ is a homogeneous function of degree one with dilation (r_1, r_2, r_1, r_2). Thus, for any $(x^T, e^T) \neq 0$, there is a positive number $\epsilon = \Gamma(x, e)$ and $z = (z_1, z_2, z_3, , z_4)^T \in S = \{(x^T, e^T)^T\colon \Gamma(x, e) = 1\}$ such that $(x_1, x_2, e_1, e_2) = (\epsilon^{r_1} z_1, \epsilon^{r_2} z_2, \epsilon^{r_1} z_3, \epsilon^{r_2} z_4)$. As a result, we only need to show that $-\dot{V}^K(x, e)$ is positive in the sphere S since, for any $(x^T, e^T) \neq 0$, there is $z \in S$, such that $\dot{V}^K(x, e) = \epsilon^{\sigma_0+k}\dot{V}^K(z)$.

Let $U_1 = \{(x^T, e^T)^T \in R^4\colon h_1(x) + h_3(x, e) > 0\}$, and $S_x = \{(x^T, 0^T)^T \in R^4, x \neq 0\}$. Then, U_1 is open since $h_1(x) + h_3(x, e)$ is continuous, and $S_x \subset U_1$. Let $S_1 = S \cap U_1$ and $S_2 = S \cap U_1^c$. Then, S_2 is compact, $S_1 \cup S_2 = S$, and $S_1 \cap S_2 = \emptyset$. Clearly $-\dot{V}^K(x, e) > 0, \forall (x, e) \in S_1$, and for all $K > 0$. Let

$$N = \min_{(x^T, e^T)^T \in S_2} h_2(e), \qquad N_0 = \min_{(x^T, e^T)^T \in S} h_3(x, e).$$

Since S_2 and S are compact, N and N_0 are finite. Clearly, $N_0 \leq 0$. Moreover, we claim $N > 0$. Otherwise, necessarily $e = 0$ since $h_2(e)$ is positive definite. That is to say that there exists a $x \neq 0$ such that $(x, 0) \in S_2 \subset U_1^c$. However, by the definition of S_x, it must also hold that $(x, 0) \in S_x \subset U_1$ which leads to a contradiction. Thus,

$$\begin{aligned} h_1(x) + Kh_2(e) + h_3(x, e) &\geq Kh_2(e) + h_3(x, e) \\ &\geq KN + N_0, \qquad (x, e) \in S_2. \end{aligned}$$

Letting $K_0 = -(N_0/N)$ shows that $-\dot{V}^K(x, e)$ is positive in S_2 for all $K > K_0$, thus completing the proof.

Combining [3, Th. 2] as mentioned above with Proposition 2 leads to the major result of this section.

Theorem 1: The zero solution of the closed-loop system (12) where $k_i > 0$, $i = 1, 2$, and conditions (14) and (15) hold, is globally finite-time stable.

Remark 2: It can be verified that the controller (4) given by Bhat and Bernstein satisfies condition (15) with $\alpha = \sigma_2/\sigma_1$.

Remark 3: It is possible to give a construction of the Lyapunov function for (11) for the special case where $k_1 = 1$, $k_2 = 1/2$. To this end, consider the following C^1 scalar function

$$V_1(e) = \frac{\gamma_1}{1+\sigma_1}\,|e_1|^{1+\sigma_1} - \gamma_2 e_1 e_2 + \frac{\sigma_1}{1+\sigma_1}\,|e_2|^{(1+\sigma_1)/\sigma_1} \tag{18}$$

where $\gamma_1 = k_2(1 + (1/\sigma_1)) > 1$ and $\gamma_2 = k_2 < 1$. It is clear that $V_1(e)$ is homogeneous C^1 function of degree $\sigma_0 = 1 + (1/\sigma_1)$ with dilation $(1/\sigma_1, 1)$. Thus, $\sigma_0 > \max\{1/\sigma_1, 1\}$. Moreover, it is possible to show that (18) is positive definite, radially unbounded, and the derivative of (18) along the trajectories of (11) with $k_1 = 1$, $k_2 = 1/2$ is negative definite. To this end, let $a = |2^{\sigma_1/(1+\sigma_1)}\gamma_2 e_1|$, $b = |2^{-\sigma_1/(1+\sigma_1)} e_2|$, $p = 1 + \sigma_1$, and $q = (1 + \sigma_1)/\sigma_1$. Using Young's inequality [2]

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1 \tag{19}$$

gives

$$|\gamma_2 e_1 e_2| = ab \leq \frac{2^{\sigma_1}\gamma_2^{1+\sigma_1}}{1+\sigma_1}\,|e_1|^{1+\sigma_1} + \frac{\sigma_1}{2(1+\sigma_1)}\,|e_2|^{(1+\sigma_1)/\sigma_1} \tag{20}$$

which implies

$$V_1(e) \geq \frac{\gamma_1 - 2^{\sigma_1}\gamma_2^{1+\sigma_1}}{1+\sigma_1}\,|e_1|^{1+\sigma_1} + \frac{\sigma_1}{2(1+\sigma_1)}\,|e_2|^{(1+\sigma_1)/\sigma_1}.$$

Therefore, (18) is positive definite and radially unbounded since $2\gamma_2^{1+\sigma_1} < 1 < \gamma_1$.

Next, consider the derivative of (18) along (11)

$$\begin{aligned} \dot{V}_1(e) = &-(\gamma_1 - k_2\gamma_2)|e_1|^{2\sigma_1} - \gamma_2 e_2^2 + (\gamma_1 + \gamma_2)|e_2||e_1|^{\sigma_1} \\ &\cdot \operatorname{sgn}(e_1 e_2) - k_2|e_1|^{\sigma_2}|e_2|^{1/\sigma_1}\operatorname{sgn}(e_1 e_2). \end{aligned} \tag{21}$$

We need to show $\dot{V}_1(e)$ is negative definite. Note that $\dot{V}_1(e)$ is homogeneous with its degree equal to two. Therefore, as suggested in the proof of [4, Prop. 1], we can limit our discussion on the closed set $S = \{z \in R^2\colon \max\{|z_1|, |z_2|\} = 1\}$. Consider the following four cases.

Case 1: When $e_1 = 1$, $|e_2| \leq 1$, from (21), we have

$$\begin{aligned} q_1(e_2) \overset{def}{=} \dot{V}|_{e_1=1, |e_2|\leq 1} &= -(\gamma_1 - \gamma_2^2) - \gamma_2 e_2^2 + (\gamma_1 + \gamma_2)e_2 \\ &\quad - \gamma_2|e_2|^{1/\sigma_1}\operatorname{sgn}(e_2). \end{aligned}$$

Since $\sigma_1 < 1$, we can differentiate q_1 with respect to e_2 to obtain

$$\frac{dq_1}{de_2} = -2\gamma_2 e_2 + (\gamma_1 + \gamma_2) - \frac{\gamma_2}{\sigma_1}\,|e_2|^{(1/\sigma_1)-1}.$$

Since $|e_2| < 1$, we have

$$\begin{aligned} \frac{dq_1}{de_2} &> -2\gamma_2 + (\gamma_1 + \gamma_2) - \frac{\gamma_2}{\sigma_1} \\ &= \gamma_2\left(1 + \frac{1}{\sigma_1}\right) - \gamma_2\left(1 + \frac{1}{\sigma_1}\right) = 0. \end{aligned}$$

Thus, q_1 will take its extreme value at either $e_2 = 1$ or $e_2 = -1$. Then,

$$q_1(e_2)|_{|e_2|=1} = -(\gamma_1 - \gamma_2^2) - \gamma_2 + (\gamma_1 + \gamma_2)\mathrm{sgn}(e_2) - \gamma_2 \mathrm{sgn}(e_2) < 0$$

yields that $q_1 < 0$ on S.

Case 2: When $|e_1| \le 1$ and $e_2 = 1$, from (21), we have

$$\begin{aligned} q_2(e_1) &\stackrel{def}{=} \dot{V}|_{|e_1|\le 1,\, e_2=1}(e) \\ &= -(\gamma_1 - \gamma_2^2)|e_1|^{2\sigma_1} - \gamma_2 + (\gamma_1 + \gamma_2)|e_1|^{\sigma_1}\mathrm{sgn}(e_1) \\ &\quad - \gamma_2 |e_1|^{\sigma_2}\mathrm{sgn}(e_1). \end{aligned}$$

Note that $4\gamma_2(\gamma_1 - \gamma_2^2) > \gamma_1^2$ since $1 < 1/\sigma_1 < 2$. Also we have $|e_1|^{\sigma_1} \le |e_1|^{\sigma_2}$ since $|e_1| \le 1$. Hence, if $e_1 \ge 0$, we have

$$q_2 \le -(\gamma_1 - \gamma_2^2)|e_1|^{2\sigma_1} + \gamma_1 |e_1|^{\sigma_1} - \gamma_2 < 0.$$

Similarly, if $e_1 < 0$, we have

$$q_2 \le -(\gamma_1 - \gamma_2^2)|e_1|^{2\sigma_1} - (\gamma_1 + \gamma_2)|e_1|^{\sigma_1} - \gamma_2 + \gamma_2 |e_1|^{\sigma_2} < 0.$$

So, $q_2 < 0$ on S.

Case 3 (when $e_1 = -1$, $|e_2| \le 1$) and Case 4 (when $|e_1| \le 1$ and $e_2 = -1$) directly follow from Cases 1 and 2, respectively, since $\dot{V}_1(e) = \dot{V}_1(-e)$.

Thus, we have shown that $\dot{V}_1 < 0$ on S, and, hence, $\dot{V}_1$ is negative definite.

IV. Second-Order Nonlinear Systems

In this section, we will further show that under our control law given in Section III, the zero solution of the closed-loop system composed of (9), (10), and the following nonlinear control system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_0(x_1, x_2) + u \\ y = x_1 \end{cases} \tag{22}$$

is locally finite-time stable where $f_0(x)$ is any C^1 function defined in a neighborhood of the origin of R^2 and satisfies $f_0(0) = 0$. To this end, we will first state the following result.

Lemma 3: Consider the following system:

$$\dot{x} = f(x) + \hat{f}(x), \qquad x \in R^n \tag{23}$$

where $f(x)$ is n-dimensional continuous homogeneous vector field of degree $k < 0$ with dilation $(r_1, \ldots, r_n)$ satisfying $f(0) = 0$, and $\hat{f}$ is also a continuous vector field satisfying $\hat{f}(0) = 0$. Assume the zero solution of $\dot{x} = f(x)$ is asymptotically stable. Then, the zero solution of (23) is locally *finite-time* stable if

$$\lim_{\epsilon \to 0^+} \hat{f}_i(\epsilon^{r_1}x_1, \ldots, \epsilon^{r_n}x_n)/\epsilon^{k+r_i} = 0, \qquad i = 1, \ldots, n \tag{24}$$

uniformly for any $x \in S^{n-1} = \{x \in R^n: \|x\| = 1\}$.

Remark 4: It has been established in [7] (also, see [13, Th. 3]) that under the conditions of Lemma 3 (without the constraint $k < 0$), the zero solution of (23) is locally asymptotically stable. Note that Lemma 3 is different from the result in [7] and [13] in that it further concludes that the zero solution of (23) is locally *finite-time* stable. Since (23) is not homogeneous, Lemma 3 cannot be directly deduced from the result of [7] or [13]. Nevertheless, the proof of Lemma 3 can be obtained in a way similar to that of Theorem 3 in [13], together with the help of Lemma 1.

Now, we are ready to state the major result of this section.

Theorem 2: Consider the control system (22) and the control law

$$\begin{cases} u = \mu(\zeta_1, \zeta_2) \\ \dot{\zeta}_1 = \zeta_2 - k_1|e_1|^{\sigma_1}\mathrm{sgn}(e_1) \\ \dot{\zeta}_2 = u - k_2|e_1|^{\sigma_2}\mathrm{sgn}(e_1) \end{cases} \tag{25}$$

where σ_1, σ_2, k_1, k_2, and $\mu(x)$ are defined as in Theorem 1. Then, the equilibrium at the origin of the closed-loop system composed of (22) and (25) is locally finite-time stable.

Proof: In the coordinates of (x, e), the closed-loop system is given by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \mu(x_1 + e_1, x_2 + e_2) + f_0(x) \\ \dot{e}_1 = e_2 - k_1|e_1|^{\sigma_1}\mathrm{sgn}(e_1) \\ \dot{e}_2 = -k_2|e_1|^{\sigma_2}\mathrm{sgn}(e_1) - f_0(x). \end{cases} \tag{26}$$

When $f_0 \equiv 0$, the system is exactly in the form of (12), which is globally finite-time stable, directly from Theorem 1.

Thus, by Lemma 3, the equilibrium at the origin of (26) is locally finite-time stable if the following perturbation vector

$$\hat{f}(x, e) = \begin{pmatrix} 0 \\ f_0(x) \\ 0 \\ -f_0(x) \end{pmatrix}$$

satisfies (24). However, by assumption, f_0 is C^1 and $f_0(0) = 0$. The mean-value inequality implies

$$f_0(\epsilon^{1/\sigma_1}x_1, \epsilon x_2) = O(\epsilon).$$

Finally, using $\sigma_2/\sigma_1 < 1$ shows that $f_0(\epsilon^{1/\sigma_1}x_1, \epsilon x_2) = o(\epsilon^{\sigma_2/\sigma_1}) = o(\epsilon^{k+r_i})$, $i = 2, 4$, and thus $\hat{f}(x, e)$ satisfies condition (24). The proof is completed.

Remark 5: Since f_0 does not have to be exactly known, Theorem 2 indicates that the output feedback homogeneous finite-time stabilizing control law has robustness property with respect to the nonlinear perturbation described by (24).

V. Conclusion

This note has addressed the problem of output feedback finite-time stabilization for a class of second-order systems. We note that, although only second-order systems are treated here, the tools used here may also be extended to handle some special class of higher order systems [8].

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H^∞ Bounds for Least-Squares Estimators

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***Abstract*—In this note, we obtain upper and lower bounds for the H^∞ norm of the Kalman filter and the recursive-least-squares (RLS) algorithm, with respect to prediction and filtered errors. These bounds can be used to study the robustness properties of such estimators. One main conclusion is that, unlike H^∞-optimal estimators which do not allow for any amplification of the disturbances, the least-squares estimators do allow for such amplification. This fact can be especially pronounced in the prediction error case, whereas in the filtered error case the energy amplification is at most four. Moreover, it is shown that the H^∞ norm for RLS is data dependent, whereas for least-mean-squares (LMS) algorithms and normalized LMS, the H^∞ norm is simply unity.**

***Index Terms*—Estimation, H^∞, least-squares, robustness.**

I. Introduction

Since its inception in the early 1960s, the Kalman filter (and the closely related recursive-least-squares (RLS) algorithm) has played a central role in estimation theory and adaptive filtering. Recently, on the other hand, there has been growing interest in (so-called) H^∞ estimation, with the belief that the resulting H^∞-optimal estimators will be more robust with respect to disturbance variation and lack of statistical knowledge of the exogenous signals. Therefore, a natural question to ask is what the robustness properties of the Kalman filter and RLS algorithm are within the H^∞ framework.

In an initial attempt to address this question, in this note we obtain upper and lower bounds on the H^∞ norm of the Kalman filter and RLS algorithm, with respect to the prediction and filtered errors of the uncorrupted output of a linear time-variant system.[1] These bounds are also of interest for several other reasons. First, they demonstrate that unlike the least-mean-squares (LMS) algorithm whose H^∞ norm is unity (independent of the input–output data) [1], the H^∞ norm of the RLS algorithm depends on the input–output data, and therefore it may be more robust or less robust with respect to different data sets. Moreover, the exact calculation of the H^∞ norm for RLS (and for the Kalman filter) requires the calculation of the induced two-norm of a linear time-variant operator, which can be quite cumbersome, and, in addition, needs all the input–output data, which may not be available in real-time scenarios. The H^∞ bounds we obtain only require simple a priori knowledge of the data, and may therefore be used as a simple check to verify whether RLS (or the Kalman filter) has the desired robustness with respect to a given application.

A brief outline of the paper is as follows. In Section II, we give general upper and lower bounds for the H^∞ norm of the Kalman filter. The proofs of the upper bounds are given in Section III and are based on certain minimization properties of least-squares estimators. The proofs of the lower bounds are given in Section IV and are essentially based on computing the energy gains for suitably chosen disturbances. Section V specializes the general results of Section II to the adaptive filtering problem and discusses its various implications. The paper concludes with Section VI.

II. A General H^∞ Bound

Consider the possibly time-variant state-space model

$$\begin{cases} x_{i+1} = F_i x_i + G_i u_i, & x_0 \\ y_i = H_i x_i + v_i, & i \geq 0 \end{cases} \tag{2.1}$$

where $F_i \in \mathcal{C}^{n\times n}$, $G_i \in \mathcal{C}^{n\times m}$ and $H_i \in \mathcal{C}^{p\times n}$ are known matrices, x_0, $\{u_i\}$, and $\{v_i\}$ are unknown quantities and $\{y_i\}$ is the measured output. Moreover, $\{v_i\}$ can be regarded as measurement noise and $\{u_i\}$ as process noise or driving disturbance. We shall be interested in estimating the uncorrupted output, $s_i = H_i x_i$.

It is well known that the Kalman filter for computing the predicted estimates of the states, denoted by $\hat{x}_i$, (i.e., $\hat{x}_i$ is the least-squares estimate of x_i, given $\{y_j,\ j < i\}$) is given by

$$\hat{x}_{i+1} = F_i \hat{x}_i + K_{p,i}(y_i - H_i \hat{x}_i) \tag{2.2}$$

where $K_{p,i} = F_i P_i H_i^* R_{e,i}^{-1}$ and $R_{e,i} = R_i + H_i P_i H_i^*$ and where P_i satisfies the Riccati recursion

$$P_{i+1} = F_i P_i F_i^* + G_i Q_i G_i^* - K_{p,i} R_{e,i} K_{p,i}^*, \qquad P_0 = \Pi_0. \tag{2.3}$$

[Note here that $\{Q_i,\ R_i\}$ and Π_0 are given positive definite weighting matrices.]

There is also a filtered form of the Kalman filter recursions for computing, $\hat{x}_{i|i}$, the least-squares estimate of x_i, given $\{y_j,\ j \leq i\}$, which is given below

$$\hat{x}_{i+1|i+1} = F_i \hat{x}_{i|i} + K_{f,i+1}(y_{i+1} - H_{i+1} F_i \hat{x}_{i|i}) \tag{2.4}$$

where $K_{f,i} = P_i H_i^* R_{e,i}^{-1}$.

Now using $\hat{x}_i$ and $\hat{x}_{i|i}$, the predicted and filtered estimation errors of the uncorrupted output, $s_i = H_i x_i$, are defined as

$$e_{p,i} = H_i x_i - H_i \hat{x}_i \triangleq H_i \tilde{x}_i$$

and

$$e_{f,i} = H_i x_i - H_i \hat{x}_{i|i} \triangleq H_i \tilde{x}_{i|i}. \tag{2.5}$$

[1]We should stress that these bounds are not for the problem of parameter estimation, for which causality is not an issue and for which the H^2 and H^∞ solutions coincide.

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