



# Leader-following coordination of multi-agent systems with coupling time delays

Jiangping Hu\*, Yiguang Hong

*Key Lab of Systems and Control, Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, China*

Received 5 December 2005; received in revised form 25 March 2006

Available online 7 September 2006

---

## Abstract

In this paper, we consider a leader-following consensus problem of a group of autonomous agents with time-varying coupling delays. Two different cases of coupling topologies are investigated. At first, a necessary and sufficient condition is proved in the case when the interconnection topology is fixed and directed. Then a sufficient condition is proposed in the case when the coupling topology is switched and balanced. Numerical examples are also given to illustrate our results.

© 2006 Elsevier B.V. All rights reserved.

*Keywords:* Leader-following; Multi-agent systems; Time delays; Lyapunov–Razumikhin function

---

## 1. Introduction

Recent years have witnessed steadily increasing recognition and attention of coordinated motion of mobile agents across a broad range of disciplines. Applications can be found in many areas such as biology or ecology (e.g., aggregation behavior of animals in Refs. [1–3]), physics (e.g., collective motion of particles in Refs. [4,5]), and engineering (e.g., formation control of robots in Refs. [6–9]). The studies of multiple autonomous agents focus on understanding the general mechanisms and interconnection rules of cooperative phenomena as well as their potential applications in various engineering problems.

In a multi-agent system, agents are usually coupled and interconnected with some simple rules including nearest-neighbor rules [4,8]. A computer graphics model to simulate collective behavior of multiple agents was presented in Ref. [10]. With a proposed simple model and neighbor-based rules, flocking and schooling were successfully simulated and analyzed for self-propelled particles in Ref. [4]. Also, self-organized aggregation behavior of particle groups with leaders becomes more and more interesting. The coordinated motion of a group of motile particles with a leader has been analyzed in Ref. [11], while leader-follower networks have been also considered in Ref. [12]. Recently, to design distributed flocking algorithms, Olfati-Saber has introduced a theoretical framework including a virtual leader/follower architecture, which is different from conventional leader/follower architecture [9].

---

\*Corresponding author.

*E-mail address:* [jphu@amss.ac.cn](mailto:jphu@amss.ac.cn) (J. Hu).

Sometimes, the coupling delays between agents have to be taken into consideration in practical problems [8,13,14]. For example, Earl and Strogatz [13] proposed a stability criterion for a network of specific oscillators with time-delayed coupling. In Ref. [8], the authors studied consensus problems of continuous-time agents with interconnection communication delays. The dynamics of each agent is first order and the graph to describe the interconnection topology of these agents is undirected.

In this paper, a leader-following consensus problem for multiple agents with coupling time delays is discussed. Here the considered dynamics of each agent is second order, coupling time delay is time varying, and the interconnection graph of the agents is directed. The convergence analysis of the consensus problem with directed graphs (or digraph for short) is more challenging than that of undirected graphs due to the complexity of directed graphs. The analysis becomes harder if time delay is involved. For time-delay systems, modeled by delayed differential equations, an effective way to deal with convergence and stability problems is Lyapunov-based; Lyapunov–Krasovskii functionals or Lyapunov–Razumikhin functions are often used in the analysis [15].

The paper is organized as follows. Section 2 presents the multi-agent model and some preliminaries. Then, two cases, fixed coupling topology and switched coupling topology, are considered. The leader-following convergence of two models in the two cases are analyzed in Sections 3 and 4, respectively. Here, Lyapunov–Razumikhin functions are employed, along with the analysis of linear matrix inequalities. Finally, some concluding remarks are given in Section 5.

By convention,  $R$  and  $Z^+$  represent the real number set and the positive integer set, respectively;  $I_n$  is an  $n \times n$  identity matrix; for any vector  $x$ ,  $x^T$  denotes its transpose;  $\|\cdot\|$  denotes Euclidean norm.

## 2. Model description

We consider a group of  $n + 1$  identical agents, in which an agent indexed by 0 is assigned as the “leader” and the other agents indexed by  $1, \dots, n$  are referred to as “follower-agents” (or “agents” when no confusion arises). The motion of the leader is independent and the motion of each follower is influenced by the leader and the other followers. A continuous-time model of the  $n$  agents is described as follows:

$$\ddot{x}_i = u_i, \quad i = 1, \dots, n, \quad (1)$$

or equivalently,

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = u_i, \end{cases} \quad (2)$$

where the state  $x_i \in R^m$  can be the position vector of agent  $i$ ,  $v_i \in R^m$  its velocity vector and  $u_i \in R^m$  its coupling inputs for  $i = 1, \dots, n$ . Denote

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in R^{mn}.$$

Without loss of generality, in the study of leader-following stability, we take  $m = 1$  for simplicity in the sequel. Then (2) can be rewritten as

$$\begin{cases} \dot{x} = v, \\ \dot{v} = u \in R^n. \end{cases} \quad (3)$$

The dynamics of the leader is described as follows:

$$\dot{x}_0 = v_0 \in R, \quad (4)$$

where  $v_0$  is the desired constant velocity.

If each agent is regarded as a node, then their coupling topology is conveniently described by a simple graph (basic concepts and notations of graph theory can be found in Refs. [16,17,8]). Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$  be a weighted digraph of order  $n$  with the set of nodes  $\mathcal{V} = \{1, 2, \dots, n\}$ , set of arcs  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ , and a weighted adjacency matrix  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$  with nonnegative elements. The node indexes belong to a finite index set  $\mathcal{I} = \{1, 2, \dots, n\}$ . An arc of  $\mathcal{G}$  is denoted by  $(i, j)$ , which starts from  $i$  and ends on  $j$ . The element  $a_{ij}$  associated with the arc of the digraph is positive, i.e.,  $a_{ij} > 0 \Leftrightarrow (i, j) \in \mathcal{E}$ . Moreover, we assume  $a_{ii} = 0$  for all  $i \in \mathcal{I}$ . The set of neighbors of node  $i$  is denoted by  $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ . A cluster is any subset  $\mathcal{J} \subset \mathcal{V}$  of the nodes of the digraph. The set of neighbors of a cluster  $\mathcal{J}$  is defined by  $\mathcal{N}_{\mathcal{J}} = \bigcup_{i \in \mathcal{J}} \mathcal{N}_i = \{j \in \mathcal{V} : i \in \mathcal{J}, (i, j) \in \mathcal{E}\}$ . A path in a digraph is a sequence  $i_0, i_1, \dots, i_f$  of distinct nodes such that  $(i_{j-1}, i_j)$  is an arc for  $j = 1, 2, \dots, f, f \in \mathbb{Z}^+$ . If there exists a path from node  $i$  to node  $j$ , we say that  $j$  is reachable from  $i$ . A digraph  $\mathcal{G}$  is strongly connected if there exists a path between any two distinct nodes. A strong component of a digraph is an induced subgraph that is maximal, subject to being strongly connected. Moreover, if  $\sum_{j \in \mathcal{N}_i} a_{ij} = \sum_{j \in \mathcal{N}_i} a_{ji}$  for all  $i = 1, \dots, n$ , then the digraph  $\mathcal{G}$  is called balanced, which was introduced in Refs. [8,17].

A diagonal matrix  $D = \text{diag}\{d_1, \dots, d_n\} \in \mathbb{R}^{n \times n}$  is a degree matrix of  $\mathcal{G}$ , whose diagonal elements  $d_i = \sum_{j \in \mathcal{N}_i} a_{ij}$  for  $i = 1, \dots, n$ . Then the Laplacian of the weighted digraph  $\mathcal{G}$  is defined as

$$L = D - A \in \mathbb{R}^{n \times n}. \tag{5}$$

To study a leader-following problem, we also concern another graph  $\tilde{\mathcal{G}}$  associated with the system consisting of  $n$  agents and one leader (labeled 0). Similarly, we define a diagonal matrix  $B \in \mathbb{R}^{n \times n}$  to be a leader adjacency matrix associated with  $\tilde{\mathcal{G}}$  with diagonal elements  $b_i$  ( $i \in \mathcal{I}$ ), where  $b_i = a_{i0}$  for some constant  $a_{i0} > 0$  if node 0 (i.e., the leader) is a neighbor of node  $i$  and  $b_i = 0$ , otherwise. For  $\tilde{\mathcal{G}}$ , if there is a path in  $\tilde{\mathcal{G}}$  from every node  $i$  in  $\mathcal{G}$  to node 0, we say that node 0 is globally reachable in  $\tilde{\mathcal{G}}$ , which is much weaker than strong connectedness.

**Example 1.** As shown in Figs. 1 and 2, both  $\tilde{\mathcal{G}}_1$  and  $\tilde{\mathcal{G}}_2$  are not strongly connected, but they have a globally reachable node 0. Suppose that the weight of each arc is 1 in both cases. Obviously,  $\mathcal{G}_2$  with  $\mathcal{V} = \{1, 2, 3, 4\}$  is balanced.

Laplacians of  $\mathcal{G}_1$  and  $\mathcal{G}_2$  as well as the leader adjacency matrices  $B_1, B_2$  are easily obtained as follows:

$$L_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad B_1 = B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The following lemma was obtained in Refs. [6,18].

**Lemma 1.** A digraph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$  has a globally reachable node if and only if for every pair of nonempty, disjoint subsets  $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$  satisfies  $\mathcal{N}_{S_1} \cup \mathcal{N}_{S_2} \neq \emptyset$ .

**Remark 1.** Let  $S_1, S_2, \dots, S_p$  be the strong components of  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  and  $\mathcal{N}_{S_i}$  be the neighbor sets for  $S_i, i = 1, \dots, p, p > 1$ . From Lemma 1, a digraph  $\mathcal{G}$  has a globally reachable node if and only if every pair of

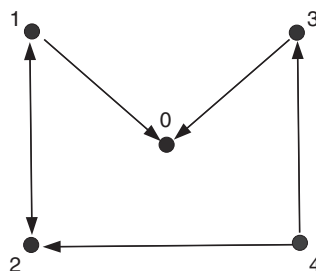


Fig. 1.  $\tilde{\mathcal{G}}_1$  and  $\mathcal{G}_1$ .

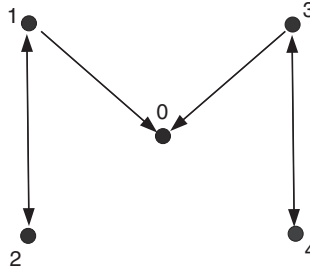


Fig. 2.  $\tilde{\mathcal{G}}_2$  and  $\mathcal{G}_2$ .

$S_i, S_j$  satisfies  $\mathcal{N}_{S_i} \cup \mathcal{N}_{S_j} \neq \emptyset$ . If the graph is strongly connected, then each node is globally reachable from every other node.

The next lemma shows an important property of Laplacian  $L$  [6].

**Lemma 2.** *The digraph  $\mathcal{G}$  has a globally reachable node if and only if Laplacian  $L$  of  $\mathcal{G}$  has a simple zero eigenvalue (with eigenvector  $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^n$ ).*

Due to the coupling delays, each agent cannot instantly get the information from others or the leader. Thus, for agent  $i$  ( $i = 1, \dots, n$ ), a neighbor-based coupling rule can be expressed as follows:

$$u_i(t) = \sum_{j \in \mathcal{N}_i(\sigma)} a_{ij}(x_j(t-r) - x_i(t-r)) + b_i(\sigma)(x_0(t-r) - x_i(t-r)) + k(v_0 - v_i(t)), \quad k > 0, \tag{6}$$

where the time-varying delay  $r(t) > 0$  is a continuously differentiable function with

$$0 < r < \tau, \tag{7}$$

$\sigma : [0, \infty) \rightarrow \mathcal{I}_\Gamma = \{1, \dots, N\}$  ( $N$  denotes the total number of all possible digraphs) is a switching signal that determines the coupling topology. The set  $\Gamma = \{\mathcal{G}_1, \dots, \mathcal{G}_N\}$  is a finite collection of graphs with a common node set  $\mathcal{V}$ . If  $\sigma$  is a constant function, then the corresponding interconnection topology is fixed. In addition,  $\mathcal{N}_i(\sigma)$  is the index set of neighbors of agent  $i$  in the digraph  $\mathcal{G}_\sigma$  while  $a_{ij}$  ( $i, j = 1, \dots, n$ ) are elements of the adjacency matrix of  $\mathcal{G}_\sigma$  and  $b_i(\sigma)$  ( $i = 1, \dots, n$ ) are the diagonal elements of the leader adjacency matrix associated with  $\tilde{\mathcal{G}}_\sigma$ .

With (6), (2) can be written in a matrix form:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -(L_\sigma + B_\sigma)x(t-r) - k(v - v_0\mathbf{1}) + B_\sigma\mathbf{1}x_0(t-r), \end{cases} \tag{8}$$

where  $L_\sigma$  is Laplacian of  $\mathcal{G}_\sigma$  and  $B_\sigma$  is the leader adjacency matrix associated with  $\tilde{\mathcal{G}}_\sigma$ .

In the sequel, we will demonstrate the convergence of the dynamics system (8); that is,  $x_i \rightarrow x_0, v_i \rightarrow v_0$  as  $t \rightarrow \infty$ .

### 3. Fixed coupling topology

In this section, we will focus on the convergence analysis of a group of dynamic agents with fixed interconnection topology. In this case, the subscript  $\sigma$  can be dropped.

Let  $\bar{x} = x - x_0\mathbf{1}$ ,  $\bar{v} = v - v_0\mathbf{1}$ . Because  $-(L + B)x(t-r) + B\mathbf{1}x_0(t-r) = -(L + B)\bar{x}(t-r)$  (invoking Lemma 2), we can rewrite system (8) as

$$\dot{\varepsilon} = C\varepsilon(t-r) + E\varepsilon(t), \tag{9}$$

where

$$\varepsilon = \begin{pmatrix} \bar{x} \\ \bar{v} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & I_n \\ 0 & -kI_n \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 \\ -H & 0 \end{pmatrix}, \quad H = L + B.$$

Before the discussion, we introduce some basic concepts or results for time-delay systems [15]. Consider the following system:

$$\begin{cases} \dot{x} = f(x_t), & t > 0, \\ x(\theta) = \varphi(\theta), & \theta \in [-\tau, 0], \end{cases} \tag{10}$$

where  $x_t(\theta) = x(t + \theta), \forall \theta \in [-\tau, 0]$  and  $f(0) = 0$ . Let  $C([-\tau, 0], \mathbb{R}^n)$  be a Banach space of continuous functions defined on an interval  $[-\tau, 0]$ , taking values in  $\mathbb{R}^n$  with the topology of uniform convergence, and with a norm  $\|\varphi\|_c = \max_{\theta \in [-\tau, 0]} \|\varphi(\theta)\|$ . The following result is for the stability of system (10) (the details can be found in Ref. [15]).

**Lemma 3** (*Lyapunov–Razumikhin theorem*). *Let  $\phi_1, \phi_2$ , and  $\phi_3$  be continuous, nonnegative, nondecreasing functions with  $\phi_1(s) > 0, \phi_2(s) > 0, \phi_3(s) > 0$  for  $s > 0$  and  $\phi_1(0) = \phi_2(0) = 0$ . For system (10), suppose that the function  $f : C([-\tau, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$  takes bounded sets of  $C([-\tau, 0], \mathbb{R}^n)$  in bounded sets of  $\mathbb{R}^n$ . If there is a continuous function  $V(t, x)$  such that*

$$\phi_1(\|x\|) \leq V(t, x) \leq \phi_2(\|x\|), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n. \tag{11}$$

*In addition, there exists a continuous nondecreasing function  $\phi(s)$  with  $\phi(s) > s, s > 0$  such that*

$$\dot{V}(t, x)|_{(10)} \leq -\phi_3(\|x\|) \quad \text{if } V(t + \theta, x(t + \theta)) < \phi(V(t, x(t))), \quad \theta \in [-\tau, 0], \tag{12}$$

*then the solution  $x = 0$  is uniformly asymptotically stable.*

Usually,  $V(t, x)$  is called Lyapunov–Razumikhin function if it satisfies (11) and (12) in Lemma 3.

**Remark 2.** Lyapunov–Razumikhin theorem indicates that it is unnecessary to require that  $\dot{V}(t, x)$  be non-positive for all initial data in order to have stability of system (10). In fact, one only needs to consider the initial data if a trajectory of Eq. (10) starting from these initial data is “diverging” (that is,  $V(t + \theta, x(t + \theta)) < \phi(V(t, x(t)))$  for all  $\theta \in [-\tau, 0]$  in (12)).

A matrix  $A$  is said to have *property SC* [19] if, for every pair of distinct integers  $h, \ell$  with  $1 \leq h, \ell \leq n$ , there is a sequence of distinct integers  $h = i_1, i_2, \dots, i_{j-1}, i_j = \ell, 1 \leq j \leq n$  such that all of the matrix entries  $a_{i_1 i_2}, a_{i_2 i_3}, \dots, a_{i_{j-1} i_j}$  are nonzero. In fact, it is obvious that, if  $\mathcal{G}$  is strongly connected, then its adjacency matrix  $A$  has *property SC*. Moreover, a matrix is called a positive stable matrix if its eigenvalues have positive real-parts. Note that  $H = L + B$  plays a key role in the convergence analysis of system (9). The following lemma shows a relationship between  $H$  and the connectedness of graph  $\mathcal{G}$  (as defined in Section 2).

**Lemma 4.** *The matrix  $H = L + B$  is positive stable if and only if node 0 is globally reachable in  $\mathcal{G}$ .*

**Proof** (*Sufficiency*). Based on Geršgorin disk theorem [19], all the eigenvalues of  $H$  are located in the union of  $n$  discs:

$$Ger(H) = \bigcup_{i=1}^n \left\{ z \in \mathbb{R}^2 : |z - d_i - b_i| \leq \sum_{j \neq i} a_{ij} \right\}.$$

However, for the graph  $\mathcal{G}$ ,  $d_i = \sum_{j \neq i} a_{ij}$ . Thus, every disc with radius  $d_i$  will be located in the right half of the complex plane, and then  $H$  has either zero eigenvalue or eigenvalue with positive real-part. Since node 0 is globally reachable, there exists at least one  $b_i > 0$ . Therefore, at least one Geršgorin circle does not pass through the origin.

The following two cases are considered to prove the sufficient condition:

*Case (i):  $\mathcal{G}$  has a globally reachable node.* Let  $S_1, \dots, S_p$  ( $p \in \mathbb{Z}^+$ ) be the strong components of  $\mathcal{G}$ . If  $p = 1$ ,  $\mathcal{G}$  is strongly connected. Then its adjacency matrix  $A$  has *property SC*. Since  $D + B$  is a diagonal matrix with nonnegative diagonal entries,  $H$  still has *property SC*. By better theorem [19], if zero is an eigenvalue of  $H$ , it is just a boundary point of  $Ger(H)$ . Therefore, every Geršgorin circle passes through zero, which leads to a contradiction. Hence, zero is not an eigenvalue of  $H$ .

If  $p > 1$ , then there is one strong component, say  $S_1$ , having no neighbor set by Lemma 1. We rearrange the indices of  $n$  agents such that the Laplacian of  $\mathcal{G}$  is taken in the form

$$L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}, \tag{13}$$

where  $L_{11} \in R^{\kappa \times \kappa}$  ( $\kappa < n$ ) is Laplacian of the component  $S_1$ . From Lemma 2, zero is a simple eigenvalue of  $L_{11}$  and  $L$ , while  $L_{22}$  is nonsingular. Since node 0 is globally reachable, then the block matrix  $B_1 \neq 0$  with  $B = \text{diag}\{B_1, B_2\}$ . Similar to the case when  $p = 1$ , we conclude that zero is not an eigenvalue of  $L_{11} + B_1$ , and is also not an eigenvalue of  $H$ .

Case (ii):  $\mathcal{G}$  has no globally reachable node. Let  $S_1, \dots, S_p$  be the strong components with  $\mathcal{N}_{S_i} = \emptyset, i = 1, \dots, p, p > 1$  by Lemma 1. Since  $\bigcup_{i=1}^p \mathcal{V}(S_i) \subset \mathcal{V}(\mathcal{G})$ , Laplacian associated with  $\mathcal{G}$  can be transformed to the following form:

$$L = \begin{pmatrix} L_{11} & & & & \\ & \ddots & & & \\ & & L_{pp} & & \\ L_{p+1,1} & \cdots & L_{p+1,p} & L_{p+1,p+1} & \end{pmatrix}, \tag{14}$$

where  $L_{ii}$  is the Laplacian associated with  $S_i$  for  $i = 1, \dots, p$ . One can easily verify that  $L_{p+1,p+1}$  is nonsingular. Since node 0 is globally reachable, then  $B_i \neq 0$  for  $i = 1, \dots, p$  where  $B_i$ , corresponding to  $L_{ii}$ , are diagonal blocks of  $B$ . Similar to the proof in Case (i), we can obtain that zero is not eigenvalue of  $H_i$  or  $H$ .

(Necessity). If node 0 is not globally reachable in  $\tilde{\mathcal{G}}$ , then we also have:

Case (i):  $\mathcal{G}$  has a globally reachable node. As discussed before, assume  $S_1$  has no neighbor set, and then we have (13), where  $L_{11} \in R^{\kappa \times \kappa}$  ( $\kappa \in Z^+$ ) is the Laplacian of  $S_1$ . Invoking Lemma 2, zero is a simple eigenvalue of  $L_{11}$  and  $L$ , while  $L_{22}$  is nonsingular. By the assumption that node 0 is not globally reachable in  $\tilde{\mathcal{G}}$ , the block matrix  $B_1 = 0$  with  $B = \text{diag}\{B_1, B_2\}$ . Therefore, zero is a simple eigenvalue of  $L_{11} + B_1$ , and is also a simple eigenvalue of  $H$ . This leads to a contradiction.

Case (ii):  $\mathcal{G}$  has no globally reachable node. As discussed before, we have (14). Since node 0 is not globally reachable in  $\tilde{\mathcal{G}}$ , there exists at least one  $B_i = 0$  for  $i = 1, \dots, p$  where  $B_i$ , corresponding to  $L_{ii}$ , are diagonal blocks of  $B$ . Thus,  $H_i$  and  $H$  have more than one zero eigenvalues. This implies a contradiction.  $\square$

Therefore, if node 0 is globally reachable in  $\tilde{\mathcal{G}}$ ,  $H$  is positive stable, and from Lyapunov theorem, there exists a positive definite matrix  $\tilde{P} \in R^{n \times n}$  such that

$$\tilde{P}H + H^T \tilde{P} = I_n. \tag{15}$$

Let  $\bar{\mu} = \max\{\text{eigenvalues of } \tilde{P}H H^T \tilde{P}\}$  and let  $\bar{\lambda}$  be the smallest eigenvalue of  $\tilde{P}$ . Now we give the main result as follows.

**Theorem 1.** For system (9), take

$$k > k^* = \frac{\bar{\mu}}{2\bar{\lambda}} + 1. \tag{16}$$

Then, when  $\tau$  is sufficiently small,

$$\lim_{t \rightarrow \infty} \varepsilon(t) = 0, \tag{17}$$

if and only if node 0 is globally reachable in  $\tilde{\mathcal{G}}$ .

**Proof (Sufficiency).** Since node 0 is globally reachable in  $\tilde{\mathcal{G}}$ ,  $H$  is positive stable and  $\tilde{P}$  is a positive definite matrix satisfying (15). Take a Lyapunov–Razumikhin function  $V(\varepsilon) = \varepsilon^T P \varepsilon$ , where

$$P = \begin{pmatrix} k\tilde{P} & \tilde{P} \\ \tilde{P} & \tilde{P} \end{pmatrix} \quad (k > 1)$$

is positive definite.

Then we consider  $\dot{V}(\varepsilon)|_{(9)}$ .

By *Leibniz–Newton* formula,

$$\begin{aligned} \varepsilon(t-r) &= \varepsilon(t) - \int_{-r}^0 \dot{\varepsilon}(t+s) \, ds \\ &= \varepsilon(t) - C \int_{-r}^0 \varepsilon(t+s) \, ds - E \int_{-2r}^{-r} \varepsilon(t+s) \, ds. \end{aligned}$$

Thus, from  $E^2 = 0$ , the delayed differential equation (9) can be rewritten as

$$\dot{\varepsilon} = F\varepsilon - EC \int_{-r}^0 \varepsilon(t+s) \, ds,$$

where  $F = C + E$ .

Note that  $2a^T b \leq a^T \Psi a + b^T \Psi^{-1} b$  holds for any appropriate positive definite matrix  $\Psi$ . Then, with  $a = -C^T E^T P \varepsilon$ ,  $b = \varepsilon(t+s)$  and  $\Psi = P^{-1}$ , we have

$$\begin{aligned} \dot{V}|_{(9)} &= \varepsilon^T (F^T P + PF) \varepsilon - 2\varepsilon^T PEC \int_{-r}^0 \varepsilon(t+s) \, ds \\ &\leq \varepsilon^T (F^T P + PF) \varepsilon + r\varepsilon^T PECP^{-1} C^T E^T P \varepsilon + \int_{-r}^0 \varepsilon^T(t+s) P \varepsilon(t+s) \, ds. \end{aligned}$$

Take  $\phi(s) = qs$  for some constant  $q > 1$ . In the case of

$$V(\varepsilon(t+\theta)) < qV(\varepsilon(t)), \quad -\tau \leq \theta \leq 0, \tag{18}$$

we have

$$\dot{V} \leq -\varepsilon^T Q \varepsilon + r\varepsilon^T (PECP^{-1} C^T E^T P + qP) \varepsilon,$$

where

$$Q = -(F^T P + PF) = \begin{pmatrix} I_n & H^T \tilde{P} \\ \tilde{P} H & 2(k-1)\tilde{P} \end{pmatrix}.$$

$Q$  is positive definite if  $k$  satisfies (16), according to Lemma 4 and Schur complements theorem [19]. Let  $\lambda_{\min}$  denote the minimum eigenvalues of  $Q$ . If we take

$$r < \tau = \frac{\lambda_{\min}}{\|PECP^{-1} C^T E^T P\| + q\|P\|}, \tag{19}$$

then  $\dot{V}(\varepsilon) \leq -\eta \varepsilon^T \varepsilon$  for some  $\eta > 0$ . Therefore, the conclusion follows by Lemma 3.

(Necessity). Since system (9) is asymptotically stable, the eigenvalues of  $F$  have negative real-parts, which implies that  $H$  is positive stable. By Lemma 4, node 0 is globally reachable in  $\tilde{\mathcal{G}}$ .  $\square$

**Remark 3.** In the proof of Theorem 1, we have obtained a finite bound of the considered time-varying delay, that is,  $\tau$  in (19), though “ $\tau$  is sufficiently small” is mentioned in Theorem 1.

**Remark 4.** Obviously, (17) still holds if the time delay is constant. Moreover, if the system (2) is free of time-delay (that is,  $r = 0$ ), then the coupling rule (6) becomes

$$u_i(t) = \sum_{j \in \mathcal{N}_i(\sigma)} a_{ij}(x_j(t) - x_i(t)) + b_i(\sigma)(x_0(t) - x_i(t)) + k(v_0 - v_i(t)),$$

which is consistent with the nearest-neighbor rules in Ref. [8].

For illustration, we give an numerical example with the interconnection graph given in Fig. 1. It is not hard to obtain

$$\bar{\mu} = 0.3139, \quad \bar{\lambda} = 0.1835, \quad k^* = 2.7106,$$

$$\lambda_{min} = 0.3325, \quad q = 1.0500, \quad \tau = 0.0334,$$

$$\bar{P} = \begin{pmatrix} 0.5379 & 0.5758 & 0.0439 & 0.0227 \\ 0.5758 & 1.1667 & 0.1091 & 0.0909 \\ 0.0439 & 0.1091 & 0.5833 & 0.0833 \\ 0.0227 & 0.0909 & 0.0833 & 0.2500 \end{pmatrix}.$$

Take  $k = 3$  and the time-varying delay  $r(t) = 0.0300|\cos(t)|$  in the simulation.

Fig. 3 shows the simulation results for both position errors and velocity errors, while Fig. 4 demonstrates that the trajectories of the four agents and the one of the leader.

#### 4. Switched coupling topology

Consider system (8) with switched coupling topology. Still taking  $\bar{x} = x - x_0\mathbf{1}, \bar{v} = v - v_0\mathbf{1}$ , we have

$$\dot{\varepsilon} = C\varepsilon(t - r) + E_\sigma\varepsilon(t), \tag{20}$$

where  $\sigma$  is the switching signal as defined in Section 2, and

$$E_\sigma = \begin{pmatrix} 0 & 0 \\ -H_\sigma & 0 \end{pmatrix}, \quad H_\sigma = L_\sigma + B_\sigma.$$

At first, we study the matrix  $H_\sigma = L_\sigma + B_\sigma$ .

**Lemma 5.** Suppose  $\mathcal{G}_\sigma$  is balanced. Then  $H_\sigma + H_\sigma^T$  is positive definite if and only if node 0 is globally reachable in  $\mathcal{G}$ .

**Proof (Necessity).** The proof is quite trivial and omitted here.

(Sufficiency). Because  $\mathcal{G}_\sigma$  is balanced, it is strongly connected if it has a globally reachable node. Then from Theorem 7 in Ref. [8],  $\frac{1}{2}(L_\sigma + L_\sigma^T)$  is a valid Laplacian matrix with single zero eigenvalue. After some manipulations, it is not difficult to obtain that  $\frac{1}{2}(L_\sigma + L_\sigma^T) + B_\sigma$  is positive definite (the details can be found in Ref. [7]) and so is  $H_\sigma + H_\sigma^T$ .

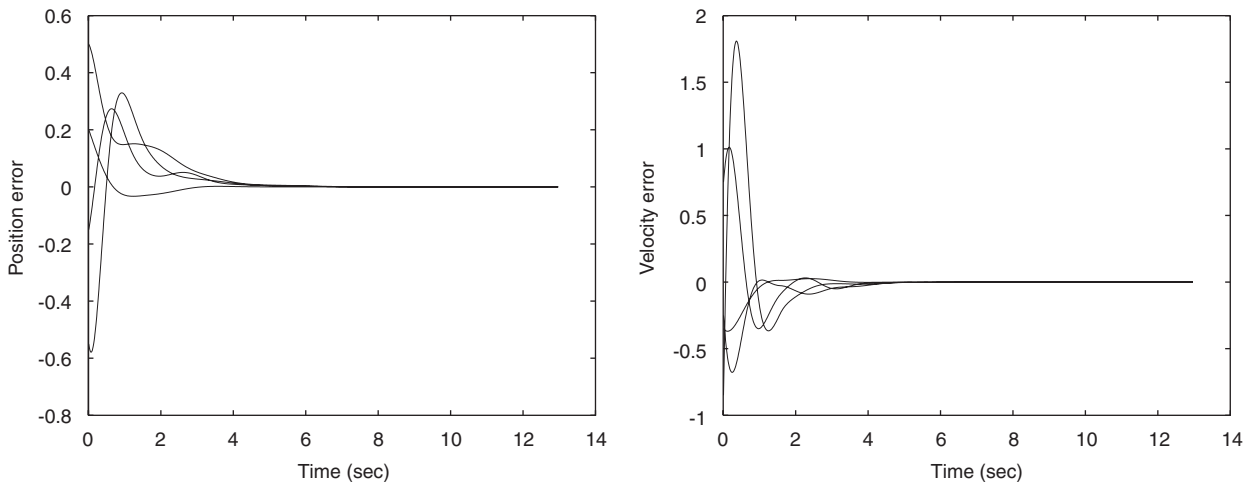


Fig. 3. Leader-following errors of four agents with the coupling topology shown in Fig. 1.



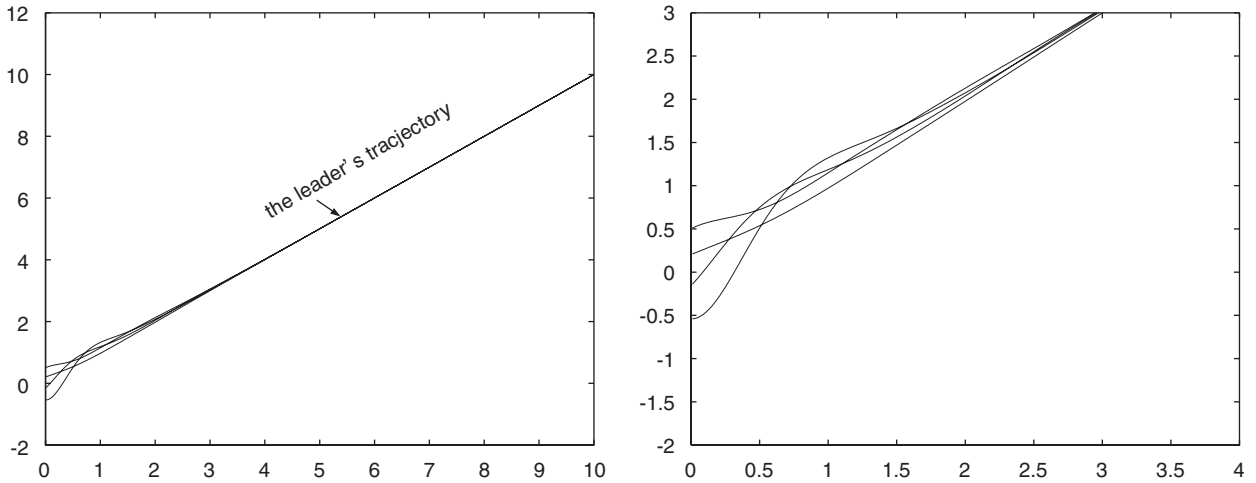


Fig. 4. Trajectories of four agents and the leader with the coupling topology shown in Fig. 1.

If  $\mathcal{G}_\sigma$  has no globally reachable node, then there is no arc between every pair of distinct strong components and we can renumber the nodes so that Laplacian associated with  $\mathcal{G}_\sigma$  has the form

$$L_\sigma = \begin{pmatrix} L_{11}(\sigma) & & & \\ & L_{22}(\sigma) & & \\ & & \ddots & \\ & & & L_{pp}(\sigma) \end{pmatrix}, \tag{21}$$

where each  $L_{ii}(\sigma)$  is Laplacian associated with a strong component  $S_i$  for  $i = 1, \dots, p, p > 1$ . By the assumption that node 0 is globally reachable in  $\mathcal{G}_\sigma$ , each diagonal block matrix  $B_i(\sigma)$ , corresponding to  $L_{ii}(\sigma)$ , is nonzero. Then, it is easy to see that  $\frac{1}{2}(L_{ii}(\sigma) + L_{ii}(\sigma^T)) + B_i(\sigma)$  is positive definite and therefore,  $H_\sigma + H_\sigma^T$  is positive definite.  $\square$

Based on the balanced graph  $\mathcal{G}_\sigma$  (with Lemma 5) and the fact that the set  $\mathcal{I}_r$  is finite, both  $\tilde{\lambda} = \min\{\text{eigenvalues of } H_\sigma + H_\sigma^T\} > 0$  and  $\tilde{\mu} = \max\{\text{eigenvalues of } H_\sigma H_\sigma^T\} > 0$  can be well defined.

**Theorem 2.** For system (20) with balanced graph  $\mathcal{G}_\sigma$ , take

$$k > k^* = \frac{\tilde{\mu}}{2\tilde{\lambda}} + 1. \tag{22}$$

If node 0 is globally reachable in  $\tilde{\mathcal{G}}_\sigma$  and  $\tau$  is sufficiently small, then

$$\lim_{t \rightarrow \infty} \varepsilon(t) = 0.$$

**Proof.** Take a Lyapunov–Razumikhin function  $V(\varepsilon) = \varepsilon^T \Phi \varepsilon$ , where

$$\Phi = \begin{pmatrix} kI_n & I_n \\ I_n & I_n \end{pmatrix} \quad (k > 1)$$

is positive definite.

Similar to the analysis in the proof of Theorem 1, we can obtain

$$\dot{V} \leq \varepsilon^T (F_\sigma^T \Phi + \Phi F_\sigma) \varepsilon + r \varepsilon^T \Phi E_\sigma C_\sigma \Phi^{-1} C_\sigma^T E_\sigma^T \Phi \varepsilon + \int_{-r}^0 \varepsilon^T(t+s) \Phi \varepsilon(t+s) ds.$$

Take  $\phi(s) = qs$  for some constant  $q > 1$ . In the case of

$$V(\varepsilon(t+\theta)) < qV(\varepsilon(t)), \quad -\tau \leq \theta \leq 0, \tag{23}$$

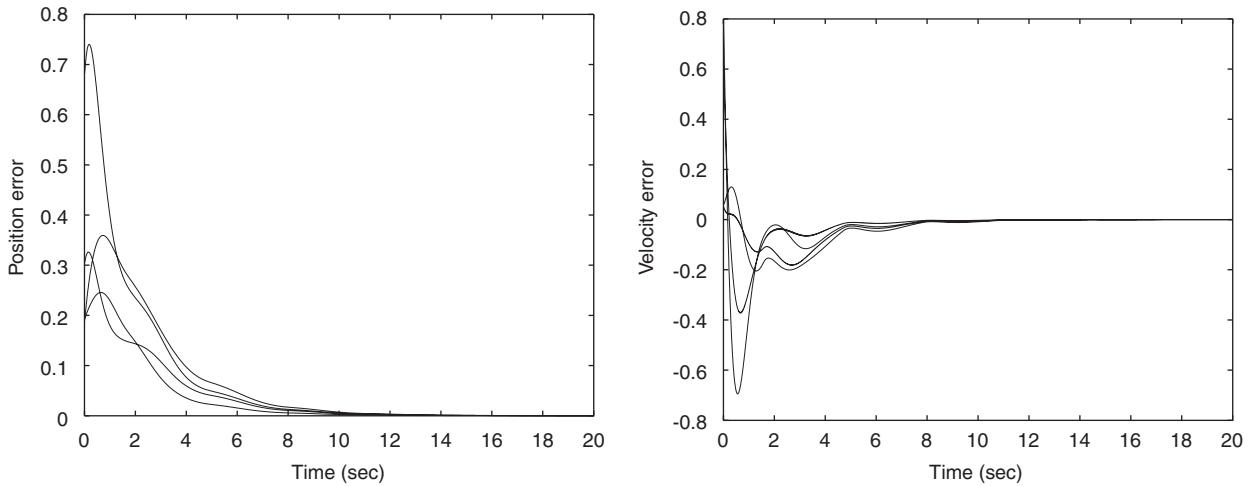


Fig. 5. Leader-following errors with two switching graphs given in Figs. 1 and 2.

we have

$$\dot{V} \leq -\varepsilon^T Q_\sigma \varepsilon + r \varepsilon^T (\Phi E_\sigma C_\sigma \Phi^{-1} C_\sigma^T E_\sigma^T \Phi + q \Phi) \varepsilon,$$

where

$$Q_\sigma = -(F_\sigma^T \Phi + \Phi F_\sigma) = \begin{pmatrix} H_\sigma^T + H_\sigma & H_\sigma^T \\ H_\sigma & 2(k-1)I_n \end{pmatrix}.$$

$Q_\sigma$  is positive definite for any value of  $\sigma$  and then  $\dot{V}(\varepsilon)$  is negative definite if we take (22) and

$$r < \tau = \frac{\lambda_{min}}{(2k/(k-1))\tilde{\mu} + \frac{1}{2}q(k+1 + \sqrt{(k-1)^2 + 4})}, \tag{24}$$

where  $\lambda_{min}$  denotes the minimum eigenvalue of all possible  $Q_\sigma$ . Thus, the conclusion is obtained according to Lemma 3.  $\square$

In the switching case, the assumption of balanced graph  $\mathcal{G}_\sigma$  is not necessary in Theorem 2. The following numerical example shows that the stability can be obtained even if the coupling topology graph is not balanced sometimes.

Here we consider there are two coupling topologies, given in Figs. 1 and 2, switching between each other, with the following switching order:  $\{\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \dots\}$ . With simple calculations, we have

$$\tilde{\lambda} = 0.5028, \quad \tilde{\mu} = 7.9257, \quad k^* = 7.8816,$$

$$\lambda_{min} = 0.4781, \quad q = 1.0500, \quad \tau = 0.0174.$$

Take  $k = 9$  and the time-varying delay  $r(t) = 0.0150|\cos(t)|$ . Then the simulation results are shown in Fig. 5.

### 5. Conclusions

This paper addressed a coordination problem of a multi-agent system with a leader. A leader moves at the constant velocity and the follower-agents follow it though there are time-varying coupling delays. When the coupling topology was fixed and directed, a necessary and sufficient condition was given. When the coupling topology was switched and balanced, a sufficient condition was presented. Moreover, several numerical simulations were shown to verify the theoretical analysis.

## Acknowledgment

This work was supported by the NNSF of China under Grants 60425307, 10472129, 50595411, and 60221301.

## References

- [1] R. Amritkar, S. Jalan, *Physica A* 321 (2003) 220.
- [2] K. Warburton, J. Lazarus, *J. Theor. Biol.* 150 (1991) 473.
- [3] C.M. Breder, *Ecology* 35 (1954) 361.
- [4] T. Vicsek, A. Czirok, E.B. Jacob, I. Cohen, O. Schochet, *Phys. Rev. Lett.* 75 (1995) 1226.
- [5] A. Czirok, T. Vicsek, *Physica A* 281 (2000) 17.
- [6] Z. Lin, B. Francis, M. Maggiore, *IEEE Trans. Autom. Control* 50 (1) (2005) 121.
- [7] Y. Hong, J. Hu, L. Gao, *Automatica* 42 (2006) 1177.
- [8] R. Olfati-Saber, R. Murray, *IEEE Trans. Autom. Control* 49 (9) (2004) 1520.
- [9] R. Olfati-Saber, *IEEE Trans. Autom. Control* 51 (3) (2006) 401.
- [10] C.W. Reynolds, *ACM SIGGRAPH '87 Conference Proceedings* vol. 21(4) 1987, p. 25.
- [11] S. Mu, T. Chu, L. Wang, *Physica A* 351 (2005) 211.
- [12] W. Wang, J.J.E. Slotine, *Biol. Cybern.* 92 (2005) 38.
- [13] M.G. Earl, S.H. Strogatz, *Phys. Rev. E* 67 (2003) 036204.
- [14] G. Kozyreff, A.G. Vladimirov, P. Mandel, *Phys. Rev. Lett.* 85 (2000) 3809.
- [15] J.K. Hale, S.M.V. Lunel, *Introduction to the Theory of Functional Differential Equations*, Applied Mathematical Sciences, vol. 99, Springer, New York, 1991.
- [16] J. Bang-Jensen, G. Gutin, *Digraphs Theory, Algorithms and Applications*, Springer, New York, 2002.
- [17] C. Godsil, G. Royle, *Algebraic Graph Theory*, Springer, New York, 2001.
- [18] L. Moreau, *IEEE Trans. Autom. Control* 50 (2) (2005) 169.
- [19] R. Horn, C. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.