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Opinion evolution analysis for short-range and long-range Deffuant–Weisbuch models

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HIGHLIGHTS

- We propose two multi-choice models to generalize the conventional DW models.
- We prove the almost sure convergence of the SMDW model.
- We prove the almost sure consensus of the LMDW model under some mild conditions.
- We demonstrate the differences between the SMDW and LMDW models with numerical simulations.

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ABSTRACT

In this paper, we propose and then analyze two generalized Deffuant–Weisbuch (DW) models. The generalized models extend the conventional DW model by taking multiple choices in two different ways. First, we demonstrate the almost sure convergence of the agent opinions for the short-range multi-choice DW dynamics when only the opinions within confidence regions may be count in. Then we analyze dynamical behavior about the long-range multi-choice DW model when some opinions out of the confidence ranges are considered with a weighted combination. Moreover, both theoretical and simulation results show that the dynamical behaviors of the two models are totally different.

1. Introduction

Recent years have witnessed a steady increase in the study of social networks and opinion dynamics [1–5] due to many potential applications in various disciplines. Various models such as the Hegselmann–Krause (HK) model [6] and the Deffuant–Weisbuch (DW) model [7] were proposed in order to understand the evolution of opinions in a group.

Due to the complexity in opinion dynamics, most of the existing results were obtained mainly based on some descriptions and simulations, but the mathematical results on convergence analysis were relatively few. Based on the zero-entries symmetry of the adjacency matrices, Ref. [8] proved the stability of one class of agent-based opinion dynamics models, including the discrete-time homogeneous HK model and the symmetric homogeneous DW model. Then [9] developed simple methods to show the convergence of the HK model, while [10] discussed empirical opinion density using the system's mean-field dynamics as the population size of the agents becomes sufficiently large. From the viewpoint of infinite flow stability of certain random matrices, Ref. [11] provided a new proof about the convergence of the HK model and gave a convergence termination time bound for this model.

Consensus and convergence phenomenon of multi-agent systems have been studied widely [12–16]. However, many techniques for the analysis of multi-agent systems (with switching topologies described by a given random switching signal function [17]) cannot apply to the DW model, because the switching topologies in the DW model are state-dependent.

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The conventional DW model is a class of bounded confidence opinion dynamics, in which each agent randomly selects one learning object (LO) to update its opinion and update by their weighted average value if the distance between these two agents' opinions is not larger than the fixed bounded confidence. However, in practice, several LOs' opinions may be chosen to update the opinion at each time.

In the paper, two multi-choice DW models are proposed to cover various opinion phenomena that may not be described by the conventional DW model. In other words, both short-range and long-range multiple choices are considered in the opinion update by selecting a group of LOs for each agent. The almost sure convergence of the short-range multi-choice DW model is discussed. What is more, the long-range multi-choice DW model is proposed to characterizing the "trust" between agents maybe with different opinions and its dynamical behavior is discussed. Then simulations are given to show the dramatic difference between the two generalized DW opinion dynamics.

2. Multi-choice DW models

In this section, we propose two multi-choice DW models.

Because the DW model is stochastic, we introduce some notations of probability theory. A probability space is usually defined as $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is a space and $\mathbf{P}$ is a normalized measure on a σ-algebra $\mathcal{F}$ of subsets of Ω (referring to Ref. [18]). The set $A \in \mathcal{F}$ is called an event. $\mathbf{P}(A)$ denotes the probability of the event A. For any sequence $\{A_t, t \geq 1\}$ of sets, we take the abbreviation "i.o". for "infinitely often", namely,

$$\{\omega : \omega \in A_t, i.o.\} = \{\cap_{t=1}^{\infty} \cup_{k=t}^{\infty} A_k\}.$$

The conventional DW model proposed in Ref. [7] is formulated as follows. Agent i has an opinion value $x_i(t) \in \mathbb{R}$ at time t. Without loss of generality, the initial opinions, $x_i(t_0)$, $1 \leq i \leq n$, are limited in $[0, 1]$ (noting that the discussion can be easily extended in any set in $\mathbb{R}$). Denote ε_0 as the confidence radius. Then the conventional DW model in Ref. [7] is described as

$$\begin{cases} x_i(t+1) = x_i(t) + \gamma_0 \mathbb{1}_{\{|x_j(t)-x_i(t)| \leq \varepsilon_0\}} \cdot (x_j(t) - x_i(t)), \\ x_j(t+1) = x_j(t) + \gamma_0 \mathbb{1}_{\{|x_j(t)-x_i(t)| \leq \varepsilon_0\}} \cdot (x_i(t) - x_j(t)), \end{cases} \tag{1}$$

where i, j are equally random selected from $\mathcal{V}$ at time t, $\gamma_0, \varepsilon_0 \in (0, 1)$ and $\mathbb{1}$ is the indicator function, that is, $\mathbb{1}_{\{\omega\}} = 1$ if ω holds and $\mathbb{1}_{\{\omega\}} = 0$ otherwise.

In fact, we can generalize the conventional DW model into the following asymmetric form,

$$x_i(t+1) = x_i(t) + \gamma_i \mathbb{1}_{\{|x_{r_i(t)}(t)-x_i(t)| \leq \varepsilon_0\}} \cdot (x_{r_i(t)}(t) - x_i(t)), \quad 1 \leq i \leq n, t \geq t_0 = 0 \tag{2}$$

where $\gamma_i \in (0, 1)$ is the combination weight of agent i, $i \in \mathcal{V}$. In other words, when agent i selects agent j, agent j may not select agent i as in the symmetric DW model (1).

Note that each agent in both (1) and (2) can only take one LO at each time, whose dynamics are based on "single-choice" opinion update. However, in many cases, each agent may learn from several LOs at the same time. For example, in a meeting or debate, a person may choose opinions from several LOs before updating his/her opinion. Therefore, we go further to provide multi-choice DW models.

The Short-range Multi-choice DW (SMDW) model is proposed to study the following scenario: each agent can select more than one agent as its LOs at each time, and make a weighted evaluation for its opinion update after removing the opinion not located in its confidence region, which can be expressed as

$$x_i(t+1) = x_i(t) + \gamma_i \sum_{j=1}^{c_i} \alpha_{ij} \mathbb{1}_{\{|x_{r_i^{(j)}(t)}(t)-x_i(t)| \leq \varepsilon_0\}} \cdot (x_{r_i^{(j)}(t)}(t) - x_i(t)) \tag{3}$$

for $1 \leq i \leq n$, $1 \leq j \leq c_i$, $t \geq t_0 = 0$, where $\gamma_i \in (0, 1)$ is the same as in (2), and $c_i \in \mathcal{V}$ is the choice number of agent i. $r_i^{(j)}(t)$ denotes the index of the LO of agent i at its j-th choice, which is a random variable uniformly distributed in $\mathcal{V}$. $\{r_i^{(j)}(t)\}$ are uncorrelated with the index sets $\{i, i \in \mathcal{V}\}$, $\{j, 1 \leq j \leq c_i\}$ and $t = 0, 1, 2, \ldots$. α_{ij} satisfies $\sum_{j=1}^{c_i} \alpha_{ij} = 1$ and $0 < \alpha_{ij} < 1$, $i \in \mathcal{V}$. Clearly, if $c_i = 1$, $\alpha_{ij} = 1$, then SMDW becomes DW (2).

However, in reality, people may update their opinions based on not only similar opinions but also quite different opinions. Hence, we propose another model called the Long-range Multi-choice DW (LMDW) model. In this model, each agent collects some selected LOs' opinions and makes a weighted summary of those opinions, and then updates its own opinion if the weighted opinion is located in its confidence region, which can be expressed as

$$x_i(t+1) = x_i(t) + \gamma_i \mathbb{1}_{\{|\sum_{j=1}^{c_i} \alpha_{ij} x_{r_i^{(j)}(t)}(t)-x_i(t)| \leq \varepsilon_0\}} \cdot \left(\sum_{j=1}^{c_i} \alpha_{ij} x_{r_i^{(j)}(t)}(t) - x_i(t) \right) \tag{4}$$

for $1 \leq i \leq n$, $1 \leq j \leq c_i$, $t \geq t_0 = 0$, where the notations are the same as those in the SMDW model. Although LMDW looks like DW or SMDW, it is totally different from them because opinions outside the confidence range of an agent may influence its opinion.

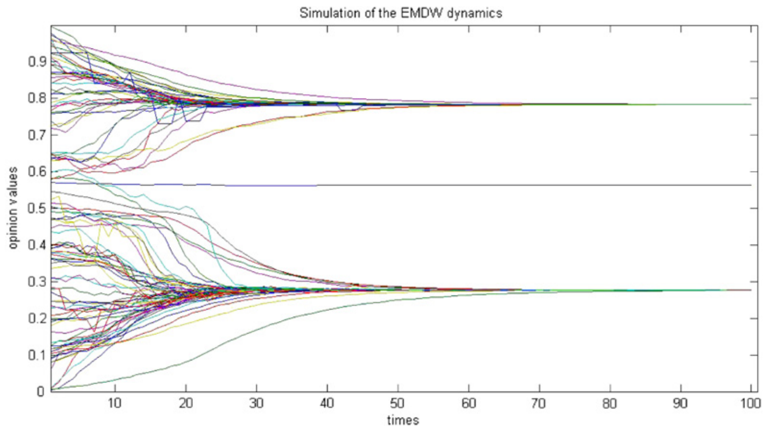


Fig. 1. Opinion evolution of 100 agents for (3).

The interaction between agents can be described by graphs. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V} = \{1, 2, \ldots, n\}$ of vertexes (or regarded as agents here) and an edge set $\mathcal{E}$ (referring to Ref. [19]). Note that the edge set $\mathcal{E}$ is time-varying during the opinion evolution. Let $\{\mathcal{F}_t\}$ be a filtration on $(\Omega, \mathcal{F})$, where the sample space Ω is composed by all the agents' selection processes $\{r_i(t), \forall t \in \mathbb{N}, i \in \mathcal{V}\}$. $|X|$ denotes the number of vertexes (or agents) in a set $X \subset \mathcal{V}$.

For $\omega \in \Omega$, both (3) and (4) can be written in a compact form as $\mathbf{x}(t+1, \omega) = W(t, \omega)\mathbf{x}(t, \omega)$, where $W(t, \omega) = (W_{ij}(t, \omega))_{1 \le i,j \le n}$ denotes the matrix corresponding to the graph describing the interconnection between agents, and $\mathbf{x}(t, \omega) = (x_1(t, \omega), \ldots, x_n(t, \omega))^T$ denotes the vector of these opinion values at time t. Define the infinite flow graph of $\{W(t, \omega)\}$ as $G^\infty : \Omega \to \mathcal{G}(\mathcal{V})$ or $G^\infty(\omega) = (\mathcal{V}, \mathcal{E}^\infty(\omega))$ with the vertex set $\mathcal{V}$ and the edge set $\mathcal{E}^\infty(\omega) = \{(i, j) | \sum_{t=0}^{\infty} (W_{ij}(t, \omega) + W_{ji}(t, \omega)) = \infty,\ i \neq j \in \mathcal{V}\}$. In the sequel, when there is no confusion, we will simply write $x_i(t)$, $\mathbf{x}(t)$, and $W_{ij}(t)$ instead of $x_i(t, \omega)$, $\mathbf{x}(t, \omega)$, and $W_{ij}(t, \omega)$. Additionally, $W(t : s)$ denotes $W(t-1)W(t-2) \cdots W(s)$.

Definition 1. $\mathbf{x}(t) \in \mathbb{R}^n$ is convergent a.s. to $\mathbf{x}^*$ if $\mathbf{P}(\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*) = 1$. Furthermore, $\mathbf{x}(t)$ achieves consensus a.s. if all the agents' opinions convergent a.s. to a common opinion value.

3. Opinion evolution of the SMDW model

In this section, we analyze the convergence for the SMDW model (3).

Fig. 1 shows that 100 agents evolve their opinions under protocol (3), where initial values are equally distributed in [0, 1] and the confidence radius is 0.2. In this simulation, the opinions of these agents converge to three different values eventually.

Before the analysis of the SMDW model, we give some concepts about $W(t)$ (referring to Refs. [11,1]). $\{W(t)\}$ is an adapted random chain (with respect to $\{\mathcal{F}_t\}$) if $\{W(t)\}$ is measurable with respect to $\mathcal{F}_{t+1}$, or equivalently, $W_{ij}(t)$ is measurable with respect to $\mathcal{F}_{t+1}$ for $i, j \in \mathcal{V}$ and $t \ge 0$. An adapted random chain $\{W(t)\}$ has the feedback property if $W_{ii}(t) \ge \gamma$ a.s. for a constant $\gamma > 0$ with $t \ge 0$ and $i \in \mathcal{V}$. A random vector process $\{\pi(t)\}$ is an absolute probability process for $\{W(t)\}$ if $E[\pi^T(t+1)W(t)|\mathcal{F}_t] = \pi^T(t)$ for $k \ge 0$, and $\pi(t)$ is a stochastic vector (that is, $\sum_{i=1}^{n} \pi_i(t) = 1, \pi_i(t) \ge 0, i \in \mathcal{V}$) a.s. for $t \ge 0$. In addition, $\{W(t)\}$ is balanced if there is a constant $\alpha > 0$ such that the expected chain $\{E[W(t)|\mathcal{F}_t]\}$ satisfies $E[W_{S\bar{S}}(t)|\mathcal{F}_t] \ge \alpha E[W_{\bar{S}S}(t)|\mathcal{F}_t]$ for any non-empty $S \subset \mathcal{V}$ and $t \ge 0$, where $\bar{S} \subset \mathcal{V}$ is the complementary set of S and $E[W_{S\bar{S}}(t)|\mathcal{F}_t] = \sum_{i \in S, j \in \bar{S}} E[W_{ij}(t)|\mathcal{F}_t]$.

The following result is useful in our analysis [11].

Lemma 1. *Let $\{W(t)\}$ be a random adapted chain having an absolute probability process $\{\pi(t)\}$ such that $\pi(t) \ge \kappa$ a.s. for some scalar $\kappa > 0$ and for all $t \ge 0$, with the feedback property and balance property of its sample paths. Then $\{W(t)\}$ is infinite flow stable namely, $\{\mathbf{x}(t)\}$ achieves convergence a.s. for any initial condition $(t_0, \mathbf{x}(t_0))$ and $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$, a.s. for $\{i, j\} \in \mathcal{E}^\infty(\omega)$.*

Denote

$$\eta^+ = \max_{i \in \{1,2,\ldots,n\}} \left\{ \gamma_i \sum_{j=1}^{c_i} \alpha_{ij} \right\}, \qquad \eta^- = \min_{i \in \{1,2,\ldots,n\}} \min_{j \in \{1,2,\ldots,c_i\}} \{\gamma_i \alpha_{ij}\}.$$

Consider a non-trivial $S \subset \mathcal{V}$ and define $d_t(S, \bar{S}) = \min_{i \in S, j \in \bar{S}} |x_i(t) - x_j(t)|$.

- If $d_t(S,\bar{S}) \le \varepsilon_0$, then

$$E[W_{S\bar{S}}(t)|\mathcal{F}_{t+1}] = \sum_{i\in S, j\in\bar{S}} E[W_{ij}(t)|\mathcal{F}_{t+1}] = \sum_{(i,j)\in\mathcal{E}_t, i\in S, j\in\bar{S}} E[W_{ij}(t)|\mathcal{F}_{t+1}]$$
$$\ge \sum_{(i,j)\in\mathcal{E}_t, i\in S, j\in\bar{S}} \eta^- = \sum_{(i,j)\in\mathcal{E}_t, i\in S, j\in\bar{S}} \eta^+\frac{\eta^-}{\eta^+} \ge \frac{\eta^-}{\eta^+} E[W_{\bar{S}S}(t)|\mathcal{F}_{t+1}].$$

- If $d_t(S,\bar{S}) > \varepsilon_0$, then $\max_{i\in S, j\in\bar{S}} E[W_{ij}(t)|\mathcal{F}_{t+1}] = 0$ because S and $\bar{S}$ are disconnected. Hence,

$$E[W_{S\bar{S}}(t)|\mathcal{F}_{t+1}] = \sum_{i\in S, j\in\bar{S}} E[W_{ij}(t)|\mathcal{F}_{t+1}] \ge \frac{\eta^-}{\eta^+} E[W_{\bar{S}S}(t)|\mathcal{F}_{t+1}].$$

Because $W_{ii}(t) \ge 1-\eta^+ > 0$ for $i\in\mathcal{V}$ and $t\ge 0$, $\{W(t)\}$ is a balanced random chain with the feedback property and uniformly bounded positive entries, that is, $W_{ij}(t) \ge \min\{1-\eta^+, \eta^-\} \ge \delta$ for $W_{ij}(t) > 0$.

Then we have the following result.

Lemma 2. *$\{W(t)\}$ has an absolute probability sequence $\{\pi(t)\}$, uniformly bounded by δ^{n-1} with $\delta = \frac{\eta^-}{n\eta^+}\min\{1-\eta^+, \eta^-\}$.*

Proof. Denote $S_k(t) = \{i\in\mathcal{V} | E[W_{ik}(t:0)|\mathcal{F}_t] > 0\}$ and $\mu_k(t) = \min_{i\in S_k(t)} E[W_{ik}(t:0)|\mathcal{F}_t]$. Clearly, $S_k(0) = \{k\}$ for $W(0:0) = I$ and $\mu_k(0) = 1$.

In fact, by

$$E[W_{ij}(t+1:0)|\mathcal{F}_{t+1}] \ge \min W_{ii}(t) E[W_{ij}(t:0)|\mathcal{F}_t] \ge \delta\mu_j(t) > 0,$$

$i\in S_j(t+1)$ if $i\in S_j(t+1)$. Hence, $S_k(t)\subset S_k(t+1)$.

Then we will prove $\mu_k(t) \ge \delta^{|S_k(t)|-1}$ for all $k\in\mathcal{V}$ and $t\ge 0$ by induction.

Suppose $\mu_k(t) \ge \delta^{|S_k(t)|-1}$. Consider $\mu_k(t+1)$ in the following cases.

Case 1: If $E[W_{S_k(t)\bar{S}_k(t)}(t)|\mathcal{F}_{t+1}] = 0$, $S_k(t)$ is disconnected with other agents at t. By $\sum_{j\in\mathcal{V}} W_{ij}(t) = \sum_{j\in S_k(t)} W_{ij}(t) = 1$ for $i\in S_k(t)$,

$$E[W_{ik}(t+1:0)|\mathcal{F}_{t+1}] \ge \mu_k(t) \sum_{s\in S_k(t)} E[W_{is}(t)|\mathcal{F}_{t+1}] = \mu_k(t), \quad \forall i\in S_k(t). \tag{5}$$

Because $E[W_{ij}(t)|\mathcal{F}_{t+1}] = E[W_{ji}(t)|\mathcal{F}_{t+1}]$ and $E[W_{S_k(t)\bar{S}_k(t)}(t)|\mathcal{F}_{t+1}] = 0$, $E[W_{\bar{S}_k(t)S_k(t)}(t)|\mathcal{F}_{t+1}] = 0$. Then, for $i\in\bar{S}_k(t)$, we have

$$\begin{aligned} 0 &\le E[W_{ik}(t+1:0)|\mathcal{F}_{t+1}] \\ &\le \sum_{s\in S_k(t)} E[W_{is}(t)|\mathcal{F}_{t+1}]E[W_{sk}(t:0)|\mathcal{F}_t] + \max_{s\in\bar{S}_k(t)}\{W_{is}(t)\} \sum_{s\in\bar{S}_k(t)} E[W_{sk}(t:0)|\mathcal{F}_t] = 0, \end{aligned}$$

where $E[W_{sk}(t:0)|\mathcal{F}_t] = 0$ due to $s\in\bar{S}_k(t)$.

Hence, $S_k(t) = S_{k+1}(t)$, and by (5), $\mu_k(t+1) \ge \mu_k(t)$. According to

$$\mu_k(t) \ge \delta^{|S_k(t)|-1} = \delta^{|S_k(t+1)|-1},$$

we have $\mu_k(t+1) \ge \delta^{|S_k(t+1)|-1}$.

Case 2: If $E[W_{S_k(t)\bar{S}_k(t)}(t)|\mathcal{F}_{t+1}] > 0$, then

$$E[W_{\bar{S}_k(t)S_k(t)}(t)|\mathcal{F}_{t+1}] \ge \frac{\eta^-}{\eta^+} E[W_{S_k(t)\bar{S}_k(t)}(t)|\mathcal{F}_{t+1}] > 0,$$

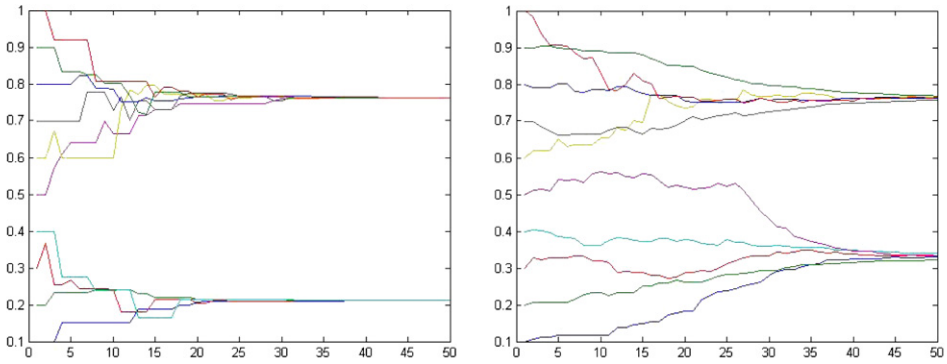
which implies $E[W_{\bar{S}_k(t)S_k(t)}(t)|\mathcal{F}_{t+1}] > 0$ since $\{W(t)\}$ is balanced. Due to the uniform boundedness of the positive entries of $E[W(t)|\mathcal{F}_{t+1}]$, there are $j\in\bar{S}_k(t)$ and $l\in S_k(t)$ such that $W_{jl}(t)\ge\delta$. Therefore,

$$E[W_{jk}(t+1:0)|\mathcal{F}_{t+1}] \ge \delta E[W_{lk}(t:0)|\mathcal{F}_t] \ge \delta\mu_k(t) \ge \delta^{|S_k(t)|}.$$

Thus, $j\in S_k(t+1)$, which implies $|S_k(t+1)| \ge |S_k(t)|+1$. Also $\mu_k(t+1) \ge \delta^{|S_k(t)|} \ge \delta^{|S_k(t+1)-1|}$.

As a result, $\frac{1}{n}\mathbf{1}^T E[W^k(t:0)|\mathcal{F}_t] \ge \frac{1}{n}|S_k(t)|\delta^{|S_k(t)|-1}$ for $i\in\mathcal{V}$, where $W^k(t)$ denotes the k-th column of $W(t)$. By $\delta < \frac{1}{n}$, we obtain $\frac{1}{n}\mathbf{1}^T E[W(t:0)|\mathcal{F}_t] \ge \delta^{n-1}\mathbf{1}^T$.

By Theorem 4.1 in Ref. [11], for the stochastic matrices $\{W(t)\}$, there exists a subsequence $\{t_r\}$ of non-negative integers such that $Q(t) = \lim_{r\to\infty} W(t_r:t) \in \mathbb{R}^{n\times n}$ exists. Further, by Theorem 4.2 in Ref. [11], the vector $\pi(t) = \frac{1}{n}\mathbf{1}^T Q(t)$ is a



(a) DW dynamics. (b) SMDW dynamics.

Fig. 2. Comparison between the DW dynamics (2) and the SMDW dynamics (3).

uniformly bounded absolute probability sequence for $\{W(t)\}$, where

$$\frac{1}{n}\mathbf{1}^T Q(t) = \frac{1}{n}\lim_{r\to\infty}\mathbf{1}^T W(t_r : t) \geq \delta^{n-1}\mathbf{1}^T$$

which implies the conclusion. □

Then we have the almost sure convergence result.

Theorem 1. *Given any initial opinions $\mathbf{x}_0 \in [0, 1]^n$, one of the following two results holds almost surely for $i, j \in \mathcal{V}$:*
(i) $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$,
(ii) $\lim_{t\to\infty} |x_i(t) - x_j(t)| > \varepsilon_0$.

Proof. By Lemmas 1 and 2, $\{W(t)\}$ is infinite flow stable in any measurable subset of Ω. In other words, all agents are almost surely convergent and moreover, agents i, j in the same infinite flow subgraph of $(\mathcal{V}, \mathcal{E})$ converge to the same value a.s.

If i, j are not in the same infinite flow subgraph, we claim $\lim_{t\to\infty} |x_i(t) - x_j(t)| > \varepsilon_0$ a.s. Otherwise, without loss of generality, there exist $i \in G_1^\infty$ and $j \in G_2^\infty$ where G_1^∞, G_2^∞ are different infinite flow subgraphs for event $\omega \in \Omega$. In fact, if $\min_{i\in G_1^\infty, j\in G_2^\infty} |x_i(t_k) - x_j(t_k)| \leq \varepsilon_0$ for some subsequence $\{t_k\}$, then by the Borel–Cantelli Theorem (Theorem 3.2.1 in Ref. [18]) and $\mathbf{P}(r_i^{(1)}(t_k) = j) = n^{-1} > 0$, $\mathbf{P}(r_i^{(1)}(t_k) = j, i.o.) = 1$. In other words, there exists an infinite subsequence $\{t_{k_l}\} \subset \{t_k\}$ almost surely such that

$$\sum_{t=0}^{\infty}(W_{ij}(t) + W_{ji}(t)) \geq \sum_{l=1}^{\infty}(W_{ij}(t_{k_l}) + W_{ji}(t_{k_l})) \geq \sum_{l=1}^{\infty}\delta^{n-1} = \infty,$$

indicating that i, j are in the same infinite flow graph, which leads to a contradiction. Thus, the conclusion follows. □

Remark 1. Since the conventional DW model (1) and the DW model (2) are special cases of the SMDW model (3), the convergence result also holds for (1) and (2). In fact, the method has been used in the convergence analysis of the HK model in Ref. [11].

A simulation is given for a comparison between the SMDW model (3) and the DW model (2). We conduct the simulations on 10 agents, whose initial opinions are equally distributed in the interval [0, 1]. Fig. 2 shows that the convergence of the agents' opinions in the SMDW dynamics becomes slower to achieve agreements than that in the DW dynamics, mainly because too many choices may influence the opinion updates. However, on the other hand, the opinion change of each agent in the SMDW model in Fig. 2(b) is much smoother than that in the DW model in Fig. 2(a). Clearly, those phenomena are consistent with the common sense.

4. Dynamics and consensus of the LMDW model

The SMDW model is like the conventional DW models, where each agent is quite narrow-minded. Although SMDW allows agents to collect more opinions, it only keeps the opinion values within the confidence ranges and discards those that are quite different. Therefore, the agents tend to form opinion groups and keep their opinions within their own groups. However, this may not be true in reality when some agents like to see the opinions quite different from theirs before updating their opinion. In this section, we analyze the dynamic behavior for the LMDW model (4), which provides a long-range interaction mechanism to count in some opinions outside the confidence range.

Denote $M_t = \max_i\{x_i(t)\}$ and $m_t = \min_i\{x_i(t)\}$. We first introduce the following lemma, whose proof is simple and therefore, omitted.

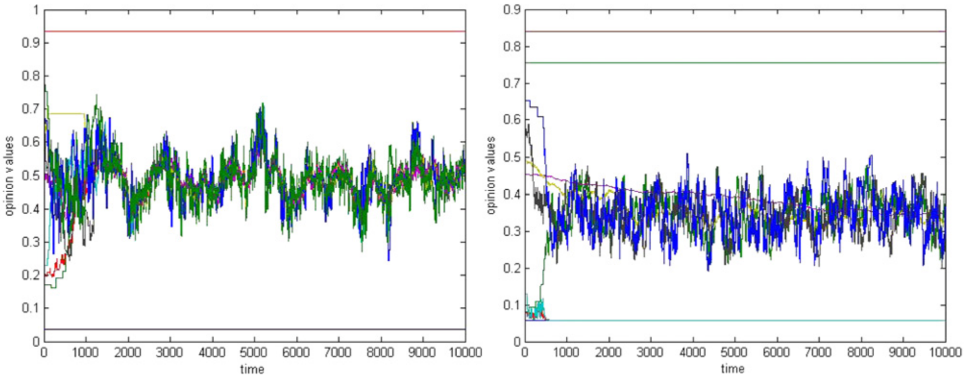


Fig. 3. Opinion oscillation for LMDW (4) with a small confidence bound.

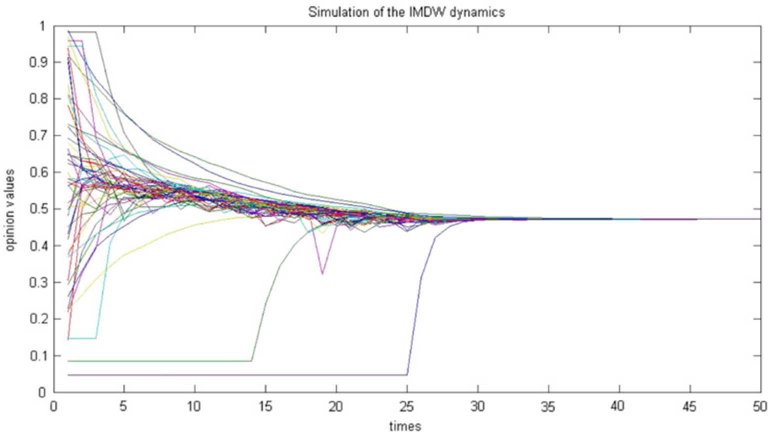


Fig. 4. Opinion evolution of 50 agents for LMDW (4).

Lemma 3. *For LMDW (4) and each event $\omega \in \Omega$, non-decreasing sequence $\{m_t\}$ converges to $\overline{m} \in [0, 1]$, and non-increasing sequence $\{M_t\}$ converges to $\underline{M} \in [0, 1]$.*

Clearly, $\overline{m} \leq \underline{M}$ for any given sample.

Although the dynamics of SMDW and LMDW look similar, their behaviors are very different. The agents in LMDW are quite social with long-range concerns: they collect various opinions, including those within their confidence region or outside the region, and then make a summary for all the opinions; then they make new opinions based on if the weighted summary are located in their confidence. Therefore, the agents in LMDW strongly interact with each other with such a long-range interaction. Two simulations in Fig. 3 display that 10 agents evolve their opinions by (4) with a confidence bound $\varepsilon_0 = 0.05$. Different from the SMDW model, opinion values of LMDW (4) cannot converge a.s., but may keep fluctuating instead, along with several straight lines, as shown in the two simulations in Fig. 3. In fact, the agent with the constant opinion (in the form of straight line) is isolated, when any weighted combination of its collected opinions is out of its confidence region for a sufficient small ε_0. Therefore, there are two kinds of the agents in such a case: the isolated agent (that sticks to its own opinion) and the fluctuating agent (that keeps changing its opinion value). It is worthwhile to point out that the fluctuation/oscillation comes from the influence of isolated constant opinions.

However, the fluctuation may disappear when we increase the confidence radius ε_0 to increase the long-range opinion exchange. As shown in Fig. 4, 50 agents evolve their opinions under LMDW dynamics (4), whose initial values are located in the interval $[0, 1]$, $\max_{1\leq i\leq n} \min_{1\leq j\leq c_i} \alpha_{ij} = 0.32$ and $\varepsilon_0 = 0.41$.

We may wonder if there is a value of ε_0 to guarantee the consensus. Actually, the next result shows almost sure consensus for the LMDW model.

Theorem 2. *For any initial values $\mathbf{x}_0 \in [0, 1]^n$ in (4) with $c_i > 1$ $(i = 1, \ldots, n)$, if*

$$\varepsilon_0 \geq \max_{1\leq i\leq n} \min_{1\leq j\leq c_i} \alpha_{ij},$$

then the opinion consensus can be achieved almost surely.

Proof. The conclusion is obvious with initial condition $M_0 - m_0 \le \varepsilon_0$. We focus on the case when $M_0 - m_0 > \varepsilon_0$.

In this case, we will prove by contradiction that $\mathbf{P}(\overline{m} < \underline{M}) > 0$ does not hold.

Conditioned on $\underline{m} < \overline{M}$, we take

$$\beta = \frac{1}{2}\min_{1\le i\le n}\min_{1\le j\le c_i}\gamma_i\alpha_{ij}\min\{\underline{M}-\overline{m}, 1-\underline{M}+\overline{m}\} > 0 \tag{6}$$

for given $\underline{m}$ and $\overline{M}$. By Lemma 3, there exists a (stopping) time $T > 0$ such that

$$0 < \max\{\overline{m}-m_t, M_t-\underline{M}\} < \beta, \quad t > T. \tag{7}$$

Denote $\zeta_i = \{j : \min_{1\le j\le c_i}\alpha_{ij}\}$ and $I_s = \{1,\ldots,c_s\}$ for $s \in \mathcal{V}$.

Because n is a finite number, by Lemma 3 there must be at least one agent whose opinion values are infinitely often located in $[0,\overline{m}]$. Denote $X = \{i \in \mathcal{V} : x_i(t) \in [0,\overline{m}]\ i.o.\}$ and $Y = \{j \in \mathcal{V} : x_j(t) \in [\underline{M}, 1]\ i.o.\}$, which are non-empty (random) subsets of $\mathcal{V}$. Denote a sample set $\{\Omega_{X,Y}\}$ as:

$$\begin{aligned}\Omega_{X,Y} = \{&(x_1(t),\ldots,x_{i_1}(t),\ldots,x_{i_{|X|}}(t),\ldots,x_{j_1}(t),\ldots,x_{j_{|Y|}}(t),\ldots,x_n(t)), t\ge 0,\\ &X=\{i_1,\ldots,i_{|X|}\}, Y=\{j_1,\ldots,j_{|Y|}\}\}.\end{aligned} \tag{8}$$

$\Omega = \cup\{\Omega_{X,Y}\}$ because there always exist agents in $[0,\overline{m}]$ and $[\underline{M}, 1]$. Then we can discuss this problem in the following two cases:

Case 1: $|X| = 1$ or $|Y| = 1$.

Without loss of generality, we assume that $X = \{1\}$ and Y is a given non-empty set, and therefore, $|Y| \ge 1$ and $\Omega_{\{1\},Y}$ is not empty. By (7) and (8), agent 1's opinion value is taken infinitely often in $(\overline{m}-\beta,\overline{m}]$ when $t > T$ for any given event ω (in fact, if $\varepsilon_0 < \beta$, we can replace $(\overline{m}-\beta,\overline{m}]$ by $(0,\overline{m}]$). In other words, $x_1(t) \in (\overline{m}-\beta,\overline{m}]$ when $t > T$. By $c_1 > 1$, we denote

$$\check{Q}_1(t) = \{r_1^{(\zeta_1)}(t) \in Y, r_1^{(k)}(t) = 1, \forall k \in I_1/\{\zeta_1\}, t > T\}$$

with $\tau_0 = \min_{t>0}\{t : \mathbb{1}_{\{\omega\in\check{Q}_1(t)\}} = 1\}$.

We claim

$$\mathbf{P}(\tau_0 < \infty) = 1. \tag{9}$$

In fact, when $t > T$, agent 1's opinion value sequences will keep their values in $[0,\overline{m}]$ for any event $\omega \in \Omega_{\{1\},Y}$. By $\mathbf{P}(r_1^{(k)}(t) = k) = \frac{1}{n} > 0$ for $k \in \mathcal{V}$ and the independence of $\{r_1^{(k)}(t)\}$, we get

$$\mathbf{P}(\check{Q}_1(t)) \ge \frac{1}{n^{c_1}} \ge \frac{1}{n^n} > 0.$$

Hence, $\sum_{t=1}^{\infty}\mathbf{P}(\check{Q}_1(t)) = \infty$. Thus, (9) holds by the Borel–Cantelli Theorem (Theorem 3.2.1 in Ref. [18]).

In fact, conditioned on $\check{Q}_1(\tau_0)$, by $\min_{1\le j\le c_1}\alpha_{1j} > -1$ and (6), we have

$$\begin{aligned}\beta &\le \frac{1}{2}\min_{1\le i\le n}\gamma_i\min_{1\le j\le c_1}\alpha_{1j}(\underline{M}-\overline{m}) \le \frac{1}{2}\min_{1\le j\le c_1}\alpha_{1j}(\underline{M}-\overline{m})\\ &\le \frac{\min\limits_{1\le j\le c_1}\alpha_{1j}}{1-\min\limits_{1\le j\le c_1}\alpha_{1j}}(\underline{M}-\overline{m}) < \frac{\min\limits_{1\le j\le c_1}\alpha_{1j}}{1-\min\limits_{1\le j\le c_1}\alpha_{1j}}(1+\underline{M}-\overline{m}),\end{aligned} \tag{10}$$

and then

$$\begin{aligned}\sum_{s=1}^{c_1}\alpha_{1s}x_{r_1^{(s)}(\tau_0)}(\tau_0) - x_1(\tau_0) &\ge \min_{1\le j\le c_1}\alpha_{1j}\underline{M} + \left(1-\min_{1\le j\le c_1}\alpha_{1j}\right)(\overline{m}-\beta) - \overline{m}\\ &= \min_{1\le j\le c_1}\alpha_{1j}(\underline{M}-\overline{m}) - \left(1-\min_{1\le j\le c_1}\alpha_{1j}\right)\beta \ge -\min_{1\le j\le c_1}\alpha_{1j} \ge -\varepsilon_0.\end{aligned} \tag{11}$$

Similarly, by (6) and $\min_{1\le j\le c_1}\alpha_{1j} < 1$, we have

$$\begin{aligned}\beta &\le \frac{1}{2}\min_{1\le i\le n}\gamma_i\min_{1\le j\le c_1}\alpha_{1j}(1-\underline{M}+\overline{m})\\ &\le \frac{1}{2}\min_{1\le j\le c_1}\alpha_{1j}(1-\underline{M}+\overline{m}) < \frac{\min\limits_{1\le j\le c_1}\alpha_{1j}}{1+\min\limits_{1\le j\le c_1}\alpha_{1j}}(1-\underline{M}+\overline{m}),\end{aligned}$$

and then

$$\sum_{s=1}^{c_1} \alpha_{1s} x_{r_1^{(s)}(\tau_0)}(\tau_0) - x_1(\tau_0) \leq \min_{1\leq j\leq c_1} \alpha_{1j}(\underline{M}+\beta) + \left(1 - \min_{1\leq j\leq c_1} \alpha_{1j}\right)\overline{m} - (\overline{m}-\beta)$$
$$= \min_{1\leq j\leq c_1} \alpha_{1j}(\underline{M}-\overline{m}) + \left(1 + \min_{1\leq j\leq c_1} \alpha_{1j}\right)\beta \leq \min_{1\leq j\leq c_1} \alpha_{1j} \leq \varepsilon_0. \tag{12}$$

Therefore, $|\sum_{s=1}^{c_1} \alpha_{1s} x_{r_1^{(s)}(\tau_0)}(\tau_0) - x_1(\tau_0)| \leq \epsilon_0$ when $\varepsilon_0 \geq \max_{1\leq i\leq n}\min_{1\leq j\leq c_i} \alpha_{ij}$ conditioned on $\check{Q}_1(\tau_0)$. Moreover, by (6) and $\min_{1\leq i\leq n} \gamma_i(1-\min_{1\leq j\leq c_1}\alpha_{1j}) < 1 - \frac{1}{2}\min_{1\leq j\leq c_1}\alpha_{1j}$, we obtain

$$\begin{aligned}
x_1(\tau_0+1) - x_1(\tau_0) &= \gamma_1\left(\sum_{s=1}^{c_1}\alpha_{1s}x_{r_1^s(\tau_0)}(\tau_0) - x_1(\tau_0)\right)\\
&\geq \left(\min_{1\leq i\leq n}\gamma_i\right)\left[\min_{1\leq j\leq c_1}\alpha_{1j}\underline{M} + \left(1-\min_{1\leq j\leq c_1}\right)(\overline{m}-\beta) - \overline{m}\right]\\
&= \left(\min_{1\leq i\leq n}\gamma_i\right)\left[\min_{1\leq j\leq c_1}\alpha_{1j}(\underline{M}-\overline{m}) - \beta\left(1-\min_{1\leq j\leq c_1}\alpha_{1j}\right)\right]\\
&\geq \left(1 - \frac{\min_{1\leq i\leq n}\gamma_i\left(1-\min_{1\leq j\leq c_1}\alpha_{1j}\right)}{2-\min_{1\leq j\leq c_1}\alpha_{1j}}\right)\min_{1\leq i\leq n}\gamma_i\min_{1\leq j\leq c_1}\alpha_{1j}(\underline{M}-\overline{m})\\
&\geq \frac{1}{2}\min_{1\leq i\leq n}\gamma_i\min_{1\leq j\leq c_1}\alpha_{1j}\min\{\underline{M}-\overline{m}, 1-\underline{M}+\overline{m}\} \geq \beta
\end{aligned} \tag{13}$$

which implies no agent in $[0, \overline{m}]$ at time τ_0+1 and leads to a contradiction with Lemma 3. By (9), $\check{Q}_1(\tau_0)$ happens a.s. for each $\omega \in \Omega_{\{1\},Y}$ and then $\mathbf{P}(\overline{m} < \underline{M}|\omega \in \Omega_{\{1\},Y}) = 0$.

Case 2: $|X| > 1$ and $|Y| > 1$.

Without loss of generality, we take $X = \{1, 2\}$ and any non-empty set Y with $|Y| \geq 2$.

By (7) and (8), opinion values of agents 1 and 2 are infinitely often in $(\overline{m}-\beta, \overline{m}]$ when $t > T$ for each given event ω, since $\beta > 0$ is a parameter independent of each agent (noting that, if $\varepsilon_0 < \beta$, we can replace $(\overline{m}-\beta, \overline{m}]$ by $(0, \overline{m}]$). Similar to Case 1, we denote

$$\check{Q}_{1,2}(t) = \{r_i^{(\zeta_i)}(t) \in Y, r_i^{(k)}(t) \in X, \forall k \in I_i/\{\zeta_i\} \text{ and } i \in X, t > T\}$$

with $\tau_0 = \min_{t>0}\{t : \mathbb{1}_{\{\omega\in\check{Q}_{1,2}(t)\}} = 1\}$.

Using the same method, we can prove $\mathbf{P}(\tau_0 < \infty) = 1$. Similarly, we can calculate

$$\left|\sum_{s=1}^{c_i}\alpha_{is}x_{r_i^{(s)}(\tau_0)}(\tau_0) - x_i(\tau_0)\right| \leq \epsilon_0 \quad \text{for } i, j \in \{1, 2\},$$

conditioned on $\check{Q}_{1,2}(\tau_0)$, and moreover, according to (4),

$$x_1(\tau_0+1) - x_1(\tau_0) \geq \beta, x_2(\tau_0+1) - x_2(\tau_0) \geq \beta \tag{14}$$

which implies that there is no agent in $[0, \overline{m}]$ about the event $\omega \in \Omega_{\{1,2\},Y}$ at time $\tau_k + 1$. This contradicts Lemma 3. Hence, $\mathbf{P}(\overline{m} < \underline{M}|\omega \in \Omega_{\{1,2\},Y}) = 0$.

For general $X = \{i_1, i_2, \ldots, i_k\}$ and $Y = \{j_1, j_2, \ldots, j_l\}$ with $1 < k, l < n$, we can still apply the above ideas to get the results similar to (9) and (14).

In sum, there are T_1 and T_2 such that $m_{T_1} > \overline{m}$ or $M_{T_2} < \underline{M}$ a.s., which contradict the definitions of $\overline{m}$ and $\underline{M}$. Hence, $\mathbf{P}(\overline{m} < \underline{M}) = 0$. Thus, the conclusion holds. □

Clearly, Theorem 2 is consistent with the phenomena given in Fig. 4, where the opinions of these agents tend to consensus with long-range opinion exchange. In the LMDW model, the weighted opinion $\bar{x}_i$ in the function $\mathbb{1}_{\{|\bar{x}_i - x_i(t)|<\varepsilon_0\}}$ can drive the agents out of the neighbor of agent i into the LO set of agent i if the confidence bound ε_0 is large enough. Therefore, the almost sure consensus occurs more easily when ε_0 is larger because the agents are more open to different opinions.

The proposed LMDW model may explain the interesting opinion phenomena that the HK or DW (and SMDW) models cannot demonstrate. When long-range opinions are taken into consideration, we see two extreme phenomena from simulations: with a large confidence radius, agents are quite open to different weighted opinions and then tend to achieve agreement; with a small confidence radius, agents do not trust others' opinions though the long-range opinions will influence agents somehow. Therefore, they either become isolated or fluctuate without any fixed opinion.

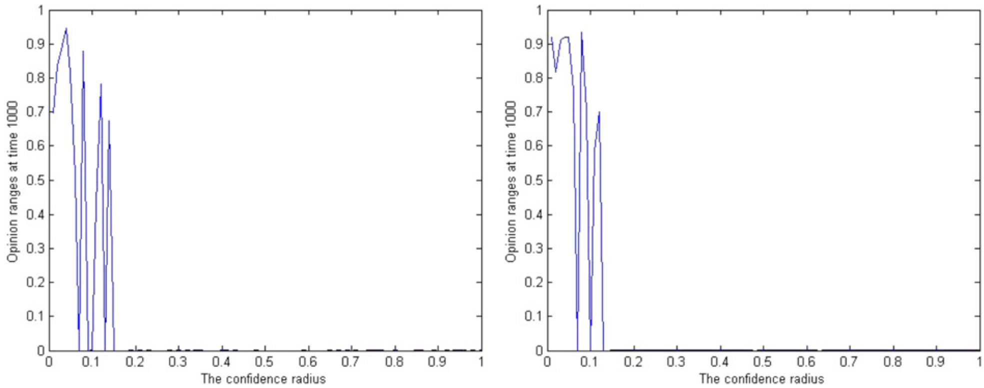


Fig. 5. Opinion difference $R(t)$ of the LMDW dynamics change with ε_0.

The opinion difference of $\mathbf{x}(t)$ is defined as $R(t) = \max_i x_i(t) - \min_i x_i(t)$. In Fig. 5, we show two numerical simulations for 10 agents with initial opinions randomly distributed in the interval [0, 1]. The LMDW model evolves with $c_i = 5$ and all $\alpha_{ij} = 0.2$. As the confidence radius ε_0 changes from 0.01 to 1 with the step 0.01, $R(t)$ changes from almost 1 to almost 0 when $\varepsilon_0 > 0.2$. The simulation result verifies Theorem 2, where all the opinions of (4) achieve consensus when ε_0 is larger than a certain ''threshold'' value. Further we can see the co-existence of oscillation the straight lines in the evolution. Therefore, for some initial conditions, there may not be isolated agents (and then no fluctuating agents), which explains why there may be consensus sometimes even when the radius is small in Fig. 5.

5. Conclusions

In the paper, we proposed two multi-choice models to generalize the conventional DW models and then investigated the dynamical behaviors of two generalized DW opinion dynamic models. We first proved the almost sure convergence of the SMDW model, and the almost sure consensus of the LMDW model under some mild conditions. We also demonstrated the differences between the SMDW and LMDW models with numerical simulations. However, many opinion dynamics problems such as phase transitions for the LMDW model and the convergence rate estimation of the opinion models remain to be solved.

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References

[1] A. Nedic, B. Touri, Multi-dimensional Hegselmann–Krause dynamics, in: Proc. IEEE Decision and Control, Hawaii, USA, Dec. 2012, pp. 68–73.
[2] S. Fortunato, V. Latora, A. Pluchino, A. Rapisarda, Vector opinion dynamics in a bounded confidence consensus model, International Journal of Modern Physics C 16 (2005) 1535–1551.
[3] G. Shi, K. Johansson, The role of persistent graphs in the agreement seeking of social networks, IEEE Journal on Selected Areas in Communications 31 (8) (2013) 1–13.
[4] J. Lorenz, Managing Complexity: Insights, Concepts, Applications, Springer, Berlin, 2008.
[5] N. Friedkin, E. Johnsen, Social influence networks and opinion change, in: E.J. Lawler, M.W. Macy (Eds.), Advances in Group Processes, JAI Press, 1999, pp. 1–29.
[6] R. Hegselmann, U. Krause, Opinion dynamics and bounded confidence models, Analysis and Simulation, Journal of Artificial Societies and Social Simulations (JASSS) 5 (2002).
[7] G. Deffuant, D. Neau, F. Amblard, G. Weisbuch, Mixing beliefs among interacting agents, Advances in Complex Systems 3 (2000) 87–98.
[8] J. Lorenz, A stabilization theorem for dynamics of continuous opinions, Physica A 335 (2005) 217–223.
[9] V. Blondel, J. Hendrickx, J. Tsitsiklisč, On Krause's multi-agent consensus model with state-dependent connectivity, IEEE Transactions on Automatic Control 54 (2009) 2586–2596.
[10] G. Como, F. Fagnani, Scaling limits for continuous opinion dynamics systems, Annals of Applied Probability 21 (4) (2011) 1537–1567.
[11] B. Touri, Product of random stochastic matrices and distributed averaging, Ph.D. Dissertation, Dept. of Industrial Engineering in the Graduate College of the University of Illinois at Urbana-Champaign (UIUC), 2011.
[12] A. Czirok, T. Vicsek, Collective behavior of interacting self-propelled particles, Physica A 281 (2006) 17–29.
[13] J. Hu, Y. Hong, Leader-following coordination of multi-agent systems with coupling time delays, Physica A 374 (2007) 853–863.
[14] G. Shi, M. Johansson, K. Johansson, How agreement and disagreement evolve over random dynamic networks, IEEE Journal on Selected Areas in Communications 31 (2013) 1061–1071.
[15] Y. Lou, Y. Hong, Target aggregation of multiple mobile agents by multiple leaders with Markovian switching topologies, Automatica 48 (2012) 559–563.
[16] H. Ishii, R. Tempo, Distributed randomized algorithms for the PageRank computation, IEEE Transactions on Automatic Control 55 (9) (2010) 1987–2002.
[17] G. Shi, K. Johansson, Randomized optimal consensus of multi-agent systems, Automatica 48 (12) (2012) 3018–3030.
[18] Y. Chow, H. Teicher, Probability Theory: Independence Interchangeability Martingales, second ed., Springer-Verlag Press, Heidelberg, Berlin, 1978.
[19] C. Godsil, G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.