Systems & Control Letters 87 (2016) 64–69

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Global robust distributed output consensus of multi-agent nonlinear systems: An internal model approach[☆]

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ARTICLE INFO

Article history:
Received 4 November 2014
Received in revised form
22 August 2015
Accepted 2 November 2015
Available online 30 November 2015

Keywords:
Nonlinear systems
Multi-agent systems
Consensus
Distributed stabilization

ABSTRACT

This paper studies the problem of *global* robust distributed output consensus of heterogeneous leader–follower multi-agent nonlinear systems by general directed output interactions. For a class of minimum-phase single-input single-output nonlinear agents having unity relative degree, it is shown that the problem is solvable by an internal model approach under certain mild conditions. A Lyapunov function based output-feedback control law is developed by converting the global output consensus into a global distributed stabilization problem for an augmented network.

1. Introduction

Recently, the consensus problem of multi-agent systems has been studied increasingly from linear to nonlinear dynamic networks; for an overview, to name but a few, see Refs. [1–5] for linear agent networks and see also Refs. [6–14] for some up-to-date studies in the context of nonlinear networks. A number of effective approaches have been developed to cope with the wide range of multi-agent coordination control problems. Previous studies on nonlinear consensus control are usually developed for normal-form agents with stable zero dynamics, see Refs. [7,8,10,11]. Notably, Ref. [11] proposed an adaptive internal model approach to a controlled bidirected network, while Refs. [12,14] studied some scenarios of global nonlinear control of general directed agent networks, where the former is by state-feedback design for a heterogeneous network and the latter is by output-feedback design for networks of identical followers. Ref. [8] considered semi-global distributed control for heterogeneous diffusively coupled nonlinear networks in a strict-feedback normal form having unity relative degree. Moreover, Refs. [9,10] developed a cyclic small-gain theorem based approach to addressing global output consensus of multi-agent nonlinear systems. In particular, for heterogeneous uncertain networks with general directed interactions, Ref. [10] developed an interesting result of reaching time-invariant agreements by output-feedback control in a cyclic-small-gain framework. In the aforementioned result, some integral control and gain assignment techniques were developed to succeed the global output consensus to a reference set-point, which also guarantees a practical output consensus when non-vanishing external disturbances exist. In view of Ref. [10], it motivates us to study the more general problem of time-varying agreement from a Lyapunov function perspective.

In the paper, for uncertain nonlinear agents in a leader–follower heterogeneous network, the global output consensus problem is formulated as forcing each agent output globally asymptotically attracted by their respective output zeroing invariant manifold (see Eq. (7) in Section 2 in this paper). The synchronous output is actually indicated by its associated output zeroing manifold. In this task, available measurements are merely provided by neighboring output interactions. The scheme engaged in this paper is comprised of two steps. The first step is the design of internal models and formulation of a certain distributed stabilization problem of some

☆ A preliminary version of the paper was presented at the 11th World Congress on Intelligent Control and Automation. This work was supported in part by the NNSF of China under Grants No. 61304009, No. 61503359, No. 61333001, and No. 61403030, and in part by the NSF Grant ECCS-1230040.
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http://dx.doi.org/10.1016/j.sysconle.2015.11.002
0167-6911/

augmented networks. The second step is to solve the stabilization problem which in turn implies the global output consensus.

For heterogeneous and minimum-phase nonlinear agents transformable into a strict-feedback normal form, it is shown that the global distributed consensus problem can be approached when their information digraph contains a directed spanning tree with the leader node as the root. Based on a modified variable gradient method, a Lyapunov function based output-feedback design is further developed to achieve the global asymptotic output consensus. The main technical challenges of the problem are two-fold. First, we deal with *global* asymptotic output consensus of heterogeneous nonlinear networks. To make the problem tractable, we need to formulate a distributed stabilization problem which should guarantee solvability of the original consensus problem. Second, because the information digraph is generally directed and the controlled nonlinear agents are heterogeneous with uncertainties, it is challenging to develop Lyapunov functions and related consensus algorithms. To address these challenges, we utilize a modified variable gradient method to construct a storage function of a general form. Then we are able to establish consensus algorithms from an output regulation viewpoint (see Ref. [15]). The main contribution of this paper is summarized as follows. A unified treatment is presented for nonlinear consensus control and a solution is developed for global asymptotic consensus control of multi-agent nonlinear systems of a normal form in directed networks. Our study partially generalizes some existing results, e.g., Refs. [1,7,10,16], from several aspects, including general directed networks and time-varying agreements.

The rest of this paper is organized as follows. In Section 2, we formulate the global output consensus problem and link it to distributed output regulation theory. Then in Section 3 we present the main result of the paper and an example is also given. Section 4 closes the paper with some concluding remarks.

2. Consensus as a distributed output regulation problem

In this section, we describe the consensus problem that is formally formulated as distributed output regulation.

Consider a group of uncertain multi-agent nonlinear systems with each one being globally transformable into a strict-feedback normal form having unity relative degree (see, e.g., Refs. [8,17,18])

$$\begin{aligned} i \in \mathcal{O} : \ \dot{z}_i &= f_i(z_i, y_i, v, w) \\ \dot{y}_i &= g_i(z_i, y_i, v, w) + b_i(w)u_i. \end{aligned} \tag{1}$$

In combination with (1), a leader node with index 0 is described by

$$\dot{v} = Sv, \qquad y_0 = q_r(v, w), \quad v \in \mathbb{R}^{n_v}. \tag{2}$$

Here $\mathcal{O} := \{1, 2, \ldots, N\}$ is to denote the follower node set, and for each agent $i \in \mathcal{O}$, $(z_i, y_i) \in \mathbb{R}^{n_i} \times \mathbb{R}$ is the state viewed as dynamic uncertainties (see Ref. [19]), $y_i \in \mathbb{R}$ is the output, $u_i \in \mathbb{R}$ is the control input, $w \in \mathbb{W}$ is the parametric uncertainties in some compact set $\mathbb{W} \subset \mathbb{R}^{n_w}$ with $\dot{w} = 0$, and $y_0 \in \mathbb{R}$ is given as the desired output reference. The function $b_i := b_i(w) \neq 0$ is the uncertain high frequency gain, continuous in w with $b_i(w) > 0$ for all $w \in \mathbb{W}$. Assume that all the functions f_i, g_i, q_r are smooth in their arguments with $f_i(0, 0, 0, w) = 0$, $g_i(0, 0, 0, w) = 0$, $q_r(0, w) = 0$ for all $i \in \mathcal{O}$ and all $w \in \mathbb{W}$. We further pose a compact set $\mathbb{V} \subset \mathbb{R}^{n_v}$ such that for each initial value $v(0) \in \mathbb{V}$ for (2), the response $v(t)$ of (2) satisfies $v(t) \in \mathbb{V}$ for all $t \geq 0$. Assume that both $\mathbb{V}$ and $\mathbb{W}$ are fixed with known boundaries. For conciseness, we denote $\mu(t) := [v^\top(t), w]^\top$ and $\mathbb{D} := \mathbb{V} \times \mathbb{W}$ referred to as node uncertainties.

The cooperative control of multi-agent systems studied in this paper is based on local measurements

$$\begin{aligned} i \in \mathcal{O} : \ e_{mi} &= \sum_{j \in \mathcal{V}} a_{ij}(y_i - y_j) = \sum_{j \in \mathcal{V}} a_{ij}(e_i - e_j) \\ &\text{with } e_0 \equiv 0, \ e_i = y_i - y_0 \end{aligned} \tag{3}$$

dictated by an information digraph[1] $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $e := [e_1, \ldots, e_N]^\top$ is regarded as regulated output. Specifically, for the group of agents (1) and (2), we aim to find a controller of the form

$$i \in \mathcal{O} : \ \dot{\xi}_i = f_{ci}(\xi_i, e_{mi}), \qquad u_i = u_{ci}(\xi_i, e_{mi}) \tag{4}$$

such that, for each initial condition $(z_i(0), y_i(0), \xi_i(0))$ in their respective entire spaces and each $(v(0), w) \in \mathbb{V} \times \mathbb{W}$, the following two conditions are satisfied (i) the trajectory of the closed-loop system exists for all $t \geq 0$ and is bounded over the time interval $[0, +\infty)$; (ii) the regulated output $e(t)$ satisfies $\lim_{t \to +\infty} e(t) = 0$.

In this paper, the concerned information digraph $\mathcal{G}$ is further posed by the following condition.

Assumption 1. The digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ contains a directed spanning tree with the leader node as the root.

Remark 1. An immediate implication of Assumption 1 is as follows. For any information digraph $\mathcal{G}$, the resultant Laplacian $\mathcal{L}$ contains a submatrix $H \in \mathbb{R}^{N \times N}$ by removing its first row and column. Moreover, the resultant matrix $-H$ is Hurwitz and there exists a diagonal matrix $R = \text{diag}(r_1, \ldots, r_N)$ with positive entries such that, for a real number $\lambda_0 > 0$, the matrix[2]

$$\lambda_0 I - RH - H^\top R \tag{5}$$

is negative definite (see [20, Theorem 2.3, pp. 134]).

In contrast to "non-networked" output regulation problems (see Ref. [15] and references therein), the output consensus can be treated as distributed output regulation with a local measurement provided in distributed sensing.

To expand our investigation, we list the following standing assumptions assuring solvability of global output consensus problem. In particular, Assumption 2 ensures the (global) solvability of the regulator equations (see [21, Assumption 7.10]), while Assumption 3 is adopted from Ref. [22], that is often used in nonlinear control theory to suffice output-feedback design as a minimum phase or output-feedback passivity condition imposed on each follower; see [23, pp. 517 & 606]; cf. strict passivity of [23, Definition 6.3] and [8, Assumption 3].

Assumption 2. For each $i \in \mathcal{O}$, there is a smooth function $\mathbf{z}_i : \mathbb{D} \to \mathbb{R}^{n_i}$ such that

$$\frac{\partial \mathbf{z}_i}{\partial v} Sv = f_i(\mathbf{z}_i(\mu), q_r(\mu), \mu).$$

Under Assumption 2, we can obtain the individual zero-error constraint input manifolds (see [21, pp. 83]) for the network of (1) and (2)

$$\mathcal{U}_i = \left\{(u_i, \mu) : \ u_i = \mathbf{u}_i(\mu), \ \mu \in \mathbb{D}\right\}, \quad i \in \mathcal{O} \tag{6}$$

where

$$\mathbf{u}_i(\mu) = b_i^{-1} \frac{\partial q_r}{\partial v} Sv - b_i^{-1} g_i(\mathbf{z}_i(\mu), q_r(\mu), \mu).$$

Clearly, $\mathbf{u}_1(\mu), \ldots, \mathbf{u}_N(\mu)$ are *heterogeneous* and all *uncertain*. Moreover, the individual global output zeroing manifolds can be given as follows: for each $i \in \mathcal{O}$

$$\mathcal{P}_i = \left\{(z_i, y_i, \mu) : \ z_i = \mathbf{z}_i(\mu), \ y_i = q_r(\mu), \ \mu \in \mathbb{D}\right\} \tag{7}$$

where agent outputs synchronize, cf. [24, Remark 3.9] and [10, Remark 4]. The manifold (7) is regarded as the synchronous invariant manifold.

[1] See Appendix at the end of this paper for graph notation.

[2] I stands for an identity matrix of suitable dimensions.

Also, under Assumption 2 and the coordinates $\bar{z}_i = z_i - \mathbf{z}_i(\mu)$ and $e_i = y_i - y_0$, we can get a translation of (1)

$$i \in \mathcal{O}: \ \dot{\bar{z}}_i = \bar{f}_i(\bar{z}_i, e_i, \mu), \qquad \dot{e}_i = b_i u_i - b_i \mathbf{u}_i(\mu) + \bar{g}_i(\bar{z}_i, e_i, \mu) \quad (8)$$

with

$$\bar{f}_i(\bar{z}_i, e_i, \mu) = f_i(\bar{z}_i + \mathbf{z}_i(\mu), e_i + q_r(\mu), \mu) - f_i(\mathbf{z}_i(\mu), q_r(\mu), \mu)$$
$$\bar{g}_i(\bar{z}_i, e_i, \mu) = g_i(\bar{z}_i + \mathbf{z}_i(\mu), e_i + q_r(\mu), \mu) - g_i(\mathbf{z}_i(\mu), q_r(\mu), \mu).$$

Assumption 3. For each $i \in \mathcal{O}$, there is a C^1 (continuously differentiable) function $V_{\bar{z}_i} : \mathbb{R}^{n_i} \to \mathbb{R}$ such that

$$\underline{\alpha}_i(\|\bar{z}_i\|) \leq V_{\bar{z}_i}(\bar{z}_i) \leq \bar{\alpha}_i(\|\bar{z}_i\|)$$
$$\dot{V}_{\bar{z}_i}|_{(8)} \leq -\alpha_i(\|\bar{z}_i\|) + \phi_i(e_i)e_i^2$$

for $\alpha_i(\cdot), \underline{\alpha}_i(\cdot), \bar{\alpha}_i(\cdot) \in \mathcal{K}_\infty$ [3] and a smooth function $\phi_i(e_i) \geq 1$ with $\alpha_i(\cdot)$ being locally quadratic in the sense that $\limsup_{s\to 0^+}(\alpha_i^{-1}(s^2)/s) < \infty$.

3. Solution to distributed output consensus

This section is devoted to elaborating a global distributed regulator design to achieve the global output consensus of (1) by an internal model approach. Because the zero-error constraint inputs $\mathcal{U}_1$ to $\mathcal{U}_N$ shown in (6) are uncertain, the internal models as dynamic compensators are indispensable to further make each $\mathcal{P}_i$ specified by (7) globally attractive by a distributed stabilization design in the spirit of general problem conversion in [24].

3.1. Converting consensus to distributed stabilization

To do the problem conversion, we first consider the internal model design. Because the zero-error constraint inputs $\mathcal{U}_1$ to $\mathcal{U}_N$ shown in (6) are uncertain, the internal models as dynamic compensators are indispensable to further make each $\mathcal{P}_i$ specified by (7) globally attractive by a distributed stabilization design.

For internal model design, we suppose the following.

Assumption 4. Each of $\mathbf{u}_i(\mu) = \mathbf{u}_i(v, w), i \in \mathcal{O}$ is polynomial in v.

Under Assumption 4, it is known that for each $\mathbf{u}_i(\mu)$, by calculating its minimal zeroing polynomial (see *page* 170 of Ref. [21]), we have an s_i-th dimensional steady-state generator with output u_i as follows:

$$\dot{\tau}_i(\mu) = \Phi_i \tau_i(\mu), \qquad \mathbf{u}_i(\mu) = \Psi_i \tau_i(\mu). \quad (9)$$

Next, for each follower agent, a linear internal model can be constructed as follows:

$$\dot{\eta}_i = M_i \eta_i + G_i u_i \quad (10)$$

where (M_i, G_i) is a controllable matrix pair with M_i being Hurwitz. In fact, such an internal model can be obtained by writing the generator (9) as follows:

$$\dot{\tau}_i(\mu) = M_i \tau_i(\mu) + G_i \Psi_i \tau_i(\mu), \qquad \mathbf{u}_i(\mu) = \Psi_i \tau_i(\mu) \quad (11)$$

where $M_i = \Phi_i - G_i \Psi_i$ for a vector $G_i \in \mathbb{R}^{s_i}$ such that M_i is Hurwitz. Then the generator (11) immediately leads to the internal model (10). We refer the reader to [21, Chapter 6] for more details.

[3] $\mathcal{K}_\infty$ is the set of continuous, strictly increasing, and unbounded functions $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$.

Remark 2. Regarding Assumption 4 as well internal model design of (10), we shall note the following. On one hand, Assumption 4 is most popular in the study of nonlinear output regulation; see, e.g., Refs. [7,12]. On the other hand, it is worth noting that if a network like (1) and (2) is homogeneous, then its associated manifolds $\mathcal{U}_i$ as (6) satisfy $\mathbf{u}_i(\mu) = 0$ for all $i \in \mathcal{O}$ (e.g., Ref. [4]). In this case, there is no need to introduce an internal model (10) for each follower agent.

For the augmented networked dynamics composed of (8) and (10), under the following coordinate and input transformations:

$$i \in \mathcal{O}: \ \bar{\eta}_i = \eta_i - \tau_i - b_i^{-1} G_i e_i, \qquad \bar{u}_i = u_i - \Psi_i \eta_i$$

we obtain

$$i \in \mathcal{O}: \begin{cases} \dot{\bar{\eta}}_i &= M_i \bar{\eta}_i + \varphi_i(\bar{z}_i, e_i, \mu) \\ \dot{\bar{z}}_i &= \bar{f}_i(\bar{z}_i, e_i, \mu) \\ \dot{e}_i &= b_i \bar{u}_i + \tilde{g}_i(\bar{z}_i, \bar{\eta}_i, e_i, \mu) \end{cases} \quad (12)$$

where

$$\varphi_i(\bar{z}_i, e_i, \mu) = b_i^{-1} M_i G_i e_i - b_i^{-1} G_i \bar{g}_i(\bar{z}_i, e_i, \mu),$$
$$\tilde{g}_i(\bar{z}_i, \bar{\eta}_i, e_i, \mu) = b_i \Psi_i(\bar{\eta}_i + b_i^{-1} G_i e_i) + \bar{g}_i(\bar{z}_i, e_i, \mu).$$

Denote $\bar{z} := [z_1^\top, \ldots, z_N^\top]^\top$ and $\bar{\eta} := [\bar{\eta}_1^\top, \ldots, \bar{\eta}_N^\top]^\top$.

Remark 3. It can be shown that (12) has an equilibrium at $(\bar{z}, \bar{\eta}, e) = (0, 0, 0)$ for all $\mu \in \mathbb{D}$. If the equilibrium can be made globally asymptotically stable, it implies the global asymptotic attractiveness of $\mathcal{P}_1$ to $\mathcal{P}_N$ by (7). Thus we are left to solve a distributed stabilization problem, i.e., *find a stabilizer of the form*

$$i \in \mathcal{O}: \ \bar{u}_i = -k_i \kappa_i(e_{mi}) \quad (13)$$

for (12), where $k_i > 0$, $\kappa_i(s) = s\bar{\kappa}_i(s)$, and $\bar{\kappa}_i(\cdot) \geq 1$ is a smooth function to be designed, in order to globally asymptotically stabilize the equilibrium $(\bar{z}, \bar{\eta}, e) = (0, 0, 0)$.

The dynamics $(\bar{z}_i, \bar{\eta}_i)$ in (12) can be viewed as dynamic uncertainties (see Ref. [19]). The following lemma (see Ref. [22]) presents a useful property for the output-feedback stabilization of (12).

Lemma 1. *For each $i \in \mathcal{O}$, consider the $\zeta_i := (\bar{z}_i, \bar{\eta}_i)$ subsystem of (12). Then under Assumption 3, for any smooth function $\Delta_i(\zeta_i) \geq 1$, there exists a C^1 function $\tilde{V}_i(\zeta_i)$ such that, for all $\mu \in \mathbb{D}$,*

$$\underline{\alpha}'_i(\|\zeta_i\|) \leq \tilde{V}_i(\zeta_i) \leq \bar{\alpha}'_i(\|\zeta_i\|), \qquad \dot{\tilde{V}}_i|_{(12)} \leq -\Delta_i(\zeta_i)\|\zeta_i\|^2 + \gamma_i(e_i)e_i^2 \quad (14)$$

with some smooth functions $\underline{\alpha}'_i(\cdot), \bar{\alpha}'_i(\cdot) \in \mathcal{K}_\infty$ and a smooth function $\gamma_i(\cdot) \geq 1$.

Remark 4. We summarize a couple of notable Lyapunov function construction techniques in the literature. The first is for bidirected (or undirected) subgraph induced by the follower nodes only. In this case, the matrix H specified in Remark 1 is positive definite, which suggests a Lyapunov function candidate

$$\breve{V}(e) = \frac{1}{2} e^\top H e \quad (15)$$

to succeed an intermediate stabilization control, cf. [11, Equation (21)]. The second is the general interaction digraphs, it is still valid to employ a quadratic function

$$\breve{V}(e) = \frac{1}{2} e^\top P e, \quad P > 0 \text{ s.t. } PH + H^\top P = I \quad (16)$$

to accomplish semi-global control (see, e.g., [6] and construction of a Lyapunov function for χ_2 subsystem in [25, Equation (16)]). For the purpose of *semi-global control*, one may also refer to [6] for a viable two-layer network design for consensus problem with local actuating disturbances and [8] for an interesting result on approaching the consensus problem by introducing an auxiliary homogeneous network of N identical copies to produce a local reference. At this point, it is noted that both the intermediate stabilization methods in [8] are accomplished in a decentralized mode and the resultant controller orders are more than that of the proposed method in this paper. In view of the above analysis, for the present global distributed output consensus control, both (15) and (16) are not amenable any more. We need to introduce a general one as (18) to cope with *general digraphs* and *global* distributed *output-feedback control*.

To proceed and for convenience, denote that

$$e_m := [e_{m1}, \ldots, e_{mN}]^\top$$
$$\varpi(e_m) := [\varpi_1(e_{m1}), \ldots, \varpi_N(e_{mN})]^\top$$
$$\varpi_i(e_{mi}) := b_i k_i \kappa_i(e_{mi}), \quad \kappa_i(s) = s\bar{\kappa}_i(s)$$

for a design function $\bar{\kappa}_i(\cdot) \geq 1$, $i \in \mathcal{O}$. By utilizing a modified variable gradient method (see, e.g., [26, Section 3.5.3]), the following lemma establishes a heuristic result on the system

$$\dot{e} = -\varpi(e_m) \quad \text{or} \quad \dot{e}_m = -H\varpi(e_m) \tag{17}$$

induced from the closed-loop system composed of (12) and (13). It is of importance to note that it may fail to construct a Lyapunov function for (17) by applying the variable gradient method in a direct manner, because H is not symmetric and $-H\varpi(e_m)$ of (17) would be not a gradient vector (see [23, pp. 120]). Nonetheless, we can introduce a modified one

$$V(e_m) = \int_0^{e_m} \varpi^\top(s) R ds \tag{18}$$

where R is specified in Remark 1.

Lemma 2. *The function (18) is a positive definite and radially unbounded, and moreover satisfies*

$$\dot{V}|_{(17)} \leq -\lambda_0' \|\varpi\|^2$$

for a constant $\lambda_0' > 0$.

Proof. First, it can be shown that for each $i \in \mathcal{O}$,

$$\int_0^{e_{mi}} \varpi_i(s) r_i ds \geq \frac{1}{2} b_i k_i r_i e_{mi}^2$$
$$\int_0^{e_{mi}} \varpi_i(s) r_i ds \leq b_i k_i r_i \bar{\kappa}_i(e_{mi}) e_{mi}^2.$$

Thus $V(e_m)$ is positive definite and radially unbounded. Next, we have

$$\dot{V}|_{(17)} = -\varpi^\top RH\varpi \leq -\frac{1}{2}\lambda_0 \|\varpi\|^2$$

by using (5). Letting $\lambda_0' = \frac{1}{2}\lambda_0$ completes the proof.

3.2. Main theorem

Theorem 3. *For the network of (1) and (2) under Assumptions 1–4, the global output consensus problem can be solved by*

$$i \in \mathcal{O}: \ \dot{\eta}_i = M_i \eta_i + G_i u_i, \quad u_i = -k_i \kappa_i(e_{mi}) + \Psi_i \eta_i. \tag{19}$$

Proof. By Remark 3, we only need to solve the stabilization problem for (12). Consider the closed-loop system composed of (12) and (13). By Lemma 2, it can be seen that $V(e_m)$ by (18) satisfies

$$\dot{V}|_{(12)+(13)} \leq -\lambda_0' \|\varpi\|^2 + \sum_i r_i \varpi_i \Big(\sum_{j\in\mathcal{V}} a_{ij}(\tilde{g}_i - \tilde{g}_j)\Big) \tag{20}$$

where $\tilde{g}_0 \equiv 0$ and $\tilde{g}_i$ is to denote $\tilde{g}_i(\bar{z}_i, \bar{\eta}_i, e_i, \mu)$ for short.

In (20), completing the squares gives

$$r_i \varpi_i \Big(\sum_{j\in\mathcal{V}} a_{ij}(\tilde{g}_i - \tilde{g}_j)\Big) \leq \frac{\epsilon}{2}\varpi_i^2 + \mathbf{I}_i \tag{21}$$

and thus we have

$$\dot{V}(e_m)|_{(12)+(13)} \leq -\Big(\lambda_0' - \frac{\epsilon}{2}\Big)\|\varpi\|^2 + \sum_i \mathbf{I}_i \tag{22}$$

where

$$\mathbf{I}_i := \frac{r_i^2}{2\epsilon}\Big(\sum_{j\in\mathcal{V}} a_{ij}(\tilde{g}_i - \tilde{g}_j)\Big)^2.$$

Further by using *Lemma* 7.8 in Ref. [21] and completing the squares, we conclude the following. For each $i \in \mathcal{O}$, there are smooth functions $\Delta_i'(\zeta_i) \geq 1$ and $\varphi_{i0}(s)$ written by $\varphi_{i0}(s) = s\varphi_{i0}^*(s)$, $\varphi_{i0}^*(s) \geq 1$ such that

$$\sum_i \mathbf{I}_i \leq \frac{1}{2\epsilon}\sum_i (\Delta_i'(\zeta_i)\|\zeta_i\|^2 + \varphi_{i0}^2(e_{mi})). \tag{23}$$

Therefore, $V(e)$ satisfies

$$\dot{V}|_{(12)+(13)} \leq -\lambda_0'\|\varpi\|^2 + \frac{1}{2\epsilon}\sum_i \big(\Delta_i'(\zeta_i)\|\zeta_i\|^2 + \varphi_{i0}^2(e_{mi})\big).$$

Next, by Lemma 1, define

$$V_\zeta(\zeta) = \sum_i \tilde{V}_i(\zeta_i)$$

which satisfies

$$\dot{V}_\zeta|_{(12)} \leq -\sum_i \Delta_i(\zeta_i)\|\zeta_i\|^2 + \sum_i \gamma_i(e_i)e_i^2.$$

In the above inequality, it can be shown that there exist smooth functions $\varphi_{i1}(s)$, $i = 1, \ldots, N$ with each one written by $\varphi_{i1}(s) = s\varphi_{i1}^*(s)$, $\varphi_{i1}^*(s) \geq 1$, such that

$$\sum_i \gamma_i(e_i)e_i^2 \leq \sum_i \varphi_{i1}^2(e_{mi}).$$

Thus, $V_\zeta(\zeta)$ satisfies

$$\dot{V}_\zeta|_{(12)} \leq -\sum_i \Delta_i(\zeta_i)\|\zeta_i\|^2 + \sum_i \varphi_{i1}^2(e_{mi}). \tag{24}$$

Finally, we can define a Lyapunov function candidate for the closed-loop system as follows:

$$U(\zeta, e_m) = V_\zeta(\zeta) + V(e_m) \tag{25}$$

which satisfies

$$\tilde{\alpha}_1(\|\zeta, e_m\|) \leq U(\zeta, e_m) \leq \tilde{\alpha}_2(\|\zeta, e_m\|) \tag{26}$$

for some functions $\tilde{\alpha}_1(\cdot), \tilde{\alpha}_2(\cdot) \in \mathcal{K}_\infty$, and by using (20), (23) and (24)

$$\dot{U}|_{(12)+(13)} \leq -\big(\lambda_0' - \frac{\epsilon}{2}\big)\|\varpi\|^2 + \sum_i \Big\{\frac{1}{2\epsilon}\varphi_{i0}^2(e_{mi}) + \varphi_{i1}^2(e_{mi}) + \Big(\frac{1}{2\epsilon}\Delta_i'(\zeta_i) - \Delta_i(\zeta_i)\Big)\|\zeta_i\|^2\Big\}.$$

In the above, to make its right side non-positive, one can choose $\epsilon, \bar{k}_i > 1$ and $\Delta_i(\cdot), \psi_i(\cdot), \bar{\kappa}_i(\cdot) \geq 1$ such that, for all their arguments

$$\lambda_0' - \frac{\epsilon}{2} \geq \frac{\lambda_0'}{2}, \qquad \Delta_i(\zeta_i) \geq \frac{1}{2\epsilon}\Delta_i'(\zeta_i) + 1$$
$$\frac{\lambda_0'}{2} b_i^2 \bar{k}_i \psi_i(e_{mi}) e_{mi}^2 \geq \varphi_{i1}^2(e_{mi}) + \frac{1}{2\epsilon}\varphi_{i0}^2(e_{mi}),$$
$$k_i \geq \sqrt{\bar{k}_i}, \ \ \bar{\kappa}_i^2(e_{mi}) \geq \psi_i(e_{mi}) + 1. \tag{27}$$

Then the above leads to

$$\dot{U}|_{(12)+(13)} \leq -\frac{\lambda_0'}{2} \sum_i b_i^2 \left(k_i^2 \bar{\kappa}_i^2(e_{mi}) e_{mi}^2 - \bar{k}_i \psi_i(e_{mi}) e_{mi}^2\right) - \sum_i \|\zeta_i\|^2$$
$$\leq -\frac{\lambda_0'}{2} \sum_i b_i^2 e_{mi}^2 - \sum_i \|\zeta_i\|^2 .$$

Thus, we obtain $\tilde{\alpha}_3(\cdot) \in \mathcal{K}_\infty$ such that

$$\dot{U}|_{(12)+(13)} \leq -\tilde{\alpha}_3 (\|\zeta, e_m\|) . \tag{28}$$

From (26) and (28), by LaSalle–Yoshizawa Theorem (see [18, pp. 492]), the equilibrium $(\bar{z}, \bar{\eta}, e) = (0, 0, 0)$ is globally uniformly asymptotically stable. Hence, the proof is complete.

Remark 5. It is of interest to note that our approach can also handle the case of information digraph with the subgraph induced by the follower agents being bidirected. It may lead to more design freedom than the method based on (15). Recently, another effective technique was proposed in [27] to construct the function in (18), *i.e.*,

$$V(e_m) = \sum_{i=1}^{N} V_i(e_{mi}) = \sum_{i=1}^{N} r_i b_i k_i \int_0^{\bar{V}_i(e_{mi})} \gamma_i(s) ds$$
$$\text{with } \bar{V}_i(e_{mi}) = e_{mi}^2$$

for a smooth design function $\gamma_i(\cdot) \geq 1$. In this manner, the controller may be given by

$$\bar{u}_i = -\gamma_i(\bar{V}_i(e_{mi})) e_{mi}. \tag{29}$$

This method is called changing supply functions based design, see [27] for more details. Disregarding the difference of the agent relative degrees, the controller (13) may have a more simplified growth nonlinearity than (29) provided by the design in [27].

3.3. An example

For an illustration, we consider a consensus problem for a heterogeneous network of agents of the FitzHugh–Nagumo (FHN) dynamics (e.g., Refs. [14,28]), which can be described as follows:

$$\begin{cases} \dot{z}_i &= \varrho_{i1}(y_i - \varrho_{i2} z_i) \\ \dot{y}_i &= y_i(y_i - \varrho_{i3})(1 - y_i) - z_i + \delta_i(v, w) + b_i u_i \\ e_i &= y_i - y_0, \quad i = 1, \ldots, 4 \end{cases} \tag{30}$$

where $\varrho_{ij} = \bar{\varrho}_{ij} + w_{ij} > 0$, $j = 1, 2, 3$ are some positive real parameters undergoing uncertainties $w_{ij} \in \mathbb{R}$ for the nominal value $\bar{\varrho}_{ij}$, (z_i, y_i) is the state, $\delta_i(v, w)$ is the input disturbance, and $y_0 = v_1$ is the reference signal generated by a harmonic oscillator

$$\dot{v} = Sv, \quad S = \text{diag}(S_1, S_2), \qquad S_j = \begin{bmatrix} 0 & \omega_j \\ -\omega_j & 0 \end{bmatrix}, \quad j = 1, 2.$$

Denote $\varrho = (\varrho_{11}, \varrho_{12}, \ldots, \varrho_{42}, \varrho_{43})$. It can be shown that Assumptions 2 and 3 are verifiable for all agents.

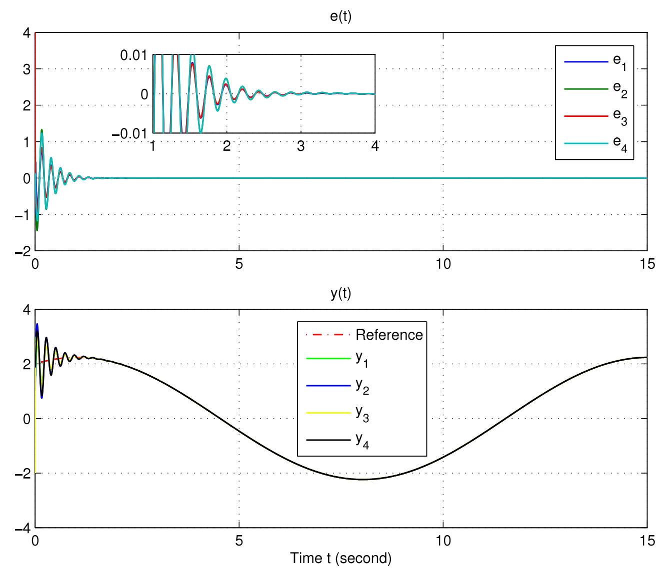

Fig. 1. Profiles of $e(t)$ and $y(t)$.

To do the simulation, we set $\omega_1 = \frac{\pi}{7}$, $\omega_2 = \frac{\pi}{10}$, and $\delta_1(v, w) = 0.5v_3$, $\delta_2(v, w) = 0.2v_4$, $\delta_3(v, w) = 0.5v_4$, $\delta_4(v, w) = 0.3v_3$. The steady-state generator (9) with $s_i = 9$ and Φ_i can be given as

$$\left[\begin{array}{c|c} 0 & I \\ \hline 0 & -\ell_4, 0, -\ell_3, 0, -\ell_2, 0, -\ell_1, 0 \end{array}\right].$$

where $\ell_1 = 14\omega_1^2 + \omega_2^2$, $\ell_2 = 49\omega_1^4 + 14\omega_1^2\omega_2^2$, $\ell_3 = 36\omega_1^6 + 49\omega_1^4\omega_2^2$, and $\ell_4 = 36\omega_1^6\omega_2^2$.

Then for each $i = 1, \ldots, 4$, we obtain the internal model (10) determined by the pair (M_i, G_i), where $M_i = \Phi_i - G_i\Psi_i$ and $G_i = [9\epsilon^{-1}, 36\epsilon^{-2}, 84\epsilon^{-3}, 126\epsilon^{-4}, 126\epsilon^{-5}, 84\epsilon^{-6}, 36\epsilon^{-7}, 9\epsilon^{-8}, \epsilon^{-9}]^\top$ and $\epsilon = 0.2$. It ensures that all M_i are Hurwitz. The information digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is chosen with $\mathcal{V} = \{0, 1, 2, 3, 4\}$ and $\mathcal{A} = [0, 0, 0, 0, 0;\ 1, 0, 1, 0, 0;\ 0, 0, 0, 1, 0;\ 1, 0, 0, 0, 1;\ 0, 0, 0, 1, 0]$. Each controller (13) is with $k_i = 25$ and $\bar{\kappa}_i(e_{mi}) = e_{mi}^6 + 3$. A simulation result is shown in Fig. 1 with $\varrho = (1, 1, 2, 2, 3, 1, 1, 2, 3, 3, 1, 1)$, $v(0) = (2, 1, 2, 3)$, the follower initial conditions are $(-2, 3)$, $(-3, 1)$, $(3, -2)$, $(-1, 2)$, respectively, and all the other initial values are set zero.

4. Conclusion

For a class of heterogeneous and minimum-phase nonlinear agents, a global distributed output consensus problem was studied by an internal model approach. A Lyapunov function based output-feedback protocol was proposed. It is interesting to note that the proposed approach is also applicable to bidirected networks.

Appendix. Graph notation

Denote the information digraph $\mathcal{G}$ associated with the leader and follower nodes by a triplet $\mathcal{G} := \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} := \{0, 1, 2, \ldots, N\}$ is the node set, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set (with self-loops excluded), and $\mathcal{A} = [a_{ij}]_{i,j=0,1,\ldots,N}$ is the weighted adjacency matrix of $(N + 1) \times (N + 1)$. An edge of $\mathcal{G}$ is denoted by an ordered pair of nodes $(j, i) \in \mathcal{E}$ with j being indicated as a neighbor of i. A directed path [29] of $\mathcal{G}$ is an ordered sequence of distinct nodes in $\mathcal{V}$ such that any consecutive nodes in the sequence correspond to an edge of the digraph. A node j is said to be connected to another node i if there is a directed path from j to i. $\mathcal{G}$ is said to contain a directed spanning tree if there is at least one node, called the root, connected to every other node. $\mathcal{A}$ is a nonnegative matrix and $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$. $\mathcal{L} = [l_{ij}]_{i,j=0,1,\ldots,N}$ with $l_{ii} = \sum_{j=0}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$ is called the Laplacian associated with $\mathcal{G}$.

References

[1] H. Grip, T. Yang, A. Saberi, A. Stoorvogel, Output synchronization for heterogeneous networks of non-introspective agents, Automatica 48 (10) (2012) 2444–2453.
[2] Y. Hong, J. Hu, L. Gao, Tracking control for multi-agent consensus with an active leader and variable topology, Automatica 42 (9) (2006) 1177–1182.
[3] H. Kim, H. Shim, J. Seo, Output consensus of heterogeneous uncertain linear multi-agent systems, IEEE Trans. Automat. Control 56 (1) (2011) 200–206.
[4] L. Scardovi, R. Sepulchre, Synchronization in networks of identical linear systems, Automatica 45 (11) (2009) 2557–2562.
[5] C. Wang, H. Ji, Leader-following consensus of multi-agent systems under directed communication topology via distributed adaptive nonlinear protocol, Systems Control Lett. 70 (2014) 23–29.
[6] X. Wang, D. Xu, Y. Hong, Consensus control of nonlinear leader–follower multi-agent systems with actuating disturbances, Systems Control Lett. 73 (2014) 58–66.
[7] Z. Ding, Consensus output regulation of a class of heterogeneous nonlinear systems, IEEE Trans. Automat. Control 58 (10) (2013) 2648–2653.
[8] A. Isidori, L. Marconi, G. Casadei, Robust output synchronization of a network of heterogeneous nonlinear agents via nonlinear regulation theory, IEEE Trans. Automat. Control 59 (10) (2014) 2680–2691.
[9] T. Liu, Z.P. Jiang, Distributed nonlinear control of mobile autonomous multi-agents, Automatica 50 (4) (2014) 1075–1086.
[10] T. Liu, Z.P. Jiang, Distributed output-feedback control of nonlinear multi-agent systems, IEEE Trans. Automat. Control 58 (11) (2013) 2912–2917.
[11] Y. Su, J. Huang, Cooperative adaptive output regulation for a class of nonlinear uncertain multi-agent systems with unknown leader, Systems Control Lett. 62 (6) (2013) 461–467.
[12] Y. Su, J. Huang, Cooperative global output regulation of heterogenous second-order nonlinear uncertain multi-agent systems, Automatica 49 (11) (2013) 3345–3350.
[13] P. Wieland, J. Wu, F. Allgöwer, On synchronous steady states and internal models of diffusively coupled systems, IEEE Trans. Automat. Control 58 (10) (2013) 2591–2602.
[14] D. Xu, Y. Hong, X. Wang, Distributed output regulation of nonlinear multi-agent systems via host internal model, IEEE Trans. Automat. Control 59 (10) (2014) 2784–2789.
[15] J. Huang, An overview of the output regulation problem, J. Systems Sci. Math. Sci. 31 (9) (2011) 1055–1081.
[16] J. Lunze, Synchronization of heterogeneous agents, IEEE Trans. Automat. Control 57 (11) (2012) 2885–2890.
[17] A. Isidori, Nonlinear Control Systems, third ed., Springer, London, 1995.
[18] M. Krstic, I. Kanellakopoulos, P.V. Kokotovic, Nonlinear and Adaptive Control Design, Wiley, New York, 1995.
[19] I. Karafyllis, Z.P. Jiang, Stability and Stabilization of Nonlinear Systems, Springer, London, 2011.
[20] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press Inc., New York, 1979.
[21] J. Huang, Nonlinear Output Regulation: Theory and Applications, SIAM, Philadelphia, 2004.
[22] D. Xu, J. Huang, Robust adaptive control of a class of nonlinear systems and its applications, IEEE Trans. Circuits Syst. I. Regul. Pap. 57 (3) (2010) 691–702.
[23] H.K. Khalil, Nonlinear Systems, third ed., Prentice Hall, New Jersey, 2002.
[24] J. Huang, Z. Chen, A general framework for tackling the output regulation problem, IEEE Trans. Automat. Control 49 (12) (2004) 2203–2218.
[25] Y. Su, J. Huang, Cooperative semi-global robust output regulation for a class of nonlinear uncertain multi-agent systems, Automatica 50 (4) (2014) 1053–1065.
[26] J.E. Slotine, W. Li, Applied Nonlinear Control, Prentice Hall, New Jersey, 1991.
[27] Y. Su, J. Huang, Cooperative global robust output regulation for lower triangular nonlinear multi-agent systems under directed graph, in: Proceedings of the 2014 American Control Conference, Portland, USA, June 4–6, 2014.
[28] G.S. Medvedev, N. Kopell, Synchronization and transient dynamics in the chains of electrically coupled FitzHugh–Nagumo oscillators, SIAM J. Appl. Math. 61 (5) (2001) 1762–1801.
[29] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, New York, 2001.