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FINITE-TIME INPUT-TO-STATE STABILITY AND APPLICATIONS TO FINITE-TIME CONTROL DESIGN*

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Abstract. This paper extends the well-known concept, Sontag's input-to-state stability (ISS), to finite-time control problems. In other words, a new concept, finite-time input-to-state stability (FTISS), is proposed and then is applied to both the analysis of finite-time stability and the design of finite-time stabilizing feedback laws of control systems. With finite-time stability, nonsmoothness has to be considered, and serious technical challenges arise in the design of finite-time controllers and the stability analysis of the closed-loop system. It is found that FTISS plays an important role as the conventional ISS in the context of asymptotic stability analysis and smooth feedback stabilization. Moreover, a robust adaptive controller is proposed to handle nonlinear systems with parametric and dynamic uncertainties by virtue of FTISS and related arguments.

Key words. finite-time control, input-to-state stability, nonsmooth feedback, Lyapunov functions, uncertainties

AMS subject classifications. 93C10, 34H15, 93D15

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1. Introduction.

Starting from the early 1980s, stability analysis and control synthesis of nonlinear systems have become more and more important following various practical demands. Many fundamental nonlinear control approaches, including feedback linearization, sliding mode, backstepping, forwarding, control Lyapunov functions, input-to-state stability (ISS), passivity-based control, and small-gain techniques, were proposed in the last few decades [5, 6, 14, 16, 17, 23, 24, 25, 27, 29, 30, 32, 35, 39, 41, 42, 46, 48, 49]. Most of these nonlinear analysis and synthesis tools focus on the design of smooth controllers for various classes of nonlinear systems.

Among the nonlinear control techniques, ISS [40] provides an effective way to tackle the stabilization of nonlinear systems or robust and adaptive control in the presence of various uncertainties arising from control engineering applications. In addition to many results on ISS stabilization design, an ISS-based modular design was proposed for adaptive control of a class of parametric strict-feedback systems in [25], where linear parametrization was considered. Moreover, dynamic uncertainties were also investigated with the help of ISS (see [13, 20, 21, 45]).

On the other hand, nonsmooth (including discontinuous and continuous but not Lipschitz continuous) control approaches have also drawn increasing attention in nonlinear control system design. There are some benefits in adopting the nonsmooth control strategy. For example, nonsmooth feedback laws can be used to stabilize sys-

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tems which, otherwise, are unstabilizable by smooth feedback. Moreover, under the action of a nonsmooth control law, a control system can possibly be forced to reach a desirable target in finite time. This approach was first studied in the literature on optimal control, and nonsmooth finite-time control design via time-invariant state or output feedback has become the focal point of several recent studies. Finite-time stability and finite-time stabilizing feedback design have been investigated for nonlinear systems or networked systems (see [2, 3, 9, 31, 28, 38, 47]). More recently, finite-time control for uncertain nonlinear systems has also been studied, and control designs have been proposed for specific classes of nonlinear systems with parametric or dynamic uncertainties using a backstepping-like procedure (see [11, 15, 12]). These finite-time controllers can also yield, in some sense, fast response and high tracking precision as well as disturbance-rejection properties because of their nonsmoothness [48, 7, 1]. Despite its potential application to practical problems, the study of finite-time stabilization is still quite underdeveloped, partially because of the lack of effective and constructive tools in nonsmooth analysis and synthesis.

The objective of this paper is to develop novel design tools for nonlinear control systems via the cross-fertilization of two fields. More specifically, our research is motivated by the fact that ISS has found powerful applications in various control problems and particularly has inspired the development of the nonlinear ISS stabilization and small-gain theorems proposed in [17, 44, 18, 19]. We intend to develop a framework for the nonsmooth finite-time control analysis and synthesis based on the finite-time variant of ISS, which we term as *finite-time ISS* (FTISS). Characterizations of FTISS are presented, and its combination with nonsmooth feedback is proposed to yield a new design tool for finite-time stabilization of nonlinear systems. The main difficulties in the analysis of FTISS and the FTISS-based design are its inherent nonsmoothness (referring to Remark 8). Furthermore, based on FTISS, a robust adaptive controller is proposed to deal with a class of uncertain nonlinear systems. In fact, the systematic application of FTISS alleviates the mathematical technicality and complexity associated with nonsmooth feedback approaches. It is our firm belief that FTISS will play a role in finite-time control, as important as what conventional ISS has played in asymptotic stability analysis and smooth feedback stabilization.

The organization of this paper is as follows. First, the concepts and preliminaries are introduced in section 2. Then in section 3, results on FTISS are presented, while issues on finite-time input-to-output stability (FTIOS) are addressed in section 4. Following that, finite-time feedback design via FTISS is reported, and adaptive finite-time control is given for a class of nonlinear systems with parametric and dynamic uncertainties in section 5. Finally, concluding remarks are given in section 6.

By convention, $|\cdot|$ denotes the absolute value; $||x||$ denotes the Euclidean norm of vector x; and $||x||_\infty = \text{ess.}\sup_{\tau\geq 0}||x(\tau)||$. Consider two functions $\chi(\cdot)$ and $\bar{\chi}(\cdot)$: $\bar{\chi}(x) = o(\chi(x))$ means $\lim_{||x||\to 0}\bar{\chi}(x)/\chi(x) = 0$; $\bar{\chi}(x) \sim \chi(x)$ means $c_1\chi(x) \leq \bar{\chi}(x) \leq c_2\chi(x)$ (when $||x||$ is sufficiently small) for some positive constants c_1 and c_2; and $\bar{\chi}(x) = O(\chi(x))$ means $|\bar{\chi}(x)| \leq c|\chi(x)|$ (when $||x||$ is sufficiently small) for some constant $c > 0$. Id denotes the identity function. $R_+ = [0, +\infty)$.

2. Concepts and preliminaries.

The investigation of finite-time stability of nonlinear systems is relatively challenging because of nonsmoothness in the system vector fields. Hence, some generalizations such as the Dini derivative have been given in this section to encompass the cases related to less smooth functions.

Let a and b ($> a$) be two real numbers, and consider a function $\phi : (a, b) \to R$. The upper Dini derivative of ϕ at $t \in (a, b)$ is defined as

$$D^+\phi(t) = \limsup_{s\to 0^+} \frac{\phi(t+s) - \phi(t)}{s}.$$

It is known that when ϕ is continuous, ϕ is nonincreasing on (a, b) if and only if $D^+\phi(t) \le 0$ for any $t \in (a, b)$ (see [36, Theorem 2.1, Appendix I] or [22] for details).

Consider the system

$$\dot{x} = f(t, x), \quad f(t, 0) = 0, \quad x \in R^n,\ t \in R, \tag{1}$$

where f is continuous with respect to (t, x). It is assumed that $x = 0$ is a unique solution (in forward time). The continuity guarantees the existence of the solution, which may not be unique. Denote the $\mathcal{S}(t, t_0, x_0)$ (or $\mathcal{S}$ for simplicity) as the set of all the solutions with initial condition $x(t_0) = x_0$ in forward time (i.e., $t \ge t_0$), and when there is no confusion, any solution in $\mathcal{S}$ is denoted simply by $x(t)$.

The following is a basic concept about finite-time stability (referring to [2, 3]).

Definition 1. *The equilibrium $x = 0$ of the system* (1) *is said to be (locally) finite-time stable if it is Lyapunov stable and (locally) finite-time convergent in a neighborhood U. By "finite-time convergence," we mean that for any initial conditions $x(t_0) = x_0 \in U$ with given t_0, there is a settling-time function*

$$T(t_0, x_0) = \inf \left\{ \hat{T} \ge t_0 \ :\ \lim_{t\to\hat{T}} x(t) = 0; \quad x(t) \equiv 0\ \forall t \ge \hat{T} \right\}$$

which is continuous with respect to (t_0, x_0) and $T(t_0, 0) = 0$ for every solution $x(t) \in \mathcal{S}$ of system (1). *When $U = R^n$, the origin is a globally finite-time stable equilibrium.*

Moreover, we consider a disturbed system, regarded as a special case of (1),

$$\dot{x}(t) = g(x(t), d(t)), \quad g(0, d) = 0, \quad x \in R^n,\ d \in R^m, \tag{2}$$

where $g(x, d)$ is continuous in (x, d), $\mathcal{S}$ denotes the solution set of system (2) with initial condition $x(t_0) = x_0$, and "disturbance" input $d(t)$ satisfies

$$d(t) \in \mathcal{M} := \{\text{all measurable functions from } R \text{ to } [-1,\ 1]^m\}. \tag{3}$$

Definition 2. *System* (2) *is uniformly globally finite-time stable (UGFTS) if it is uniformly stable (that is, for each $\varepsilon > 0$ and for some K_∞-function $\delta(\varepsilon)$, $||x(t)|| \le \varepsilon$ ($\forall x(t) \in \mathcal{S}$) holds for any given $d \in \mathcal{M}$, $||x_0|| \le \delta$, and $t \ge t_0$) and uniformly finite-time convergent (that is, system* (2) *is finite-time convergent for any given $d(t)$, and, for any initial condition x_0, there is a finite positive number $\bar{T}$, depending on x_0, such that $||x(t)|| = 0\ \forall d \in \mathcal{M}$ when $t \ge \bar{T}$). Moreover, system* (2) *is strongly uniformly globally finite-time stable (SUGFTS) if it is UGFTS and the settling time T_d is continuous with respect to x_0 (for any initial condition x_0) uniformly in $d \in \mathcal{M}$.*

Remark 1. If system (1) is time invariant, we can always take $t_0 = 0$. A time-invariant finite-time convergent system must be non-Lipschitz due to the nonuniqueness in the backward time at the origin although its settling time T is a continuous function of the initial state x_0 (see [2] for more details). Note that although system (2) takes a time-varying form, we can still always take $t_0 = 0$ as we did for time-invariant systems. In fact, we consider two cases: $x(t_0^1) = x_0$ and $x(t_0^2) = x_0$. Clearly, system (2) with $x(t_0^1)=x_0$ and any given $d^1(t)\in\mathcal{M}$ is equivalent to system (2) with $x(t_0^2)=x_0$ and $d^2(t)=d^1(t+t_0^1-t_0^2)\in\mathcal{M}$.

In this paper, we mainly consider time-invariant systems with input variables. Based on the discussion given in Remark 1, we will take the initial moment $t_0=0$.

Then we introduce new concepts to combine finite-time control with ISS ideas. Consider a system

$$\dot z = f(z,v),\quad f(0,0)=0,\quad z\in R^n,\ v\in R^m, \tag{4}$$

where f is continuous with respect to (z,v) and the input, $v:R_+\to R^m$, is measurable and locally *essentially bounded* (that is, bounded except a set of measure 0 [37]).

A function $\gamma:R_+\to R_+$ is said to be a K-function if it is strictly increasing and continuous with $\gamma(0)=0$, and it is a K_∞-function if it is a K-function and $\gamma(s)\to\infty$ as $s\to\infty$. To study finite-time issues, we further consider generalized functions. A function $\varphi:R_+\to R_+$ is said to be a generalized K-function if it is continuous with $\varphi(0)=0$ and satisfies

$$\begin{cases}\varphi(s_1)>\varphi(s_2) & \text{if } \varphi(s_1)>0,\ s_1>s_2,\\ \varphi(s_1)=\varphi(s_2) & \text{if } \varphi(s_1)=0,\ s_1>s_2.\end{cases}$$

Note that a (conventional) K-function is a generalized K-function. A function β : $R_+\times R_+\to R_+$ is a generalized KL-function (GKL-function) if, for each fixed $t\geq 0$, the function $s\mapsto\beta(s,t)$ is a generalized K-function, and for each fixed $s\geq 0$, the function $t\mapsto\beta(s,t)$ is continuous and decreases to zero as $t\to T$ for some $T\leq\infty$.

Definition 3. *System* (4) *is said to be locally FTISS (with v as the input) if there exist some neighborhoods U of $0\in R^n$ and U_v of $0\in R^m$ such that, for initial state $z(0)=z_0\in U$ and measurable and locally essentially bounded input $v\in U_v$, each solution $z(t)\in\mathcal{S}$ is defined for $t\geq 0$ and satisfies*

$$||z(t)||\leq\beta(||z_0||,t)+\gamma(||v||_\infty), \tag{5}$$

where γ is a K-function and β is a GKL-function with $\beta(r,t)\equiv 0$ when $t\geq\hat T(r)$ with $\hat T(r)$ continuous with respect to r and $\hat T(0)=0$. When $U=R^n$ and $U_v=R^m$, the system is said to be (globally) FTISS.

Obviously, FTISS implies ISS. Note that the main difference between ISS and FTISS is that β is a GKL-function. In what follows, we mainly consider global FTISS, and we write "finite-time input-to-state stability" instead of "global finite-time input-to-state stability" if there is no confusion. In fact, it is rather easy to extend our results to cover the local case.

Remark 2. It should be mentioned that there is an equivalent definition of FTISS. In fact, it is mathematically equivalent to replace (5) by

$$||z(t)||\leq\max\{\bar\beta(||z(0)||,t),\ \bar\gamma(||v||_\infty)\}, \tag{6}$$

where $\bar\beta$ is a GKL-function and γ is a K-function since (5) implies (6) by taking $\bar\beta=2\beta$ and $\bar\gamma=2\beta$.

Remark 3. When $v\equiv 0$, system (4) with $z(0)=z_0$ is finite-time stable with continuous settling-time function denoted by $T(z_0)$ (see [2]). Clearly, the existence of a function $T(z_0)$ continuous in z_0 is equivalent to the existence of a function $\hat T(r)$ defined in Definition 3. In fact, with taking $r=||z_0||$, the existence of $\hat T(r)$ leads to that of $T(z_0)$, and conversely, the conclusion also holds by noting that $T(z_0)$ is continuous and $\{z_0\in R^n : ||z_0||\leq r\}$ is a compact set.

Remark 4. If a system is FTISS with v as the input, it is FTISS with input $w = \chi(v)$ for a continuous function χ with $\chi(0) = 0$. Moreover, given a homeomorphism χ, a system is FTISS with v as the input if and only if it is so with w as the input.

Example 1. Consider the system $\dot{z} = -z^{1/3} - z^3 + v^2$. Take

$$\beta(s,t) = \begin{cases} \left(s^{\frac{2}{3}} - \frac{1}{3}t\right)^{\frac{3}{2}} & \text{if} \quad 0 \le t \le T = 3s^{\frac{2}{3}}, \\ 0, & \text{otherwise}, \end{cases} \quad \gamma(s) = 2s^2,$$

where T is continuous in s. Therefore, the system is FTISS with input $v(t)$.

Example 2. FTISS implies the finite-time stability when $v = 0$, but the converse may not be true, even in the bounded-input-bounded-state (BIBS) case. Consider

$$\dot{z} = -(1 + \sin v) z^{\frac{1}{3}},$$

which is BIBS and finite-time stable when $v = 0$. However, taking $v = 3\pi/2$ makes the system not finite-time stable; therefore, this system is not FTISS.

Remark 5. As a natural extension of FTISS, system (4) is said to be finite-time input-to-state practically stable (FTISpS) with v as the input if, for any initial state $z(0) = z_0$ and bounded input $v(t)$,

$$||z(t)|| \le \beta(||z_0||, t) + \gamma(||v||_\infty) + c_0, \quad c_0 \ge 0, \tag{7}$$

where γ is a K-function and β is a GKL-function with $\beta(r, \tau) \equiv 0$ when $\tau \ge T$ with T continuous with respect to r. When constant $c_0 = 0$, FTISpS becomes FTISS. Clearly, FTISpS implies input-to-state practical stability (ISpS) [17]. Conversely, if system (4) is ISpS, then, according to [17], $||z(t)|| \le \hat{\beta}(||z_0||, t) + \gamma(||v||_\infty) + \hat{c}_0$ with a constant $\hat{c}_0 \ge 0$, a K-function γ, and a KL-function $\hat{\beta}$. Since $\hat{\beta}$ is continuous, for any $\epsilon > 0$, there is $T \ge 0$, depending on z_0 and ϵ, such that $\hat{\beta}(||z_0||, t) \le \epsilon$ when $t \ge T$. Therefore, there is a GKL-function β with $\hat{\beta}(||z_0||, t) \le \beta(||z_0||, t) + \epsilon$ and $\beta(||z_0||, t) = 0$ when $t \ge T$ such that (7) holds with $c_0 = \hat{c}_0 + \epsilon$. Thus, FTISpS is equivalent to ISpS. By using the same reasoning, an ISS system is always FTISpS.

For subsequent use, we introduce some useful inequalities.

LEMMA 1 (see [17]). *For any K-function γ, any K_∞-function ρ such that $\rho - Id$ is a K_∞ function, and any nonnegative real numbers a and b, we have*

$$\gamma(a + b) \le \gamma(\rho(a)) + \gamma(\rho \circ (\rho - Id)^{-1}(b)), \tag{8}$$

where $\circ$ denotes the composition of the functions.

LEMMA 2. *For any continuous function $g(x, z)$, there are continuous nonnegative functions $g_1(x)$ and $g_2(z)$ such that*

$$|g(x, z)| \le g_1(x) + g_2(z). \tag{9}$$

Moreover, both functions $g_1(x)$ and $g_2(z)$ vanish at zero when $g(0, 0) = 0$.

Note that inequality (9) has been widely used in the research work of others (see, for instance, [34, Equations (22) and (14)]).

LEMMA 3 (Young's inequality [8]). *Let $\phi(x)$ be a strictly increasing continuous function. Then*

$$ab \le \int_0^a \phi(x)\, dx + \int_0^b \phi^{-1}(x) dx, \tag{10}$$

where ϕ^{-1} is the inverse function of ϕ. Particularly,

$$ab \le \frac{a^{1+c}}{1+c} + \frac{c\, b^{1+\frac{1}{c}}}{1+c}, \quad a \ge 0,\ b \ge 0,\ c > 0. \tag{11}$$

LEMMA 4 (Jensen's inequality [8]). *With $b_i \ge 0$ $(1 \le i \le n)$,*

$$\left(\sum_{i=1}^{n} b_i^{c_2}\right)^{\frac{1}{c_2}} \le \left(\sum_{i=1}^{n} b_i^{c_1}\right)^{\frac{1}{c_1}}, \quad 0 < c_1 < c_2. \tag{12}$$

3. FTISS and its properties.

In this section, we will discuss the stability properties related to FTISS.

Let $V(z) : R \times R^n \to R$ be a continuous function. Thus,

$$D^+V(z(t))|_{(4)} = \limsup_{s \to 0^+} \frac{V(z(t+s)) - V(z(t))}{s},$$

where $z(t) \in \mathcal{S}$ is a solution of system (4) with any given $v(t)$. For simplicity, denote

$$\dot{V}(z(t))|_{(4)} := D^+V(z(t))|_{(4)}. \tag{13}$$

A continuous function $V(z)$ is called an ISS-Lyapunov function for system (4) if there exist K_∞-functions ϕ_1 and ϕ_2 and K-functions ϕ_3 and ϕ_4 such that

(A1)

$$\phi_1(||z||) \le V(z) \le \phi_2(||z||) \quad \forall z \in R^n. \tag{14}$$

(A2) For any solution $z(t)$ and input $v(t)$ of system (4), it holds that

$$||z(t)|| \ge \phi_4(||v(t)||) \quad \Longrightarrow \quad \dot{V}(z(t))|_{(4)} \le -\phi_3(||z(t)||) \quad \forall t \ge 0. \tag{15}$$

Remark 6. If $V(z)$ is C^1 (i.e., continuously differentiable), as shown in [43], (A2) holds if and only if there exist K_∞-functions ϕ_5 and ϕ_6 such that, for any input $v(t)$ and solution $z(t)$ with $t \ge 0$,

$$\dot{V}(z(t))|_{(4)} \le -\phi_5(||z(t)||) + \phi_6(||v(t)||). \tag{16}$$

DEFINITION 4. *A continuous function $V(z)$ is called an FTISS-Lyapunov function for system* (4) *if it is an ISS-Lyapunov function with conditions* (A1) *and* (A2) *and $\phi_3(||z||) \sim V(z)^a$ for some positive constant $a < 1$.*

To study FTISS and its related properties, we introduce the following concepts.

DEFINITION 5. *System* (4) *is said to be weakly robustly finite-time stable if there is a smooth function φ satisfying $\bar{\rho}(||z||) \le \varphi(z) \le \hat{\rho}(||z||)$ for some K_∞-functions $\bar{\rho}$ and $\hat{\rho}$ so that the system*

$$\dot{z}(t) = f(z(t), d(t)\varphi(z(t))) = g(z(t), d(t)),\ g(0, d(t)) = 0, \tag{17}$$

where g is continuous with respect to (z, d) and is UGFTS for any $d \in \mathcal{M}$ given in (3). *System* (4) *is said to be strongly finite-time stable if it is weakly robustly finite-time stable and* (17) *is SUGFTS.*

Although weakly robust finite-time stability of (4) implies UGFTS of (17), it cannot guarantee that the settling time $T_d(x)$ is continuous with respect to x *uniformly* in $d(t) \in \mathcal{M}$, noting that $\mathcal{M}$ is not a compact set (referring to Definition 2).

DEFINITION 6. *System* (4) *is said to be robustly finite-time stable if there are a* K_∞*-function* ρ *and a* GKL*-function* β *such that, for every solution* $z(t)$ *of system* (4) *with any input function* $v(t)$ *satisfying* $||v(t)|| \le \rho(||z(t)||)$*, it holds that*

$$||z(t)|| \le \beta(||z(0)||, t) \quad \forall t \ge 0 \tag{18}$$

with $\beta(r,t) \equiv 0$ *when* $t \ge T(r) = \inf\{\tilde{T} \ : \ \beta(r,\tilde{T}) = 0 \ \forall t \ge \hat{T}, ||v(t)|| \le \rho(||z(t)||)\}$.

Then we have the first main result of this section.

THEOREM 1. *The following statements hold:*

(a) *System* (4) *is FTISS with* v *as the input if it has an FTISS-Lyapunov function;*

(b) *The FTISS of system* (4) *implies the weakly robust finite-time stability.*

Proof. (a): Assume system (4) admits a FTISS-Lyapunov function $V(z)$ with (14) and (15) and $z(t) \in \mathcal{S}$ is a solution with initial condition z_0 and input function $v(t)$. Then $\mathcal{V} = \{z : V(z) \le \phi_2(\phi_4(||v||_\infty))\}$ is an invariant set (which was shown in Lemma 2.14 in [43]). Let $t_* := \inf\{t \ge 0 : z(t) \in \mathcal{V}\} \le \infty$, which implies

$$||z(t)|| \le \gamma(||v||_\infty), \quad \gamma = \phi_1^{-1} \circ \phi_2 \circ \phi_4 \quad \forall t > t_*. \tag{19}$$

When $t \le t_*$, $z(t) \notin \mathcal{V}$, and therefore, $||z(t)|| \ge \phi_4(||v(t)||)$. According to Definition 4, $||z(t)|| \ge \phi_4(||v(t)||)$ implies $\dot{V}(z(t))|_{(4)} \le -\phi_3(||z(t)||)$ for any solution $z(t)$ with input $v(t)$. Because $\phi_3(||z||) \sim V(z)^a$ for a positive constant $a < 1$, it is not hard to obtain

$$||z(t)|| \le \beta(||z_0||, t) \quad \forall t \le t_* \tag{20}$$

for a GKL-function β with $\beta(r,t) \equiv 0 \ \forall t \ge T$ (with T continuous with respect to r). Combining (19) and (20) gives

$$||z(t)|| \le \beta(||z_0||, t) + \gamma(||v||_\infty) \quad \forall t \ge 0,$$

and then the conclusion follows.

(b): From Remark 2, the FTISS of system (4) holds if and only if there exist a GKL-function $\bar{\beta}$ and a K-function γ such that $||z(t)|| \le \max\{\bar{\beta}(||z_0||, t), \gamma(||v||_\infty)\}$ with $\bar{\beta}(r,t) \equiv 0$ when $t \ge \hat{T}$ with $\hat{T}(r)$ continuous in r.

It is not hard to find a GKL-function $\beta(r,t) \ge \bar{\beta}(r,t)$ with $\beta(||z_0||, t) \equiv 0$ (when $t \ge \hat{T}$) and $\beta(r/2, t) \equiv 0$ when $t \ge T/2$ if $\beta(r,t) \equiv 0$ when $t \ge T$ for any $r \ge 0, T \ge 0$ (so the function $t \mapsto \beta(r,t)$ may be concave). Obviously,

$$||z(t)|| \le \max\{\beta(||z_0||, t), \gamma(||v||_\infty)\}. \tag{21}$$

According to the analysis for ISS given in Lemma 2.12 of [43], γ in (21) can be assumed to be a K_∞-function. Moreover, we have a continuous (or smooth everywhere except possibly at the origin) function φ and a K_∞-function $\bar{\varphi}$ such that

$$\bar{\varphi}(||z||) \le \varphi(z) \le \gamma^{-1}\left(\frac{\beta_0^{-1}(||z||)}{4}\right) \quad \forall z \in R^n$$

with $\beta_0(r) = \beta(r,0)$ when $\beta(r,0) \ne 0$.

To obtain the conclusion, we need to prove the uniform finite-time stability of the following system in the form of (17)

$$\dot{z}(t) = f(z(t), d(t)\varphi(z(t))) = g(z(t), d(t)) \quad \forall d \in \mathcal{M}. \tag{22}$$

Consider the uniform finite-time convergence at first. Let $\zeta(t, z_0, d) = z(t)|_{v(t)=d(t)\varphi(z)} \in \mathcal{S}$ denote any solution of (22) with $z(0) = z_0$ and "disturbance" $d \in \mathcal{M}$.

Still invoking the proof of Lemma 2.12 of [43], we similarly obtain that

$$\gamma\big(||d\varphi(\zeta(t, z_0, d))||\big) \le \gamma\big(\varphi(\zeta(t, z_0, d))\big) \le \frac{||z_0||}{2}, \quad t \ge 0 \tag{23}$$

and, moreover, that for any $r > 0$, there is a $T_r \ge 0$ so that $\beta(r, t) \le \frac{r}{2}$ $\forall t \ge T_r$, which implies

$$||\zeta(t, z_0, d)|| \le \frac{r}{2}, \quad ||z_0|| \le r \ \forall t \ge T_r \text{ and } \forall d \in \mathcal{M} \tag{24}$$

due to (21).

Then there is a sequence of time instants $0 = T_0 \le T_1 \dots T_i \le T_{i+1} \dots,\ i = 1, 2, \dots$ such that

$$||\zeta(t, z_0, d)|| = ||\zeta\big(t - T_{i-1}, z(T_{i-1}), d\big)|| \le \frac{r}{2^i}, \quad ||z(T_{i-1})|| \le \frac{r}{2^{i-1}},\ t \ge T_i = \sum_{j=0}^{i-1} T_{r/2^j},$$

where $z(T_{i-1})$ is a shorthand for $\zeta(T_{i-1}, z_0, d)$ with satisfying $\lim_{i\to\infty} T_i \le \bar{T} := 2\hat{T}$ because

$$||\zeta\big(\tau, z(T_{i-1}), d\big)|| \le \max\left\{\beta(||z(T_{i-1})||, \tau), \frac{||z(T_{i-1})||}{2}\right\}$$

by (21) and (23), and $\beta(||z(T_{i-1})||, \tau)$ with β defined in (21) vanishes when $\tau \ge \hat{T}/2^{i-1}$ (and then we can take $T_{r/2^i} \le \hat{T}/2^i,\ i = 1, 2, \dots,$ which implies $\lim_{t\to\infty} T_i \le 2\hat{T}$). This shows that the origin of system (22) is uniformly finite-time convergent.

In addition, (21) and (24) imply $||\zeta(t, z_0, d)|| \le \beta(||z_0||, 0)$ for all $t \ge 0$, $z_0 \in R^n$, and $d \in \mathcal{M}$, which yields the uniform stability. Thus, system (22) is weakly robustly finite-time stable according to Definition 5. ∎

Remark 7. It is an interesting topic for future research to know whether the converse statement of (a) in Theorem 1 holds, that is, FTISS implies the existence of a (continuous) FTISS-Lyapunov function. However, an FTISS-Lyapunov function does exist under the condition of strong finite-time stability (see Theorem 2).

When system (4) is strongly finite-time stable, system (17) is SUGFTS. Then we denote $T_d(\bar{z})$ as the (continuous) settling-time function for system (17) with initial state $\bar{z}$ and a given $d \in \mathcal{M}$ (recalling Definition 1 and Remark 1). Furthermore, from Definition 2, there is a finite number $\bar{T}(\bar{z})$ such that $T_d(\bar{z}) < \bar{T}$ $\forall d \in \mathcal{M}$. Therefore, we can define

$$T_*(\bar{z}) = \sup_{d\in\mathcal{M}} T_d(\bar{z}) \le \bar{T} \tag{25}$$

for system (17) with initial state $\bar{z} \in R^n$. Obviously, $T_*(0) = 0$, and there are K_∞-functions $\tilde{\phi}_1$ (which can be taken with $\tilde{\phi}_1 \le \inf_{||z||=r} T_*(z)$) and $\tilde{\phi}_2$ (with $\tilde{\phi}_2 \ge \sup_{||z||=r} T_*(z)$) such that $\tilde{\phi}_1(||z||) \le T_*(z) \le \tilde{\phi}_2(||z||)$ for any $z \in R^n$.

Furthermore, we have the following lemma.

LEMMA 5. *If system* (17) *is SUGFTS,* $T_*(\bar{z})$ *is continuous with respect to* $\bar{z}$.

Proof. To prove that $T_*(\bar{z})$ is continuous at $\bar{z}$, we need to prove that, for any $\bar{\epsilon} > 0$, there is $\delta > 0$ such that $|T_*(z) - T_*(\bar{z})| \le \bar{\epsilon}$ when $||z - \bar{z}|| \le \delta$.

Here are two cases: (i) $T_*(\bar{z}) - T_*(z) \geq 0$: Obviously,

$$T_*(\bar{z}) - T_*(z) = T_*(\bar{z}) - T_{\bar{d}}(\bar{z}) + T_{\bar{d}}(\bar{z}) - T_{\bar{d}}(z) + T_{\bar{d}}(z) - T_*(z) \tag{26}$$

for any $\bar{d} \in \mathcal{M}$, which implies

$$0 \leq T_*(\bar{z}) - T_*(z) \leq T_*(\bar{z}) - T_{\bar{d}}(\bar{z}) + T_{\bar{d}}(\bar{z}) - T_{\bar{d}}(z)$$

because $T_{\bar{d}}(z) - T_*(z) \leq 0$ for any $\bar{d} \in \mathcal{M}$.

According to (25), there is $\bar{d} \in \mathcal{M}$ such that $T_*(\bar{z}) - T_{\bar{d}}(\bar{z}) < \bar{\epsilon}/2$. Because $T_{\bar{d}}$ is continuous by Definition 1, then there is $\delta_0 > 0$ such that $||z - \bar{z}|| \leq \delta_0$ and $|T_{\bar{d}}(z) - T_{\bar{d}}(\bar{z})| \leq \bar{\epsilon}/2$. It follows that

$$0 \leq T_*(\bar{z}) - T_*(z) \leq \bar{\epsilon}. \tag{27}$$

(ii) $T_*(z) - T_*(\bar{z}) > 0$: Similarly, we have

$$T_*(z) - T_*(\bar{z}) = T_*(z) - T_d(z) + T_d(z) - T_d(\bar{z}) + T_d(\bar{z}) - T_*(\bar{z}) \tag{28}$$

which holds for any $d \in \mathcal{M}$. Moreover, for each z, there is some $d_z \in M$ such that $T_*(z) - T_{d_z}(z) < \bar{\epsilon}/2$. Based on (25) and (28), we have

$$0 < T_*(z) - T_*(\bar{z}) \leq \frac{\bar{\epsilon}}{2} + T_{d_z}(z) - T_{d_z}(\bar{z}). \tag{29}$$

Moreover, since $T_d(z)$ is continuous (with respect to z) uniformly in $d \in \mathcal{M}$, there is $\delta' > 0$ such that $||z - \bar{z}|| \leq \delta'$ and $|T_{d_z}(z) - T_{d_z}(\bar{z})| \leq \bar{\epsilon}/2$, which leads to

$$0 < T_*(z) - T_*(\bar{z}) \leq \bar{\epsilon}. \tag{30}$$

Taking $\delta = \min\{\delta_0, \delta'\}$ yields $|T_*(z) - T_*(\bar{z})| \leq \bar{\epsilon}$ by combining (27) and (30). □

Now, we are ready to prove the second main result of this section.

THEOREM 2. *The following statements hold:*

(i) *The strong finite-time stability of system* (4) *implies that it admits an FTISS-Lyapunov function.*

(ii) *The existence of an FTISS-Lyapunov function implies the robust finite-time stability.*

(iii) *The robust finite-time stability of system* (4) *implies the weakly robust finite-time stability.*

Proof. (i): According to Definition 5, we need to consider only the SUGFTS of system (17). Denote any solution $z(t) \in \mathcal{S}$ for any given $d \in \mathcal{M}$. Note that $T_d(z(\tau)) = t - \tau + T_d(z(t))$ (referring to [2]), and then

$$T_*(z(\tau)) = \sup_{\forall d(s),\ s \in [\tau, \infty)} T_d(z(\tau)) \geq t - \tau + \sup_{\forall d(s),\ s \in [\tau + t, \infty)} T_d(z(t)) = t - \tau + T_*(z(t)).$$

Therefore,

$$\dot{T}_*|_{(17)} := D^+ T_*|_{(17)} = \lim_{t \to \tau^+} \frac{T_*(z(t)) - T_*(z(\tau))}{t - \tau} \leq \lim_{t \to \tau^+} \frac{\tau - t}{t - \tau} = -1. \tag{31}$$

By Lemma 5, take a Lyapunov function candidate $V(z(t)) = (T_*(z(t)))^{1/(1-a)}$ with $0 < a < 1$, which is continuous and positive definite with $V(0) = 0$. Then

$$\dot{V}(z(t))|_{(17)} = \frac{T_*^{\frac{a}{1-a}} \dot{T}_*}{1-a}|_{(17)} \leq -\frac{1}{1-a} T_*^{\frac{a}{1-a}} = -cV(z(t))^a, \quad c = \frac{1}{1-a}. \tag{32}$$

From Definition 4, the conclusion of (i) follows readily by setting $\phi_3 = c\phi_1^a$ and $\phi_4 = \hat{\rho}^{-1}$.

(ii): Assume $V(z)$ is an FTISS-Lyapunov function as given in Definition 4. Then there is a K_∞-function ρ (larger than or equal to ϕ_4^{-1}) such that $\dot{V}(z(t)) \le -\phi_3(z(t))$ when $\phi_3(z) \sim V(z)^a$ (for $0 < a < 1$) and $||v(t)|| \le \rho(||z(t)||)$ (for any solution $z(t)$ of system (4)). Clearly, there is a settling time $T(z_0) \ge 0$ such that $V(z(t)) \equiv 0$ (or equivalently $z(t) \equiv 0$) when $t \ge T$. It is not hard to see that there is a GKL-function $\beta(r,t)$ such that $V(z(t)) \le \beta(||z_0||, t)$ for any $z(t) \in \mathcal{S}$ and that $\beta(r,t) \equiv 0$ when $t \ge \hat{T}(r) = \sup\{T(z_0), \, ||z_0|| = r\}$ (referring to Remark 3), which implies the robust finite-time stability.

(iii): For any K_∞-function ρ, there exist a smooth function $\varphi : R^n \to R_+$ and a K_∞-function $\bar{\rho}$ such that $\bar{\rho}(||z||) \le \varphi(z) \le \rho(||z||)$. Note that $d(t)\varphi(z)$ with $d \in \mathcal{M}$ is just a particular type of the feedback bounded by $\hat{\rho} = \rho$. Then it is easy to see system (17) is UGFTS, and the conclusion follows by Definition 5. ∎

It was shown in [2] that finite-time stability of $\dot{z} = f(z)$ implies there is a continuous Lyapunov function $V(z)$ such that $\dot{V}(z(t)) \le -cV(z(t))^a, \; c > 0, \; 0 < a < 1$. Part (i) of Theorem 2 can be viewed as an extension by considering a family of nonsmooth systems with disturbance $d(t) \in \mathcal{M}$. The analysis becomes much harder partially because $\mathcal{M}$ is not compact. If there is no $d(t)$, [2] also showed that $T(z)$ is continuous if and only if $T(0)$ is continuous. However, such a statement does not hold in the presence of $d(t)$.

Example 3. Revisiting Example 1, it is not hard to verify that $V(z) = z^2$ is a smooth finite-time ISS-Lyapunov function for system $\dot{z} = -z^{-1/3} - z^3 + v^2$ because $\dot{V}(z(t)) \le -V(z(t))^{\frac{2}{3}}/2$ if $|z(t)| \ge |v(t)|^2$.

4. Finite-time input-to-output stability and small-gain theorem. In this section, we consider finite-time input-to-output stability (FTIOS) and the related small-gain results. In fact, the "finite-time" discussion can be extended to systems with outputs. Consider the system

$$\begin{cases} \dot{x} = f(x,u), & x \in R^n, \; x(0) = x_0, \\ y = h(x,u), & y \in R^l, \;\; u \in R^m, \end{cases} \tag{33}$$

where n, l, and m are positive integers, y denotes the output variables, and f and h are continuous with $h(0,0) = 0$ and $f(0,0) = 0$.

Definition 7. *System* (33) *is said to be FTIOS if there exist a GKL-function β and a K-function γ such that, for any initial condition $x(0) = x_0$, each measurable and locally essentially bounded input $u(t)$ on $[0,\infty)$, and each t in the right maximal interval of the definition of the corresponding solution of system* (33), *we have*

$$||y(t)|| \le \beta(||x_0||, t) + \gamma(||u||_\infty), \tag{34}$$

where $\beta(r,t) \equiv 0$ when $t \ge T(r)$ with $T(0) = 0$ and T continuous with respect to r.

Definition 8. *System* (33) *is said to be finite-time strongly detectable (FTSD) if there exist a GKL-function β^0 and a K-function γ^0 such that, for each measurable and locally essentially bounded input $u(t)$ defined on $[0,T_u)$ with $T_u \le \infty$, the solution $x(t)$ of system* (33) *right maximally defined on $[0,T_x)$ $(T_x \le T_u)$ satisfies*

$$||x(t)|| \le \beta^0(||x_0||, t) + \gamma^0(||(u^T, y^T)^T||_\infty) \quad \forall t \in [0, T_x), \tag{35}$$

where $\beta^0(r,t) \equiv 0$ when $t \ge T(r)$ with $T(0) = 0$ and T continuous with respect to r.

Next we show a relationship between FTISS and FTIOS for system (33).

THEOREM 3. *System* (33) *is FTISS with u as the input if and only if it is FTIOS and FTSD.*

Proof. The "only if" part is as follows: Due to FTISS of system (33),

$$||x(t)|| \leq \beta(||x_0||, t) + \gamma(||u||_\infty)$$

with $\beta(r,t) = 0,\ t \geq T$ for some $T(r) \geq 0$. Then FTSD can be easily obtained.

Moreover, it is not hard to see that there are K-functions γ_x and γ_u such that $||h(x,u)|| \leq \gamma_x(||x||) + \gamma_u(||u||)$ by Lemma 1. With $||x(t)|| \leq \max\{2\beta(||x_0||,t), 2\gamma(||u||_\infty)\}$ (see Remark 2), FTIOS follows readily because

$$||y(t)|| \leq \gamma_x \circ 2\beta(||x_0||, t) + (\gamma_x \circ 2\gamma + \gamma_u)(||u||_\infty).$$

The "if" part is as follows: There are two GKL-functions β and β^0 and two K-functions γ and γ^0 such that, for $t \geq t_* \geq 0$,

$$||y(t)|| \leq \beta(||x(t_*)||, t - t_*) + \gamma(||u||_\infty), \tag{36}$$

$$||x(t)|| \leq \beta^0(||x(t_*)||, t - t_*) + \gamma^0(||(u,y)||_\infty), \tag{37}$$

where $\beta(r,t) = 0$ for $t \geq T(r)$ and $\beta^0(r,t) = 0$ for $t \geq T'(r)$ with T and T' continuous with respect to r.

By taking $t_* = t/2$ in (37) and plugging (36) with $t_* = 0$,

$$||x(t)|| \leq \beta^0\left(||x\left(\frac{t}{2}\right)||, \frac{t}{2}\right) + \gamma^0\left(||u||_\infty + \beta\left(||x(0)||, \frac{t}{2}\right) + \gamma(||u||_\infty)\right). \tag{38}$$

By Lemma 1 for any K_∞-function ρ with $\rho - Id$ being a K_∞-function, it follows that

$$\gamma^0\left(||u||_\infty + \beta\left(||x(0)||, \frac{t}{2}\right) + \gamma(||u||_\infty)\right) \leq \gamma^0\left(\rho\left(\beta(||x(0)||, \frac{t}{2}\right)\right) + \bar{\gamma}(||u||_\infty),$$

where $\bar{\gamma} = \gamma^0\left(\rho \circ (\rho - Id)^{-1} \circ (Id + \gamma)\right)$ is a K_∞-function.

Moreover,

$$\begin{aligned}\left\|x\left(\frac{t}{2}\right)\right\| &\leq \beta^0\left(||x(0)||, \frac{t}{2}\right) + \gamma^0(||u||_\infty + \beta(||x(0)||, 0) + \gamma(||u||_\infty)) \\ &\leq \beta^0\left(||x(0)||, \frac{t}{2}\right) + \gamma^0(\rho(\beta(||x(0)||, 0))) + \bar{\gamma}(||u||_\infty).\end{aligned}$$

Again according to Lemma 1,

$$\beta^0\left(\left\|x\left(\frac{t}{2}\right)\right\|, \frac{t}{2}\right) \leq \beta^0\left(\rho\left(\beta^0\left(||x(0)||, \frac{t}{2}\right)\right) + \gamma^0(\rho(\beta(||x(0)||, 0))), \frac{t}{2}\right) + \hat{\gamma}(||u||_\infty), \tag{39}$$

where

$$\hat{\gamma}(||u||_\infty) = \beta^0\left(\rho \circ (\rho - Id)^{-1} \circ \bar{\gamma}(||u||_\infty), 0\right) \geq \beta^0\left(\rho \circ (\rho - Id)^{-1} \circ \bar{\gamma}(||u||_\infty), \frac{t}{2}\right).$$

Therefore, by some manipulations after substituting (39) into (38),

$$||x(t)|| \leq \beta_*(||x(0)||, t) + \gamma_*(||u||_\infty) \tag{40}$$

with $\gamma_* = \bar{\gamma} + \hat{\gamma}$ and a GKL-function

$$\beta_*(||x(0)||, t) = \beta^0\left(\rho\left(\beta^0\left(||x(0)||, \frac{t}{2}\right)\right) + \gamma^0(\rho(\beta(||x(0)||, 0))), \frac{t}{2}\right) + \gamma^0\left(\rho(\beta(||x(0)||, t))\right),$$

where $\beta_*(r, t) = 0$ when $t \geq T^0$ for some constant T^0, depending only on r, due to the finite-time convergence properties of β^0 and β. Thus, system (33) is FTISS. ∎

Based on the FTIOS, we show a finite-time small-gain result. Consider interconnected systems in the following form:

$$\dot{x} = f(x, y_z, u_x), \quad y_x = h_x(x, y_z, u_x), \tag{41}$$

$$\dot{z} = g(z, y_x, u_z), \quad y_z = h_z(z, y_x, u_z), \tag{42}$$

where $x \in R^{n_x}$; $z \in R^{n_z}$; $u_x \in R^{n_1}$; $u_z \in R^{n_2}$; $y_x \in R^{m_x}$; $y_z \in R^{m_z}$; f, g, h_x, and h_z are continuous, and (y_x, y_z) is the unique solution of equations

$$y_x = h_x(x, h_z(z, y_x, u_z), u_x), \quad y_z = h_z(z, h_x(x, y_z, u_x), u_z).$$

THEOREM 4. *Suppose systems* (41) *and* (42) *are FTIOS with* (y_z, u_x) *and* (y_x, u_z) *as input and* y_x *and* y_z *as output, respectively, satisfying:*

$$\begin{cases} ||y_x(t)|| \leq \beta_1(||x_0||, t) + \gamma_1^y(||y_z||_\infty) + \gamma_1^u(||u_x||_\infty), \\ ||y_z(t)|| \leq \beta_2(||z_0||, t) + \gamma_2^y(||y_x||_\infty) + \gamma_2^u(||u_z||_\infty) \end{cases} \tag{43}$$

for suitable functions $\beta_i, \gamma_i^y,\ \gamma_i^u,\ i = 1, 2$. *Also suppose* (41) *and* (42) *are FTSD. If there are two* K_∞*-functions* $\rho_i,\ i = 1, 2$ *satisfying*

$$(Id + \rho_2) \circ \gamma_2^y \circ (Id + \rho_1) \circ \gamma_1^y(s) \leq s, \qquad s \geq 0 \tag{44}$$

(or equivalently, $(Id + \rho_1) \circ \gamma_1^y \circ (Id + \rho_2) \circ \gamma_2^y(s) \leq s$*), then system* (41)–(42) *with* $u = (u_x, u_z)$ *as the input and* $y = (y_x, y_z)$ *as the output is FTIOS.*

Proof. It follows directly from the constructive proof given in [17] (Proof of Theorem 2.1). The KL-functions $\hat{\beta}_i$ and β_i $(i = 1, 2)$ in the proof of Theorem 2.1 of [17] can be constructed to become GKL-functions, which leads to our conclusion. ∎

As explained in [18, 41, 44], the small-gain condition (44) can be simplified as $\gamma_2^y \circ \gamma_1^y(s) < s\ \forall s > 0$ whenever we use "max" instead of "+" in the definitions of FTIOS and FTISS (see Remark 2, for example).

COROLLARY 1. *Suppose the z-subsystem of the following system*

$$\begin{cases} \dot{x} = f(x, v), \\ \dot{z} = g(x, z, v), \end{cases} \tag{45}$$

where f *and* g *are continuous, is FTISS with* (x, v) *as input and the x-subsystem of system* (45) *is FTISS with input* v. *Then system* (45) *is FTISS with input* v.

The corollary can be obtained from Theorem 4 by taking $y_x = x,\ y_z = z$, and $u_x = u_z = v$. In this case, $\gamma_1^y \equiv 0$, and thus, the small-gain condition (44) holds.

5. Finite-time input-to-state stabilization. In this section, we will consider the (adaptive) finite-time control design for the systems with or without uncertainties by virtue of FTISS, FTIOS, and their properties.

5.1. Finite-time stabilization. For practical Lyapunov-based design, we usually construct a Lipschitz continuous (or even smooth) Lyapunov function for a finite-time stable system, although we are not sure if a finite-time stable system admits a smooth Lyapunov function (see [2]). However, the assumption of the existence of a Lipschitz continuous (or C^1) Lyapunov function is not very restrictive in the finite-time control design because we can try to obtain such Lyapunov functions using "control" input during feedback design [2, 11, 15]. In fact, most of the existing results on finite-time stabilization reveal the existence of smooth Lyapunov function. Here, we consider a C^1 Lyapunov function in the study of FTISS-based control design. In this case, (15) and (16) are equivalent by Remark 6.

We first give a lemma for the following analysis.

LEMMA 6. *Suppose there are a C^1 Lyapunov function $V(z) = \sum_{i=1}^n V_i(z_i^0)$ with $z_i^0 = (0, \dots, 0, z_i, 0, \dots, 0)^T$, a positive definite function φ_1, and a K-function φ_2 such that*

$$\dot{V}(z(t))|_{(4)} \le -\varphi_1(z(t)) + \varphi_2(||v(t)||), \tag{46}$$

where $\varphi_1(z) \sim \sum_{i=1}^n \bar{c}_i V_i(z_i^0)^{a_i}$, $\bar{c}_i > 0$, and $0 < a_i < 1$ for $i = 1, \dots, n$. Then system (4) is FTISS with v as the input.

Proof. We give the proof only for $n = 2$ because the general case can be treated analogously. When $\varphi_1(z) \ge 2\varphi_2(||v||)$,

$$\dot{V}(z(t)) \le -\frac{1}{2}\varphi_1(z(t)), \tag{47}$$

which makes system (4) asymptotically stable. In some neighborhood of $z = 0$, $\dot{V} \le -\bar{c}_0(V_1^{a_1} + V_2^{a_2})$ with $\bar{c}_0 = \min\{\bar{c}_1/2, \bar{c}_2/2\} > 0$ since $\varphi_1(z_1, z_2) \sim \bar{c}_1 V_1(z_1)^{a_1} + \bar{c}_i V_2(z_2)^{a_2}$. Without loss of generality, we assume $a_1 \ge a_2$, which implies $V_2^{a_1} \le V_2^{a_2}$ locally around the origin. By (12), we locally have

$$V_1^{a_1} + V_2^{a_2} \ge V_1^{a_1} + V_2^{a_1} \ge (V_1 + V_2)^{a_1} = V^{a_1},$$

and then $\dot{V}(z(t)) \le -\bar{c}_0 V(z(t))^{a_1}$, which implies (local) finite-time convergence of $z = 0$. Therefore, the global finite-time stability of $z = 0$ of system (4) can be obtained due to the local finite-time convergence and global asymptotic stability based on (47). Then, with arguments similar to the part ($2 \Rightarrow 1$) in the proof of Theorem 1, the conclusion follows. ∎

Then we consider a system of the form

$$\begin{cases} \dot{z} = g(x, z), \\ \dot{x} = f(x, z) + v, \quad (v, x, z) \in R \times R \times R^l, \end{cases} \tag{48}$$

where f and g are continuous functions with $f(0, 0) = 0$ and $g(0, 0) = 0$.

For simplicity, denote

$$x^{\smile p} := |x|^p sgn(x), \quad x \in R, \tag{49}$$

where $sgn(\cdot)$ is the sign function. $x^{\smile p} \cdot x = |x|^{p+1}$ and $x^{\smile p}$ is C^1 if $p \ge 1$.

THEOREM 5. *Suppose the z-subsystem of (48) is FTISS with respect to the input function $h(x, z) = x^{\smile p} - \mu(z)^{\smile p}$ for some $p > 1$, where $\mu(z)^{\smile p}$ is C^1 with respect to z and $\mu(0) = 0$. Moreover, suppose that the z-subsystem admits a C^1 Lyapunov function V_z and that there is a constant $q > p$ such that*

$$\dot{V}_z(z)|_{\dot{z}=g(x,z)} \le -\gamma_1(V_z(z)) + \gamma_2(|h(x, z)|), \quad \gamma_2(|h|) = O(|h|^{1+\frac{1}{q}}), \tag{50}$$

where γ_i $(i = 1, 2)$ are K-functions with $\gamma_1(r) \sim r^a$ for some constant $0 < a < 1$,

$$|f(x,z)| \le \hat{f}(h) + \bar{f}(z), \quad \hat{f}(h) = O(|h|^{\frac{1}{p}}),\ \bar{f}(z) = O(\gamma_1(V_z(z))^{\frac{1}{1+q}}), \tag{51}$$

and

$$\left|\frac{\partial \mu(z)^{\smile p}}{\partial z} g(x,z)\right| \le \hat{g}_1(h) + \hat{g}_2(z) \tag{52}$$

with nonnegative functions $\hat{g}_1(h) = O(|h|)$ and $\hat{g}_2(z) = O(\gamma_1(V_z(z))^{\frac{p}{1+p}})$. Then there is a continuous function $\mu_*(h)$ with $\mu_*(0) = 0$ and $\mu_*^{\smile q}$ of class C^1 with respect to (x, z) such that system (48) is FTISS with w as the input by taking

$$v(h(x,z), w) = \mu_*(h) + w. \tag{53}$$

Remark 8. It is worth noting that the z-subsystem is not assumed to be FTISS with respect to x but with respect to the (virtual) input $h(x,z) = x^{\smile p} - \mu(z)^{\smile p}$. This represents one of the main differences between C^1 stabilizing control and nonsmooth finite-time stabilizing control. It is not hard to show that $|x - \mu(z)| \le 2|h(x,z)|^{\frac{1}{p}}$ and $|h(x,z)| \le 2^p|x - \mu(z)|^p$. With these inequalities, $\dot{z} = g(x,z)$ is FTISS with $\bar{h} = x - \mu(z)$ as the input if and only if it is FTISS with h as its input. Additionally, according to Remark 4, if a system is FTISS with h as the input, it is also FTISS with the input $h^{\smile s}$ for any constant $s > 0$.

Proof. Consider the following positive definite Lyapunov function

$$V(x,z) = \int_0^{V_z(z)} \bar{\rho}(s)ds + W_*(x,z) \tag{54}$$

with

$$W_*(x,z) = \int_{\mu(z)}^{x} h(s,z)\,ds = \frac{|x|^{p+1} + p|\mu(z)|^{p+1}}{p+1} - x\mu(z)^{\smile p},$$

where $\bar{\rho} : R_+ \to R_+$ is an increasing and positive continuous function to be determined later and W_* is positive for $h \ne 0$ and is C^1 with respect to z because $\mu^{\smile p}$ is so. Then $\dot{V}(x,z) \le -\bar{\rho}(V_z(z))\gamma_1(V_z(z)) + \bar{\rho}(V_z(z))\gamma_2(|h|) + \dot{W}_*(x,z)$, where

$$\dot{W}_* = \frac{\partial W_*}{\partial x}\dot{x} + \frac{\partial W_*}{\partial \mu(z)^{\smile p}}\frac{\partial \mu(z)^{\smile p}}{\partial z}\dot{z} = h(x,z)[f(x,z)+v] - [x - \mu(z)]\frac{\partial \mu(z)^{\smile p}}{\partial z}g(x,z).$$

According to (51), $q > p$, Remark 8, and Young's inequality (11),

$$hf(x,z) \le |h|[\hat{f}(h) + \bar{f}(z)] \le |h|^{1+\frac{1}{q}}\bar{f}_1(h) + \alpha_f(z) \tag{55}$$

for nonnegative continuous functions $\bar{f}_1(h)$ and $\alpha_f(z) = O(\gamma_1(V_z(z)))$.

By Remark 8, $|x - \mu(z)| \le 2|h|^{1/p}$. Then, from Lemmas 2 and 3 as well as (52),

$$[x - \mu(z)]\frac{\partial \mu(z)^{\smile p}}{\partial z}g(x,z) \le 2|h|^{\frac{1}{p}}[\hat{g}_1(h) + \hat{g}_2(z)] \le |h|^{1+\frac{1}{q}}\bar{g}_2(h) + \alpha_g(z)$$

with suitable nonnegative functions $\bar{g}_2(h)$ and $\alpha_g(z) = O(\gamma_1(V_z(z)))$.

Select a function $\bar{\rho}$ such that $\bar{\rho}(V_z(z))\gamma_1(V_z(z)) > 2(\alpha_f + \alpha_g)(z)$, noting that $(\alpha_f + \alpha_g)(z) = O(\gamma_1(V_z(z)))$.

Moreover, recalling Young's inequality (10) by taking $a = \bar{\rho}(V_z)$, $b = \gamma_2(|h|)$, and K-function ϕ with $\int_0^a \phi(s)\,ds \le a\gamma_1(\bar{\rho}^{-1}(a))/4$, there is a K_∞-function $\gamma_0(b) \ge \int_0^b \phi^{-1}(s)ds$ such that

$$\bar{\rho}(V_z(z))\gamma_2(|h|) \le \frac{1}{4}\bar{\rho}(V_z(z))\gamma_1(V_z) + \gamma_0(\gamma_2(|h|)).$$

Then, with (50), there is a nonnegative function $\bar{g}_1(h)$ such that

$$\gamma_0(\gamma_2(|h|)) \le |h|^{1+\frac{1}{q}}\bar{g}_1(h). \tag{56}$$

After some manipulations,

$$\dot{V}(x,z) \le -\frac{3}{4}\bar{\rho}(V_z(z))\gamma_1(V_z(z)) + hv + |h|^{1+\frac{1}{q}}[\bar{f}_1(h) + \bar{g}_1(h) + \bar{g}_2(h)] + \alpha_f(z) + \alpha_g(z).$$

Take the control law (53) with

$$\mu_*(h) = -h^{\smile\frac{1}{q}}\Phi(h), \tag{57}$$

where Φ is C^1 and dominates $1 + \bar{g}_1(h) + \bar{g}_2(h) + \bar{f}_1(h)$, or equivalently,

$$h\mu_*(h) + |h|^{1+\frac{1}{q}}\left[\bar{f}_1(h) + \sum_{i=1}^{2}\bar{g}_i(h)\right] \le -|h|^{1+\frac{1}{q}}.$$

Obviously, $\mu_*(h)^{\smile q}$ is C^1. Therefore, according to Young's inequality,

$$\dot{V}(x,z) \le -\frac{1}{4}\bar{\rho}(V_z)\gamma_1(V_z(z)) - |h|^{1+\frac{1}{q}} + |w|^{1+q}.$$

Since $\bar{\rho}(0) > 0$ and $q > p$,

$$\bar{\rho}(V_z(z))\gamma_1(V_z(z)) \sim V_z(z)^a \sim \left(\int_0^{V_z(z)}\bar{\rho}(s)ds\right)^a \quad \text{as } z \to 0$$

and $|h|^{1+1/p} \sim W_*(h)$ or, equivalently,

$$|h|^{1+\frac{1}{q}} \sim W_*(h)^{a_0} \quad \text{as } h \to 0; \qquad a_0 = \frac{p(1+q)}{q(1+p)} < 1.$$

Then, by Lemma 6, the conclusion follows. ∎

The proof of Theorem 5 is consistent with the design procedure given in our papers [11, 12]. Note that the controller v depends only on $h(x,z)$, which may be viewed as an output variable. In this sense, v may be regarded as an "output" or "partial-state" feedback law.

5.2. Adaptive finite-time stabilization. Consider the system

$$\dot{X} = f(X) + g(X)u, \quad X \in R^N, \tag{58}$$

where f and g are continuous with uncertainties (for example, unknown parameters and uncertain nonlinearities). It is said to be (globally) adaptively finite-time stabilizable if there is a continuous (partial-state) feedback law of the form

$$\begin{cases} u = \mu(X^1,\hat{\sigma}), & \mu(0,\hat{\sigma}) = 0,\ X(0) = X_0, \\ \dot{\hat{\sigma}} = \nu(X^1,\hat{\sigma}), & \hat{\sigma}(0) = \sigma_0, \end{cases} \tag{59}$$

where μ and ν are continuous, $X^1 \in R^n$ ($n \le N$) is a part of X (maybe measurable output), and $\hat{\sigma} \in R$ is an auxiliary variable to deal with uncertainties such that $X = 0$ of system (58) is globally finite-time convergent and $\hat{\sigma}(t)$ keeps bounded for any initial condition (X_0, σ_0); namely, there is a continuous settling-time function $T(X_0, \sigma_0)$ with $T(0, \sigma_0) = 0$ such that any solution $(X(t)^T, \hat{\sigma}(t))^T \in \mathcal{S}(t, X_0, \sigma_0)$ (the solution set of system (58) with (59) since the solution may not be unique) is bounded and satisfies

$$\lim_{t\to T} X(t, x_0, \sigma_0) = 0; \quad X(t, x_0, \sigma_0) = 0 \ \forall t > T.$$

Note that $\hat{\sigma}$ is not required to converge to the real value in the design since it is not included in X. Instead, the limit value of $\hat{\sigma}$ usually depends on the initial condition (X_0, σ_0). If $X^1 = X$, the feedback becomes full state, and if there is no uncertainty in system (58), the update law $\dot{\hat{\sigma}} = \nu$ can be removed and the control design reduces to the case discussed in the last subsection.

Lemma 7. *Consider a system*

$$\dot{X} = \hat{f}(X, \sigma, \hat{\sigma}), \quad \hat{f}(0, \sigma, \hat{\sigma}) = 0 \in R^n \tag{60}$$

with continuous function $\hat{f}$, unknown constant σ, and its estimate $\hat{\sigma}(t)$. There is a C^1 positive definite function $V(X)$

$$\dot{V}(X)|_{(60)} \le -\gamma_1(V(X)) + (\sigma - \hat{\sigma})\hat{\nu}(X, \hat{\sigma}) + \hat{\eta}(X, \hat{\sigma})[\hat{\nu}(X, \hat{\sigma}) - \dot{\hat{\sigma}}]$$

with continuous functions $\hat{\eta}$ and $\hat{\nu} = o(\gamma_1(V(X))) \ge 0$ for any bounded $\hat{\sigma}$ and K_∞-function $\gamma_1(V(X)) \sim \sum_{i=1}^n V(X_i^0)^{a_i}$ for $0 < a_i < 1$ and $X_i^0 = (0, \ldots, 0, X_i, 0, \ldots, 0)^T$, $i = 1, \ldots, n$. Then with dynamics $\dot{\hat{\sigma}} = \hat{\nu}(X, \hat{\sigma})$, $X(t)$ becomes 0 in finite time.

This result was similarly obtained in [11] for adaptive finite-time analysis. Here we give only a sketch of its proof. Take $\tilde{\sigma} = \sigma - \hat{\sigma}$ and $V_*(X, \tilde{\sigma}) = V(X) + \tilde{\sigma}^2/2$. Then

$$\dot{V}_*(X, \tilde{\sigma})|_{(60),\ \dot{\hat{\sigma}} = \hat{\nu}} = \frac{\partial V_*(X, \tilde{\sigma})}{\partial X}\dot{X} + \frac{\partial V_*(X, \tilde{\sigma})}{\partial \tilde{\sigma}}\dot{\tilde{\sigma}} \le -\gamma_1(V(X)).$$

Following the routine analysis of the conventional adaptive stabilization (referring to [25, 27]) gives the asymptotic stability of system (60) and $|\tilde{\sigma}(t)| \le \hat{c}_0$ for a constant $\hat{c}_0$. Therefore, $\dot{V}(X) = \dot{V}_* - \tilde{\sigma}\hat{\nu} \le -\gamma_1(V(X)) + \hat{c}_0\hat{\nu}$. Since $\hat{\nu} = o(\gamma_1(V(X)))$ for any $\hat{\sigma}$, there is a constant $r(\hat{c}_0) > 0$ such that $\hat{c}_0\hat{\nu} \le \gamma_1(V(X))/2$ once $V(X) \le r$ (referring to [11] for details); namely, if $V(X(0)) \le r$, $\dot{V} \le -\gamma_1(V(X))/2$ and $V(X(t)) \le r \ \forall t \ge 0$. If $V(X(0)) > r$, then $V(X(t)) \le r$ in finite time. Otherwise, due to $V_* \ge V > r$,

$$V_*(X(0), \hat{\sigma}(0)) \ge V_*(X(\tau), \hat{\sigma}(\tau)) + \int_0^\tau \gamma_1(V(X))ds \ge \tau\gamma_1(r),$$

which leads to a contradiction when $t \ge V_*(X(0), \hat{\sigma}(0))/\gamma_1(r)$. Thus, $\dot{V}(X) \le -\gamma_1(V(X))/2$ will hold in finite time. Because $\gamma_1(V(X)) \sim \sum_{i=1}^n V(X_i^0)^{a_i}$, X will be 0 in finite time by Lemma 6.

Example 4. Consider the system

$$\begin{cases} \dot{x} = -x^{1/3} + x\tilde{\sigma}, \\ \dot{\tilde{\sigma}} = -x^2, \end{cases}$$

where $\tilde{\sigma}$-subsystem can be viewed as an update law. Borrowing arguments from [11], it directly follows that $x(t)$ converges to zero in finite time. However, system

$\dot{x} = -x^{1/3} + x\tilde{\sigma}$ is not FTISS with $\tilde{\sigma}$ as the input, but it is so if $\bar{\nu} = x\tilde{\sigma}$ is regarded as its input. This observation, that is, FTISS depends on the selection of input variable, will be applied to an adaptive finite-time design procedure in what follows.

Consider the system

$$\begin{cases} \dot{z} = g(x, z), \\ \dot{x} = f(x, z) + v, \quad (v, x, z) \in R \times R \times R^l, \end{cases} \tag{61}$$

where f and g are *unknown* continuous functions with $g(0, 0) = 0$ and $f(0, 0) = 0$.

Theorem 5 cannot be applied straightforwardly to the uncertain system (61), but, with similar proof ideas, the next result can be viewed as its extension to the adaptive case.

THEOREM 6. *Suppose the z-subsystem of* (61) *admits a C^1 Lyapunov function V_z and that there are two constants $q > p > 1$ and an unknown constant σ such that $h(x, z) = x^{\smile p} - \mu(z)^{\smile p}$ is C^1,*

$$|f(x, z)| \leq \tilde{f}_1(h) + \sigma\tilde{f}_2(h) + \tilde{f}(z), \tilde{f}(z) = O(\gamma_1(V_z(z))^{\frac{1}{1+q}}), \tilde{f}_i(h) = O(|h|^{\frac{1}{p}}), i = 1, 2, \tag{62}$$

$$\left|\frac{\partial\mu^{\smile p}}{\partial z} g(x, z)\right| \leq \tilde{g}_1(h) + \sigma\tilde{g}_2(h) + \tilde{g}(z), \tilde{g}(z) = O(\gamma_1(V_z(z))^{\frac{p}{1+p}}), \tilde{g}_i(h) = O(|h|), i = 1, 2, \tag{63}$$

and

$$\dot{V}_z(z) \leq -\gamma_1(V_z(z)) + \gamma_2(|h(x, z)|) + (\sigma - \hat{\sigma})\nu(x, z) + \eta(x, z)[\nu(x, z) - \dot{\hat{\sigma}}], \tag{64}$$

where $\hat{\sigma}$ is the estimate of σ, $\nu = o(|h|^{1+\frac{1}{q}} + \gamma_1(V_z(z))) \geq 0$, and γ_1 and γ_2 are K-functions with $\gamma_2(|h|) = O(|h|^{1+\frac{1}{q}})$ and $\gamma_1(r) \sim r^a$ for some constant $0 < a < 1$. Then there is a continuous function $\mu_*(h, \hat{\sigma})$ with $\mu_*(0, \hat{\sigma}) = 0$ and $\mu_*^{\smile q}$ of class C^1 such that system (61) *is FTISS by taking*

$$v(h, \hat{\sigma}, w) = \mu_*(h, \hat{\sigma}) + w. \tag{65}$$

Moreover, if we take $w = 0$ and $\dot{\hat{\sigma}} = \hat{\nu}(x, z)$ for a suitably selected function $\hat{\nu}$, the origin $(x, z) = (0, 0)$ of system (61) *is finite-time stabilizable.*

Proof. Take a Lyapunov function

$$V(x, z) = \int_0^{V_z(z)} \bar{\rho}(s)ds + W_*(x, z), \quad W_*(x, z) = \int_{\mu(z)}^{x} h(s, z)\, ds,$$

where $\bar{\rho} : R_+ \to R_+$ is a continuous and increasing function with $\bar{\rho}(0) \geq 1$ to be determined later. As usual, denote $\tilde{\sigma}(t) = \sigma - \hat{\sigma}(t)$.

According to (62), $q > p$, and (11), we have

$$hf(x, z) \leq |h|^{1+\frac{1}{q}}[\bar{f}_1(h) + \sigma\bar{f}_2(h)] + \alpha_f(z)$$

for nonnegative continuous functions $\bar{f}_i(h)$ $(i = 1, 2)$ and $\alpha_f(z) = O(\gamma_1(V_z(z)))$. Similarly, from Remark 8, (63), and (11), we obtain

$$(x - \mu)\frac{\partial\mu^{\smile p}}{\partial z} g(x, z) \leq |h|^{1+\frac{1}{q}}[\bar{g}_1(h) + \sigma\bar{g}_2(h)] + \alpha_g(z)$$

for some nonnegative functions $\bar{g}_i(h)$ $(i = 1, 2)$ and $\alpha_g(z) = O(\gamma_1(V_z(z)))$.

Take

$$\hat{\eta} = \bar{\rho}\eta - \frac{\partial W_*}{\partial \hat{\sigma}}, \quad \hat{\nu} = \bar{\rho}\nu + |h|^{1+\frac{1}{q}}[\bar{f}_2(h) + \bar{g}_2(h)] = o(|h|^{1+\frac{1}{q}} + \gamma_1(V_z(z))).$$

Then, from (11),

$$\bar{\rho}\eta[\bar{f}_2(h) + \bar{g}_2(h)] + \bar{\rho}\nu\frac{\partial W_*}{\partial \hat{\sigma}} \le |h|^{1+\frac{1}{q}}[\hat{f}_1(h) + \hat{\sigma}\hat{f}_2(h)] + \alpha_e(z)$$

with suitable nonnegative functions $\hat{f}_i(h)$ $(i = 1, 2)$ and $\alpha_e(z) = \gamma_1(V_z(z))/4$.

Select a function $\bar{\rho}$ such that $\bar{\rho}(V_z(z))\gamma_1(V_z(z)) > 2\bar{\alpha}_0(z)$ since $\bar{\alpha}_0(z) = \alpha_f + \alpha_g + \alpha_e = O(\gamma_1(V_z(z)))$. By Young's inequality, it follows that

$$\bar{\rho}(V_z)\gamma_2(|h|) \le \frac{1}{4}\bar{\rho}(V_z)\gamma_1(V_z) + |h|^{1+\frac{1}{q}}\bar{g}_0(h).$$

With similar arguments given in the proof of Theorem 5,

$$\dot{V} \le -\frac{3}{4}\bar{\rho}(V_z)\gamma_1(V_z) + hu + |h|^{1+\frac{1}{q}}[\bar{f}_1 + \hat{f}_1 + \bar{g}_1 + \bar{g}_0 + \sigma(\bar{f}_2 + \bar{g}_2) + \hat{\sigma}\hat{f}_2] + \bar{\alpha}_0 + \tilde{\sigma}\hat{\nu} + \hat{\eta}[\hat{\nu} - \dot{\hat{\sigma}}].$$

Take the control law

$$\mu_*(h, \hat{\sigma}) = -h^{\smile\frac{1}{q}}\Phi(h, \hat{\sigma}), \tag{66}$$

where Φ is C^1 and dominates $1 + \bar{f}_1(h) + \hat{f}_1(h) + \bar{g}_1(h) + \bar{g}_0(h) + \hat{\sigma}[\bar{f}_2(h) + \bar{g}_2(h) + \hat{f}_2(h)]$, or equivalently,

$$h\mu_* + |h|^{1+\frac{1}{q}}[\bar{f}_1 + \hat{f}_1 + \bar{g}_1 + \bar{g}_0 + \sigma(\bar{f}_2 + \bar{g}_2) + \hat{\sigma}\hat{f}_2] \le -|h|^{1+\frac{1}{q}} + \tilde{\sigma}|h|^{1+\frac{1}{q}}(\bar{f}_2 + \bar{g}_2).$$

Using Young's inequality again,

$$\dot{V}(x, z) \le -\frac{1}{2}\bar{\rho}(V_z)\gamma_1(V_z(z)) - |h|^{1+\frac{1}{q}} + |w|^{1+q} + \bar{\nu}_w,$$

where $\bar{\nu}_w = \tilde{\sigma}\hat{\nu} + \hat{\eta}[\hat{\nu} - \dot{\hat{\sigma}}]$ and $\mu_*(h)^{\smile q}$ is C^1. Similar to the analysis given in the proof of Theorem 5, we obtain the FTISS of system (61) under (65) with $(\bar{\nu}_w, w)$ as the input.

Moreover, set $w = 0$ and $\dot{\hat{\sigma}} = \hat{\nu} = -\dot{\tilde{\sigma}}$. Then $\dot{V}(h, z) \le -\frac{1}{2}\bar{\rho}(V_z)\gamma_1(V_z(z)) - |h|^{1+\frac{1}{q}} + \tilde{\sigma}\hat{\nu}$, which implies the conclusion by Lemma 7. □

It is hard to solve the adaptive stabilization problem for general nonlinear systems. Here we focus on the following class of systems:

$$\begin{cases} \dot{\zeta}_i = \psi_i(x_1, \ldots, x_i, \zeta_i), \quad 1 \le i \le n, \\ \dot{x}_i = x_{i+1} + f_i(x_1, \ldots, x_i, \zeta_i), \quad 1 \le i \le n-1, \\ \dot{x}_n = u + f_n(x_1, \ldots, x_n, \zeta_n), \end{cases} \tag{67}$$

where $u \in R$ is the control input, $x = (x_1, \ldots, x_n)^T \in R^n$ is the measured portion of the state, and $\zeta = (\zeta_1^T, \ldots, \zeta_n^T)^T \in R^{n_0}$ is the remaining unmeasured state, referred to as *dynamic uncertainty*, which often arises from many engineering applications (see [13, 20, 21, 45] and references therein). For each $i = 1, 2, \ldots, n$, f_i is an unknown and Lipschitz continuous function, and ψ_i is piecewise continuous with respect to ζ_i and Lipschitz continuous with respect to $(x_1, \ldots, x_i)$. For simplicity, we assume

$\psi_i(0,\ldots,0)=0$. Nonlinear systems transformable into (67) have been studied extensively over the last decades from both theoretical and practical viewpoints (see, e.g., [24, 25, 27]).

The following assumptions are made regarding system (67) with dynamic and parametric uncertainties:

(B1) $\dot{\zeta}_i=\psi_i(x_1,\ldots,x_i,\zeta_i)$ is FTISS with $(x_1,\ldots,x_i)$ as the input, along with an FTISS-Lyapunov function $U_i(\zeta_i)$ that satisfies

$$\dot{U}_i(\zeta_i)\le -\gamma_i(U_i(\zeta_i))+\gamma_i^0(||(x_1,\ldots,x_i)||), \tag{68}$$

with K_∞-functions γ_i and γ_i^0, where $\gamma_i(r)\sim r^{\hat{a}_i}$ for some positive constant $\hat{a}_i<1$ and $\gamma_i^0=O(||(x_1,\ldots,x_i)||^{1+\hat{p}})$ for some constant $\hat{p}>1$.

(B2) For each $i=1,\ldots,n$, there are an unknown parameter $\theta>0$ and C^1 known nonnegative functions κ_{i1} and κ_{i2} vanishing at the origin such that

$$|f_i(x_1,\ldots,x_i,\zeta_i,t)|\le \kappa_i(\zeta_i,x_1,\ldots,x_i,\theta):=\kappa_{i1}(\zeta_i)+\theta\kappa_{i2}(x_1,\ldots,x_i), \tag{69}$$

where $\kappa_{i1}=O(\gamma_i(U_i)^{\frac{1}{1+\hat{q}}})$ for a positive constant $\hat{q}\le 1$.

Here we show a constructive procedure for adaptive finite-time control of system (67), based heavily on the repeated usage of the technique given in the proof of Theorems 6 and 5. In fact, the recursive design procedure is similar to that given, for example, in [45, 33, 19, 20, 11, 12]. Therefore, without loss of the main ideas, some tedious details are omitted due to space limitations.

Choose $0<\bar{a}<1$ such that

$$\begin{cases} r_1=1>r_2>\cdots>r_i:=r_{i-1}-\bar{a}>0, \quad \hat{q}<\dfrac{(1-\bar{a})r_n}{r_n-\bar{a}},\\ p_0=1, \quad (p_{i-1}+1)r_i=(p_i+1)r_{i+1}, \quad 1<p_i<\hat{p}\ \ (i=1,\ldots,n-1)\end{cases} \tag{70}$$

which can be satisfied as $\bar{a}$ is sufficiently small (noting that $r_i=p_{i-1}=1$ and $i=1,\ldots,n$ if $\bar{a}=0$). Take

$$q_i=\frac{r_ip_{i-1}}{r_{i+1}}, \quad q_i>q_{i-1}\ (i=2,\ldots,n). \tag{71}$$

Clearly, $1\le r_ip_{i-1}<r_{i+1}p_i,\ p_{i-1}<q_i<p_i$ for $i=1,\ldots,n-1$.

As in [12], denote $\sigma=\max\{1,\theta^{\frac{2-2\bar{a}}{r_n}}\}\ge\theta$, which is an unknown parameter since θ is unknown. Denote $\hat{\sigma}$ as an estimate of σ. Define "virtual controller" μ_j $(j=1,\ldots,n)$ recursively as follows:

$$\begin{cases}\mu_0=0, \quad \mu_j(x_1,\ldots,x_{j-1},\hat{\sigma})=-h_j^{\smile\frac{1}{q_j}}\Phi_j(x_1,\ldots,x_j,\hat{\sigma}),\\ h_j=x_j^{\smile p_{j-1}}-\mu_{j-1}^{\smile p_{j-1}}, \quad 1\le j\le n,\end{cases} \tag{72}$$

where Φ_j $(1\le j\le n)$ is a C^1 positive function to be determined later. Take

$$V_i(x_1,\ldots,x_i,\zeta_1,\ldots,\zeta_i)=\sum_{j=1}^{i}U_j(\zeta_j)+\bar{V}_i(x_1,\ldots,x_i), \quad \bar{V}_i=\sum_{j=1}^{i}\int_0^{W_j(x_1,\ldots,x_j)}\rho_j(s)ds, \tag{73}$$

where

$$W_j(x_1,\ldots,x_j)=\int_{\mu_{j-1}}^{x_j}[s^{\smile p_{j-1}}-\mu_{j-1}^{\smile p_{j-1}}(x_1,\ldots,x_{j-1})]\,ds, \quad j=1,\ldots,n, \tag{74}$$

and $\rho_j(s)\ge 1$ $(j=1,\ldots,n)$ is an increasing function for $s\ge 0$ to be determined.

Step 1: Consider the subsystem of (67):

$$\begin{cases} \dot{\zeta}_1 = \psi_1(x_1, \zeta_1), \\ \dot{x}_1 = x_2 + f_1(x_1, \zeta_1). \end{cases} \tag{75}$$

Denote $\alpha_1 = \psi_1$ and $Z_1 = \zeta_1$ for the recursive procedure. Take $h_1 = x_1$ and $W_1(x_1) = x_1^2/2$ (see (72) and (74)). Then

$$\dot{V}_1(\zeta_i, x_1)|_{(75)} \le \rho_1\left(\frac{x_1^2}{2}\right)[x_1 x_2 + |x_1|\kappa_1(\zeta_1, x_1, \theta)] - \gamma_1(U_1(\zeta_1)) + \gamma_1^0(|x_1|).$$

Based on assumptions (B1) and (B2), the conditions (64), (62), and (63) are satisfied. Following the proof of Theorem 6 with some manipulations (referring to [11, 12] for details, too), there are $\rho_1(W_1)$ and a "virtual controller" $\mu_1(x_1, \hat{\sigma})$ in the form of (72) as μ_* constructed in the proof of Theorem 6 such that

$$\dot{V}_1(Z_1, x_1)|_{(75)} \le -l_1[\rho_1(W_1)W_1^{\frac{2-\bar{a}}{2}} + \gamma_1(U_1)] + 2|h_2|^{1+\frac{1}{q_2}} + \bar{\nu}_2, \tag{76}$$

where $l_1 > 0$ and $\bar{\nu}_2 = \tilde{\sigma}\nu_1$ with $\nu_1(x_1) = \rho_1(W_1)|x_1|\kappa_{12}(x_1)$ and h_2 given in (72). Thus, system (75) is FTISS with input $(\bar{\nu}_2, h_2)$.

Step i ($2 \le i < n$): Suppose we have designed a function $\mu_{i-1}(h_1, \ldots, h_{i-1})$ so that

$$\begin{cases} \dot{Z}_{i-1} = \alpha_{i-1}(h_{i-1}, Z_{i-1}), \\ \dot{x}_{i-1} = x_i + \hat{f}_{i-1}, \quad \hat{f}_{i-1}(h_{i-1}, Z_{i-1}) = f_{i-1}(x_1, \ldots, x_{i-1}, \zeta_{i-1}), \end{cases} \tag{77}$$

where $Z_{i-1} = (\zeta_1^T, x_1, \ldots, \zeta_{i-2}^T, x_{i-2}, \zeta_{i-1}^T)^T$, is FTISS with $(\bar{\nu}_i, h_i)$ as its input with

$$\dot{V}_{i-1}(Z_{i-1}, x_{i-1})|_{(77)} \le -l_{i-1}\left[\sum_{j=1}^{i-1}\rho_j(W_j)W_j^{\frac{2-\bar{a}}{2}} + \sum_{j=1}^{i-1}\gamma_j(U_j)\right] + 2|h_i|^{1+\frac{1}{q_i}} + \bar{\nu}_i \tag{78}$$

for a suitably constant $l_{i-1} > 0$, h_i in the form of (72), and

$$\bar{\nu}_i(x_1, \ldots, x_{i-1}, \tilde{\sigma}, \dot{\hat{\sigma}}) := \tilde{\sigma}\nu_{i-1}(x_1, \ldots, x_{i-1}, \hat{\sigma}) - \sum_{j=1}^{i-1}\rho_j(W_j)\frac{\partial W_j}{\partial \hat{\sigma}}[\nu_{i-1} - \dot{\hat{\sigma}}]$$

with a continuous function ν_{i-1}. Clearly, (76) is a special form of (78) by $\frac{\partial W_1}{\partial \hat{\sigma}} = 0$.

Take $Z_i = (Z_{i-1}^T, x_{i-1}, \zeta_i^T)^T$, and rewrite $\dot{Z}_i = \alpha_i(h_i, Z_i)$ for a vector-valued function α_i. After manipulations with (B1), (B2), (70), (71), and (73),

$$\kappa_{i1} = O(\gamma_i(U_i)^{\frac{1}{1+q_i}}), \quad \kappa_{i2}(x_1, \ldots, x_i)|_{h_i \equiv 0} = O(\bar{V}_{i-1}^{\frac{1}{1+q_i}}), \tag{79}$$

and, moveover, because μ_{i-1} is independent of $\zeta_1, \ldots, \zeta_i$,

$$\left|\frac{\partial \mu_{i-1}^{\smile p_{i-1}}}{\partial Z_i}\alpha_i\right| = \left|\sum_{j=1}^{i-1}\frac{\partial \mu_{i-1}^{\smile p_{i-1}}}{\partial x_j}(x_{j+1} + f_j(h_j, Z_j))\right|. \tag{80}$$

Since the ζ_i-subsystem, $\dot{\zeta}_i = \psi_i(x_1, \ldots, x_i, \zeta_i)$, is FTISS with input $(x_1, \ldots, x_i)$, it is FTISS with $(x_1, \ldots, x_{i-1}, h_i)$ as the input according to Remark 4 (by noting that there is a global homeomorphism between $(x_1, \ldots, x_i)$ and $(x_1, \ldots, x_{i-1}, h_i)$). Therefore, $\dot{Z}_i = \alpha_i(h_i, Z_i)$ is FTISS with $(\bar{\nu}_i, h_i)$ as its input in light of Corollary 1 with taking $x = \zeta_i$ and $z = (Z_{i-1}^T, x_{i-1})^T$. Then the considered subsystem of (67) is expressed in a compact form:

$$\begin{cases} \dot{Z}_i = \alpha_i(h_i, Z_i), \\ \dot{x}_i = x_{i+1} + \hat{f}_i(h_i, Z_i), \quad \hat{f}_i(h_i, Z_i) = f_i(x_1, \ldots, x_i, \zeta_i). \end{cases} \tag{81}$$

Recalling the proof of Theorem 6 along with (70), (79), and (80), there is a function $\rho_i(s) \geq 1$ and a "virtual controller" $\mu_i(x_1, \ldots, x_i, \hat{\sigma})$, constructed in the form of (72) as μ_* done in the proof of Theorem 6 such that

$$\dot{V}_i(Z_i, x_i)|_{(81)} \leq -l_i \left[\sum_{j=1}^{i} \rho_j(W_j) W_j^{\frac{2-\bar{a}}{2}} + \sum_{j=1}^{i} \gamma_j(U_j) \right] + 2|h_{i+1}|^{1+\frac{1}{q_i}} + \bar{\nu}_{i+1} \tag{82}$$

with a constant $l_i > 0$ and the "input" function $\bar{\nu}_{i+1} = \tilde{\sigma}\nu_i - \sum_{j=1}^{i} \frac{\partial W_j}{\partial \hat{\sigma}}[\nu_i - \dot{\hat{\sigma}}]$ for a selected nonnegative function $\nu_i(x_1, \ldots, x_i, \hat{\sigma})$. Then system (81) is FTISS with $(\bar{\nu}_{i+1}, h_{i+1})$ as its input.

Step n: From (82), it holds that

$$\dot{V}_n(Z_n, x_n)|_{(67)} \leq -l_n \left[\sum_{j=1}^{n} \rho_j(W_j) W_j^{\frac{2-\bar{a}}{2}} + \sum_{j=1}^{n} \gamma_j(U_j(\zeta_j)) \right] + 2|h_{n+1}|^{1+\frac{1}{q_i}} + \bar{\nu}_{n+1},\ l_n > 0.$$

With $u = \mu_n(x, \hat{\sigma})$ and $\dot{\hat{\sigma}} = \nu_n(x, \hat{\sigma})$ for a suitably constructed function ν_n and μ_n given in (72), the adaptive finite-time stabilization of system (67) is obtained from Theorem 6 (or Lemma 7) by noting that $w = h_{n+1} = 0$ (for $x_{n+1} = u$), $\hat{\nu} = \bar{\nu}_{n+1} = \tilde{\sigma}\nu_n$, and $\sum_{j=1}^{n} \rho_j(W_j) W_j^{\frac{2-\bar{a}}{2}} \sim \bar{V}_n^{\frac{2-\bar{a}}{2}}$ $(i = 1, \ldots, n)$ (referring to the proof of Theorem 5).

Summarizing the above discussions, we have the following theorem.

THEOREM 7. *System* (67) *with assumptions* (B1) *and* (B2) *is adaptively finite-time stabilizable.*

Theorem 7 is consistent with existing finite-time control results. If there are no ζ_i-subsystem and no parametric uncertainties in system (67), the result is consistent with the finite-time stabilizing control design shown in [10], and if there is no ζ_i-subsystem (i.e., no dynamic uncertainties), a similar adaptive finite-time controller is proposed in [11]. Moreover, [12] considered a simple dynamic uncertainty case, where there is no ζ_i-subsystem when $i \geq 2$.

Example 5. Consider the system

$$\begin{cases} \dot{\zeta}_1 = -\zeta_1^{5/7} + x_1, \\ \dot{\zeta}_2 = -\zeta_2^{3/5} + \sin(x_2 - x_1), \\ \dot{x}_1 = x_2 + \theta \sin x_1 \cos \zeta_1, \\ \dot{x}_2 = u - \zeta_2^2 \sin x_2, \end{cases}$$

where $\theta = 1$ is an unknown parameter and ζ_i $(i = 1, 2)$ are unmeasurable variables.

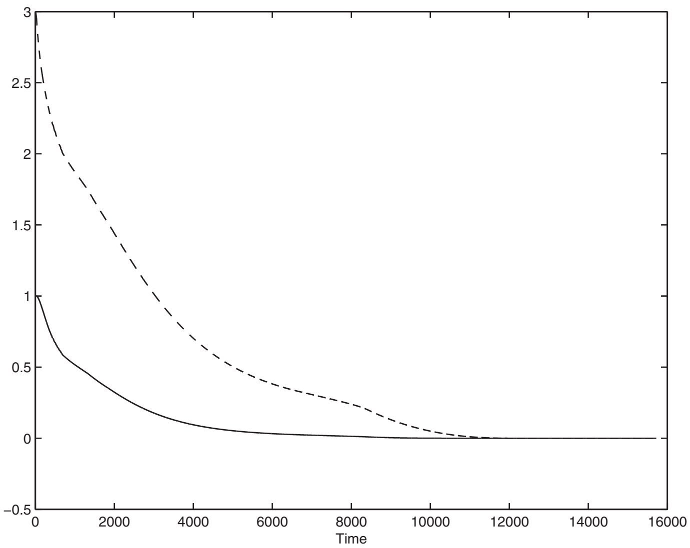

FIG. 1. *Trajectories of* ζ_1 *(solid line) and* ζ_2 *(dashed line).*

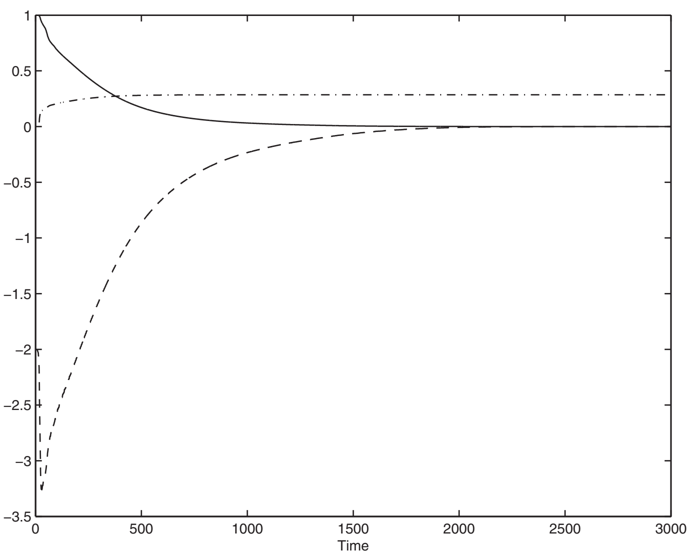

FIG. 2. *Trajectories of* x_1 *(solid line),* x_2 *(dashed line), and* $\hat{\sigma}$ *(dotted line).*

Set $r_1 = 1$ and $r_2 = 7/9$. Following the procedure described as above, the adaptive finite-time controller can be taken as

$$\begin{cases} u = -4(x_2^{\frac{9}{7}} + 2x_1\Phi_1^{\frac{9}{7}})^{\frac{5}{9}}(3 + 2\Phi_1 + 2\Phi_1^2 + 2\hat{\sigma}x_1^{\frac{4}{9}}), \\ \dot{\hat{\sigma}} = x_1^{\frac{16}{9}} + 4x_1^{\frac{4}{9}}(x_2^{\frac{9}{7}} + 2x_1\Phi_1^{\frac{9}{7}})^{\frac{14}{9}}, \end{cases} \qquad \Phi_1 = \hat{\sigma}^2 x_1^2/9 + 2.$$

With initial conditions $\zeta_1(0) = 1$, $\zeta_2(0) = 3$, $x_1(0) = 1$, $x_2(0) = -2$, and $\hat{\sigma}(0) = 0$, Figures 1 and 2 show the trajectories of ζ_i, x_i $(i = 1, 2)$ and $\hat{\sigma}$, respectively.

6. Conclusions. In this paper, we have developed a new framework for tackling finite-time control problems in nonlinear uncertain systems. First, Sontag's original notion of ISS was extended to FTISS, along with various characterizations of FTISS. Then the notion of FTIOS and a finite-time small-gain theorem were presented. Furthermore, a design tool for propagating the FTISS property was also proposed for a class of interconnected nonlinear systems with "partial-state" feedback. The proposed framework was complemented by control applications including finite-time stabilization and adaptive finite-time control, which generalizes several previously proposed adaptive control results to a broader class of nonlinear uncertain systems. Further characterizations and applications of FTISS are under investigation.

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