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Consensus control of nonlinear leader–follower multi-agent systems with actuating disturbances[☆]

Xinghu Wang[a], Dabo Xu[b,*], Yiguang Hong[a]

[a] Key Laboratory of Systems and Control, Institute of Systems Science, Chinese Academy of Sciences, Beijing 100190, China

[b] School of Automation, Nanjing University of Science & Technology, Nanjing 210094, China

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ABSTRACT

This paper studies a semi-global asymptotic consensus problem of nonlinear multi-agent systems with local actuating disturbances. For a modest nonlinear scenario, a consensus protocol is proposed based on a viable two-layer network. The consensus problem is treated as distributed output regulation, which is resolved by a joint decomposition of the zero-error constraint inputs and a configuration of a flexible internal model network. An illustrative example is also given to show the efficiency of the two-layer networked design.

1. Introduction

In the past few years, cooperative control of multi-agent systems has gained increasing research interest and a major development. One of the central problems is that of controlling all agents in order to make their outputs converge to a common output trajectory. This problem usually refers to as consensus or output synchronization. Since the fundamental study of consensus protocols for single-integrator agents (see [1]), a number of effective techniques have been proposed for multi-agent systems from linear to nonlinear in many directions, see for instances [2–7] and references therein. At the present stage, to the best of our knowledge, there is a lack of particular studies on consensus control with external disturbances appearing at individual agent dynamics. Relevant results may be found in [8–10]. In particular, [9,10] proposed an observer-based control by viewing the disturbances as exosystem outputs and [8] studied the problem by a non-smooth control technique. It is noted that the methods developed in the aforementioned literature rely on the absolute state information in addition to the neighbor information. Also, the result of [8] is a practical consensus design, not leading to asymptotic consensus. In view of these existing studies, consensus control with exogenous disturbances deserves further investigation, particularly for nonlinear multi-agent networks.

It is known that, to cope with asymptotic tracking and/or disturbance rejection of uncertain systems, in the terminology of robust output regulation, the device of internal model plays an indispensable role, see [11–16]. Recently, a lot of efforts have been made to distributed control of the leader–follower multi-agent systems with uncertainties by applying the output regulation theory; see, for instances [17–19]. In accordance with these results, an individual internal model should be embedded in each local controller to succeed the consensus with node uncertainties and disturbances. Recall that in output regulation, the exosystem is often used to model references and disturbances as well. In the usual centralized or decentralized setup, there is no need to treat them separately. Nonetheless, regarding multi-agent systems in the distributed fashion, things are basically inconsistent due to limited interactions or communications. To adapt this situation, the leader in leader–follower type networks can be viewed as an exosystem to produce references relating to certain collective behaviors. Contrary to references, local disturbances certainly have a negative effect to the control goal. These two types of signals thus have opposite effects with respect to the control goal. This fact basically motivates us to develop more flexible strategies that can manage

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* Corresponding author. Tel.: +86 25 8431 5872.
E-mail addresses: wxh@amss.ac.cn (X. Wang), dabo.xu@ymail.com, xu.dabo@gmail.com (D. Xu), yghong@iss.ac.cn (Y. Hong).

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them in some separable and general manner to realize the necessary input compensation for all controlled agents.

In the paper, we study an asymptotic consensus design for a class of locally disturbed homogeneous multi-agent systems; refer to [20,21] for consensus control of homogeneous networks in the absence of disturbances. Our main objective is to demonstrate how a networked internal model can be implemented in a more general network to solve the problem. In the problem, each controller is required to asymptotically reject its local disturbances, meanwhile to reach consensus as chief control goal. To do the task, we employ a novel configurable internal model network, distinguished from the output-interaction network. It is shown that some agents only need an internal model to reject its local disturbance. The new treatment still allows us to engage a scheme of converting the consensus problem into a tractable distributed stabilization problem of an augmented network. For an arbitrary initial region, the latter stabilization problem is further solved by a linear high-gain type distributed stabilizer.

The proposed design belongs to two-layer networked control in a broadened level than the usual distributed consensus control. Particular attentions will be paid to the links among local controllers. A relevant two-layer networked control was studied in [22]. Regarding the consensus protocol developed in the present study, three main reasons are summarized to explain why we do so. First, in most existing results (e.g. [17–19]), the internal model communication has not been carefully considered, especially when communication is "cheap". Our study at least leads to an interesting alternative. Second, by the internal model based consensus control, the controller network itself has a certain "consensus" property, since it contains internal models that are essentially "copies" to the controlled agents. This observation actually has evoked the interest of building up links among local controllers for the purpose of sharing useful information. Third, from the viewpoint of computing the consensus protocol, reducing its complexity certainly makes sense. In summary, our study can provide flexible strategies especially when control complexity matters and communication is comforted.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given. In Section 3, the configurable internal model network is elaborated and the main result is presented. In Section 4, an example is given for an illustration. Finally in Section 5, the paper is closed with some concluding remarks. The graph notations used throughout the paper are put in the Appendix.

Notations: throughout the paper, for any column vectors $x_1, \ldots, x_n$, $(x_1, \ldots, x_n)$ denotes $\left[x_1^\top, \ldots, x_n^\top\right]^\top$ if no confusion arises in the context; for a real number $\rho > 0$, $\mathcal{B}_\rho^x = \{x \in \mathbb{R}^n : \|x\| \le \rho\}$; for a real number $c > 0$ and a smooth positive definite and radially unbounded function $W : \mathbb{R}^n \to \mathbb{R}$, $\Omega_c(W) := \{x \in \mathbb{R}^n : W(x) \le c\}$.

2. Preliminaries

The focus of this paper is on leader-following consensus of nonlinear multi-agent systems with local disturbances, consisting of a leader and a set of controlled nonlinear uncertain systems. The leader with node index 0 is described by

$$\dot{v}_r = S_r v_r, \qquad y_0 = q(v_r, w) \tag{1}$$

where $v_r \in \mathbb{R}^{n_{v_r}}$ is the state and $y_0 \in \mathbb{R}$ is the desirable reference characterizing the collective output behavior. The follower agents are assumed to be globally transformable into the form

$$i \in \mathcal{O} : \quad \begin{cases} \dot{z}_i = f(z_i, y_i, w) \\ \dot{y}_i = g(z_i, y_i, w) + \delta_i(v_i, w) + u_i \end{cases} \tag{2}$$

where $\mathcal{O} := \{1, \ldots, N\}$ denotes the follower node set and for each agent $i \in \mathcal{O}$, $(z_i, y_i) \in \mathbb{R}^n$ is the state, $y_i \in \mathbb{R}$ is the output, $u_i \in \mathbb{R}$ is the control input, $w \in \mathbb{W}$ is the parameter uncertainty in a known compact set $\mathbb{W} \subset \mathbb{R}^{n_w}$ and $\delta_i(v_i, w)$ is the local actuating disturbance of agent i with v_i governed by some local disturbance source

$$\dot{v}_i = S_i v_i, \quad v_i \in \mathbb{R}^{n_{v_i}}. \tag{3}$$

It is noticed that the local disturbance sources and the leader are divided, different from the usual. We also assume that each of the matrices S_r and S_i, $i = 1, \ldots, N$ has distinct eigenvalues lying on the imaginary axis such that (1) and (3) can generate the fundamental sinusoidal/step type signals; cf. *Assumption* A2.2 in [14] and *Assumption* 3 in [16]. This type of disturbances have been studied in the framework of output regulation, see, for instances, [11,15]. The functions f, g, q and δ_i are polynomials in their arguments, satisfying

$f(0, 0, w) = 0, g(0, 0, w) = 0, q(0, w) = 0, \delta_i(0, w) = 0, \forall w \in \mathbb{W}.$

We denote $z := (z_1, \ldots, z_N) \in \mathbb{R}^{N(n-1)}$, $y := (y_1, \ldots, y_N) \in \mathbb{R}^N$.

Remark 1. The follower agent in (2) is called a strict-feedback uncertain nonlinear system having unity relative degree [23,24] that is basic and popular in nonlinear control, see, e.g. [25,17,18,26]. The consensus of this class of nonlinear multi-agent systems can cover interesting synchronization problems of a number of benchmark nonlinear oscillators, including Lorenz systems, FitzHugh–Nagumo (FHN) systems (that will be discussed in Section 4), etc., see [26].

Let $v = (v_r, v_1, \ldots, v_N)$ with initial condition $v(0)$ starting in a known compact region $\mathbb{V}$. Clearly, we have another compact set $\mathbb{V}'$ such that for any $v(0) \in \mathbb{V}$, its response $v(t) := v(t, v(0))$ satisfies $v(t) \in \mathbb{V}'$ for all $t \ge 0$. It is denoted that $\mathbb{D} := \mathbb{V}' \times \mathbb{W}$. Denote the regulated output $e = (e_1, \ldots, e_N)$ where

$$i \in \mathcal{O} : \quad e_i = y_i - y_0$$

which is usually unavailable to every follower agent. In practice, determined by an output-interaction graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with $\mathcal{V} = \{0, 1, 2, \ldots, N\}$ (turn to Appendix at the end of the paper for graph notations), agent i has a neighbor-based measurement e_{mi} as follows:

$$i \in \mathcal{O} : \quad e_{mi} = \sum_{j \in \mathcal{V}} a_{ij}(y_i - y_j) = \sum_{j \in \mathcal{V}} a_{ij}(e_i - e_j) \tag{4}$$

with $e_0 := 0$.

Besides the graph $\mathcal{G}$ (referred to as network sometimes in the paper), another graph $\mathcal{G}^c = \{\mathcal{O}, \mathcal{E}^c\}$, to denote an internal model network, is configured to indicate the internal model communications, to be elaborated in Section 3.1. One of the main objectives of the paper is to implement a communication network in a more extended distributed control setup than the single-layer one where only $\mathcal{G}$ is considered.

Specifically, we study a semi-global leader-following consensus problem of the systems (1) and (2) in the presence of the disturbance (3), formulated as follows. *For any graph $\mathcal{G}^c$ (as the internal model network admitting certain physical requirements, if any) and for any compact sets $\mathcal{B}_\rho^z$ and $\mathcal{B}_\rho^y$, find a smooth distributed controller*[1]

$$i \in \mathcal{O} : \quad \begin{cases} \dot{\xi}_i = h_{\xi i}(\xi_i, \xi_j, u_i, j \in \mathcal{N}_i^c) \\ u_i = u_{ci}(\xi_i, \xi_j, e_{mi}, j \in \mathcal{N}_i^c) \end{cases} \tag{5}$$

together with a specified compact set $\mathcal{B}_{\rho'}^{\xi'}$, $\xi' := (\xi_1, \ldots, \xi_N)$, such that, for each $(v(0), w, z(0), y(0), \xi'(0)) \in \mathcal{B}'$, $\mathcal{B}' := \mathbb{V} \times \mathbb{W} \times \mathcal{B}_\rho^z \times \mathcal{B}_\rho^y \times \mathcal{B}_{\rho'}^{\xi'}$, both the following conditions hold

(i) *the trajectory of the closed-loop system composed of (1), (2) and (5) exists for all $t \ge 0$ and is bounded over $[0, +\infty)$;*

[1] $\mathcal{N}_i^c := \{j \in \mathcal{O} : (j, i) \in \mathcal{E}^c\}$ means the neighbor set of agent i in the graph $\mathcal{G}^c$.

(ii) *every output y_i synchronizes with the reference y_0 at the infinity, i.e.*, $\lim_{t\to\infty} e(t) = 0$.

The above problem is formulated as distributed output regulation for leader–follower networks. It leads to a general time-varying agreement. A relevant study can be found in [27] for a disturbance-free network, where a solution with $\mathcal{G}^c$ containing a single host agent (see Section 3.1 for the definition) has been addressed. In the rest of this section, we list some basic assumptions as those in [19,27].

Assumption 1. The output-interaction digraph $\mathcal{G}$ contains a directed spanning tree with the leader as the root.

Assumption 2. There exists a smooth function $\mathbf{z}(v_r, w)$, which is polynomial in v_r and satisfies $\mathbf{z}(0, w) = 0$, such that

$$\frac{\partial \mathbf{z}(v_r, w)}{\partial v_r} S_r v_r = f(\mathbf{z}(v_r, w), q(v_r, w), w), \quad (v_r, w) \in \mathbb{D}. \tag{6}$$

Under Assumption 2, performing the coordinate transformation $\bar{z}_i = z_i - \mathbf{z}(v_r, w)$ gives a translated inverse dynamics

$$\dot{\bar{z}}_i = \bar{f}(\bar{z}_i, e_i, v, w) \tag{7}$$

where $\bar{f}(\bar{z}_i, e_i, v, w) = f(\bar{z}_i + \mathbf{z}(v_r, w), e_i + q(v_r, w), w) - f(\mathbf{z}(v_r, w), q(v_r, w), w)$. To ensure stabilizability of an augmented network (18) in the next section, we need another condition imposed on (7).

Assumption 3. There is a smooth function $V(\bar{z}_i)$ such that

$$\underline{\alpha}(\|\bar{z}_i\|) \le V(\bar{z}_i) \le \overline{\alpha}(\|\bar{z}_i\|)$$
$$\frac{\partial V(\bar{z}_i)}{\partial \bar{z}_i} \bar{f}(\bar{z}_i, 0, v, w) \le -\varepsilon_1 \|\bar{z}_i\|^2, \quad \forall (v, w) \in \mathbb{D}$$

for some smooth functions $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot) \in \mathcal{K}_\infty$ and a real number $\varepsilon_1 > 0$.

Remark 2. At this point, we note that Assumption 1 is a standing as well as necessary condition that is imposed on the interaction graph for multi-agent system control, e.g. see *Assumption* 6 in [19]. Assumptions 2 and 3 are most often used in nonlinear output regulation. In particular, Assumption 2 ensures the global solvability of the so-called regulator equations (REs); see *Remark* 3.10 and *Assumption* 7.10 in [13], *Assumption* 4 in [16], and also *Assumption* 3 in [19]. Assumption 3 restricts the agent (2) to some minimum-phase systems, which is used to suffice the output feedback control; cf. *Assumption* 6 in [16] and *Assumption* 5 in [19].

To examine Assumption 2, an effective way is by using the power series approach, referring to *Chapter* 4.3 of [13] for details. According to *Lemma* 4.12 and *Lemma* 4.13 of [13], Assumption 2 can be verified by two steps. First, consider the condition: *for each $w \in \mathbb{W}$, none of eigenvalues of the matrix $\frac{\partial f}{\partial z_i}(0, 0, w)$ coincides with any $\lambda \in \Lambda_l$, $l = 1, 2, \ldots$, with*

$$\Lambda_l := \{\lambda : \lambda = l_1\lambda_1 + \cdots + l_{n_{v_r}}\lambda_{n_{v_r}},\ l_1 + \cdots + l_{n_{v_r}} = l,$$
$$l_1, \ldots, l_{n_{v_r}} = 0, 1, 2, \ldots, l\} \tag{8}$$

where $\lambda_1, \ldots, \lambda_{n_{v_r}}$ are the eigenvalues of the matrix S_r. If so, by *Lemma* 4.12 and *Lemma* 4.13 in [13], a unique series solution to Eq. (6) can be obtained with $\mathbf{z}(v_r, w) = \sum_{l\ge 1} Z_l(w) v_r^{[l]}$ with $Z_l(w)$ satisfying a certain Sylvester matrix equation (specified by *Equation* 4.63 in [13]) and $v_r^{[l]}$ being defined as (see p. 118 of [13])

$$v_r^{[l]} := (v_{r1}^l, v_{r1}^{l-1}v_{r2}, \ldots, v_{r1}^{l-1}v_{rn_{v_r}}, v_{r1}^{l-2}v_{r2}^2, v_{r1}^{l-2}v_{r2}v_{r3}, \ldots,$$
$$v_{r1}^{l-2}v_{r2}v_{rn_{v_r}}, \ldots, v_{rn_{v_r}}^l)$$

where $v_r := (v_{r1}, \ldots, v_{rn_{v_r}})$. Second, examine the existence of an integer $l_0 \ge 1$ so that $Z_l(w) = 0$, $\forall l = l_0, l_0+1, l_0+2, \ldots$. If l_0 exists, an exact solution satisfying Assumption 2 can be obtained, see *Remarks* 4.14 and 4.15 of [13]. To show an example, note that both Assumptions 2 and 3 are verifiable for the system (2) with the following manageable condition: *The nonlinearities of (2) satisfy*

$$f(z_i, y_i, w) = Az_i + f_0(y_i, w), \qquad g(z_i, y_i, w) = Bz_i + g_0(y_i, w)$$

where A and B are matrices of appropriate dimension with A being Hurwitz, and functions f_0 and g_0 are polynomials in y_i. Indeed, the above condition has been used in [17] for a multi-agent systems control problem and the 2nd order FHN systems are exactly of this case, see [28,27].

Also note that as direct consequences of Assumption 2, by solving the associated REs of agents (1) and (2), it gives

$$i \in \mathcal{O}: \quad \mathbf{u}_i(v, w) = \mathbf{u}_{ri}(v_r, w) + \mathbf{u}_{di}(v_i, w) \tag{9}$$

where $\mathbf{u}_{di}(v_i, w) = -\delta_i(v_i, w)$, $i = 1, \ldots, N$ and $\mathbf{u}_{r1}(v_r, w) = \cdots = \mathbf{u}_{rN}(v_r, w) = \mathbf{u}_r(v_r, w)$ with

$$\mathbf{u}_r(v_r, w) = \frac{\partial q(v_r, w)}{\partial v_r} S_r v_r - g(\mathbf{z}(v_r, w), q(v_r, w), w).$$

Clearly, $\mathbf{u}_r(v_r, w)$ and $\mathbf{u}_{di}(v_i, w)$ are polynomials in v_r and v_i, respectively. Consequently, according to the *Proposition* in [29], $\mathbf{u}_r(v_r, w)$ and $\mathbf{u}_{di}(v_i, w)$, $i \in \mathcal{O}$ have the following minimal zeroing polynomials

$$p_r(\lambda) = \lambda^{s_r} - \varrho_{r1} - \varrho_{r2}\lambda - \cdots - \varrho_{rs_r}\lambda^{s_r-1}$$
$$p_i(\lambda) = \lambda^{s_i} - \varrho_{i1} - \varrho_{i2}\lambda - \cdots - \varrho_{is_i}\lambda^{s_i-1} \tag{10}$$

for some real numbers $\varrho_{r1}, \ldots, \varrho_{rs_r}$ and $\varrho_{i1}, \ldots, \varrho_{is_i}$.

3. Main result

In this section, we elaborate a solution to the consensus problem. In particular, we demonstrate how to attach a configurable internal model network to the output-interaction network to constitute a general two-layer network; see Fig. 1 for a fast grasp of the network topology.

3.1. Configuration of internal model network

Here we aim to introduce a configurable internal model network to cope with the leader-following consensus in the presence of actuating disturbances. The internal model network $\mathcal{G}^c = \{\mathcal{O}, \mathcal{E}^c\}$ is to describe the information communication among the internal models. Let us start by giving the description of the internal model network concerned in the paper.

Definition 1. An internal model network $\mathcal{G}^c = \{\mathcal{O}, \mathcal{E}^c\}$ is a digraph satisfying the following conditions:

(i) $\mathcal{O}$ is the union of two disjoint subsets $\mathcal{H}$ and $\mathcal{H}'$ with $\mathcal{H}$ nonempty, i.e., $\mathcal{O} = \mathcal{H} \cup \mathcal{H}'$ and $\mathcal{H} \cap \mathcal{H}' = \emptyset$ ($\emptyset$ denotes the empty set);
(ii) For each $i \in \mathcal{H}$, there exists no $j \in \mathcal{O}$ satisfying $(j, i) \in \mathcal{E}^c$;
(iii) For each $i \in \mathcal{H}'$, there is a unique $j \in \mathcal{H}$ satisfying $(j, i) \in \mathcal{E}^c$ and no $j \in \mathcal{H}'$ satisfying $(j, i) \in \mathcal{E}^c$.

Throughout the paper, for $i \in \mathcal{H}'$, we denote $\jmath_i$ as the index such that $(\jmath_i, i) \in \mathcal{E}^c$.

By Definition 1, the agent with its index belonging to the set $\mathcal{H}$ is called the host agent, and its associated internal model is called host internal model. It will be shown later that for an internal model network $\mathcal{G}^c$, each host internal model can produce a jointly decomposable output. All the jointly decomposable outputs are distributed along with $\mathcal{G}^c$. In particular, each agent in $\mathcal{H}'$ only needs an internal model to cope with its local actuating disturbances.

The resultant topology of the closed-loop system consists of two layers; see Fig. 1 again. What makes our design interesting is

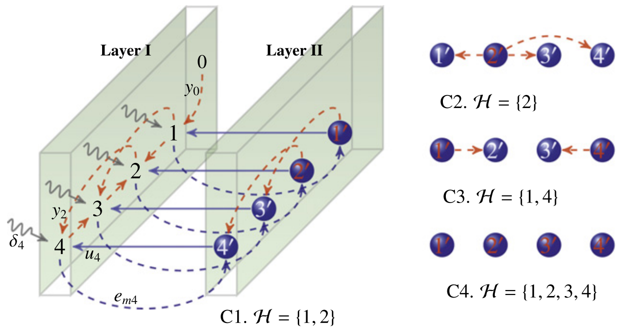

Fig. 1. Illustration of two-layer networked design and internal model network candidates. ($0, 1, \ldots, 4$ are plant nodes; $1', \ldots, 4'$ are controller nodes with the red colored to indicate host internal models. Layer I is the output-interaction network $\mathcal{G}$ with $\mathcal{V} = \{0, 1, 2, 3, 4\}$ and $\mathcal{E} = \{(0, 1), (1, 2), (1, 3), (2, 4), (3, 2), (4, 3)\}$; Layer II is the designed internal model network.) (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

two-fold. One is that $\mathcal{G}^c$ is introduced that can be different from $\mathcal{G}$. The other is that the information transmission along with $\mathcal{G}^c$ is non-trivial, i.e., their steady-state values are generally persistent. The host agent set $\mathcal{H}$ can be an arbitrary non-empty subset of $\mathcal{O}$, which results in two extreme situations. One is $\mathcal{H} = \mathcal{O}$ and $\mathcal{E}^c = \emptyset$, which means that an internal model needs to be implemented for each controller without communications, e.g. the candidate C4 in Fig. 1. This configuration of internal model network has been studied in [18,19]. The other one is $\mathcal{H} = \{\jmath\}$ and $\mathcal{E}^c = \{(\jmath, i) \in \jmath \times \mathcal{O} : i \in \mathcal{O}, i \neq \jmath\}$ for some index $\jmath \in \mathcal{O}$, e.g. C2 in Fig. 1 with $\jmath = 2$. In the latter case, our configuration reduces to the case of a host internal model approach discussed in [27] in the absence of local disturbances. As a result, our configuration of internal model network indicates a notable design freedom and the choice of the internal model network is generally not unique if there is no particular physical requirement.

Remark 3. The internal model network can be judiciously selected to fulfill certain physical communication requirements. It manifests that $\mathcal{G}^c$ can be a subgraph of $\mathcal{G}$, e.g. C1 in Fig. 1, $\mathcal{E}^c \subset \mathcal{E}$. In this situation, for each agent $i \in \mathcal{O}$, the local controller for agent i is only based on the neighboring information of agent i in the sense of output-interaction graph $\mathcal{G}$. Thus, the designed controller is a distributed one. Notice that, in addition to the neighbor-based measurement e_{mi}, using information exchanges among controller states has been issued in [20,21] along with the same output-interaction digraph $\mathcal{G}$ that is essentially to assist the stabilization design. Again, it is emphasized that the communications were used for the purpose of stabilization in the aforementioned results. Here they are however used for the networked internal model construction.

Keeping the above notations in mind, we now consider the internal model construction that is networked along with a fixed digraph. To fulfill this construction, we need to characterize the common information to be shared and communicated in the internal model network. For this purpose, we first present a useful lemma.

Lemma 1. *For the network (1) and (2) under Assumption 2, suppose that each pair of polynomials (p_r, p_i), $i \in \mathcal{O}$ is generically coprime. Then for each $i \in \mathcal{O}$, there is an observable steady-state generator*

$$\dot{\tau}_i(v, w) = \Phi_i^c \tau_i(v, w), \qquad \mathbf{u}_i(v, w) = \Psi_i^c \tau_i(v, w) \tag{11}$$

with output u_i (in the sense of Definition 6.2 in [13]), being jointly decomposable in the following sense:

$$\forall i \in \mathcal{O} : \quad \exists\, \Gamma_i \in \mathbb{R}^{1\times(s_r+s_i)} \text{ s.t. } \mathbf{u}_r(v_r, w) = \Gamma_i \tau_i(v, w). \tag{12}$$

Proof. To show the lemma, from polynomials in (9), define $\tau_r(v_r, w) = (\tau_{r1}(v_r, w), \ldots, \tau_{rs_r}(v_r, w))$ and $\tau_{di}(v_i, w) = (\tau_{di1}(v_i, w), \ldots, \tau_{dis_i}(v_i, w))$, $i \in \mathcal{O}$ which can be given by

$$\tau_{r1}(v_r, w) = \mathbf{u}_r(v_r, w)$$

$$\tau_{rj}(v_r, w) = \frac{\partial \tau_{r,j-1}(v_r, w)}{\partial v_r} S_r v_r, \quad j = 2, \ldots, s_r$$

and for each $i \in \mathcal{O}$

$$\tau_{di1}(v_i, w) = \mathbf{u}_{di}(v_i, w)$$

$$\tau_{dij}(v_i, w) = \frac{\partial \tau_{di,j-1}(v_i, w)}{\partial v_i} S_i v_i, \quad j = 2, \ldots, s_i.$$

Then, we have

$$\dot{\tau}_r(v_r, w) = \Phi_r \tau_r(v_r, w), \qquad \mathbf{u}_r(v_r, w) = \Psi_r \tau_r(v_r, w)$$
$$\dot{\tau}_{di}(v_i, w) = \Phi_i \tau_{di}(v_i, w), \qquad \mathbf{u}_{di}(v_i, w) = \Psi_i \tau_{di}(v_i, w)$$

where

$$\Phi_r = \left[\begin{array}{c|c} 0 & I_{s_r-1} \\ \hline \varrho_{r1} & \varrho_{r2}, \ldots, \varrho_{rs_r} \end{array}\right]_{s_r\times s_r}, \quad \Psi_r = \begin{bmatrix} 1 & 0_{1\times(s_r-1)} \end{bmatrix}$$

$$\Phi_i = \left[\begin{array}{c|c} 0 & I_{s_i-1} \\ \hline \varrho_{i1} & \varrho_{i2}, \ldots, \varrho_{is_i} \end{array}\right]_{s_i\times s_i}, \quad \Psi_i = \begin{bmatrix} 1 & 0_{1\times(s_i-1)} \end{bmatrix}.$$

Recalling (9), defining $\tau_i(v, w) = (\tau_r(v_r, w), \tau_{di}(v_i, w))$ leads to the steady-state generator (11) with

$$\Phi_i^c = \begin{bmatrix} \Phi_r & 0 \\ 0 & \Phi_i \end{bmatrix}, \qquad \Psi_i^c = \begin{bmatrix} \Psi_r & \Psi_i \end{bmatrix}.$$

Thus, letting

$$\Gamma_i = \begin{bmatrix} \Psi_r & 0_{1\times s_i} \end{bmatrix}$$

which is independent of $w \in \mathbb{W}$, confirms (12). Finally, the observability of the generator (11) can be verified by *Lemma* 6.17 in [13] together with the fact that the pair of polynomials $p_r(\lambda)$ and $p_i(\lambda)$ is coprime, which completes the proof.

Remark 4. Lemma 1 means that the steady-state generators of the form (11) have a non-empty *intersection*, given by

$$\dot{\tau}_r(v_r, w) = \Phi_r \tau_r(v_r, w), \qquad \mathbf{u}_r(v_r, w) = \Psi_r \tau_r(v_r, w). \tag{13}$$

We refer to *Definition* 3.1 in [30] for a notion of system intersections, which can be adopted to understand the forthcoming networked steady-state generator. Roughly speaking, to reach consensus, each agent needs to embed a dynamic compensator (called internal model) to reproduce this common signal. Thus, all the internal models share a common information, which leads to a repeat of the internal model dynamics in a partial manner for all agents. In view of (13), we find interest in constructing some networked steady-state generator which can consequently facilitate some networked internal model design. In this way, by introducing a suitable networked internal model, it is possible to realize distributed compensation and therefore avoid possible repeats in local internal model design. The above observation motivates the materials in the rest of this subsection.

Based on the joint decomposition of (12), we can obtain a networked steady-state generator (cf. [22]) and a networked internal model both specified by $\mathcal{G}^c$, described by the following lemma.

Lemma 2. *Suppose that all conditions in Lemma 1 are satisfied. For any digraph $\mathcal{G}^c$ specified by Definition 1, there is a networked* *steady-state generator described by*

$$i \in \mathcal{O}: \quad \begin{cases} \dot{\tau}_i'(v, w) = \Phi_i' \tau_i'(v, w) \\ \psi_i\big(\mathbf{u}_i(v, w), \tau_{J_i}'(v, w)\big) = \Psi_i' \tau_i'(v, w) \end{cases} \tag{14}$$

where (Φ_i', Ψ_i') *is an observable matrix pair, and* ψ_i *is a smooth function satisfying, for some matrix* Γ_{J_i}'

$$\psi_i\big(\mathbf{u}_i(v, w), \tau_{J_i}'(v, w)\big) = \begin{cases} \mathbf{u}_i(v, w) & \text{when } i \in \mathcal{H} \\ \mathbf{u}_i(v, w) - \Gamma_{J_i}' \tau_{J_i}'(v, w) & \text{when } i \in \mathcal{H}'. \end{cases} \tag{15}$$

Furthermore, each agent has an internal model confined to $\mathcal{G}^c$*, described by*

$$i \in \mathcal{O}: \quad \dot{\eta}_i = M_i \eta_i + Q_i \psi_i(u_i, \eta_{J_i}) \tag{16}$$

for a controllable pair (M_i, Q_i) *with* M_i *being Hurwitz.*

Proof. To show (14), we need to find $\tau_i'(v, w)$ and the pair (Ψ_i', Φ_i') for $i \in \mathcal{O}$, and Γ_{J_i}' for $i \in \mathcal{H}'$. As in the proof of Lemma 1, let

$$\begin{aligned} &\tau_i'(v, w) = \tau_i(v, w), \quad \Phi_i' = \Phi_i^c, \quad \Psi_i' = \Psi_i^c, \quad i \in \mathcal{H} \\ &\tau_i'(v, w) = \tau_{di}(v, w), \quad \Phi_i' = \Phi_i, \quad \Psi_i' = \Psi_i, \\ &\Gamma_{J_i}' = \Gamma_{J_i}, \quad i \in \mathcal{H}'. \end{aligned}$$

Clearly, (Ψ_i', Φ_i') is observable. Then the networked steady-state generator (14) specified by graph $\mathcal{G}^c$ can be obtained.

To show the internal model design, put the steady-state generator (14) in the following form

$$\begin{aligned} &\dot{\tau}_i'(v_i, w) = M_i \tau_i'(v_i, w) + Q_i \psi_i(\mathbf{u}_i(v, w), \tau_{J_i}'(v, w)) \\ &\psi_i(\mathbf{u}_i(v, w), \tau_{J_i}'(v, w)) = \Psi_i' \tau_i'(v_i, w) \end{aligned} \tag{17}$$

where

$$M_i = \Phi_i' - Q_i \Psi_i'$$

with Q_i being chosen such that M_i is Hurwitz, which can be ensured by the observability of the pair (Ψ_i', Φ_i'). Then an internal model with output $\psi_i(u_i, \eta_{J_i})$ of the form (16) follows immediately. The proof is complete.

Remark 5. From the definition of ψ_i by (15), for each host agent J, the output $\Gamma_J' \eta_J$ is produced to compensate the steady-state information $\mathbf{u}_r(v_r, w)$ to achieve the leader-following consensus. Then it can be transmitted to some other follower agents, along with a directed path from J to them, specified by $\mathcal{G}^c$.

Remark 6. It should be noted that the networked steady-state generator (14) can be given in some other form. For example, let

$$T_i = \begin{bmatrix} \Psi_i^c \\ \Psi_i^c \Phi_i^c \\ \cdots \\ \Psi_i^c (\Phi_i^c)^{s_r + s_i - 1} \end{bmatrix}.$$

Since (11) is observable, T_i is nonsingular. Thus, let

$$\begin{aligned} &\tau_i'(v, w) = T_i \tau_i(v, w), \quad \Phi_i' = T_i \Phi_i^c T_i^{-1}, \\ &\Psi_i' = \Psi_i^c T_i^{-1}, \quad i \in \mathcal{H} \\ &\tau_i'(v, w) = \tau_{di}(v, w), \quad \Phi_i' = \Phi_i, \quad \Psi_i' = \Psi_i, \\ &\Gamma_{J_i}' = \Gamma_{J_i} T_{J_i}^{-1}, \quad i \in \mathcal{H}' \end{aligned}$$

which leads to another networked steady-state generator candidate with (Ψ_i', Φ_i') taking Brunovsky observable normal form.

With the designed internal model network, we have an augmented network defined by attaching (16) to (2). In the spirit of [14], we next derive a tractable augmented dynamics and achieve the problem conversion. To this end, let

$$i \in \mathcal{O}: \quad \begin{cases} \bar{\eta}_i = \eta_i - \tau_i'(v, w) - Q_i e_i \\ e_i = y_i - q(v_r, w) \\ \bar{u}_i = u_i - \Gamma_{J_i}' \eta_{J_i} - \Psi_i' \eta_i \end{cases}$$

with $\Gamma_{J_i}' := 0$ for $i \in \mathcal{H}$. Then it gives a (translated) augmented network

$$i \in \mathcal{O}: \quad \begin{cases} \dot{\bar{z}}_i = \bar{f}(\bar{z}_i, e_i, v, w) \\ \dot{\bar{\eta}}_i = M_i \bar{\eta}_i + \varphi_i(\bar{z}_i, \bar{\eta}_{J_i}, e_i, e_J, v, w) \\ \dot{e}_i = \breve{g}_i(\bar{z}_i, \bar{\eta}_i, \bar{\eta}_{J_i}, e_i, e_{J_i}, v, w) + \bar{u}_i \end{cases} \tag{18}$$

where

$$\varphi_i(\bar{z}_i, \bar{\eta}_{J_i}, e_i, e_{J_i}, v, w) = \begin{cases} M_i Q_i e_i - Q_i \bar{g}(\bar{z}_i, e_i, v, w) & \text{when } i \in \mathcal{H} \\ M_i Q_i e_i - Q_i \Gamma_{J_i}' (\bar{\eta}_{J_i} + Q_{J_i} e_{J_i}) \\ \quad - Q_i \bar{g}(\bar{z}_i, e_i, v, w) & \text{when } i \in \mathcal{H}' \end{cases}$$

$$\breve{g}_i(\bar{z}_i, \bar{\eta}_i, \bar{\eta}_{J_i}, e_i, e_{J_i}, v, w) = \begin{cases} \bar{g}(\bar{z}_i, e_i, v, w) + \Psi_i'(\bar{\eta}_i + Q_i e_i) & \text{when } i \in \mathcal{H} \\ \bar{g}(\bar{z}_i, e_i, v, w) + \Gamma_{J_i}' (\bar{\eta}_{J_i} + Q_{J_i} e_{J_i}) \\ \quad + \Psi_i'(\bar{\eta}_i + Q_i e_i) & \text{when } i \in \mathcal{H}' \end{cases}$$

$$\begin{aligned} \bar{g}(\bar{z}_i, e_i, v, w) \ = \ & g\big(\bar{z}_i + \mathbf{z}(v_r, w), e_i + q(v_r, w), w\big) \\ & - g(\mathbf{z}(v_r, w), q(v_r, w), w). \end{aligned}$$

It can be shown that the unforced augmented dynamics (18) has an equilibrium at

$$(\bar{z}, \bar{\eta}, e) = (0, 0, 0), \quad \forall (v, w) \in \mathbb{D}$$

with $\bar{z} := (\bar{z}_1, \ldots, \bar{z}_N)$ and $\bar{\eta} := (\bar{\eta}_1, \ldots, \bar{\eta}_N)$.

Remark 7. In this remark, we show that the consensus problem can be converted into a tractable stabilization problem of the augmented network (18). Note that, by subadditivity property of norm

$$\begin{aligned} &\|\bar{z}_i\| \le \|z_i\| + \|\mathbf{z}(v_r, w)\| \\ &\|\bar{\eta}_i\| \le \|\eta_i\| + \|\tau_i'(v_i, w)\| + \|Q_i\|\big(|y_i| + |q(v_r, w)|\big) \\ &|e_i| \le |y_i| + |q(v_r, w)|. \end{aligned}$$

Since $\mathbf{z}(v_r, w)$, $q(v_r, w)$, $\tau_r(v_r, w)$ and $\tau_i(v_i, w)$ are bounded over $(v, w) \in \mathbb{D}$, for any compact sets $\mathcal{B}_\rho^z$, $\mathcal{B}_\rho^y$, $\mathcal{B}_{\rho'}^\eta$ with $\eta := (\eta_1, \ldots, \eta_N)$, there exists a real number $\rho^* > 0$ such that, for each $(z(0), y(0), \eta(0))$ subject to

$$\|z(0)\| \le \rho, \quad \|y(0)\| \le \rho, \quad \|\eta(0)\| \le \rho'$$

it follows

$$\|\bar{z}(0)\| \le \rho^*, \quad \|\bar{\eta}(0)\| \le \rho^*, \quad \|e(0)\| \le \rho^*.$$

Thus, by a general property of *Corollary* 7.4 in [13], the semi-global leader-following consensus problem of (1) and (2) in the presence of the disturbance (3) can be solved if we can solve the following problem: *for any digraph* $\mathcal{G}^c$ *(specified by Definition 1), and any compact sets* $\mathcal{B}_\rho^{\bar{z}}$, $\mathcal{B}_\rho^{\bar{\eta}}$ *and* $\mathcal{B}_\rho^e$*, we can find a distributed stabilizer*

$$i \in \mathcal{O}: \quad \bar{u}_i = -k_i e_{mi}, \ k_i > 0 \tag{19}$$

for (18) such that the equilibrium $(\bar{z}, \bar{\eta}, e) = (0, 0, 0)$ *of the closed-loop system composed of (18) and (19) is uniformly asymptotically stable with a basin of attraction containing* $\mathcal{B}_\rho^{\bar{z}} \times \mathcal{B}_\rho^{\bar{\eta}} \times \mathcal{B}_\rho^e$. The latter problem is called the semi-global distributed stabilization of the augmented network (18). As a result, we are left to solve the semi-global stabilization problem to solve the original consensus problem.

Remark 8. In (18), by a direct application of *Lemma* 7.8 in [13], we have, for all $(v, w) \in \mathbb{D}$,

$$\begin{aligned}&\left\|\varphi_i(\bar z_i, \bar\eta_{J_i}, e_i, e_{J_i}, v, w)\right\|^2\\ &\le \begin{cases} p_{i1}(\bar z_i)\,\|\bar z_i\|^2 + p_{i2}(e_i)e_i^2 & \text{when } i\in\mathcal{H}\\ p_0(\left\|\bar\eta_{J_i}\right\|^2 + e_{J_i}^2) & \\ \quad + p_{i1}(\bar z_i)\,\|\bar z_i\|^2 + p_{i2}(e_i)e_i^2 & \text{when } i\in\mathcal{H}'\end{cases}\\ &|\breve g_i(\bar z_i, \bar\eta_i, \bar\eta_{J_i}, e_i, e_{J_i}, v, w)|^2\\ &\le \begin{cases} p_0\,\|\bar\eta_i\|^2 + p_{i3}(\bar z_i)\,\|\bar z_i\|^2 + p_{i4}(e_i)e_i^2 & \text{when } i\in\mathcal{H}\\ p_0(\|\bar\eta_i\|^2 + \left\|\bar\eta_{J_i}\right\|^2 + e_{J_i}^2) & \\ \quad + p_{i3}(\bar z_i)\,\|\bar z_i\|^2 + p_{i4}(e_i)e_i^2 & \text{when } i\in\mathcal{H}'\end{cases}\end{aligned} \tag{20}$$

for a real number $p_0 > 0$ and smooth functions $p_{ij}(\cdot) \ge 1$, $j = 1, 2, 3, 4$, $i = 1, \ldots, N$. The above property is useful in the proof of Proposition 1 to be given later.

3.2. Distributed stabilization of augmented network

Before stating the stabilization solution, as preparations, we show some useful dissipation properties for (18). First for the $\bar z_i$ subsystem with $V(\bar z_i)$ by Assumption 3, let

$$V_{\bar z}(\bar z) = \sum_{i\in\mathcal{O}} V(\bar z_i)$$

which manifests the following

$$\underline{\alpha}'(\|\bar z\|) \le V_{\bar z}(\bar z) \le \overline{\alpha}'(\|\bar z\|)$$
$$\sum_{i\in\mathcal{O}} \frac{\partial V_{\bar z}(\bar z)}{\partial \bar z_i}\bar f(\bar z_i, 0, v, w) \le -\varepsilon_1\|\bar z\|^2$$

for some smooth functions $\underline{\alpha}'(\cdot), \overline{\alpha}'(\cdot) \in \mathcal{K}_\infty$ and the real number $\varepsilon_1 > 0$ being specified by Assumption 3.

Next, for each $i \in \mathcal{O}$, consider the $\bar\eta_i$ subsystem. Because each M_i is Hurwitz, define

$$V_{\bar\eta_i}(\bar\eta_i) = \bar\eta_i^\top P_{1i}\bar\eta_i$$

where the matrix P_{1i} is positive definite satisfying $P_{1i}M_i + M_i^\top P_{1i} = -2I$. It can be seen that

$$\begin{aligned}\dot V_{\bar\eta_i}|_{(18)} &= -2\,\|\bar\eta_i\|^2 + 2\bar\eta_i^\top P_{1i}\varphi_i(\bar z_i, \bar\eta_J, e_i, e_J, v, w)\\ &\le \begin{cases} -\,\|\bar\eta_i\|^2 + \|P_{1i}\|^2\left(p_{i1}(\bar z_i)\,\|\bar z_i\|^2 + p_{i2}(e_i)e_i^2\right)\\ \quad \text{when } i\in\mathcal{H}\\ -\,\|\bar\eta_i\|^2 + \|P_{1i}\|^2\left(p_0(\left\|\bar\eta_{J_i}\right\|^2 + e_{J_i}^2)\right.\\ \quad \left.+\,p_{i1}(\bar z_i)\,\|\bar z_i\|^2 + p_{i2}(e_i)e_i^2\right)\\ \quad \text{when } i\in\mathcal{H}'.\end{cases}\end{aligned}$$

Thus, by letting

$$V_{\bar\eta}(\bar\eta) = \sum_{i\in\mathcal{H}} \bar p_0 V_{\bar\eta_i} + \sum_{i\in\mathcal{H}'} V_{\bar\eta_i},$$
$$\bar p_0 := p_0 \sum_{j\in\{j\in\mathcal{H}' : (i,j)\in\mathcal{E}^c\}} \|P_{1j}\|^2 + 1$$

we have

$$\dot V_{\bar\eta}|_{(18)} \le -\,\|\bar\eta\|^2 + \bar p_1(\bar z)\,\|\bar z\|^2 + \bar p_2(e)\,\|e\|^2$$

where

$$\bar p_1(\bar z) = \sum_{i\in\mathcal{H}} \bar p_0\,\|P_{1i}\|^2\,p_{i1}(\bar z_i) + \sum_{i\in\mathcal{H}'} \|P_{1i}\|^2\,p_{i1}(\bar z_i)$$
$$\bar p_2(e) = \sum_{i\in\mathcal{H}} \bar p_0\,\|P_{1i}\|^2\,p_{i2}(e_i) + \sum_{i\in\mathcal{H}'} \|P_{1i}\|^2\,(p_0 + p_{i2}(e_i)).$$

With above preparations, we are ready to derive a stabilizability result for the augmented network (18).

Proposition 1. *Suppose that the network (18) satisfies Assumptions 1 and 3. Then, for any digraph $\mathcal{G}^c$ specified by Definition 1 and for any compact sets $\mathcal{B}_\rho^{\bar z}$, $\mathcal{B}_\rho^{\bar\eta}$, $\mathcal{B}_\rho^{e}$, there exist a real number $k > 0$ (depending on ρ) and a distributed controller of the following form*

$$i \in \mathcal{O}: \quad \bar u_i = -ke_{mi} \tag{21}$$

such that the equilibrium point $(\bar z, \bar\eta, e) = (0, 0, 0)$ of the closed-loop system composed of (18) and (21) is uniformly asymptotically stable with a basin of attraction containing $\mathcal{B}_\rho^{\bar z} \times \mathcal{B}_\rho^{\bar\eta} \times \mathcal{B}_\rho^{e}$.

Proof. First consider the composite e subsystem. Because all the eigenvalues of matrix H (defined in the Appendix) have positive real parts, the matrix equation $P_2H + H^\top P_2 = I$ admits a positive definite matrix P_2. Let

$$V_e(e) = e^\top P_2 e.$$

Then, substituting (21) in (18) and using (20) gives

$$\begin{aligned}\dot V_e|_{(18)+(21)} &= -2ke^\top P_2He + 2e^\top P_2\left[\breve g_1, \ldots, \breve g_N\right]^\top\\ &= -k\,\|e\|^2 + 2e^\top P_2\left[\breve g_1, \ldots, \breve g_N\right]^\top\\ &\le -k\,\|e\|^2 + p_0'\,\|\bar\eta\|^2 + \bar p_3(\bar z)\,\|\bar z\|^2 + \bar p_4(e)\,\|e\|^2\end{aligned}$$

where

$$p_0' = 2Np_0, \qquad \bar p_3(\bar z) = \sum_{i\in\mathcal{O}} p_{i3}(\bar z_i)$$
$$\bar p_4(e) = \sum_{i\in\mathcal{O}} p_{i4}(e_i) + Np_0 + \|P_2\|^2.$$

Because each of $V_{\bar z}, V_{\bar\eta}, V_e$ is positive definite and radially unbounded, for any $\mathcal{B}_\rho^{\bar z}$, $\mathcal{B}_\rho^{\bar\eta}$ and $\mathcal{B}_\rho^{e}$, there exists a real number $\bar\rho > 0$ such that

$$\mathcal{B}_\rho^{\bar z} \subset \Omega_{\bar\rho}(V_{\bar z}), \qquad \mathcal{B}_\rho^{\bar\eta} \subset \Omega_{\bar\rho}(V_{\bar\eta}), \qquad \mathcal{B}_\rho^{e} \subset \Omega_{\bar\rho}(V_e).$$

Clearly, each of $\Omega_{\bar\rho}(V_{\bar z}), \Omega_{\bar\rho}(V_{\bar\eta}), \Omega_{\bar\rho}(V_e)$ is compact set. Consequently, by *Lemma* 7.8 in [13], there exist a real number $c_1 > 0$ and a smooth function $\varpi(e) \ge 1$ such that, for all $\bar z \in \Omega_{3\bar\rho}(V_{\bar z})$

$$\sum_{i\in\mathcal{O}} \left\|\frac{\partial V_{\bar z}(\bar z)}{\partial \bar z_i}\right\| \le c_1\|\bar z\|$$
$$\sum_{i\in\mathcal{O}} \left\|\bar f(\bar z_i, e_i, v, w) - \bar f(\bar z_i, 0, v, w)\right\| \le \varpi(e)\,\|e\|$$
$$\bar p_1(\bar z) \le c_1, \qquad \bar p_3(\bar z) \le c_1.$$

Furthermore, letting

$$\varepsilon_1' = \frac{1}{2}\varepsilon_1, \qquad \varpi'(e) = \frac{c_1^2}{2\varepsilon_1}\varpi^2(e)$$

yields

$$\dot V_{\bar z}|_{(18)} \le -\varepsilon_1'\,\|\bar z\|^2 + \varpi'(e)\,\|e\|^2, \quad \bar z \in \Omega_{3\bar\rho}(V_{\bar z}),\ e \in \mathbb{R}^N.$$

Next, consider the following Lyapunov function candidate

$$W(\bar z, \bar\eta, e) = V_{\bar z}(\bar z) + c_2V_{\bar\eta}(\bar\eta) + c_2c_3V_e(e)$$

where

$$c_2 = \min\left\{1, \frac{\varepsilon_1'}{2c_1(1+c_3)}\right\}, \qquad c_3 = \min\left\{1, \frac{1}{2p_0'}\right\}.$$

Clearly, $\Omega_{3\bar\rho}(W)$ is a compact set satisfying

$$\Omega_{3\bar\rho}(W) \subset \Omega_{3\bar\rho}(V_{\bar z}) \times \Omega_{3\bar\rho'}(V_{\bar\eta}) \times \Omega_{3\bar\rho''}(V_e)$$

with $\bar\rho' = \frac{\bar\rho}{c_2}$ and $\bar\rho'' = \frac{\bar\rho}{c_2c_3}$. Accordingly, there exists a real number $c_4 > 0$ such that

$$\varpi'(e) + c_2\bar p_2(e) + c_2c_3\bar p_4(e) \le c_4, \quad \forall e \in \Omega_{3\bar\rho''}(V_e).$$

As a result, on $\Omega_{3\bar{\rho}}(W)$, we have

$$\dot{W}|_{(18)+(21)} \le -\frac{\varepsilon_1'}{2}\|\bar{z}\|^2 - \frac{c_2}{2}\|\bar{\eta}\|^2 - \left(c_2c_3k - c_4\right)\|e\|^2 .$$

Choose

$$k^* = \frac{c_4+1}{c_2c_3}.$$

Then, for each $k \ge k^*$, on $\Omega_{3\bar{\rho}}(W)$, we have

$$\dot{W}|_{(18)+(21)} \le -\frac{\varepsilon_1'}{2}\|\bar{z}\|^2 - \frac{c_2}{2}\|\bar{\eta}\|^2 - \|e\|^2 .$$

Because

$$\mathcal{B}_\rho^{\bar{z}} \times \mathcal{B}_\rho^{\bar{\eta}} \times \mathcal{B}_\rho^{e} \subset \Omega_{\bar{\rho}}(V_{\bar{z}}) \times \Omega_{\bar{\rho}}(V_{\bar{\eta}}) \times \Omega_{\bar{\rho}}(e) \subset \Omega_{3\bar{\rho}}(W)$$

for each initial condition $(\bar{z}(0), \bar{\eta}(0), e(0)) \in \mathcal{B}_\rho^{\bar{z}} \times \mathcal{B}_\rho^{\bar{\eta}} \times \mathcal{B}_\rho^{e}$, $(\bar{z}(t), \bar{\eta}(t), e(t)) \in \Omega_{3\bar{\rho}}(W)$.

Moreover, we have $\lim_{t\to\infty}(\bar{z}(t), \bar{\eta}(t), e(t)) = (0, 0, 0)$. The proof is thus complete.

By Remark 7 and Proposition 1, we state the main result of the paper.

Theorem 3. *Suppose that the network (1) and (2) satisfies Assumptions 1–3 and each pair of polynomials (p_r, p_i), $i \in \mathcal{O}$ is generically coprime. Then, for any digraph $\mathcal{G}^c$ specified by Definition 1 and for any real numbers $\rho, \rho' > 0$, there exist a real number $k > 0$ (depending on ρ and ρ') and a distributed controller of the form*

$$i \in \mathcal{O}: \quad \begin{cases} \dot{\eta}_i = M_i\eta_i + Q_i\psi_i(u_i, \eta_{J_i}) \\ u_i = -ke_{mi} + \Gamma_{J_i}'\eta_{J_i} + \Psi_i'\eta_i \end{cases} \tag{22}$$

such that, for each initial condition $(z(0), y(0), \eta(0)) \in \mathcal{B}_\rho^z \times \mathcal{B}_\rho^y \times \mathcal{B}_{\rho'}^\eta$, with $\eta := (\eta_1, \dots, \eta_N)$, the trajectory of the closed-loop system composed of (1), (2) and (22) exists and is bounded over $[0, +\infty)$ and meanwhile, $\lim_{t\to\infty} e(t) = 0$.

Remark 9. Note that when $\delta_i(v_i, w) = 0$, $i = 1, \dots, N$, i.e., the local disturbances vanish, we have $\mathbf{u}_i(v, w) = \mathbf{u}_r(v_r, w)$. In this situation, focusing on the internal model network and in comparison with the single-host internal model studied in [27], the proposed networked design is extended in two directions. First, the network candidates are carefully presented in the consensus design. Second, local actuating disturbances are issued in the plant network. Also note that the recent result of [19] proposed an applicable approach to solving a general setting of the problem by constructing a separate internal model for each agent without internal model communications (e.g. C4 in Fig. 1). Indeed, [19] is applicable to the present problem. By [19], we note that the resultant order of the composite internal models is $Ns_r + \sum_{i=1}^N s_i$, whereas the dimension of the composite one consisting of (16) is $|\mathcal{H}|s_r + \sum_{i=1}^N s_i$. Here $|\mathcal{H}|$ denotes the number of host agents in set $\mathcal{H}$. Thus by contrast, on the one hand, a notable reduction of controller order is reached, especially when N and s_r (defined in (9)) are large and $|\mathcal{H}|$ is small. On the other hand, to reduce the computation complexity of the consensus protocol, it is fair to note that certain additional communication costs come out when transmitting the signals $\Gamma_J'\eta_J, J \in \mathcal{H}$.

To close this section, we summarize the proposed protocol by Theorem 3 as follows. If the assumptions are verifiable, one can perform the following procedure to reach at (22).

Step 1: Set the host agent set $\mathcal{H}$ and the internal model network $\mathcal{G}^c$ (in line with Definition 1), taking care of physical requirements or similar others on controller network requirements, if any;

Step 2: Construct the internal model of the form (16) for each agent by Lemma 2;

Step 3: Convert the problem into a stabilization problem of the network (18) by Remark 7;

Step 4: Design the distributed controller (21) according to the proof of Proposition 1 and obtain (22).

4. Example

For an illustration, we consider a consensus design of a group of FHN type agents with local actuating disturbances, adopted from [26]. The interaction graph $\mathcal{G}$ is specified in Fig. 1 with $N = 4$ and unity weights which verifies Assumption 1. The controlled agents are described by

$$\begin{cases} \dot{x}_{i1} = x_{i1} - \frac{1}{3}x_{i1}^3 - x_{i2} + x_{i3} + F_i(t) + F_{ui} \\ \dot{x}_{i2} = \sigma_5(x_{i1} + \sigma_1 - \sigma_2 x_{i2}) \\ \dot{x}_{i3} = \sigma_6(-x_{i1} + \sigma_3 - \sigma_4 x_{i3}), \quad i = 1, \dots, 4 \end{cases} \tag{23}$$

where $x_i = (x_{i1}, x_{i2}, x_{i3})$ is the state, F_{ui} is the control input, $\sigma(w) = (\sigma_1, \dots, \sigma_6)$ is a parameter vector undergoing some uncertainties with $\sigma_2, \sigma_4, \sigma_5, \sigma_6$ being positive entries, i.e. $\sigma = \bar{\sigma} + w$ with $\bar{\sigma}$ being the nominal value and $w \in \mathbb{R}^6$ being uncertainties, $F_i(t) = A_{mi}\sin(\omega_i t + \phi_i) + d_i$ is the local disturbance in control channel with unknown (A_{mi}, ϕ_i, d_i). For simplicity, we assume $\sigma_1 = \sigma_3 = 0$. Then by letting

$$z_{i1} = x_{i2}, \qquad z_{i2} = x_{i3}, \qquad y_i = x_{i1}, \qquad u_i = F_{ui}, \quad i = 1, \dots, 4$$

we can write (23) in the form of (2) as follows:

$$\begin{cases} \dot{z}_{i1} = \sigma_5(y_i - \sigma_2 z_{i1}) \\ \dot{z}_{i2} = \sigma_6(-y_i - \sigma_4 z_{i2}) \\ \dot{y}_i = y_i - \frac{1}{3}y_i^3 - z_{i1} + z_{i2} + F_i(t) + u_i, \quad i = 1, \dots, 4 \end{cases} \tag{24}$$

with $(z_i, y_i) = (z_{i1}, z_{i2}, y_i)$. The leader (1) with node index 0 can be formulated by a harmonic oscillator

$$\dot{v}_{r1} = \omega_r v_{r2}, \qquad \dot{v}_{r2} = -\omega_r v_{r1}, \qquad \omega_r > 0$$

with $v_r = (v_{r1}, v_{r2})$ being its state and $y_0 = v_{r1}$ being the desired reference. Defining the following differential equation

$$\dot{v}_{i1} = \omega_i v_{i2}, \qquad \dot{v}_{i2} = -\omega_i v_{i1}, \qquad \dot{v}_{i3} = 0, \quad \omega_i > 0$$

with $v_i = (v_{i1}, v_{i2}, v_{i3})$ and $v_i(0) = (A_{mi}\sin\phi_i, A_{mi}\cos\phi_i, d_i)$, it can be seen that $F_i(t) = v_{i1} + v_{i3}$.

By solving the REs, we have $\mathbf{u}_i(v, w) = \mathbf{u}_r(v_r, w) + \mathbf{u}_{di}(v_i, w)$ with

$$\mathbf{z}(v_r, w) = \begin{bmatrix} \mathbf{z}_1(v_r, w) \\ \mathbf{z}_2(v_r, w) \end{bmatrix} = \begin{bmatrix} \dfrac{\sigma_5^2\sigma_2}{\omega_r^2 + \sigma_5^2\sigma_2^2} v_{r1} - \dfrac{\omega_r\sigma_5}{\omega_r^2 + \sigma_5^2\sigma_2^2} v_{r2} \\ -\dfrac{\sigma_6^2\sigma_4}{\omega_r^2 + \sigma_6^2\sigma_4^2} v_{r1} + \dfrac{\omega_r\sigma_6}{\omega_r^2 + \sigma_6^2\sigma_4^2} v_{r2} \end{bmatrix}$$

$$\mathbf{u}_r(v_r, w) = -v_{r1} + \omega_r v_{r2} + \frac{1}{3}v_{r1}^3 + \mathbf{z}_1(v_r, w) - \mathbf{z}_2(v_r, w)$$

$$\mathbf{u}_{di}(v_i, w) = -v_{i1} - v_{i3}$$

which verifies Assumption 2. Because $\sigma_5\sigma_2 > 0$ and $\sigma_6\sigma_4 > 0$, Assumption 3 is also satisfied. Moreover, the minimal zeroing polynomials in (9) are given with

$$p_r(\lambda) = \lambda^4 + 10\omega_r^2\lambda^2 + 9\omega_r^4$$

$$p_1(\lambda) = \lambda^2 + \omega_1^2, \quad p_2(\lambda) = \lambda^2 + \omega_2^2$$

$$p_3(\lambda) = \lambda^2 + \omega_3^2, \quad p_4(\lambda) = \lambda(\lambda^2 + \omega_4^2).$$

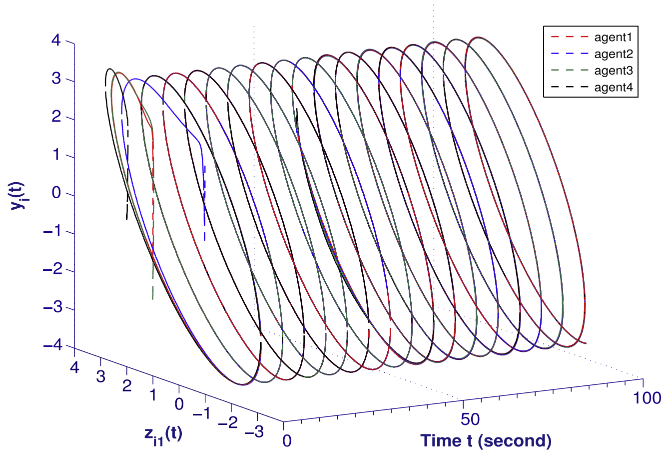

Fig. 2. 3D plot of agent responses with axis (t, z_{i1}, y_i).

In addition, Assumption 3 is also verifiable. It is concluded that the problem can be solved by Theorem 3.

To do the simulation, we choose $(\omega_r, \omega_1, \omega_2, \omega_3, \omega_4) = (\frac{\pi}{3}, \frac{\pi}{5}, \frac{\pi}{7}, \frac{\pi}{10}, \frac{\pi}{8})$ which ensures the coprime condition in Theorem 3. Choose the network $\mathcal{G}^c$ specified by C1 in Fig. 1 and in line with (16) to design the internal models. We place the poles of M_1 and M_2 at $\{-1 \pm \imath, -2 \pm \imath, -3 \pm \imath\}$, M_3 at $\{-2, -3\}$, and M_4 at $\{-1, -2, -3\}$. Choose $(\sigma_2, \sigma_4, \sigma_5, \sigma_6) = (1, 1, 1, 1)$ and the agent initial values $\{(2, 3), (1, 0, -1), (-1, 1, 2), (1, 0, -2), (2, -1, 0)\}$. The disturbances are $\{F_1(t); \ldots; F_4(t)\} = \{3\sin(\omega_1 t); 2\sin(\omega_2 t); 2\sin(\omega_3 t); \sin(\omega_4 t) - 5\}$ for $0 \leq t \leq 50$. To test the efficiency of the networked internal model, the disturbances vary at $t = 50$ to be $\{F_1(t); \ldots; F_4(t)\} = \{35\sin(\omega_1 t); 50\sin(\omega_2 t); 60\sin(\omega_3 t); 40\sin(\omega_4 t) + 20\}$. All the other initial conditions are set to be zero. A simulation result is shown in Fig. 2 with $k = 80$ chosen as the stabilizer gain to ensure the numerical performance. From the simulation, it is observed that the consensus can still be achieved under the same controller even when a vast variation of the disturbance parameters was made.

5. Conclusion

We have studied the leader-following consensus problem with an asymptotic rejection of actuating disturbances. A two-layer networked design was proposed by configuring an internal model network in addition to the output-interaction network. The consensus control problem was solved in the framework of distributed output regulation. Finally, an illustrative example was given to confirm the efficiency of the proposed two-layer networked design.

Appendix. Graph notation

The interaction among the leader and follower nodes can be described by an output-interaction (directed) graph $\mathcal{G} := \{\mathcal{V}, \mathcal{E}\}$ where $\mathcal{V} := \{0, 1, 2, \ldots, N\}$ is the node set, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. A subgraph of a graph $\mathcal{G}$ is a graph $\mathcal{G}_s = \{\mathcal{V}_s, \mathcal{E}_s\}$ such that $\mathcal{V}_s \subset \mathcal{G}$, $\mathcal{E}_s \subset \mathcal{E}$. An edge of $\mathcal{G}$ from node j to node i is denoted by the ordered pair (j, i), where node j is said to be a neighbor of node i. A directed path from node i_1 to node i_k is an ordered sequence of distinct nodes $i_1, \ldots, i_k \in \mathcal{V}$ such that the pairs $(i_1, i_2), \ldots, (i_{k-1}, i_k)$ belong to the edge set. A node i is said to be reachable from another node j if there is a directed path from node j to node i. $\mathcal{G}$ is said to contain a directed spanning tree if there is at least one node, called the root, from which every other node is reachable. The weighted adjacency matrix $\mathcal{A} = [a_{ij}]_{i,j=0,1,\ldots,N}$ of the graph $\mathcal{G}$ is a nonnegative matrix satisfying $a_{ii} = 0$ and $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$, $i \neq j$. Associated with adjacency matrix $\mathcal{A}$, the Laplacian of $\mathcal{G}$ is defined by $\mathcal{L} = [l_{ij}]_{i,j=0,1,\ldots,N}$ with $l_{ii} = \sum_{j\in\mathcal{V}} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$. Note that self-loops and edges from followers to the leader are of no interest to our problem. Let H denote the $N \times N$ submatrix obtained by removing the first row and column of the Laplacian $\mathcal{L}$ of $\mathcal{G}$. Note that under Assumption 1, all the eigenvalues of H have positive real parts. We refer to [32] for more details on graph notations used in the paper.

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