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Semi-global output consensus of a group of linear systems in the presence of external disturbances and actuator saturation: An output regulation approach

Zhiyun Zhao[1], Yiguang Hong[2] and Zongli Lin[3,*,†]

[1] *Department of Automation, and Key Laboratory of System Control and Information Processing of the Ministry of Education, Shanghai Jiao Tong University, Shanghai 200240, China*
[2] *Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*
[3] *Charles L. Brown Department of Electrical and Computer Engineering, University of Virginia, Charlottesville, VA 22904-4743 USA*

SUMMARY

This paper studies the problem of semi-global leader-following output consensus of a multi-agent system. The output of each follower agent in the system, described by a same general linear system subject to external disturbances and actuator saturation, is to track the output of the leader, described by a linear system, which also generates disturbances as the exosystem does in the classical output regulation problem. Conditions on the agent dynamics are identified, under which a low-gain feedback-based linear state-control algorithm is constructed for each follower agent such that the output consensus is achieved when the communication topology among the agents is a digraph containing no loop, and the leader is reachable from any follower agent. We also extend the results to the non-identical disturbance case. In this case, conditions based on both the agent dynamics and the communication topology are identified, under which a low-gain feedback-based linear state-control algorithm is constructed for each follower agent such that the leader-following output consensus is achieved when the communication topology among the follower agents is a strongly connected and detailed balanced digraph, and the leader is a neighbor of at least one follower. In addition, under some further conditions on the agent dynamics, the control algorithm is adapted so as to achieve semi-global leader-following output consensus for a jointly connected undirected graph and the leader reachable from at least one follower. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The consensus problem for multi-agent systems has drawn much attention in recent years because of its many applications, such as cooperative control [1, 2], energy consumption control in smart grids [3], and formation of vehicles [4], and in part to the theoretical challenges it poses. In a leader-following output consensus problem, the output of each follower agent is required to track that of the leader agent through the use of information of its neighbors. This is different from the general leader-following consensus problem (see, for example, [5–10]), where the full state, not just the output, of each follower agent is required to track that of the leader agent.

*Correspondence to: Zongli Lin, Charles L. Brown Department of Electrical and Computer Engineering, University of Virginia, Charlottesville, VA 22904-4743, USA.
†E-mail: zl5y@virginia.edu

Several results have been obtained from the solution of the output consensus problem [11–15]. In particular, reference [11] proposes both a dynamic state feedback-control algorithm and a dynamic measurement output feedback-control algorithm by devising a distributed observer network. It is proven that, under certain conditions on the agent dynamics, the proposed control algorithms achieve output consensus over a jointly connected communication topology. Reference [12] considers robust output consensus for a class of heterogeneous uncertain linear multi-agent systems. It establishes the solvability of the problem and follows an internal model approach to construct both a dynamic state feedback-control law and a dynamic output feedback-control law. These feedback laws achieve leader-following output consensus when the communication topology among the agents is a digraph that contains no loop, and the leader is globally reachable. Reference [13] studies the leaderless output consensus problems for the same agent dynamics as in [11]. Reference [14] points out that the synchronized output regulation problem of a multi-leader system can be decoupled into an output regulation problem and an asymptotic stability problem. A distributed synchronous protocol is then constructed to solve the problem. Reference [15] considers the problem of output synchronization for heterogeneous networks of non-introspective agents, which have no knowledge of their own state or output separate from what is received from the network.

A feature that is largely missing from the existing work on output consensus is the consideration of actuator saturation, which has been extensively studied in the context of individual systems. In particular, the semi-global output regulation problem of an individual system has been studied in [16]. It is known that a linear system subject to input saturation cannot achieve global output regulation. The conditions are identified under which the semi-global output regulation problem is solvable with both state-feedback and error-feedback laws. However, very few results have been obtained for output consensus (or regulation) of multi-agent systems subject to actuator saturation. Only recently has some solvability conditions based on both the agent dynamics the communication topology among the agents been provided in reference [17].

In this paper, we study the problem of leader-following output consensus of a group of linear follower agents in the presence of external disturbance and actuator saturation. The linear leader agent generates the disturbances and the reference output as the exosystem does in the classical output regulation problem. We establish conditions based on the agent dynamics, under which we construct a low-gain feedback-based control algorithm for each follower agent. We show that these control algorithms achieve semi-global leader-following output consensus when the communication topology among the agents is a digraph containing no loop, and the leader is reachable from any follower agent.

We then extend the results to the non-identical disturbance case. In this case, we establish conditions based on both the agent dynamics and the communication topology among the agents, under which we construct a low-gain feedback-control algorithm for each follower agent and show that these control algorithms achieve semi-global leader-following output consensus when the communication topology among the agents is a strongly connected and detailed balanced digraph, and the leader is a neighbor of at least one follower. In addition, we identify further conditions on the agent dynamics under which the control algorithm can also achieve semi-global output consensus when the communication topology is a jointly connected undirected graph, and the leader is reachable for at least one follower.

An outline of this paper is as follows. In Section 2, we state the problem of semi-global leader-following output consensus of a linear multi-agent system in the presence of external disturbances and actuator saturation and recall basic definitions and relevant results in graph theory. In Section 3, we establish the solvability conditions for the problem and construct linear state feedback-control laws to achieve semi-global output consensus for the identical disturbance case. In Section 4, we obtain the results for the non-identical disturbance case. In Section 5, we give simulation results to illustrate the theoretical results. We draw a brief conclusion to the paper in Section 6.

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2. PROBLEM STATEMENT AND PRELIMINARIES

Consider a multi-agent system consisting of N follower agents and one exosystem (or leader). Each follower agent is described by the dynamics of a linear system,

$$\dot{x}_i = Ax_i + B\sigma(u_i) + E_i w, \quad y_i = Cx_i, \quad i = 1, 2, \cdots, N, \tag{1}$$

where $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^p$, and $u_i \in \mathbb{R}^m$ are, respectively, the state, output, and control input of agent i; $E_i w$ is the disturbance to be described later; and $\sigma(s)$ is a saturation function defined as $\sigma(s) = \text{sign}(s) \min\{\Delta, |s|\}$ for a constant $\Delta > 0$. Here, we have slightly abused the notation by using σ to denote both a scalar-valued and a vector-valued saturation function. That is, for $s = [s_1, s_2, \cdots, s_m]^T$, $\sigma(s) = [\sigma(s_1), \sigma(s_2), \cdots, \sigma(s_m)]^T$.

Note that the agent dynamics in (1) are, in general, not identical because $E_i, i = 1, 2, \cdots, N$, may be different. When the agent dynamics are identical, that is, when $E_i = E, \ i = 1, 2, \cdots, N$, (1) simplifies to

$$\dot{x}_i = Ax_i + B\sigma(u_i) + Ew, \quad y_i = Cx_i, \quad i = 1, 2, \cdots, N. \tag{2}$$

In the sequel, we will refer to this case as *the identical disturbance case* because the disturbances enter each follower agent in a same way. Correspondingly, the case with $E_i, \ i = 1, 2, \cdots, N$, being not identical, is referred to as *the non-identical disturbance case* because the disturbances may enter each follower agent differently.

We make the following assumption on the follower agents.

Assumption 2.1
The pair (A, B) is stabilizable, and all eigenvalues of A are in the closed left-half plane.

Remark 2.1
Assumption 2.1 is required to achieve semi-global leader-following output consensus, as described in the succeeding paragraphs. In the absence of this assumption, only local results can be obtained [18].

The dynamics of the exosystem is also described by a linear system,

$$\dot{w} = Sw, \quad y = -Qw, \tag{3}$$

where $w \in \mathbb{R}^r$ and $y \in \mathbb{R}^q$ are the state and output of the exosystem. Let $w(0) \in \mathcal{W}_0$, where $\mathcal{W}_0$ is a bounded set in $\mathbb{R}^r$. The exosystem generates both the reference signal y, which is sent for the followers to track via a neighbor-based multi-agent network, and the signals $E_i w$ to disturb the followers, as an exosystem does in the classical output regulation problem. In the sequel, for convenience, we will use 'leader', instead of 'exosystem', to emphasize the tracking of the reference signal (that is, the output y of the leader) by the followers.

The output consensus problem studied in this paper is stated as follows. Consider a multi-agent system consisting of the follower agents (2) and the leader (3) operating on an underlying communication network. Denote $e_i = y_i - y, i = 1, 2, \cdots, N$, as the error between the output of agent i and that of the leader. For any *a priori* given bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$ and $\mathcal{W}_0 \subset \mathbb{R}^r$, construct a linear state feedback-control law u_i for each follower, which only uses local information, such that all these state feedback-control laws together achieve leader-following output consensus, that is, for all $x_i(0) \in \mathcal{X}_0, i = 1, 2, \cdots, N$, and $w(0) \in \mathcal{W}_0$,

$$\lim_{t\to\infty} e_i(t) = \lim_{t\to\infty} (y_i(t) - y(t)) = 0, \ \ i = 1, 2, \cdots, N.$$

Remark 2.2
There are some results on the leader-following consensus with actuator saturation (see [19] and references therein), but very few results were obtained for the output consensus with actuator saturation. Only recently was the semi-global output consensus problem of linear multi-agent systems

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in the presence of external disturbances and actuator saturation studied in [17]. The results to be presented in the current paper are different to those in [17] in several ways. First, in the identical disturbance case, [17] establishes conditions based on the agent dynamics and the communication topology among agents jointly. Thus, when the topology changes, the coefficients in control law, calculated from the solvability conditions, have to be recalculated. On the other hand, we establish conditions based only on the agent dynamics and analyze the trajectory of each agent rather than that of the whole multi-agent system. Second, in the non-identical disturbance case, [17] requires the output matrices of the follower agents and the leader to be invertible, which we do not require. Third, in [17], the communication topology among the follower agents is an undirected switching graph. In our paper, the topology is a digraph without loop for the identical disturbance case; the topology among the follower agents is either a strongly connected and detailed balanced digraph or a jointly-connected undirected graph for the non-identical disturbance case. Note that our jointly connected undirected graph assumption is much more relaxed than that of [17].

Graphs are often used to represent the communication topology among agents. A digraph (or directed graph) $\mathcal{G}$ consists of a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{\nu_1, \nu_2, \cdots, \nu_N\}$ is a nonempty set of nodes, each denoting a follower agent, and $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ is a set of edges, each denoting an ordered pairs of nodes. An edge (ν_i, ν_j) in a digraph denotes that ν_j (i.e., agent j) has access to the information of ν_i. When an edge (ν_i, ν_j) exists, ν_i is said to be a father agent of ν_j and ν_j be a child agent of ν_i. Denote the set of father agents of ν_i as $\mathcal{N}_i = \{\nu_j : (\nu_j, \nu_i) \in \mathcal{E}\}$. Let $|\mathcal{N}_i|$ be the cardinality of the set $\mathcal{N}_i, i = 1, 2, \cdots, N$. In an undirected graph, $(\nu_i, \nu_j) \in \mathcal{E}$ implies $(\nu_j, \nu_i) \in \mathcal{E}$. A sequence of edges of the form $(\nu_1, \nu_2), (\nu_2, \nu_3), \cdots, (\nu_k, \nu_{k+1})$ in a digraph is called a directed path between ν_1 and ν_{k+1}. If $\nu_1 = \nu_{k+1}$, then such a directed path is called a loop. If there exists a directed path between ν_i and ν_j, then ν_i is said to be reachable form ν_j. Let d_{ij} be the length of the longest directed path between ν_i and agent j. Let the leader agent be labeled as ν_0 and $K = \max\{d_{0i}\} \leqslant N$.

Let $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ be the adjacency matrix associated with $\mathcal{G}$, where $a_{ij} = 1$ if $(\nu_j, \nu_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Here, we assume that $a_{ii} = 0$ for all $i = 1, 2, \cdots, N$. Let $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ be the Laplacian matrix associated with $\mathcal{A}$, where $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ when $i \neq j$.

The communication between follower ν_i and the leader ν_0 is denoted as a_{i0}, where $a_{i0} = 1$ if ν_i has access to the information of the leader and $a_{i0} = 0$ otherwise. Denote $\mathcal{L}_0 = \mathrm{diag}\{a_{10}, a_{20}, \cdots, a_{N0}\}$. For an integer $k \leqslant N$, denote $D_k = \{\nu_i : d_{0i} = k\}$ and $|D_k|$ as its the cardinality.

The communication topology among the follower agents is said to be jointly connected, if there exist an infinite sequence of contiguous, nonempty and uniformly bounded time-intervals $[t_k, t_{k+1}), k = 1, 2, \cdots$, such that for each time-interval $[t_k, t_{k+1})$, there are l_k topologies switching over the time-subintervals $\left[t_k^l, t_k^{l+1}\right), l = 0, 1, \cdots, l_k - 1$, with $t_k^0 = t_k$, $t_k^{l_k} = t_{k+1}$ and $t_k^{l+1} - t_k^l \geqslant \tau$, for some constant $\tau > 0$; in each time-subinterval $\left[t_k^l, t_k^{l+1}\right)$, the communication topology among the follower agents is a fixed graph, while in each time-interval $[t_k, t_{k+1})$, the union of the l_k graphs over the l_k time-subintervals is a connected graph.

We also recall the following property of a parameterized algebraic Riccati equation, which will be needed in the design of low-gain feedback laws.

Lemma 2.1 ([18]*)*
Let Assumption 2.1 hold. Let γ be any given positive scalar. Then, for each $\varepsilon \in (0, 1]$, there exists a unique matrix $P(\varepsilon)$ that solves the algebraic Riccati equation (ARE)

$$A^T P + PA - 2\gamma PBB^T P + \varepsilon I = 0. \tag{4}$$

Moreover, $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$.

Lemma 2.2
Let $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^m$ be positive definite matrices. Denote the smallest eigenvalue of X as λ_X. Then for any column vector $x \in \mathbb{R}^{nm}$ and $\gamma \in (0, \lambda_X]$, we have

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$$x^T\left(X^2 \otimes Y\right)x \geqslant \gamma x^T(X \otimes Y)x.$$

Lemma 2.2 is a more general form of Lemma 3 in [19] and can be established based on the same analysis as in [19].

We will present our output consensus results for the identical disturbance case and non-identical disturbance case separately in the following two sections.

3. THE IDENTICAL DISTURBANCE CASE

In this section, we consider the semi-global output consensus for system (2), under the following assumption on the communication topology.

Assumption 3.1
The digraph $\mathcal{G}$ contains no loop and agent v_0 is reachable from any follower agent v_i, $i = 1, 2, \cdots, N$.

This assumption has been widely used in the study of output consensus (e.g., [12]).

The following two assumptions on the followers and the leader define the solvability of the semi-global output consensus problem for the multi-agent system (2)–(3).

Assumption 3.2
There exist matrices Π, Γ, and H that solve the following linear matrix equations:

$$\Pi S = A\Pi + B\Gamma + E, \quad C\Pi = -Q, \quad H\Pi = \Gamma. \tag{5}$$

Assumption 3.3
There exist a positive number $\delta < \Delta$ and a $T \geqslant 0$ such that $\|\Gamma w\|_{\infty,T} \leqslant \delta$ for all w with $w(0) \in \mathcal{W}_0$, where $\|\Gamma w\|_{\infty,T} := \sup_{t \geqslant T}\|\Gamma w\|_\infty$.

Under Assumptions 2.1 and 3.2, we construct the following state feedback-control algorithm for each follower agent,

$$u_i = -\frac{1}{|\mathcal{N}_i|}B^T P\left(\sum_{j\in\mathcal{N}_i\setminus\{0\}} a_{ij}(x_i - x_j) + a_{i0}(x_i - \Pi w)\right) + \frac{1}{|\mathcal{N}_i|}\left(\sum_{j\in\mathcal{N}_i\setminus\{0\}} Hx_j + a_{i0}\Gamma w\right),$$
$$i = 1, 2, \cdots, N, \tag{6}$$

where Π, H, and Γ are a solution to the linear matrix (5), whose existence is guaranteed by Assumption 3.2, and $P := P(\varepsilon)$ is the solution to the ARE (4), with $\gamma = \frac{1}{2}$, as given in Lemma 2.1.

We have the following result on semi-global leader-following output consensus of the multi-agent system (2)–(3).

Theorem 3.1
Let Assumptions 2.1, 3.1, 3.2, and 3.3 hold. Then, under the linear state feedback control laws (6), the group of follower agents (2) and the leader agent (3) achieve semi-global leader-following output consensus. That is, for any *a priori* given bounded set $\mathcal{X}_0 \subset \mathrm{R}^n$, there exists an $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{t\to\infty} e_i(t) = 0, \quad i = 1, 2, \cdots, N,$$

hold for all $x_i(0) \in \mathcal{X}_0, i = 1, 2, \cdots, N$, and $w(0) \in \mathcal{W}_0$.

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Proof

According to the length of the longest directed path between each follower agent and the leader agent, we classify the follower agents into different sets $D_1, D_2, D_3, \cdots$. We also assume that the elements in each set D_i are listed in the ascending order.

We first consider the evolution of follower agent i, $i \in D_1$. Recall that D_1 consists of follower agents whose longest directed paths to the leader agent are of length 1. In this case, the leader agent is the only father agent of each agent in D_1; therefore,

$$u_i = -B^T P(x_i - \Pi w) + \Gamma w,\ i \in D_1.$$

Denoting the error state $\bar{x}_i = x_i - \Pi w, i \in D_1$, we have

$$\begin{aligned}\dot{\bar{x}}_i &= Ax_i + B\sigma(u_i) + Hw - \Pi S w \\ &= A\bar{x}_i + B\left(\sigma\left(-B^T P\bar{x}_i + \Gamma w\right) - \Gamma w\right),\ i \in D_1.\end{aligned} \tag{7}$$

Notice that $\bar{x}_i(0)$ belongs to a bounded set because $\mathcal{X}_0$ and $\mathcal{W}_0$ are both bounded. Moreover, $\bar{x}_i(T) \in \mathcal{U}_T^1, i \in D_1$, for some bounded set $\mathcal{U}_T^1$, which is independent of ε, since $\bar{x}_i(0)$ is bounded and $\bar{x}_i(T), i \in D_1$, is determined by a linear differential equation with bounded inputs $\sigma(u_i)$ and $\Gamma\omega$.

Construct a Lyapunov function,

$$V_1 = \sum_{i \in D_1} \bar{x}_i^T P \bar{x}_i,$$

which is positive definite because P is positive definite.

Let $c_1 > 0$ be a constant such that

$$c_1 \geqslant \sup_{\bar{x}_i \in \mathcal{U}_T^1, i \in D_1, \varepsilon \in (0,1]} V_1.$$

Such a c_1 exists because $\mathcal{U}_T^1$ is bounded and independent of ε.

Let $L_{V_1}(c_1) := \{(\bar{x}_i, i \in D_1) : V_1 \leqslant c_1\}$, where we have used $(\bar{x}_i, i \in D_1)$ to denote a vector whose elements $\bar{x}_i$, $i \in D_1$, are ordered according to the order in D_1. Let $\varepsilon_1 \in (0, 1]$ be such that, for all $\varepsilon \in (0, \varepsilon_1]$, $(\bar{x}_i, i \in D_1) \in L_{V_1}(c_1)$ implies that

$$\left\|-B^T P\bar{x}_i + \Gamma w\right\|_{\infty,T} \leqslant \Delta,\ i \in D_1.$$

The existence of such an ε_1 is due to the facts that $\lim_{\varepsilon\to 0} P(\varepsilon) = 0$ and $\|\Gamma w\|_{\infty,T} \leqslant \delta$.

Hence, for $t \geqslant T$ and for $(\bar{x}_i, i \in D_1) \in L_{V_1}(c_1)$, system (7) can be written as

$$\dot{\bar{x}}_i = A\bar{x}_i - BB^T P\bar{x}_i,\ i \in D_1, \tag{8}$$

and the derivative of V_1 along the trajectory of (8) inside $L_{V_1}(c_1)$ can be evaluated as

$$\begin{aligned}\dot{V}_1 &= \sum_{i \in D_1} \bar{x}_i^T \left(PA + A^T P - 2PBB^T P\right)\bar{x}_i \\ &= -\sum_{i \in D_1} \varepsilon \bar{x}_i^T \bar{x}_i - \bar{x}_i^T PBB^T P\bar{x}_i \\ &< 0,\ \forall (\bar{x}_i, i \in D_1) \in L_{V_1}(c_1) \setminus \{0\}, \varepsilon \in (0, \varepsilon_1].\end{aligned}$$

This implies that $\lim_{t\to\infty} \bar{x}_i(t) = 0, i \in D_1$, which in turn implies that

$$\begin{aligned}\lim_{t\to\infty} e_i(t) &= \lim_{t\to\infty} (y_i(t) - y_0(t)) \\ &= \lim_{t\to\infty} C\,(x_i(t) - \Pi w(t)) \\ &= \lim_{t\to\infty} C\bar{x}_i(t) \\ &= 0,\ i \in D_1.\end{aligned}$$

Moreover,

$$\lim_{t\to\infty} u_i(t) = \lim_{t\to\infty} \left(-B^T P \bar{x}_i(t) + \Gamma w(t)\right)$$
$$= \Gamma w(t),\ i \in D_1.$$

Here and later in the paper, in similar situations, we have, by an abuse of notation, used $\lim_{t\to\infty} u_i(t) = \Gamma w(t)$ to mean that signal $u(t)$ approaches $\Gamma w(t)$ as time passes by, that is, $\lim_{t\to\infty}(u_i(t) - \Gamma w(t)) = 0$.

Because $\lim_{t\to\infty} \bar{x}_i(t) = 0, i \in D_1$, for any positive number $\delta_1 < \Delta - \delta$, there exists a $T_1(\varepsilon) \geqslant T$ such that

$$\left\| B\left(H\bar{x}_i + B^T P \bar{x}_i\right)\right\|_{\infty,T_1} < \varepsilon^2,\ \|H\bar{x}_i\|_{\infty,T_1} < \delta_1,\ i \in D_1,$$

and hence,

$$\|Hx_i\|_{\infty,T_1} < \Delta,\ \|B(Hx_i - \sigma(u_i))\|_{\infty,T_1} < \varepsilon^2,\ i \in D_1.$$

We then consider the evolution of agent $i \in D_2$. Because $\mathcal{G}$ contains no loop, besides the leader agent, all the father agents of agent $i, i \in D_2$, are in D_1, that is, $\mathcal{N}_i \subset \{\nu_0\} \cup D_1$.

In this case, let

$$\bar{x}_i = \frac{1}{|\mathcal{N}_i|}\left(\sum_{j\in\mathcal{N}_i\backslash\{0\}} (x_i - x_j) + a_{i0}(x_i - \Pi w)\right), i \in D_2.$$

Then we have

$$\dot{\bar{x}}_i = A\bar{x}_i + B\left(\sigma(u_i) - \frac{1}{|\mathcal{N}_i|}\left(\sum_{j\in\mathcal{N}_i\backslash\{0\}} \sigma(u_j) + a_{i0}\Gamma w\right)\right),\ i \in D_2, \tag{9}$$

and

$$u_i = -B^T P\bar{x}_i + \frac{1}{|\mathcal{N}_i|}\left(\sum_{j\in\mathcal{N}_i\backslash\{0\}} Hx_j + a_{i0}\Gamma w\right), i \in D_2.$$

Recall that $\bar{x}_i(0)$ belongs to a bounded set and for $(\bar{x}_i, i \in D_1) \in L_{V_1}(c_1)$, $\| -B^T P\bar{x}_i + \Gamma w\|_{\infty,T} \leqslant \Delta$, which implies that

$$\left\|\frac{1}{|\mathcal{N}_i|}\left(\sum_{j\in\mathcal{N}_i\backslash\{0\}} \sigma(u_j) + a_{i0}\Gamma w\right)\right\|_{\infty,T} \leqslant \Delta.$$

Notice that $\bar{x}_i(T) \in \mathcal{U}_T^2, i \in D_2$, for some bounded set $\mathcal{U}_T^2$, which is independent of ε, because $\bar{x}_i(0)$ is bounded and $\bar{x}_i(T), i \in D_2$ is determined by a linear differential equation with bounded inputs $\sigma(u_i)$ and $\frac{1}{|\mathcal{N}_i|}\left(\sum_{j\in\mathcal{N}_i\backslash\{0\}} \sigma(u_j) + a_{i0}\Gamma w\right)$.

Construct a Lyapunov function,

$$V_2 = \sum_{i\in D_2} \bar{x}_i^T P \bar{x}_i,$$

which is positive definite because P is positive definite.

Let $c_2 > 0$ be a constant such that

$$c_2 \geqslant \sup_{\bar{x}_i \in \mathcal{U}_T^2, i\in D_2, \varepsilon\in(0,1]} V_2.$$

Such a c_2 exists since $\mathcal{U}_T^2$ is bounded and independent of ε.

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Let $L_{V_2}(c_2) := \{(\bar{x}_i, i \in D_2) : V_2 \leqslant c_2\}$. Let $\varepsilon_2 \in (0, \varepsilon_1]$ be such that, for all $\varepsilon \in (0, \varepsilon_2]$, $(\bar{x}_i, i \in D_2) \in L_{V_2}(c_2)$ implies that

$$\left\| -B^T P \bar{x}_i + \frac{1}{|\mathcal{N}_i|} \left(\sum_{j \in \mathcal{N}_i \setminus \{0\}} H x_j + a_{i0} \Gamma w \right) \right\|_{\infty, T_1} \leqslant \Delta, i \in D_2.$$

The existence of such an ε_2 is due to the facts that $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$, $\|\Gamma w\|_{\infty,T} \leqslant \delta$ and $\|H x_i\|_{\infty,T_1} < \Delta, i \in D_1$.

Let $\partial L_{V_2}(c_2)$ be the boundary of $L_{V_2}(c_2)$ and $\lambda_{\max}(P)$ be the largest eigenvalue of P. Then, for $(\bar{x}_i, i \in D_2) \in \partial L_{V_2}(c_2)$, we have

$$\sum_{i \in D_2} \lambda_{\max}(P) \|\bar{x}_i\|^2 \geqslant \sum_{i \in D_2} \bar{x}_i^T P \bar{x}_i = c_2,$$

which implies that

$$\sum_{i \in D_2} \|\bar{x}_i\| \geqslant \frac{c_2}{\lambda_{\max}(P)}, \ (\bar{x}_i, i \in D_2) \in \partial L_{V_2}(c_2).$$

Let $\varepsilon_2^* \in (0, \varepsilon_2]$ be such that, for all $\varepsilon \in (0, \varepsilon_2^*]$,

$$\begin{aligned} \sum_{i \in D_2} \|\bar{x}_i \geqslant & \frac{c_2}{\lambda_{\max}(P)} \\ & \geqslant 4|D_2|, \ (\bar{x}_i, i \in D_2) \in \partial L_{V_2}(c_2). \end{aligned}$$

The existence of such an ε_2^* is due to the fact that $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$.

Hence, for $t \geqslant T_1$ and for $(\bar{x}_i, i \in D_2) \in L_{V_2}(c_2)$, system (9) can be written as

$$\dot{\bar{x}}_i = \left(A - BB^T P\right) \bar{x}_i - e_i, \ i \in D_2, \tag{10}$$

where

$$e_i = \frac{1}{|\mathcal{N}_i| - a_{i0}} B \sum_{j \in \mathcal{N}_i \setminus \{0\}} (H x_j - \sigma(u_j)), \ i \in D_2.$$

Clearly, $|e_i| < \varepsilon^2, i \in D_2$, because $\|B(H x_i - \sigma(u_i))\|_{\infty,T_1} < \varepsilon^2, i \in D_1$.

The derivative of V_2 along the trajectory of (10) on the boundary of $L_{V_2}(c_2)$ can be evaluated as

$$\begin{aligned} \dot{V}_2 &= \sum_{i \in D_2} \bar{x}_i^T \left(PA + A^T P - 2PBB^T P\right) \bar{x}_i - 2 \sum_{i \in D_2} \bar{x}_i^T e_i \\ &= - \sum_{i \in D_2} \varepsilon \bar{x}_i^T \bar{x}_i - \bar{x}_i^T PBB^T P \bar{x}_i - 2 \sum_{i \in D_2} \bar{x}_i^T e_i \\ &\leqslant - \sum_{i \in D_2} \varepsilon \|\bar{x}_i\|^2 + 2 \sum_{i \in D_2} \|\bar{x}_i\| \|e_i\| \\ &\leqslant - \sum_{i \in D_2} \varepsilon \|\bar{x}_i\|^2 + \sum_{i \in D_2} \frac{\varepsilon}{2} \|\bar{x}_i\|^2 + \frac{2}{\varepsilon} \|e_i\|^2 \\ &\leqslant - \sum_{i \in D_2} \left(\frac{\varepsilon}{2} \|\bar{x}_i\|^2 - \frac{2}{\varepsilon} \|e_i\|^2 \right) \\ &< - \sum_{i \in D_2} \frac{\varepsilon}{2} \left(\|\bar{x}_i\|^2 - 4\varepsilon^2 \right) \\ &< 0, \ \forall (\bar{x}_i, i \in D_2) \in \partial L_{V_2}(c_2), \varepsilon \in (0, \varepsilon_2^*], \end{aligned}$$

where we have used the facts that $|e_i| < \varepsilon^2$ and $\sum_{i \in D_2} \|\bar{x}_i\| \geqslant 4|D_2|, \forall \varepsilon \in (0, \varepsilon_2^*]$. This implies that $L_{V_2}(c_2)$ is an invariant set, and (10) holds for all $t \geqslant T_1$. Because $\dot{\bar{x}}_i = \left(A - BB^T P\right)\bar{x}_i, i \in D_2$ is asymptotically stable and $\lim_{t\to\infty} \bar{x}_i(t) = \lim_{t\to\infty}(x_i(t) - \Pi w(t)) = 0, i \in D_1$, that is, $\lim_{t\to\infty}(Hx_i(t) - \Gamma w(t)) = 0, i \in D_1$, and $\lim_{t\to\infty} u_i(t) = \Gamma w(t), i \in D_1$, we can conclude that

$$\lim_{t\to\infty} \bar{x}_i(t) = 0, \ i \in D_2,$$

which implies that

$$\begin{aligned}
\lim_{t\to\infty} e_i(t) &= \lim_{t\to\infty} (y_i(t) - y_0(t)) \\
&= \lim_{t\to\infty} C\,(x_i(t) - \Pi w(t)) \\
&= \lim_{t\to\infty} C\bar{x}_i(t) \\
&= 0, \ i \in D_2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\lim_{t\to\infty} u_i(t) &= \lim_{t\to\infty} \left(-B^T P\bar{x}_i(t) + \frac{1}{|\mathcal{N}_i|} \left(\sum_{j \in \mathcal{N}_i \setminus \{0\}} Hx_j(t) + a_{i0}\Gamma w(t) \right) \right) \\
&= \Gamma w(t), \ i \in D_2.
\end{aligned}$$

Because $\lim_{t\to\infty} \bar{x}_i(t) = 0, i \in D_2$, for any positive number $\delta_2 < \Delta - \delta_1^*$, $\delta_1^* = \max\{\delta_1, \delta\}$, there exists a $T_2(\varepsilon) \geqslant T_1$ such that

$$\left\| B\left(H\bar{x}_i + B^T P\bar{x}_i\right)\right\|_{\infty, T_1} < \varepsilon^2, \ \|H\bar{x}_i\|_{\infty, T_2} < \delta_2, \ i \in D_2;$$

hence,

$$\begin{aligned}
\|Hx_i\|_{\infty, T_2} &= \left\| H\bar{x}_i + \frac{1}{|\mathcal{N}_i|}\left(\sum_{j \in \mathcal{N}_i \setminus \{0\}} Hx_i + a_{i0}\Gamma\omega \right) \right\|_{\infty, T_2} \\
&< \Delta,
\end{aligned}$$

and

$$\|B(Hx_i - \sigma(u_i))\|_{\infty, T_1} < \varepsilon^2, \ i \in D_2.$$

Following a similar analysis as given for the evolution of agent $i \in D_2$, we can prove that for $x_i, i \in D_k, k \leqslant N$, there exists an ε_k^*, such that $\forall \varepsilon \in (0, \varepsilon_k^*]$,

$$\lim_{t\to\infty} e_i(t) = 0, \ i \in D_k,$$

and $\lim_{t\to\infty} u_i(t) = \Gamma w(t)$, for any $i \in D_k, k \leqslant N$.

Therefore, with $\varepsilon^* = \varepsilon_k^*$, we have that, for any $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{t\to\infty} e_i(t) = 0, \ i = 1, 2, \cdots, N.$$

This completes the proof. □

Remark 3.1

A result related to that in Theorem 3.1 earlier was given in [17]. In [17], the solvability conditions were based on both the agent dynamics and the communication topology among the agents jointly. Thus, the coefficients in the control laws need to be recalculated when the communication topology changes. Moreover, the result in [17] was obtained with matrices C and Q assumed to be invertible. On the contrary, in our Theorem 3.1, the coefficients in the control laws do not need be recalculated when the communication topology changes, and no invertibility assumption is required on either matrix C and Q.

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In the special case when $\Gamma = 0$, we can get the following result.

Assumption 3.4
The undirected graph $\mathcal{G}$ is connected, and the leader is a neighbor of at least one follower.

Denote $M = \mathcal{L} + \mathcal{L}_0$. Then we can order the eigenvalues of M as $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$ because of the following lemma.

Lemma 3.1
[20] Let Assumption 3.4 hold. Then M is symmetric and positive definite.

Theorem 3.2
Let Assumptions 2.1, 3.3, and 3.4 hold. Then, the group of follower agents (2) and the leader agent (3) achieve semi-global leader-following output consensus under the control only based on relative measurements:

$$u_i = -B^T P \left(\sum_{j=1}^{N} a_{ij}(x_i - x_j) + a_{i0}(x_i - \Pi w) \right), \quad i = 1, 2, \cdots, N, \tag{11}$$

where P is the solution of ARE (4) in Lemma 2.1 with $\gamma \leqslant \lambda_1$, if there exists a matrix Π that solves the following linear matrix equations:

$$\Pi S = A\Pi + E, \quad C\Pi + Q = 0. \tag{12}$$

That is, for any *a priori* given a bounded set $\mathcal{X}_0 \subset \mathrm{R}^n$, there is an ε^* such that, for any $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{t \to \infty} e_i(t) = 0, \quad i = 1, 2, \cdots, N,$$

hold for all $x_i(0) \in \mathcal{X}_0, i = 1, 2, \cdots, N$, and $w(0) \in \mathcal{W}_0$.

Proof
With $\bar{x}_i = x_i - \Pi w, i = 1, 2, \cdots, N$, we have

$$\begin{aligned} \dot{\bar{x}}_i &= Ax_i + B\sigma(u_i) + Ew - \Pi S w \\ &= A\bar{x}_i + B\sigma(u_i), \quad i = 1, 2, \cdots, N. \end{aligned} \tag{13}$$

Notice that $\bar{x}_i(0)$ belongs to a bounded set because $\mathcal{X}_0$ and $\mathcal{W}_0$ are both bounded. Therefore, $\bar{x}_i(T) \in \mathcal{U}_T$, for some bounded set $\mathcal{U}_T$, which is independent of ε because $\bar{x}_i(0)$ is bounded and $\bar{x}_i(T)$ is determined by a linear differential equation with a bounded input $\sigma(u_i)$.

Let $\bar{x} = \left[\bar{x}_1^T, \bar{x}_2^T, \cdots, \bar{x}_N^T\right]^T$, then

$$\dot{\bar{x}} = (I_N \otimes A)\bar{x} + (I_N \otimes B)\sigma\left(-\left(M \otimes B^T P\right)\bar{x}\right). \tag{14}$$

Construct a Lyapunov function

$$V(\bar{x}) = \frac{1}{2}\bar{x}^T (M \otimes P)\bar{x},$$

which is positive definite for M and P are positive definite.

Let $c > 0$ be a constant such that

$$c \geqslant \sup_{\bar{x}_i \in \mathcal{U}_T, i=1,2,\cdots,N, \varepsilon \in (0,1]} V(\bar{x}).$$

Such a c exists because $\mathcal{U}_T$ is bounded and independent of ε.

Let $L_V(c) := \left\{\bar{x} \in \mathbb{R}^{Nn} : V(\bar{x}) \leqslant c\right\}$ and let $\varepsilon^* \in (0, 1]$ be such that, for each $\varepsilon \in (0, \varepsilon^*]$, $\bar{x} \in L_V(c)$ implies that

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$$\left\| B^T P \sum_{j=1}^{N} a_{ij}(x_i - x_j) + a_{i0}(x_i - \Pi w) \right\| \leqslant \Delta,\ i = 1, 2, \cdots, N.$$

The derivative of V along the trajectory of (14) inside $L_V(c)$ can be evaluated, in view of Lemmas 2.1, 2.2, and 3.1, as follows:

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\bar{x}^T \left(M \otimes \left(PA + A^T P\right)\right)\bar{x} - \bar{x}^T \left(M^2 \otimes BB^T P\right)\bar{x} \\
&= -\frac{\varepsilon}{2}\bar{x}^T (M \otimes I_n)\bar{x} + \gamma \bar{x}^T \left(M \otimes BB^T P\right)\bar{x} - \bar{x}^T \left(M^2 \otimes BB^T P\right)\bar{x} \\
&\leqslant -\frac{\varepsilon}{2}\bar{x}^T (M \otimes I_n)\bar{x}, \\
&< 0,\ \forall \bar{x} \in L_V(c) \setminus \{0\}, \varepsilon \in (0, \varepsilon^*].
\end{aligned}$$

This implies that

$$\begin{aligned}
\lim_{t\to\infty}(x_i(t) - \Pi w(t)) &= \lim_{t\to\infty}\bar{x}_i(t) \\
&= 0,\ i = 1, 2, \cdots, N.
\end{aligned}$$

Therefore, we can conclude that

$$\begin{aligned}
\lim_{t\to\infty} e_i(t) &= \lim_{t\to\infty} y_i(t) - y_0(t) \\
&= \lim_{t\to\infty} C(x_i(t) - \Pi_i w(t)) \\
&= \lim_{t\to\infty} C\bar{x}_i(t) \\
&= 0,\ i = 1, 2, \cdots, N.
\end{aligned}$$

This completes the proof. □

4. THE NON-IDENTICAL DISTURBANCE CASE

In this section, we go further and study output consensus for the non-identical disturbance case. In the following two subsections, we study the output consensus problem under the communication topologies represented by two different graphs, the detailed balanced graph, and the jointly connected graph.

4.1. *Output consensus over a directed communication topology*

In this subsection, we consider the situation when the communication topology among the follower agents is a strongly connected and detailed balanced digraph. A digraph is detailed balanced if there exist some real numbers $v_i > 0, i = 1, 2, \cdots, N$, such that $v_i a_{ij} = v_j a_{ji}$, for all $i, j = 1, 2, \cdots, N$ [21]. Denote $v_{\min} = \min\{v_i\}$, $v_{\max} = \max\{v_i\}$, and $R = \mathrm{diag}\{v_1, v_2, \cdots, v_N\}$.

Assumption 4.1
The fixed digraph $\mathcal{G}$ is strongly connected and detailed balanced, and the leader is a neighbor of at least one follower.

Lemma 4.1
Let Assumption 4.1 hold and denote $M = \mathcal{L} + \mathcal{L}_0$. Then all eigenvalues of M are on the open right-half plane, and the matrix $RM + M^T R = 2M^T R$ is positive definite.

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In the aforementioned lemma, the fact that all eigenvalues of M are on the open right-half plane is established in [22], and the fact that $RM + M^T R = 2M^T R$ is positive definite can be established based on the analysis given in the proof of Lemma 4 in [20]. According to Lemma 4.1, RM and $M^T R$ have the same eigenvalues, which can be ordered as $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$.

The following two assumptions on the follower agents and the leader define the solvability of the semi-global output consensus problem for the multi-agent system composing of (1) and (3).

Assumption 4.2
There exist a family of matrices Π_i and Γ_i that solve the following linear matrix equations:

$$\Pi_i S = A\Pi_i + a_{i0}B\Gamma_i + E_i, \quad C\Pi_i + Q = 0, \quad i = 1, 2, \cdots, N. \tag{15}$$

Let

$$\Pi = \begin{bmatrix} \Pi_1 & 0 & \cdots & 0 \\ 0 & \Pi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Pi_N \end{bmatrix}, \ \Gamma = \begin{bmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_N \end{bmatrix}, \ E = \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_N \end{bmatrix}.$$

Then linear matrix (15) can be written in the following compact form:

$$\Pi(I_N \otimes S) = (I_N \otimes A)\Pi + (\mathcal{L}_0 \otimes B)\Gamma + E, \quad (I_N \otimes C)\Pi + (I_N \otimes Q) = 0. \tag{16}$$

Assumption 4.3
There exist a positive number $\delta < \Delta$ and a time $T \geqslant 0$ such that $\|\Gamma_i w\|_{\infty,T} \leqslant \delta$, $i = 1, 2, \cdots, N$, for all w with $w(0) \in \mathcal{W}_0$, where $\|\Gamma_i w\|_{\infty,T} := \sup_{t \geqslant T} \|\Gamma_i w\|_\infty$.

Under Assumptions 2.1 and 4.2, we construct the following state feedback control algorithm for each follower agent,

$$u_i = -B^T P \left(\sum_{j=1}^{N} a_{ij}((x_i - x_j) - (\Pi_i - \Pi_j)w) + a_{i0}(x_i - \Pi_i w) \right) + a_{i0}\Gamma w, \ i = 1, 2, \cdots, N, \tag{17}$$

where Π_i and Γ_i are solutions to the linear matrix (15), whose existence is guaranteed by Assumption 4.2, and $P := P(\varepsilon)$ is the solution to the ARE (4) as given in Lemma 2.1 with $\gamma \in \left(0, \min\left\{\frac{\lambda_1}{v_{\max}}, \lambda_1\right\}\right]$.

We have the following result on semi-global leader-following output consensus of the multi-agent system consisting of the follower agents (1) and the leader agent (3).

Theorem 4.1
Let Assumptions 2.1, 4.1, 4.2, and 4.3 hold. Then, under the linear state feedback control laws (17), the group of follower agents (1) and the leader agent (3) achieve semi-global leader-following output consensus. That is, for any *a priori* given bounded set $\mathcal{X}_0 \subset \mathbb{R}^n$, there exists an $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{t \to \infty} e_i(t) = 0, \quad i = 1, 2, \cdots, N,$$

hold for all $x_i(0) \in \mathcal{X}_0$, $i = 1, 2, \cdots, N$, and $w(0) \in \mathcal{W}_0$.

Proof
Denote $\bar{x}_i = x_i - \Pi_i w, i = 1, 2, \cdots, N$; then we have

$$\begin{aligned}\dot{\bar{x}}_i &= \dot{x}_i - \Pi_i \dot{w} \\ &= Ax_i + B\sigma(u_i) + E_i - \Pi_i S w \\ &= A\bar{x}_i + B(\sigma(u_i) - a_{i0}\Gamma_i w), i = 1, 2, \cdots, N.\end{aligned} \tag{18}$$

Notice that $\bar{x}_i(0), i = 1, 2, \cdots, N$, belong to a bounded set because $\mathcal{X}_0$ and $\mathcal{W}_0$ are both bounded. Moreover, $\bar{x}_i(T) \in \mathcal{U}_T, i = 1, 2, \cdots, N$, for some bounded set $\mathcal{U}_T$, which is independent of ε, because $\bar{x}_i(0)$ is bounded and $\bar{x}_i(T), i = 1, 2, \cdots, N$, is determined by a linear differential equation with bounded inputs $\sigma(u_i)$ and $a_{i0}\Gamma_i \omega$.

Let $\bar{x} = \left[\bar{x}_1^T, \bar{x}_2^T, \cdots, \bar{x}_N^T\right]^T$, $\bar{w} = [w^T, w^T, \cdots, w^T]^T \in \mathbb{R}^{Nr}$, and $u = \left[u_1^T, u_2^T, \cdots, u_N^T\right]^T$. Then, (18) and control law (17) can respectively be written as

$$\dot{\bar{x}} = (I_N \otimes A)\bar{x} + (I_N \otimes B)(\sigma(u) - (\mathcal{L}_0 \otimes \Gamma)\bar{w}), \tag{19}$$

and

$$u = -\left(M \otimes B^T P\right)\bar{x} + (\mathcal{L}_0 \otimes \Gamma)\bar{w}. \tag{20}$$

Construct a Lyapunov function,

$$V(\bar{x}) = \bar{x}^T (RMR \otimes P)\bar{x},$$

which is positive definite because P, R, and MR are positive definite.

Let $c > 0$ be a constant such that

$$c \geqslant \sup_{\bar{x}_i \in \mathcal{U}_T, i=1,2,\cdots,N, \varepsilon \in (0,1]} V(\bar{x}).$$

Such a c exists because $\mathcal{U}_T$ is bounded and independent of ε.

Let $L_V(c) = \left\{\bar{x} \in \mathbb{R}^{Nn} : V(\bar{x}) \leqslant c\right\}$. Let $\varepsilon^* \in (0, 1]$ be such that, for all $\varepsilon \in (0, \varepsilon^*]$, $\bar{x} \in L_V(c)$ implies that

$$\begin{aligned}&\left\| B^T P \left(\sum_{j=1}^{N} a_{ij}((x_i - x_j) - (\Pi_i - \Pi_j)w) + a_{i0}(x_i - \Pi_i w) \right) - a_{i0}\Gamma_i w \right\|_{\infty,T} \\ &\quad \leqslant \Delta,\ i = 1, 2, \cdots, N.\end{aligned}$$

The existence of such an ε is due to the facts that $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$ and $\|\Gamma_i w\|_{\infty,T} \leqslant \delta, i = 1, 2, \cdots, N$.

Hence, for $t \geqslant T$ and for $\bar{x} \in L_V(c)$, the closed-loop system (19)–(20) can be written as

$$\begin{aligned}\dot{\bar{x}} &= (I_N \otimes A)\bar{x} + (I_N \otimes B)\left(-\left(M \otimes B^T P\right)\bar{x} + (\mathcal{L}_0 \otimes \Gamma)\bar{w} - (\mathcal{L}_0 \otimes \Gamma)\bar{w}\right) \\ &= (I_N \otimes A)\bar{x} - \left(M \otimes BB^T P\right)\bar{x},\end{aligned} \tag{21}$$

and the derivative of V along the trajectory of (21) inside $L_V(c)$ can be evaluated, in view of Lemmas 2.2 and 4.1, as follows:

$$\begin{aligned}\dot{V} &= \bar{x}^T \left(RMR \otimes \left(A^T P + PA\right)\right)\bar{x} - 2\bar{x}^T \left((MR)^2 \otimes PBB^T P\right)\bar{x} \\ &= -\varepsilon \bar{x}^T (RMR \otimes I_N)\bar{x} + 2\gamma \bar{x}^T \left(RMR \otimes PBB^T P\right)\bar{x} - 2\bar{x}^T \left((MR)^2 \otimes PBB^T P\right)\bar{x} \\ &\leqslant -\varepsilon \bar{x}^T (RMR \otimes I_N)\bar{x} \\ &< 0,\ \forall \bar{x} \in L_V(c) \setminus \{0\}, \varepsilon \in (0, \varepsilon^*].\end{aligned}$$

This implies that closed-loop system is asymptotically stable at $\bar{x} = 0$ with $L_V(c)$ included in the domain of attraction, and hence,

$$\lim_{t\to\infty} \bar{x}_i(t) = 0, i = 1, 2, \cdots, N,$$

which, in turn, implies that

$$\begin{aligned}\lim_{t\to\infty} e_i(t) &= \lim_{t\to\infty} y_i(t) - y_0(t)\\ &= \lim_{t\to\infty} C(x_i(t) - \Pi_i w(t))\\ &= \lim_{t\to\infty} C\bar{x}_i(t)\\ &= 0,\ i = 1, 2, \cdots, N.\end{aligned}$$

This completes the proof. □

Remark 4.1
In [17], Theorem 1 establishes semi-global output consensus under a switching undirected communication topology and with the assumption that both matrices C and Q are invertible. Our result in Theorem 4.1 does not require the invertibility of these two matrices.

4.2. *Output consensus over jointly connected undirected graph*

In this subsection, we consider the communication topology that satisfies the following assumption.

Assumption 4.4
The communication topology among the follower agents is a jointly connected undirected graph over time-intervals $[t_k, t_{k+1}), k = 1, 2, \cdots$, and in any given time-interval $[t_k, t_{k+1})$, the leader is a neighbor of at least one follower with a constant communication weight between each follower agent and the leader.

Furthermore, under Assumption 4.4, in each time-subinterval $\left[t_k^l, t_k^{l+1}\right)$, the communication weight between two distinct follower agents is denoted by $a_{ij}\left(t_k^l\right), i, j = 1, 2\cdots, N, l = 0, 1, \cdots, l_k - 1$, and in each time-interval $[t_k, t_{k+1})$, the communication weight between the follower agent ν_i and the leader agent ν_0 is denoted by $a_{i0}(t_k), i = 1, 2, \cdots, N$. In each time-subinterval $\left[t_k^l, t_k^{l+1}\right)$, denote $M_k^l = \mathcal{L}_k^l + \mathcal{L}_{0,k}$, where $\mathcal{L}_k^l$ is the Laplacian matrix associated with the graph $\mathcal{G}_k^l$ that represents the communication topology in the time-subinterval $\left[t_k^l, t_k^{l+1}\right)$ and $\mathcal{L}_{0,k} = \mathrm{diag}\{a_{10}(t_k), a_{20}(t_k), \cdots, a_{N0}(t_k)\}$. Note that, $\mathcal{G}_k^l$ is fixed in each time-subinterval $\left[t_k^l, t_k^{l+1}\right)$. Denote $\Lambda_k^l = \left\{i : \lambda_{k,i}^l \neq 0, i \in [1, N]\right\}$, where $\lambda_{k,i}^l \geqslant 0, i = 1, 2, \cdots, N$, are eigenvalues of M_k^l. Obviously, $\lambda_{\min} = \min\left\{\lambda_{k,i}^l, i \in \Lambda_k^l, l = 0, 1, \cdots, l_k - 1, k = 1, 2, \cdots\right\}$ is a positive value due to the finite number of all possible graphs and Assumption 4.4. Denote the union graph of the l_k graphs over the l_k time-subintervals in $[t_k, t_{k+1})$ as $\mathcal{G}_k$, that is, $\mathcal{G}_k = \mathcal{G}_k^0 \bigcup \mathcal{G}_k^1 \bigcup \cdots \bigcup \mathcal{G}_k^{l_k-1}$, and the Laplacian matrix associated with $\mathcal{G}_k$ as $\mathcal{L}_k$. Then the matrix

$$M_k = \mathcal{L}_k + \mathcal{L}_{0,k},$$

is a positive definite constant matrix in each time interval $[t_k, t_{k+1})$ according to Assumption 4.4 [23]. Thus, the eigenvalues of M_k can be ordered as $0 < \lambda_{k,1} \leqslant \lambda_{k,2} \leqslant \cdots \leqslant \lambda_{k,N}$. Denote $\Lambda_k = \left\{i : \lambda_{k,i} \neq 0, i \in [1, N]\right\}$. Clearly, $\Lambda_k = \{1, 2, \cdots, N\}$.

The following result was established in [24].

Lemma 4.2
Under Assumption 4.4, for each $k = 1, 2, \cdots$,

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$$\Lambda_k^0 \cup \Lambda_k^1 \cup \cdots \cup \Lambda_k^{l_k-1} = \Lambda_k.$$

The following assumption, which was inspired by [24], requires that each agent is marginally stable.

Assumption 4.5
There exist a $P(\varepsilon)$ that satisfies ARE (4) and the following Riccati and Lyapunov inequalities

$$PA + A^T P - 2\lambda_{\min} PBB^T P + \lambda_{\min} I < 0, \tag{22}$$

$$PA + A^T P \leqslant 0. \tag{23}$$

The following two assumptions on the follower agents and the leader agent define the solvability of the semi-global output consensus problem for the multi-agent system (1)–(3).

Assumption 4.6
For each $k = 1, 2, \cdots$, there exist a family of matrices $\Pi_i(t_k)$ and $\Gamma_i(t_k)$ that solve the following linear matrix equations:

$$\Pi_i(t_k)S = A\Pi_i(t_k) + a_{i0}(t_k)B\Gamma_i(t_k) + E_i, \quad C\Pi_i(t_k) + Q = 0, \quad i = 1, 2, \cdots, N. \tag{24}$$

Let

$$\Pi(t_k) = \begin{bmatrix} \Pi_1(t_k) & 0 & \cdots & 0 \\ 0 & \Pi_2(t_k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Pi_N(t_k) \end{bmatrix}, \quad \Gamma(t_k) = \begin{bmatrix} \Gamma_1(t_k) & 0 & \cdots & 0 \\ 0 & \Gamma_2(t_k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_N(t_k) \end{bmatrix}.$$

Then, linear matrix (24) can be written in the following compact form:

$$\Pi(t_k)(I_N \otimes S) = (I_N \otimes A)\Pi(t_k) + (\mathcal{L}_{0,k} \otimes B)\Gamma(t_k) + E, \quad (I_N \otimes C)\Pi(t_k) + (I_N \otimes Q) = 0. \tag{25}$$

Assumption 4.7
There exist a positive number $\delta < \Delta$ and a time $T \geqslant 0$ such that, for each $k = 1, 2, \cdots$, $\|\Gamma_i(t_k)w\|_{\infty,T} \leqslant \delta$, $i = 1, 2, \cdots, N$, for all w with $w(0) \in \mathcal{W}_0$, where $\|\Gamma_i(t_k)w\|_{\infty,T} := \sup_{t \geqslant T} \|\Gamma_i(t_k)w\|_\infty$.

Under Assumptions 2.1, 4.5, and 4.6, we construct the following state feedback control algorithm for each follower agent:

$$\begin{aligned} u_i = -B^T P &\left(\sum_{j=1}^{N} a_{ij}\left(t_k^l\right)((x_i - x_j) - (\Pi_i(t_k) - \Pi_j(t_k))w) + a_{i0}(t_k)(x_i - \Pi_i(t_k)w) \right) \\ &+ a_{i0}(t_k)\Gamma_i(t_k)w, \ i = 1, 2, \cdots, N, \end{aligned} \tag{26}$$

where $\Pi_i(t_k)$ and $\Gamma_i(t_k), i = 1, 2, \cdots, N$, are solutions to the linear matrix (24) in time-interval $[t_k, t_{k+1})$, whose existence is guaranteed by Assumption 4.6, and $P := P(\varepsilon)$ is the solution to the ARE (4) and inequalities (22) and (23) as given in Assumption 4.5.

We have the following result on semi-global leader-following output consensus of the multi-agent system consisting of the follower agents (1) and the leader agent (3).

Theorem 4.2
Let Assumptions 2.1, 4.4, 4.5, 4.6, and 4.7 hold. Then, under the linear state feedback control laws (26), the group of follower agents (1) and the leader agent (3) achieve semi-global leader-following output consensus. That is, for any *a priori* given bounded set $\mathcal{X}_0 \subset \mathbb{R}^n$, there exists an $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{t \to \infty} e_i(t) = 0, \quad i = 1, 2, \cdots, N,$$

hold for all $x_i(0) \in \mathcal{X}_0, i = 1, 2, \cdots, N$, and $w(0) \in \mathcal{W}_0$.

Proof
Within each time-interval $[t_k, t_{k+1})$, denote $\bar{x}_i = x_i - \Pi_i(t_k)w, i = 1, 2, \cdots, N$, then we have

$$\dot{\bar{x}}_i = A\bar{x}_i + B\left(\sigma(u_i) - a_{i0}(t_k)\,\Gamma_i(t_k)w\right),\ i = 1, 2, \cdots, N.$$

Notice that $\bar{x}_i(0), i = 1, 2, \cdots, N$, belongs to a bounded set because $\mathcal{X}_0$ and $\mathcal{W}_0$ are both bounded. Moreover, $\bar{x}_i(T) \in \mathcal{U}_T, i = 1, 2, \cdots, N$, for some bounded set $\mathcal{U}_T$, which is independent of ε, because $\bar{x}_i(0)$ is bounded and $\bar{x}_i(T), i = 1, 2, \cdots, N$, is determined by a linear differential equation with bounded inputs $\sigma(u_i)$ and $\Gamma_i(t_k)\omega$.

Let $\bar{x} = \left[\bar{x}_1^T, \bar{x}_2^T, \cdots, \bar{x}_N^T\right]^T$, $\bar{w} = \left[w^T, w^T, \cdots, w^T\right]^T \in \mathbb{R}^{Nr}$, and $u = \left[u_1^T, u_2^T, \cdots, u_N^T\right]^T$, then in each time-interval $[t_k, t_{k+1})$, system (18) and control law (26) can be respectively written as

$$\dot{\bar{x}} = (I_N \otimes A)\bar{x} + (I_N \otimes B)(\sigma(u) - (\mathcal{L}_{0,k} \otimes \Gamma(t_k))\bar{w}), \tag{27}$$

and

$$u = -\left(M_k^l \otimes B^T P\right)\bar{x} + (\mathcal{L}_{0,k} \otimes \Gamma(t_k))\bar{w}, \tag{28}$$

Construct a Lyapunov function

$$V(\bar{x}) = \bar{x}^T (I_N \otimes P)\bar{x},$$

which is positive definite because P is positive definite.

Let $c > 0$ be a constant such that

$$c \geqslant \sup_{\bar{x}_i \in \mathcal{U}_T, i=1,2,\cdots,N, \varepsilon \in (0,1]} V(\bar{x}).$$

Such a c exists because $\mathcal{U}_T$ is bounded and independent of ε.

Let $L_V(c) := \left\{\bar{x} \in \mathbb{R}^{Nn} : V(\bar{x}) \leqslant c\right\}$. Let $\varepsilon^* \in (0, 1]$ be such that, for all $\varepsilon \in (0, \varepsilon^*]$, $\bar{x} \in L_V(c)$ implies that

$$\left\| B^T P \left(\sum_{j=1}^{N} a_{ij}\left(t_k^l\right)\left((x_i - x_j) - (\Pi_i(t_k) - \Pi_j(t_k))w\right) + a_{i0}(t_k)\left(x_i - \Pi_i(t_k)w\right)\right) \right.$$
$$\left. -a_{i0}(t_k)\,\Gamma_i(t_k)w(t) \right\|_{\infty,T} \leqslant \Delta,\ i = 1, 2, \cdots, N.$$

The existence of such an ε is due to the facts that $\lim_{\varepsilon \to 0} P(\varepsilon) = 0$ and $\|\Gamma_i(t_k)w\|_{\infty,T} \leqslant \delta, i =$

$1, 2, \cdots, N$.

Hence, for $t \geqslant T$ and for $\bar{x} \in L_V(c)$, the closed-loop system (27)–(28) can be written as

$$\begin{aligned}\dot{\bar{x}} &= (I_N \otimes A)\bar{x} + (I_N \otimes B)\left(-\left(M_k^l \otimes B^T P\right)\bar{x} + (\mathcal{L}_{0,k} \otimes \Gamma(t_k))\bar{w} - (\mathcal{L}_{0,k} \otimes \Gamma(t_k)\bar{w}\right) \\ &= (I_N \otimes A)\bar{x} - \left(M_k^l \otimes BB^T P\right)\bar{x}.\end{aligned} \tag{29}$$

Notice that for a symmetry matrix M_k^l, there exists an orthogonal matrix $T_k^l \in \mathbb{R}^{N \times N}$ such that

$$M_k^l = \left(T_k^l\right)^T \operatorname{diag}\left\{\lambda_{k,1}^l, \lambda_{k,2}^l, \cdots, \lambda_{k,N}^l\right\} T_k^l,$$

where $\lambda_{k,1}^l, \lambda_{k,2}^l, \cdots, \lambda_{k,N}^l$ are eigenvalues of M_k^l.

Let $\tilde{x} = \left(T_k^l \otimes I_N\right)\bar{x}$; then in each time-interval $[t_k, t_{k+1})$, the derivative of V along the trajectory of (29) inside $L_V(c)$ can be evaluated as

$$\begin{aligned}\dot{V}(\bar{x}) &= \bar{x}^T\left(I_N \otimes \left(A^T P + PA\right)\right)\bar{x} - 2\bar{x}^T\left(M_k^l \otimes PBB^T P\right)\bar{x} \\ &= \bar{x}^T\left(\left(T_k^l\right)^T T_k^l \otimes \left(A^T P + PA\right)\right)\bar{x} \\ &\quad - 2\bar{x}^T\left(\left(T_k^l\right)^T \operatorname{diag}\left\{\lambda_{k,1}^l, \lambda_{k,2}^l, \cdots, \lambda_{k,N}^l\right\} T_k^l \otimes PBB^T P\right)\bar{x} \\ &= \bar{x}^T\left(\left(T_k^l\right)^T \otimes I_N\right)\left(I_N \otimes \left(A^T P + PA\right)\right)\left(T_k^l \otimes I_N\right)\bar{x} \\ &\quad - 2\bar{x}^T\left(\left(T_k^l\right)^T \otimes I_N\right)\left(\operatorname{diag}\left\{\lambda_{k,1}^l, \lambda_{k,2}^l, \cdots, \lambda_{k,N}^l\right\} \otimes PBB^T P\right)\left(T_k^l \otimes I_N\right)\bar{x} \\ &= \sum_{l=0}^{l_k-1}\sum_{i \in \Lambda_k^l} \tilde{x}_i^T\left(\left(A^T P + PA\right) - 2\lambda_{k,i}^l PBB^T P\right)\tilde{x}_i \\ &\leqslant \sum_{l=0}^{l_k-1}\sum_{i \in \Lambda_k^l} \tilde{x}_i^T\left(\left(A^T P + PA\right) - 2\lambda_{\min} PBB^T P\right)\tilde{x}_i \\ &\leqslant -\sum_{l=0}^{l_k-1}\sum_{i \in \Lambda_k^l} \lambda_{\min}\tilde{x}_i^T\tilde{x}_i \\ &\leqslant 0,\end{aligned}$$

which implies that the time function $V(\bar{x}(t))$ converges as $t \to \infty$. Consequently, $V(\bar{x}(t_k))$ converges as $k \to \infty$. Thus, by the Cauchy's convergence criterion, we have that, for any $\delta > 0$, there exists a positive number m_δ such that, for any $k \geqslant m_\delta$,

$$0 \leqslant V(\bar{x}(t_k)) - V(\bar{x}(t_{k+1})) = -\int_{t_k}^{t_{k+1}} \dot{V}(\bar{x}(\upsilon))d\upsilon < \delta.$$

Therefore, we have, for each $k \geqslant m_\delta$,

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$$
\begin{aligned}
\delta &> -\int_{t_k}^{t_{k+1}} \dot{V}(\bar{x}(\upsilon))d\upsilon \\
&= -\int_{t_k^0}^{t_k^1} \dot{V}(\bar{x}(\upsilon))d\upsilon - \cdots - \int_{t_k^{l_k-1}}^{t_k^{l_k}} \dot{V}(\bar{x}(\upsilon))d\upsilon \\
&= -\int_{t_k^0}^{t_k^1} \bar{x}^T(\upsilon)\left(I_N \otimes \left(A^T P + PA\right) - 2M_k^0 \otimes PBB^T P\right)\bar{x}(\upsilon)d\upsilon \\
&\quad -\int_{t_k^1}^{t_k^2} \bar{x}^T(\upsilon)\left(I_N \otimes \left(A^T P + PA\right) - 2M_k^1 \otimes PBB^T P\right)\bar{x}(\upsilon)d\upsilon \\
&\quad -\cdots \\
&\quad -\int_{t_k^{l_k-1}}^{t_k^{l_k}} \bar{x}^T(\upsilon)\left(I_N \otimes \left(A^T P + PA\right) - 2M_k^{l_k-1} \otimes PBB^T P\right)\bar{x}(\upsilon)d\upsilon \\
&\geqslant -\int_{t_k^0}^{t_k^0+\tau} \bar{x}^T(\upsilon)\left(I_N \otimes \left(A^T P + PA\right) - 2M_k^0 \otimes PBB^T P\right)\bar{x}(\upsilon)d\upsilon \\
&\quad -\int_{t_k^1}^{t_k^1+\tau} \bar{x}^T(\upsilon)\left(I_N \otimes \left(A^T P + PA\right) - 2M_k^1 \otimes PBB^T P\right)\bar{x}(\upsilon)d\upsilon \\
&\quad -\cdots \\
&\quad -\int_{t_k^{l_k-1}}^{t_k^{l_k-1}+\tau} \bar{x}^T(\upsilon)\left(I_N \otimes \left(A^T P + PA\right) - 2M_k^{l_k-1} \otimes PBB^T P\right)\bar{x}(\upsilon)d\upsilon.
\end{aligned}
$$

In the above derivation, we have used the fact that

$$
I_N \otimes \left(A^T P + PA\right) - 2M_k^l \otimes PBB^T P \leqslant 0,\ l = 0, 1, \cdots, l_k - 1, k = 1, 2, \cdots.
$$

Recalling that $\tilde{x} = \left(T_k^l \otimes I_N\right)\bar{x}$, we have, for each $l = 0, 1 \cdots, l_k - 1$,

$$
\begin{aligned}
\delta &> -\int_{t_k^l}^{t_k^l+\tau} \bar{x}^T(\upsilon)\left(I_N \otimes \left(A^T P + PA\right) - 2M_k^l \otimes PBB^T P\right)\bar{x}(\upsilon)d\upsilon \\
&\geqslant -\int_{t_k^l}^{t_k^l+\tau} \bar{x}^T(\upsilon)\left(\left(T_k^l\right)^T \otimes I_N\right)\left(I_N \otimes \left(A^T P + PA\right)\right)\left(T_k^l \otimes I_N\right)\bar{x}(\upsilon)d\upsilon \\
&\quad - 2\bar{x}^T(\upsilon)\left(\left(T_k^l\right)^T \otimes I_N\right)\left(\operatorname{diag}\left\{\lambda_{k,1}^l, \lambda_{k,2}^l, \cdots, \lambda_{k,N}^l\right\} \otimes PBB^T P\right)\left(T_k^l \otimes I_N\right)\bar{x}(\upsilon)d\upsilon \\
&\geqslant -\int_{t_k^l}^{t_k^l+\tau} \sum_{i\in\Lambda_k^l} \tilde{x}_i^T(\upsilon)\left(\left(A^T P + PA\right) - 2\lambda_{k,i}^l PBB^T P\right)\tilde{x}_i(\upsilon)d\upsilon \\
&\geqslant -\int_{t_k^l}^{t_k^l+\tau} \sum_{i\in\Lambda_k^l} \tilde{x}_i^T(\upsilon)\left(\left(A^T P + PA\right) - 2\lambda_{\min} PBB^T P\right)\tilde{x}_i(\upsilon)d\upsilon \\
&\geqslant \int_{t_k^l}^{t_k^l+\tau} \sum_{i\in\Lambda_k^l} \lambda_{\min}\tilde{x}_i^T(\upsilon)\tilde{x}_i(\upsilon)d\upsilon,
\end{aligned}
$$

it follows that, for each $l = 0, 1, \cdots, l_k - 1,\ k = 1, 2, \cdots$,

$$
\lim_{t\to\infty} \sum_{i\in\Lambda_k^l} \lambda_{\min}\tilde{x}_i^T(t)\tilde{x}_i(t) = 0.
$$

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Therefore, in a time-interval $[t_k, t_{k+1})$, we have

$$\lim_{t\to\infty} \lambda_{\min}\left(\sum_{i\in\Lambda_k^0} \tilde{x}_i^T(t)\tilde{x}_i(t) + \sum_{i\in\Lambda_k^1} \tilde{x}_i^T(t)\tilde{x}_i(t) + \cdots + \sum_{i\in\Lambda_k^{l_k-1}} \tilde{x}_i^T(t)\tilde{x}_i(t)\right) = 0. \tag{30}$$

Recalling that, by Assumption 4.4 and Lemma 4.2, we can rewrite (30) as

$$\lim_{t\to\infty} \sum_{i=1}^{N} \alpha_i \tilde{x}_i^T(t)\tilde{x}_i(t) = 0, \tag{31}$$

where $\alpha_i, i = 1, 2, \cdots, N$, are some positive scalars.

This implies that, for any $\bar{x}(0) \in L_V(c)$,

$$\lim_{t\to\infty} \bar{x}_i(t) = 0, \quad i = 1, 2, \cdots, N,$$

which, in turn, implies that

$$\begin{aligned}\lim_{t\to\infty} e_i(t) &= \lim_{t\to\infty} y_i(t) - y_0(t)\\ &= \lim_{t\to\infty} C(x_i(t) - \Pi_i(t)w(t))\\ &= \lim_{t\to\infty} C\bar{x}_i(t)\\ &= 0,\ i = 1, 2, \cdots, N.\end{aligned}$$

This completes the proof. □

Remark 4.2
In [17], the communication topology among the follower agents needs to be connected all the time. Here, we only need it to be jointly connected without requiring it to be connected all the time.

5. SIMULATION RESULTS

In this section, we give a few examples to illustrate our results.

Example 1
Consider a group of four follower agents, whose dynamics are given by (2) with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad E = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 2 & -1 \end{bmatrix},\quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and $\Delta = 35$. Clearly, Assumption 2.1 is satisfied.

The dynamics of the leader is given by (3) with

$$S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

The communication topology among the agents is represented by a digraph shown in Figure 1(a), which satisfies Assumption 3.1. Associated with the communication topology, we obtain the entries of adjacency matrix with $a_{10} = a_{30} = 1, a_{21} = 1, a_{31} = 1, a_{41} = 1, a_{43} = 1$, and all others being zero.

The linear matrix equations in Assumption 3.2 can be solved to obtain the following solution:

$$\Pi = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix},\quad \Gamma = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix},\quad H = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

Let Assumption 3.3 be satisfied with $T = 0$ and $\delta = 10$.

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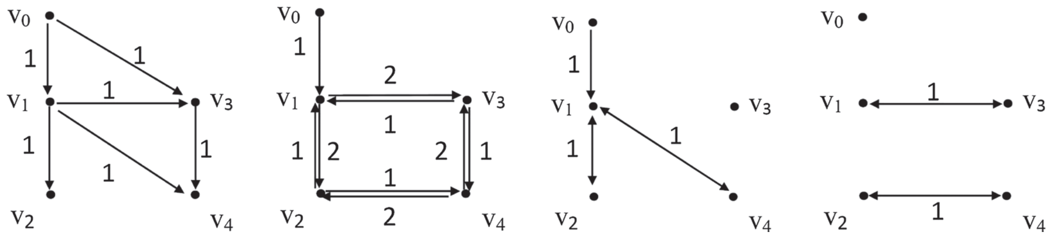
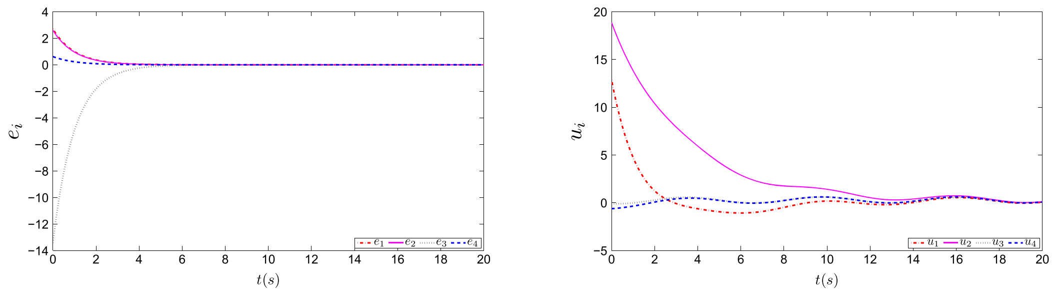


(a) $\mathcal{G}$ (b) $\mathcal{G}$ (c) $\mathcal{G}_1$ (d) $\mathcal{G}_2$

Figure 1. The communication topologies among agents.



(a) The differences $e_i, i = 1, 2, 3, 4$, between the outputs of the follower agents and the reference output.

(b) The inputs of follower agents $u_i, i = 1, 2, 3, 4$.

Figure 2. Example 1: The evolution of the agents with $\varepsilon = 0.1$.

In the simulation, choose the initial values of each agent as

$$x(0) = \begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & x_4(0) \end{bmatrix} = \begin{bmatrix} -0.1 & 2.1 & 2 & -14 \\ 20 & -3 & -60 & 0.3 \end{bmatrix},$$

and $w(0) = [0.1 \quad 0.2 \quad 0.3]^T$.

Thus, all assumptions of Theorem 3.1 are satisfied. Letting $\varepsilon = 0.1$, we can obtain

$$P(0.1) = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.3162 \end{bmatrix},$$

and construct feedback laws according to (6).

Figures 2(a) and (b) respectively show the evolutions of the differences between the outputs of follower agents and the reference output as specified by the leader agent, and the inputs of follower agents. In Figure 2(a), we see that output consensus is achieved for the given initial conditions under the control law (6), as indicated by Theorem 3.1. A large number of simulation results show that, for arbitrarily large initial conditions, output consensus can always be achieved by sufficiently decreasing the value of ε. This verifies that semi-global output consensus is achieved.

Example 2

Consider a group of 4 follower agents, whose dynamics are given by (1) with

$$E_1 = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 2 & -1 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & -2 & -1 \\ -1 & 1 & 0 \end{bmatrix},\ E_3 = \begin{bmatrix} 0 & -2 & -1 \\ -1 & 2 & 0 \end{bmatrix},\ E_4 = \begin{bmatrix} 0 & -2 & -1 \\ -1 & 3 & 0 \end{bmatrix},$$

and Δ, A, B, and C as given in Example 1. Let the dynamics of the leader agent be also as given in Example 1.

The communication topology among the agents is shown in Figure 1(b), which satisfies Assumption 4.1. Associated with the communication topology, we obtain $a_{10} = 1$, $a_{20} = a_{30} = a_{40} = 0$, and $\lambda_1 = 0.2215$.

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The linear matrix equations in Assumption 4.2 can be solved to obtain the following solution:

$$\Pi_1 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix},\ \Pi_2 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix},\ \Pi_3 = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & -1 \end{bmatrix},$$
$$\Pi_4 = \begin{bmatrix} -1 & -1 & -1 \\ 3 & 1 & -1 \end{bmatrix},\ \Gamma_1 = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix},\ \Gamma_2 = \Gamma_3 = \Gamma_4 = 0.$$

Let Assumption 4.3 be satisfied with $T = 0$ and $\delta = 10$.

Thus, all assumptions of Theorem 4.1 are satisfied. Letting $\gamma = 0.125 < \lambda_1$ and $\varepsilon = 0.001$, we obtain,

$$P(0.001) = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.06325 \end{bmatrix},$$

and construct feedback laws according to (17).

In the simulation, we choose the initial values of each agent randomly as

$$x(0) = \begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & x_4(0) \end{bmatrix} = \begin{bmatrix} 1.09 & -20 & -1.4 & 20 \\ 40 & -3 & 30 & 1.3 \end{bmatrix},$$

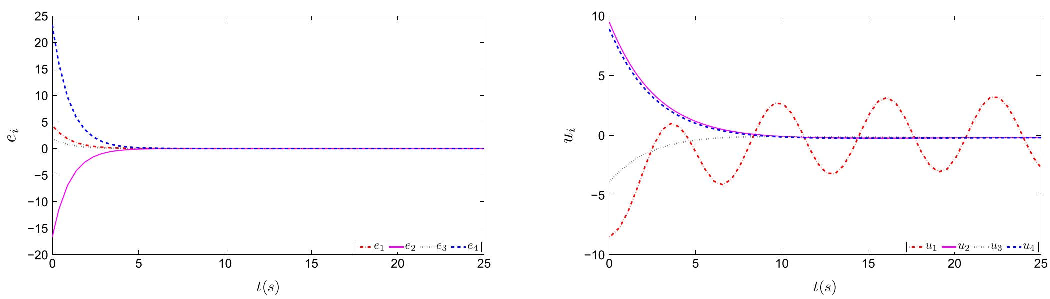
and $w(0) = [1 \quad 2 \quad 0.3]^T$.

Figures 3(a) and (b) respectively show the evolutions of the differences between the outputs of follower agents and the (reference) output of the leader and the inputs of follower agents. In Figure 3(a), we see that output consensus is achieved for the given initial conditions under the control law (17), as indicated by Theorem 4.1. Further simulation shows that, for arbitrarily large initial conditions, output consensus can always be achieved by sufficiently decreasing the value of ε. This verifies that semi-global output consensus can be achieved.

Example 3

Reconsider the multi-agent system in Example 2 with $\Delta = 150$, but the communication topology among the agents switches at time $t = 1, 2, \cdots$ between the two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ as shown in Figures 1(c) and (d). It can be easily verified that this topology satisfies Assumption 4.4. Also, it is clear that Assumption 4.5 is satisfied.

According to the communication topology, we obtain that $a_{10} = 1$, $a_{20} = a_{30} = a_{40} = 0$, and $\lambda_{\min} = 0.2679$. The same solution $\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 to the matrix equations in Assumption 4.6 can be obtained as in Example 2.



(a) The differences $e_i, i = 1, 2, 3, 4$, between the outputs of the follower agents and the reference output.

(b) The inputs of follower agents $u_i, i = 1, 2, 3, 4$.

Figure 3. Example 2: The evolution of the agents with $\gamma = 0.125$ and $\varepsilon = 0.001$.

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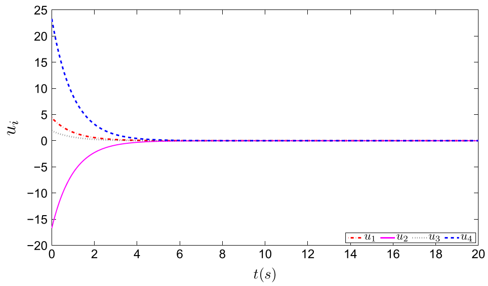
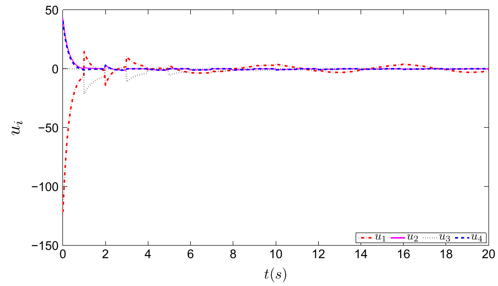


(a) The differences $e_i, i = 1, 2, 3, 4$, between the outputs of the follower agents and the reference output.



(b) The inputs of follower agents $u_i, i = 1, 2, 3, 4$.

Figure 4. Example 3: The evolution of the agents with $\gamma = 0.2$ and $\varepsilon = 0.4$.

Let Assumption 4.7 be satisfied with $T = 0$ and $\delta = 10$.

Thus, all assumptions of Theorem 4.2 are satisfied. Letting $\varepsilon = 0.4$ and $\gamma = 0.2$, we have

$$P(0.1) = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}$$

and construct feedback laws according to (26).

Figures 4(a) and (b) respectively show the evolutions of the differences between the outputs of follower agents and the reference output as specified by the leader agent, and the inputs of follower agents, with the same initial conditions as in the simulation in Example 2. In Figure 4(a), we see that output consensus is also achieved for the given initial conditions under the control law (26), as indicated by Theorem 4.2. Further simulation shows that, for arbitrarily large initial condition, output consensus can always be achieved by sufficiently decreasing the value of ε. This verifies that semi-global output consensus can be achieved.

6. CONCLUSIONS

In this paper, we studied semi-global leader-following output consensus of a group of linear systems in the presence of external disturbances and actuator saturation. Both the case of disturbance entering all follower agents the same way, which we referred to as the identical disturbance case, and the case of disturbance entering the follower agents differently, referred to as the non-identical disturbance case, are considered. In the identical disturbance case, we established the solvability conditions based solely on agent dynamics and constructed a linear state feedback control algorithm by using low-gain feedback design for each follower agent. We provided a new digraph condition to guarantee semi-global leader-following output consensus. In the non-identical disturbance case, we established the conditions based on both the agent dynamics and the communication topology jointly. A family of linear state feedback control algorithms was constructed to achieve semi-global output consensus when the communication topology is a strongly connected and detailed balanced digraph and the leader is a neighbor of at least one follower. We also proved that under some further conditions on the agent dynamics, the control algorithms also achieve semi-global output consensus when the communication topology among the follower agents is a jointly connected undirected graph, and the leader is a neighbor of at least one follower.

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