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Controllability analysis of multi-agent systems with directed and weighted interconnection

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In this article, we investigate the controllability of multi-agent systems with leaders as control inputs, where the interconnection is directed and weighted. We employ weight-balanced partition to classify the interconnection graphs, and study the controllable subspaces with given nontrivial weight-balanced partition. We also provide two necessary and sufficient graph conditions on structural controllability and strong structural controllability. Moreover, we consider the effect of the zero row-sum restrictions of the system matrices on structural controllability.

Keywords: controllability; partition; weight-balanced partition; structural controllability; strong structural controllability

1. Introduction

In recent years, multi-agent systems have received huge attention due to the broad applications (Jadbabaie, Lin, and Morse 2003; Fax and Murray 2004; Ren and Beard 2005; Lin and Jia 2010). As one of important problems, coordination control of leader–follower problems has been studied widely with different formulations, where the leader information can be viewed as the control input of the whole multi-agent system. Consensus tracking control for multi-agent systems with stationary or active leaders was considered in Jadbabaie et al. (2003) and Hong, Hu, and Gao (2006). Moreover, multi-leader coordination such as target set aggregation and containment was also considered with deterministic or random switching topologies, where multiple leaders can be viewed as multi-input coordination to drive the whole group of agents (Shi and Hong 2009; Cao and Ren 2010; Lou and Hong 2012).

The controllability study of multi-agent systems aims at driving follower agents to achieve any positions from any initial states with the control inputs provided by the leaders. It is closely related to the leader-follower coordination. The concept of controllability of multi-agent systems was proposed in Tanner (2004), where a sufficient and necessary condition based on eigenvalue and eigenvector of the system matrix was provided to make the follower subsystem controlled by a single leader. In Rahmani, Ji, Mesbahi, and Egerstedt (2009), the controllability was investigated by virtue of equitable partition from graph-theoretic perspective, and it was shown that systems symmetric with respect to a single leader are not completely controllable. In Martini, Egerstedt, and Bicchi (2010), relaxed equitable partition was employed for controllability, with the conclusion that a single-leader follower system is completely controllable if and only if each partition cell contains only one node, and the controllable subspace was also discussed in the uncontrollable case. In Ji, Wang, Lin, and Wang (2009), interesting results were given for checking controllability. In addition, other interesting problems were also considered, including controllability analysis with time-delays (Ji, Wang, Lin, and Wang 2010), homogeneous systems (Twu, Egerstedt, and Martini 2010), switching topologies (Liu, Chu, Wang, and Xie 2008) and link failures (Jafari, Ajorlou, and Aghdam 2011).

Structural controllability and strong structural controllability are also important problems in the study of controllability of multi-agent systems, which characterise the graph conditions of making systems controllable by re-designing weights of arcs when original systems are uncontrollable and of systems being controllable for any non-zero weights, respectively. The graph condition of structural controllability for a single-leader system with fixed topology was investigated in Zamani and Lin (2009), while similar problems were considered for high-order systems in Partovi, Lin, and Ji (2010).

Most aforementioned results mainly were concerned with undirected interconnection systems, where all weights between agents are taken as ones.

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In general cases, the controllability of multi-agent systems depends on both interconnection topology and the weights between agents. The objective of this article is to investigate the controllability of directed and weighted multi-leader systems. The main contributions include: (i) we employ a new partition concept on interconnection graph to characterise the controllability of directed and weighted systems, since the existing techniques, for example, equitable partition (Rahmani et al. 2009) and relaxed equitable partition (Martini et al. 2010), are not appropriate to deal with weighted systems; (ii) we give the necessary and sufficient conditions of structural controllability and strong structural controllability characterised by partitions on interconnection graphs. Moreover, to the best of our knowledge, we first consider strong structural controllability for multi-agent systems; (iii) we discuss the effect of the zero-row sum restrictions of the system matrices on the structural controllability of the considered multi-agent system.

This article is organised as follows. Section 2 provides problem formulation. Then, Section 3 gives controllability analysis based on the newly proposed concept of weight-balanced partition on the interconnection graph. Necessary and sufficient conditions of structurably controllable (SC) and strong SC (SSC) systems are provided in Sections 4 and 5, respectively. The effect of the zero row-sum restrictions of the system matrices on structural controllability was discussed in Section 6. Finally, some concluding remarks are given in Section 7.

2. Problem formulation

In this article, we consider a multi-agent system consisting of $n+l$ agents in $\mathbb{R}$. The interconnection topology between agents can be characterised by a directed graph (or digraph) $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with the set of nodes $\mathcal{V}=\{1,2,\ldots,n+l\}$, set of arcs $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$. $(i,j)\in\mathcal{E}$ if and only if agent j is a neighbour of agent i. $F=[a_{ij}]_{(n+l)\times(n+l)}$ denotes the adjacency matrix of $\mathcal{G}$, where $a_{ij}\neq 0$ if and only if $(i,j)\in\mathcal{E}$. There is no self-loop in $\mathcal{G}$, that is, $(i,i)\notin\mathcal{E}$ for all i. The set of neighbours of node i is denoted by $\mathcal{N}_i=\{j\in\mathcal{V}|(i,j)\in\mathcal{E}\}$. A path from i to j in $\mathcal{G}$ is a sequence $i_0,i_1,\ldots,i_{r-1},i_r$ of nodes with $(i_s,i_{s+1})\in\mathcal{E}$ for $0\leq s\leq r-1$, where $i_0=i$ and $i_r=j$ with i and j called the initial node and the end node of this path, respectively. Graph $\mathcal{G}$ is said to be a directed tree if there is a node i_0 such that there exists one and only one path from i_0 to i for any $i\neq i_0$. Graph $\mathcal{G}$ is said to be an inverse-tree if the graph formed by reversing all the directions of arcs of $\mathcal{G}$ is a directed tree. Node j_0 is called the centre node of graph $\mathcal{G}$ if there exists at least one path from j to j_0 for any $j\neq j_0$. Graph $\mathcal{G}$ is said to be undirected if $(i,j)\in\mathcal{E}$ if and only if $(j,i)\in\mathcal{E}$ and $a_{ij}=a_{ji}$ (referring to Godsil and Royle (2001) for details).

Suppose that a subset of the agents are able to make decisions on their own, which can be regarded as leaders. Without loss of generality, we assume agents labelled $n+1,\ldots,n+l$, are leaders (so there are l leaders) and denote their position vector as $u=(u_{n+1},\ldots,u_{n+l})^T$. Correspondingly, the position vector of followers is denoted by $x=(x_1,\ldots,x_n)^T$. Denote $\mathcal{I}=\{1,2,\ldots,n\}$ and $\mathcal{L}=\{n+1,n+2,\ldots,n+l\}$. In the dynamic evolution, all followers obey the well-known neighbour rule (Jadbabaie et al. 2003; Ren and Beard 2005);

$$\dot{x}_i(t)=\sum_{j\in\mathcal{N}_i\bigcap\mathcal{I}}a_{ij}\big(x_j(t)-x_i(t)\big)+\sum_{j\in\mathcal{N}_i\bigcap\mathcal{L}}a_{ij}\big(u_j(t)-x_i(t)\big),\quad i\in\mathcal{I},\tag{1}$$

while the leaders can take any actions on their own. In this article, we assume $(j,i)\notin\mathcal{E}$ for all $j\in\mathcal{L}$ and $i\in\mathcal{V}$. By undirected graph $\mathcal{G}$, we mean that its induced subgraph of $\mathcal{G}$ consisting of all follower nodes is undirected.

Denote $L\in\mathbb{R}^{n\times n}$ as

$$L_{ij}=\begin{cases}\sum_{j=1}^{n}a_{ij} & \text{if } i=j,\\ -a_{ij} & \text{otherwise.}\end{cases}$$

Then system (1) can be rewritten in a compact form:

$$\dot{x}(t)=Ax(t)+Bu(t),\quad x(t)\in\mathbb{R}^n,\quad u(t)\in\mathbb{R}^l,\tag{2}$$

where $B=(b_{ik})\in\mathbb{R}^{n\times l}$ with $b_{ik}=a_{i(n+k)}$ $(1\leq i\leq n,\ 1\leq k\leq l)$ and $A=-L-\mathrm{diag}\{d_1,\ldots,d_n\}$ with $d_i=\sum_{k=1}^{l}b_{ik}$ $(1\leq i\leq n)$. (A,B) is called the matrix pair (MP) of system (2).

The objective of this article is, based on the partition on interconnection graph, to characterise graph conditions on the controllability of system (2), including complete controllability, structural controllability and strong structural controllability. In addition, we say that system (2) is controllable if it is completely controllable for simplicity.

Given a matrix $H=[h_1,\ldots,h_p]\in\mathbb{R}^{n\times p}$, the column space $\mathcal{R}(H)$ is defined as $\mathrm{span}\{h_1,\ldots,h_p\}$. The controllable subspace $\mathcal{C}$ of linear time-invariant (LTI) system (2) is the column space $\mathcal{R}(C)$, where $C=[B,AB,\ldots,A^{n-1}B]$ is the controllability matrix (CM) of system (2). It is well known that system (2) is controllable if and only if (A,B) satisfies either of the

following conditions (Kailath 1980):

(i) rank $[B, AB, \ldots, A^{n-1}B] = n$; (3)

(ii) rank $[\lambda I_n - A, B] = n$ for any $\lambda \in \mathbb{C}$. (4)

3. Graph-based controllability conditions

In this section, we investigate the controllability and related properties of system (2) with a fixed interconnection digraph $\mathcal{G}$.

3.1 *Partition and controllability*

Rahmani et al. (2009) and Martini et al. (2010) proposed equitable partition and relaxed equitable partition to check the controllability of leader–follower systems, respectively, but the two concepts cannot be directly applied to multi-agent systems with weighted interconnection graphs. In this section, we employ the concept of weight-balanced partition on follower nodes $\mathcal{I}$ to investigate the controllability of system (2).

For $i \in \mathcal{I}$, $k \in \mathcal{L}$, denote $d(i,k)$ as the length of the shortest path from i to k, if there is at least one path from i to k in $\mathcal{G}$ and $d(i,k) = \infty$ otherwise. Define $\mathcal{P}_\infty = \{i \in \mathcal{I} | d(i,k) = \infty \text{ for all } k \in \mathcal{L}\}$.

Definition 3.1:

(i) A distance-based partition $\hat{\mathcal{P}} = \{\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_{\hat{\upsilon}}, \mathcal{P}_\infty\}$ on $\mathcal{I}$ is defined according to i, j belong to $\hat{\mathcal{P}}_r$ $(1 \le r \le \hat{\upsilon})$ if and only if $d(i,k) = d(j,k)$ for all $k \in \mathcal{L}$. Moreover, $\hat{\mathcal{P}}$ is said to be connected if $\mathcal{P}_\infty = \emptyset$ and chain-like (Figure 1) if $\mathcal{P}_\infty = \emptyset$ and $\hat{\upsilon} = n$.

(ii) A weight-balanced partition $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_\upsilon, \mathcal{P}_\infty\}$ on $\mathcal{I}$ is defined according to i, j belong to $\mathcal{P}_r$ $(1 \le r \le \upsilon)$ if and only if $b_{ik} = b_{jk}$ for $1 \le k \le l$ and $\sum_{p \in \mathcal{P}_\iota} a_{ip} = \sum_{p \in \mathcal{P}_\iota} a_{jp}$ for all $\iota \ne r$. $\mathcal{P}_1, \ldots, \mathcal{P}_\upsilon$ are said to be cells of $\mathcal{P}$. In addition, $\mathcal{P}$ is said to be nontrivial if either $\mathcal{P}_\infty \ne \emptyset$ or $\mathcal{P}_\infty = \emptyset$ but $\upsilon < n$.

Remark 3.2: For a single-leader graph with identical weights, the concept of weight-balanced partition in Definition 3.1 is the same as the leader-invariant relaxed equitable partition proposed in Martini et al. (2010).

Given a set K. Let $\mathcal{K}_1 = \{K_{11}, \ldots, K_{1r_1}\}$, $\mathcal{K}_2$ and $\mathcal{K} = \{K_1, \ldots, K_r\}$ be three partitions of K.

Definition 3.3 (Borůvka 1976)**:** $\mathcal{K}$ is said to a covering of $\mathcal{K}_1$ if, for any $1 \le s_1 \le r_1$, there exists some $1 \le s \le r$ such that $K_{1s_1} \subseteq K_s$. $\mathcal{K}$ is said to a common covering of $\mathcal{K}_1$ and $\mathcal{K}_2$ if $\mathcal{K}$ is a covering of $\mathcal{K}_1$ as well of $\mathcal{K}_2$. $\mathcal{K}$ is said to be a finest common covering of $\mathcal{K}_1$ and $\mathcal{K}_2$ if $\mathcal{K}$ is a common covering of $\mathcal{K}_1$ and $\mathcal{K}_2$ and every common covering of $\mathcal{K}_1$ and $\mathcal{K}_2$ is also a covering of $\mathcal{K}$.

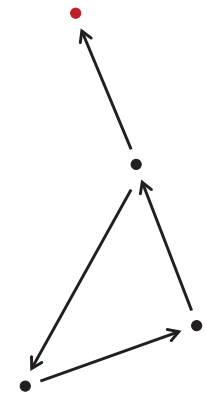

Figure 1. Chain-like graph.

For any graph with n follower nodes and l leader nodes, there is a weight-balanced partition on $\mathcal{I}$, say $\{\{i_1\}, \ldots, \{i_r\}, \mathcal{P}_\infty\}$, where $\{i_1, \ldots, i_r\} = \mathcal{I} \backslash \mathcal{P}_\infty$. Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are two weight-balanced partitions on $\mathcal{I}$. Let $\mathcal{P}$ be the finest common covering of $\mathcal{P}_1$ and $\mathcal{P}_2$. It is not hard to verify that $\mathcal{P}$ is also weight-balanced. As a result, the weight-balanced partition on $\mathcal{I}$ is unique in the sense that the number of its cells is minimal. In the following, denote $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_\upsilon, \mathcal{P}_\infty\}$ as the unique one with the minimal cardinality. Without loss of generality, in this section we assume that nodes labelled $1, 2, \ldots, |\mathcal{P}_1|$ belong to cell $\mathcal{P}_1$ and nodes labelled $\sum_{k=1}^{r} |\mathcal{P}_k| + 1, \ldots, \sum_{k=1}^{r+1} |\mathcal{P}_k|$ belong to cell $\mathcal{P}_{r+1}$, $1 \le r \le \upsilon - 1$. Then we have the following result.

Theorem 3.4: *System* (2) *is uncontrollable if $\mathcal{G}$ has a nontrivial weight-balanced partition $\mathcal{P}$ and the dimension of its controllable subspace* rank C *satisfies* rank $C \le \upsilon$.

Proof: We first show that rank $C \le \upsilon$ if $\mathcal{P}_\infty = \emptyset$ and $\upsilon < n$. If i, j belong to the same cell, then $b_{ik} = b_{jk}$ for $1 \le k \le l$. Set $B = [b_1, \ldots, b_l]$. By the weight balance condition in Definition 3.1, $(Ab_k)_i = (Ab_k)_j$. Then by recursion, we have $(A^r b_k)_i = (A^r b_k)_j$ for $1 \le k \le l$ and $1 \le r \le n-1$. Thus, the CM C is in the following form:

$$\begin{pmatrix} *\mathbf{e}_1 & \cdots & *\mathbf{e}_1 \\ \vdots & \ddots & \vdots \\ *\mathbf{e}_\upsilon & \cdots & *\mathbf{e}_\upsilon \end{pmatrix}$$

where $\mathbf{e}_r$ $(1 \le r \le \upsilon)$ is a vector of ones with $|\mathcal{P}_r|$ components. Clearly, rank $C \le \upsilon < n$, and hence system (2) is uncontrollable. The controllable subspace $\mathcal{C}$ satisfies

$$\mathcal{C} \subseteq \operatorname{span}\left\{ \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ \mathbf{e}_\upsilon \end{pmatrix} \right\},$$

which implies the conclusion. If $\mathcal{P}_\infty \neq \emptyset$, we can prove the conclusion by similar arguments given above. Thus, the conclusion follows. □

To further characterise controllability, quotient graph and relaxed quotient graph have been proposed for undirected graphs with weight ones in Rahmani et al. (2009) and Martini et al. (2010), respectively (also referring to Godsil and Royle (2001)). Next, we give the corresponding concept based on weight-balanced partition for weighted and directed graphs and reveal the close relation between system (2) and its quotient system.

Definition 3.5: The quotient graph $\mathcal{G}/\mathcal{P} = (\mathcal{V}/\mathcal{P}, \mathcal{E}/\mathcal{P})$ of graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined as follows:

(1) $\mathcal{V}/\mathcal{P} := \{\bar{i} \mid i \in \mathcal{I}\} \bigcup \mathcal{L}$, where $\bar{i}$ denotes the cell of $i \in \mathcal{I}$.
(2) $\mathcal{E}/\mathcal{P} := \{(\bar{i}, \bar{j}) \mid (i,j) \in \mathcal{E} \bigcap (\mathcal{I} \times \mathcal{I})$ and i, j do not belong to same cell$\} \bigcup \{(\bar{i}, k) \mid (i,k) \in \mathcal{E} \bigcap (\mathcal{I} \times \mathcal{L})\}$.

Moreover, the adjacency matrix $F/\mathcal{P}$ of the quotient graph is defined as follows: $(F/\mathcal{P})_{\bar{i}k} = a_{ik}$ and $(F/\mathcal{P})_{\bar{i}\bar{j}} = \sum_{r \in \bar{j}} a_{ir}$ for all $i \in \mathcal{I}$, $k \in \mathcal{L}$ and $\bar{i} \neq \bar{j}$.

In Figure 2, all weights are ones except the two labelled 2 and -1. According to Definition 3.1 (ii), the quotient graph is well defined. The system with the quotient graph $\mathcal{G}/\mathcal{P}$ is called the quotient system of system (2). An immediate conclusion can be obtained by combining Theorem 3.4 with Definition 3.5.

Corollary 3.6: *The dimension of controllability subspace of system* (2) *is identical to that of the quotient system of system* (2). *In particular, the controllable subspace of system* (2) *with* $\mathcal{P}_\infty = \emptyset$

$$\mathcal{C} = \text{span}\left\{ \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ \mathbf{e}_\upsilon \end{pmatrix} \right\}$$

if and only if the quotient system of system (2) *is controllable.*

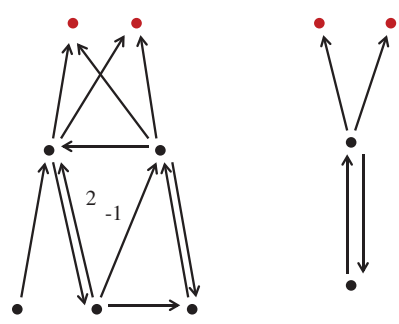

Figure 2. Original graph and its quotient graph.

3.2 *Symmetric systems*

In this subsection, we consider an important special case of the preceding subsection, symmetric systems.

Definition 3.7: System (2) is said to be symmetric with respect to leaders $\mathcal{L}$ if there exists a nonidentity permutation ψ on $\mathcal{I}$ such that

(1) $a_{ij} = a_{\psi(i)\psi(j)}$ for all $i, j \in \mathcal{I}$ and
(2) $b_{ik} = b_{\psi(i)k}$ for all $i \in \mathcal{I}$ and $1 \leq k \leq l$.

According to Definition 3.7, we can easily spoil the symmetry by re-designing non-zero weights to make them not equal each other. Then, we shall show that the symmetry with respect to the control nodes is a special case of existing a weight-balanced partition. In fact, it is not hard to obtain the following result.

Lemma 3.8: *System* (2) *is symmetric with respect to leaders* $\mathcal{L}$ *if and only if there is a nonidentity permutation matrix* J *satisfying*

$$JA = AJ, \quad JB = B \tag{5}$$

The single-leader symmetry was first proposed in Rahmani et al. (2009), and Definition 3.7 (referring to Lemma 3.8) extends it to multi-leader cases. In the single-leader case, the permutation matrix J satisfies (5) if and only if $JA = AJ$, which is consistent with the result given in Rahmani et al. (2009).

Example 3.9: The system consists of followers 1, 2, 3, 4 and leaders 5, 6 with all weights being ones (Figure 3). It is easy to verify that the system is symmetric with respect to leaders 5 and 6. The nonidentity permutation is the one for which $\psi(1) = 2$, $\psi(2) = 1$, $\psi(3) = 4$ and $\psi(4) = 3$.

The nonidentity permutation ψ in Definition 3.7 can be expressed as the product of permutations $\psi = \psi_1 \psi_2 \cdots \psi_\tau$ uniquely in the sense that τ is minimal. We denote $|\psi_r|$ by the order of permutation ψ_r $(1 \leq r \leq \tau)$, that is, the least positive integer m for which ψ_r^m is the identity permutation. Define a

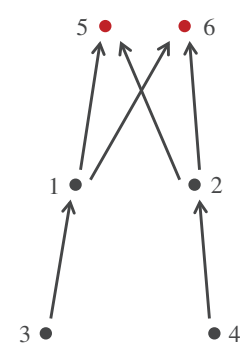

Figure 3. Symmetric with respect to leaders 5 and 6.

partition on $\mathcal{I}$ as follows: i, j belong to the same cell if and only if there exist some $1 \le r \le \tau$ and a positive integer m such that $\psi_r^m(i) = j$. The newly defined partition on $\mathcal{I}$ is nontrivial and weight-balanced. According to Theorem 3.4, we have the following result.

Corollary 3.10: *System* (2) *with* $\mathcal{P}_\infty = \emptyset$ *is uncontrollable if it is symmetric with respect to leaders and the controllable subspace*

$$\mathcal{C} \subseteq \operatorname{span}\left\{ \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ \mathbf{e}_\tau \end{pmatrix} \right\},$$

where $\mathbf{e}_r$ *is a vector of ones with* $|\psi_r|$ *components,* $1 \le r \le \tau$.

3.3 *Flow invariance*

For illustration and understanding, we show some properties related to weight-balanced partition of graph $\mathcal{G}$. Given a partition $\mathcal{Q} = \{\mathcal{Q}_1, \ldots, \mathcal{Q}_\nu\}$ on $\mathcal{I}$, we define vector $x \in \mathbb{R}^n$ and subspace $\Lambda_\mathcal{Q}$ based on partition $\mathcal{Q}$ as follows:

$$\begin{aligned} &x_i = x_j \quad \text{whenever } i, j \in \mathcal{Q}_r,\ 1 \le r \le \nu; \\ &\Lambda_\mathcal{Q} = \{x \in \mathbb{R}^n | x \text{ is of the form (6)}\}. \end{aligned} \tag{6}$$

System (2) is said to be flow invariant with respect to subspace $\Lambda_\mathcal{Q}$ if, for any continuous l-valued function $u(t)$, $x(t) \in \Lambda_\mathcal{Q}$ for any $t \ge 0$ provided that $x(0) \in \Lambda_\mathcal{Q}$. The solution of system (2) is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)\mathrm{d}s, \quad t \ge 0. \tag{7}$$

Suppose the partition $\mathcal{Q}$ is weight-balanced and $x(0) \in \Lambda_\mathcal{Q}$. Then by the weight balance condition in Definition 3.1, all columns of $e^{A(t-s)}B$ are in the form (6) for any $t \ge s \ge 0$, which implies that the last term in (7) is of the form (6). Therefore, $x(t)$ is of the form (6) since $e^{At}x(0)$ is also in the form of (6), that is, system (2) is flow invariant with respect to subspace $\Lambda_\mathcal{Q}$. Conversely, suppose system (2) is flow invariant with respect to subspace $\Lambda_\mathcal{Q}$. Taking $u(t) \equiv 0$, $e^{At}x(0)$ is in the form (6) for any $t \ge 0$ once $x(0) \in \Lambda_\mathcal{Q}$. Then by taking the first derivative of $e^{At}x(0)$ at $t = 0$, we have

$$Ax(0) \text{ is of the form (6) once } x(0) \in \Lambda_\mathcal{Q}. \tag{8}$$

Then (8) implies that the last term in (7) is of the form (6) for any continuous function $u(t)$. By taking the first derivative of the last term in (7) at $t = 0$, we have

$$Bu(0) \text{ is of the form (6) for any } u(0). \tag{9}$$

(8) and (9) imply that the partition $\mathcal{Q}$ is weight-balanced. Thus, we obtain the next theorem to show that the existence of weight-balanced partition is equivalent to the flow invariance of system (2).

Theorem 3.11: *System* (2) *is flow invariant with respect to subspace* $\Lambda_\mathcal{Q}$ *if and only if the partition* $\mathcal{Q}$ *is weight-balanced.*

4. Structural controllability

In the preceding section, we showed that system (2) is uncontrollable if there exists a nontrivial weight-balanced partition on follower nodes. However, in practice, maybe we have to find an effective method to make an uncontrollable system controllable. Since controllability of multi-agent systems depends not only on interconnection topology but also the weights between agents, we can re-design the weights between agents, or add some new control nodes and new arcs to make an uncontrollable system controllable.

In this section, we first consider the method of re-designing weights. Then, we discuss the method to make the system SC maybe by adding new control nodes and/or arcs.

The concept of structural controllability has been proposed for LTI single-input systems in Lin (1974) and the results were extended to multi-input cases in Mayeda (1981). Structural controllability discussed in Lin (1974) shows the capacity to make the system controllable by selecting values for all free parameters, which are the weights of all arcs in our problem. The re-assigned value of the weight of an arc is zero means that the interconnection is cut off. This case is excluded in our problem. Now, we give the corresponding concept of structural controllability for system (2).

Definition 4.1: System (2) is said to be SC if there exist weights $a_{ij} \ne 0$, $(i, j) \in \mathcal{E}$ to make system (2) controllable.

Remark 4.2: The structural controllability has been considered for multi-agent systems in Zamani and Lin (2009) and Partovi et al. (2010), mainly based on the existing results for LTI systems in Lin (1974) and Mayeda (1981), where all the weights including a_{ii}, i.e. A_{ii}, $i \in \mathcal{I}$, are allowed to be re-designed. Different from the existing results, here zero row-sum restriction (i.e. $a_{ii} = -\sum_{j \in \mathcal{N}_i} a_{ij}$) is forced, which makes the parameters a_{ii}, $i \in \mathcal{I}$ not free but determined by other parameters. Thus, the results on structural controllability for LTI systems in Lin (1974) and Mayeda (1981) cannot be directly applied to our problem.

Remark 4.3: If the interconnection between agents is allowed to cut off, a new definition of SC can be proposed as follows: system (2) is said to be SC if there

exist weights a_{ij}, $(i,j)\in\mathcal{E}$ such that system (2) is controllable. Similarly, we can prove that under this new definition of structural controllability, the following Theorem 4.6 also holds.

Sometimes, we also say a graph is SC if the system (2) corresponding to this graph is SC. According to the equivalent condition (4), directed chain graphs with the end node as the unique leader are controllable for any non-zero weights and then directed chain graphs are SC.

Before representing the main result of this section, we give two lemmas.

Lemma 4.4: *Consider chain-like graph $\mathcal{G}_1$ with p nodes and a single leader i_p and graph $\mathcal{G}_2$ with q nodes and a single leader j_q. Denote the node with the farthest distance to leader i_p as i_1 and define $\mathcal{G}_0 = \mathcal{G}_1 \bigcup \mathcal{G}_2 \bigcup (j_q, i_1)$, where i_p is the only leader of $\mathcal{G}_0$ (Figure 4). Then $\mathcal{G}_0$ is SC if and only if $\mathcal{G}_2$ is SC.*

Proof: Let (A_2, h_2) and $(\hat{A}, \hat{h})$ be the MPs of graphs $\mathcal{G}_2$ and $\mathcal{G}_0$, respectively. It is not hard to find that

$$\begin{aligned}&\operatorname{rank}(\lambda I_{q-1} - A_2, h_2) + p\\ &\quad = \operatorname{rank}(\lambda I_{p+q-1} - \hat{A}, \hat{h}) \quad \text{for any } \lambda \in \mathbb{C}.\end{aligned}$$

Thus, the conclusion follows from equivalent condition (4). □

Lemma 4.5: *Let $\mathcal{G}_5$ be a single-leader graph constructed by merging the two leaders of $\mathcal{G}_3$ and $\mathcal{G}_4$ into the only one leader of $\mathcal{G}_5$, where $\mathcal{G}_3$ and $\mathcal{G}_4$ are single-leader graphs (Figure 5). If $\mathcal{G}_5$ is SC, then $\mathcal{G}_3$ and $\mathcal{G}_4$ are SC. Moreover, if there exist positive weights to make $\mathcal{G}_3$ and $\mathcal{G}_4$ controllable, then there also exist positive weights to make $\mathcal{G}_5$ controllable.*

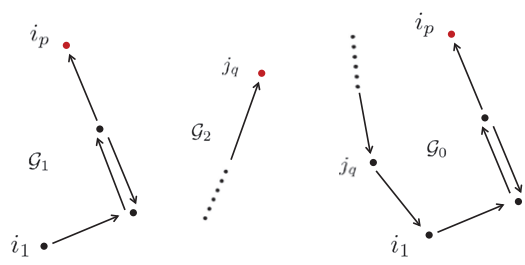

Figure 4. The graph operation in Lemma 4.4.

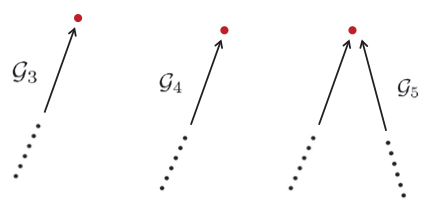

Figure 5. The graph operation in Lemma 4.5.

Proof: Let $(A_3, h_3) \in \mathbb{R}^{n_1\times(n_1+1)}$ and $(A_4, h_4) \in \mathbb{R}^{n_2\times(n_2+1)}$ be the MPs of graphs $\mathcal{G}_3$ and $\mathcal{G}_4$, respectively. By re-labelling the nodes appropriately, we suppose the MP of $\mathcal{G}_5$ takes the form of

$$\begin{pmatrix} A_3 & 0 & h_3 \\ 0 & A_4 & h_4 \end{pmatrix}.$$

Thus, the CM $\hat{C}$ of $\mathcal{G}_5$ is in the form of

$$\begin{aligned}\hat{C} &= \hat{C}(A_3, h_3; A_4, h_4)\\ &= \begin{pmatrix} h_3 & A_3h_3 & \cdots & A_3^{n_1-1}h_3 & A_3^{n_1}h_3 & \cdots & A_3^{n_1+n_2-1}h_3 \\ h_4 & A_4h_4 & \cdots & A_4^{n_1-1}h_4 & A_4^{n_1}h_4 & \cdots & A_4^{n_1+n_2-1}h_4 \end{pmatrix}\\ &:= \begin{pmatrix} \hat{C}_1 & \hat{C}_2 \\ \hat{C}_3 & \hat{C}_4 \end{pmatrix},\end{aligned}$$

where $\hat{C}_1 \in \mathbb{R}^{n_1\times n_1}$, $\hat{C}_2 \in \mathbb{R}^{n_1\times n_2}$, $\hat{C}_3 \in \mathbb{R}^{n_2\times n_1}$ and $\hat{C}_4 \in \mathbb{R}^{n_2\times n_2}$.

Let $\lambda^{n_1} + \omega_{n_1-1}\lambda^{n_1-1} + \cdots + \omega_1\lambda + \omega_0$ be the characteristic polynomial of matrix A_3. By Hamilton–Caylay theorem, $\sum_{r=0}^{n_1-1} \omega_r A_3^r + A_3^{n_1} = 0$. Thus, every column of $\hat{C}_2$ can be expressed as a linear combination of columns of $\hat{C}_1$, that is, there is a matrix $S \in \mathbb{R}^{n_1\times n_2}$ such that $\hat{C}_2 = \hat{C}_1 S$, and then $\det\hat{C} = \det\hat{C}_1 \times \det(\hat{C}_4 - \hat{C}_3 S)$. In fact, every entry of S is a polynomial function with variables $\{\omega_r, 0 \le r \le n_1 - 1\}$, for example, $S_{r1} = -\omega_{r-1}$ for $1 \le r \le n_1$, $S_{12} = \omega_{n_1-1}\omega_0$, etc.

If $\mathcal{G}_5$ is SC, there exist weights to make $\det\hat{C} \neq 0$, which implies $\det\hat{C}_1 \neq 0$, and then $\mathcal{G}_3$ is SC. Similarly, we can show that $\mathcal{G}_4$ is also SC if $\mathcal{G}_5$ is SC.

To prove the latter conclusion, we first present three claims:

(i) A SC graph has a connected partition.
(ii) Both $\mathcal{G}_3$ and $\mathcal{G}_4$ have connected partitions if and only if $\mathcal{G}_5$ has a connected partition.
(iii) If $\mathcal{G}_3$ ($\mathcal{G}_4$) with positive weights has a connected partition, then A_3 (A_4) is Hurwitz.

If a graph has no connected partition, that is, there is at least one follower node that has no path to all leader nodes, then the graph is uncontrollable for any weights. The second claim can be verified by Definition 3.1, while the third one was obtained as Lemma 4 in Hu and Hong (2007).

Suppose there exist positive weights to make $\mathcal{G}_3$ and $\mathcal{G}_4$ controllable, i.e. $\det\hat{C}_1 = \det(h_3, A_3h_3, \ldots, A_3^{n_1-1}h_3) \neq 0$ and $\det(h_4, A_4h_4, \ldots, A_4^{n_2-1}h_4) \neq 0$ (A_3, h_3, A_4, h_4 with re-designed positive weights). By claims (i) and (iii), A_4 is Hurwitz and then $\det\hat{C}_4 = (\det A_4)^{n_1}\det(h_4, A_4h_4, \ldots, A_4^{n_2-1}h_4) \neq 0$.

Re-define

$$\hat{C}(\sigma) = \hat{C}(\sigma A_3, \sigma h_3; A_4, h_4) := \begin{pmatrix} \hat{C}_1(\sigma) & \hat{C}_2(\sigma) \\ \hat{C}_3 & \hat{C}_4 \end{pmatrix}.$$

We can choose a sufficiently small $\sigma > 0$ such that all the entries of σA_3 are sufficiently small, and then all entries of $S(\sigma)$ are sufficiently small, where $S(\sigma)$ is the matrix for which $\hat{C}_2(\sigma) = \hat{C}_1(\sigma)S(\sigma)$. Thus, $\det(\hat{C}_4 - \hat{C}_3 S(\sigma)) \neq 0$ for sufficiently small $\sigma > 0$.

Thus,

$$\begin{aligned}\det \hat{C}(\sigma) &= \det \hat{C}_1(\sigma)\det(\hat{C}_4 - \hat{C}_3 S(\sigma)) \\ &= \sigma^{\frac{n_1(n_1+1)}{2}} \det \hat{C}_1 \det(\hat{C}_4 - \hat{C}_3 S(\sigma)) \neq 0\end{aligned}$$

implies that $\mathcal{G}_5$ is SC, where the re-designed positive weights for $\mathcal{G}_5$ are the ones in $(\sigma A_3, \sigma b_1; A_4, b_2)$ with sufficiently small σ. We complete the proof. □

Then we introduce the main result of this section, a necessary and sufficient graph condition for structural controllability.

Theorem 4.6: *System* (2) *is SC if and only if there exists a connected partition in* $\mathcal{G}$.

Proof: The necessity is the claim (i) in Lemma 4.5. Here we only focus on the sufficiency.

First, we prove the conclusion for the single-leader system (2) with inverse-tree interconnection graph $\mathcal{T}$, where the leader is the centre node of the inverse-tree. Since the single-leader directed chains are SC and $\mathcal{T}$ can be constructed by directed chains applying the two graph operations in Lemmas 4.4 and 4.5, $\mathcal{T}$ is SC by Lemmas 4.4 and 4.5.

Suppose graph $\mathcal{G}$ consisting of n followers and $l > 1$ leaders has a connected partition. Then, we can find that $\mathcal{G}$ contains p ($p \leq l$) disjoint inverse-trees $\mathcal{T}_1, \ldots, \mathcal{T}_p$, where the p inverse-trees cover all nodes of $\mathcal{G}$ and every inverse-tree has only one leader. Consequently, by what we have proven for inverse-trees, there exist p weights $\mathbf{a}_1, \ldots, \mathbf{a}_p$ (each weight $\mathbf{a}_r$ consists of multiple values corresponding to the arcs of graph $\mathcal{T}_r$) such that $\mathcal{T}_1(\mathbf{a}_1), \ldots, \mathcal{T}_p(\mathbf{a}_p)$ (with re-designed weights) are controllable, respectively. In other words, $\sum_{1 \leq r \leq \binom{np}{n}} |\rho_r| > 0$, where $\left\{\rho_r, 1 \leq r \leq \binom{np}{n}\right\}$ are the determinants of all $n \times n$ submatrices of the CM of the union graph of the p inverse-trees with weights $\mathbf{a}_1, \ldots, \mathbf{a}_p$. Let all the weights of arcs $\mathcal{E}(\mathcal{G}) \backslash \bigcup_{r=1}^{p} \mathcal{E}(\mathcal{T}_r)$ be ε and $\left\{\rho_r(\varepsilon), 1 \leq r \leq \binom{np}{n}\right\}$ the determinants of all $n \times n$ submatrices of the CM of graph $\mathcal{G}$ with weights $\mathbf{a}_1, \ldots, \mathbf{a}_p$ and ε's. When $\varepsilon > 0$ is sufficiently small, $\sum_{1 \leq r \leq \binom{np}{n}} |\rho_r(\varepsilon)| > 0$ since $\rho_r(\varepsilon)$ is a continuous function of ε and $\rho_r(0) = \rho_r$, which implies that $\mathcal{G}$ is controllable with weights $\mathbf{a}_1, \ldots, \mathbf{a}_p$ and sufficiently small ε. Thus, the conclusion follows. □

Remark 4.7: From the proofs of Lemmas 4.4 and 4.5, we can find that there exist positive weights such that system (2) is controllable if graph $\mathcal{G}$ has a connected partition.

Remark 4.8: When we say that an undirected graph with arc set $\mathcal{E}$ is SC, it means that there exist weights $a_{ij} \neq 0$, $(i,j) \in \mathcal{E}$ satisfying $a_{ij} = a_{ji}$ such that the graph is controllable. Note that Theorem 4.6 holds for directed graphs does not imply that it holds for undirected graphs. In fact, although undirected graphs can be viewed as a special case of directed graphs, the re-designed weights for undirected graphs must satisfy $a_{ij} = a_{ji}$. In other words, designing weights for undirected graphs has certain weight restrictions while it does not for directed graphs. In fact, we also could prove that undirected graph $\mathcal{G}$ is SC if and only if there exists a connected partition in $\mathcal{G}$ by considering trees instead of inverse-trees and following the same proof lines in Theorem 4.6.

Next, we consider the following question: given a graph that is not SC, how many new control nodes and new arcs between the new control nodes and follower nodes are needed to make the new graph SC? Consider a digraph $\mathcal{G}$ with follower nodes $\mathcal{I}$ and leader nodes $\mathcal{L}$, which is not SC. Define set $\mathcal{I}^* := \{i \in \mathcal{I} |$ there is no path from i to all leaders in $\mathcal{L}\}$, which is nonempty by Theorem 4.6. To make $\mathcal{G}$ SC, we can add only one control node labelled $n+l+1$ and new arcs set $\{(i, n+l+1), i \in \mathcal{I}^*\}$ by Theorem 4.6. In other words, the number of the new arcs needed is $|\mathcal{I}^*|$.

At the end of this section, we discuss the following question: given a graph $\mathcal{H}$ consisting of n follower nodes, how many control nodes and new arcs between the control nodes and follower nodes are needed to make the new graph SC? Similar problems have been investigated for LTI systems in Liu, Slotine, and Barabasi (2011), where the minimum number of control nodes (i.e. leaders) required to make LTI system SC is closely related with the size maximum matching in the interconnection graph (referring to the supplementary information of Liu et al. (2011) for details). Consider a directed graph $\mathcal{H}$ with n nodes (follower nodes). Let $\hat{\mathcal{I}}_1, \ldots, \hat{\mathcal{I}}_\phi$ be all the node sets satisfying the following property: for any $i \in \mathcal{I} \backslash \hat{\mathcal{I}}_r$ $(1 \leq r \leq \phi)$, there exists a path from i to some node in $\hat{\mathcal{I}}_r$. By Theorem 4.6, it suffices to add only one control node labelled $n+1$. Then, we can easily obtain the following result.

Theorem 4.9: *The minimum number of new arcs required to be added in* $\mathcal{H}$ *to make the new graph SC is*

$$\min_{1 \leq r \leq \phi} |\hat{\mathcal{I}}_r|$$

and the new arcs needed to be added is $\{(i, n+1), i \in \hat{\mathcal{I}}_{\hat{r}}\}$, *where* $\hat{r} \in \arg\min_{1 \leq r \leq \phi} |\hat{\mathcal{I}}_r|$. *Moreover, the minimum number of new arcs required to be added in an undirected graph is the number of its connected components.*

5. Strong structural controllability

If a LTI system is SC, then the set of all parameter values that make the system SC is open and dense in the space of all MPs (Lin 1974). However, sometimes we do not allow that the considered system is uncontrollable with some parameters, and therefore, SSC, was introduced and discussed in Mayeda and Yamada (1979). A LTI system is said to be SSC if the system is controllable provided that the re-designed values for all free parameters of this system are not zeros. Here, we introduce the corresponding concept for system (2).

Definition 5.1: System (2) is said to be SSC if system (2) is controllable for any weights $a_{ij} \neq 0$, $(i,j) \in \mathcal{E}$.

Clearly, every SSC system is also SC. It is not hard to find that the single-leader system (2) with a chain-like partition is SSC. In fact, we can obtain the following result.

Theorem 5.2: *The single-leader system* (2) *is SSC if and only if there exists a chain-like partition in* $\mathcal{G}$.

Proof: The sufficiency part can be proven by the equivalent condition (4). Here, we only prove the necessity part. Since SSC systems are also SC, $\mathcal{P}_\infty = \emptyset$ by Theorem 4.6.

Given an inverse-tree $\mathcal{T}$. According to the distance from the follower nodes to the leader, we can partition followers nodes into disjoint sets $\mathcal{I}_1, \mathcal{I}_2, \ldots$, where the distance from the nodes in $\mathcal{I}_r$ to the single leader is r, $r \geq 1$. We shall prove that if $|\mathcal{I}_1| \geq 2$, then there are two positive weights $\mathbf{a}_1$, $\mathbf{a}_2$ such that the determinants of CMs of $\mathcal{T}$ associated with $\mathbf{a}_1$, $\mathbf{a}_2$ are positive and negative, respectively. This conclusion is proven by induction on n.

For inverse-tree $\mathcal{T}$, by deleting all the nodes in $\mathcal{I}_r$ and all the arcs with initial nodes in $\mathcal{I}_r$, $r \geq 2$, we can get a star graph (Figure 6). Therefore, we first prove the above conclusion for star graphs. Given a star graph with p follower nodes and weights $\alpha_1, \ldots, \alpha_p$. It is easy to find that the CM of the star graph is a multiple of a Vandermonde matrix, whose determinant is

$$\alpha_1 \alpha_2 \cdots \alpha_p \prod_{1 \leq r < k \leq p} (\alpha_r - \alpha_k),$$

which is positive if $\alpha_1 > \cdots > \alpha_{p-1} > \alpha_p > 0$, and is negative if $\alpha_1 > \cdots > \alpha_{p-2} > \alpha_p > \alpha_{p-1} > 0$.

Suppose this conclusion is true for inverse-trees with n follower nodes. Consider the inverse-tree $\mathcal{T}_{n+1}$ with $n+1$ follower nodes. Select one node, say $n+1$, which has the farthest distance among the follower nodes and the leader $n+2$. Without loss of generality, let the neighbour of $n+1$ be n and the weight between them be δ. Denote the subgraph of $\mathcal{T}_{n+1}$ by deleting node $n+1$ and arc $(n+1, n)$ as $\mathcal{T}_n$, which is also an inverse-tree. Let C_n and C_{n+1} be the CMs of $\mathcal{T}_n$ and $\mathcal{T}_{n+1}$, respectively. By the induction assumption, there exist positive weights $\mathbf{a}_1$, $\mathbf{a}_2$ such that $\det C_n(\mathbf{a}_1) > 0$ and $\det C_n(\mathbf{a}_2) < 0$, respectively.

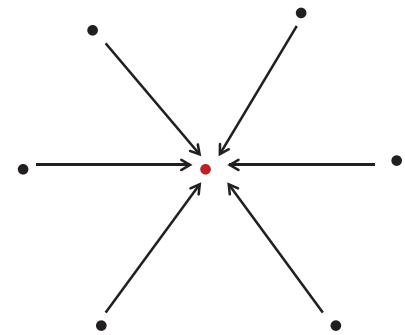

Figure 6. Star graph.

The MP of the inverse-tree $\mathcal{T}_{n+1}$ can be written in the form of

$$\begin{pmatrix} \tilde{A} & \mathbf{0} & \tilde{h} \\ \mathbf{c}_1^T & -\delta & 0 \end{pmatrix}$$

where $\mathbf{c}_1 = (0, \ldots, 0, \delta)^T \in \mathbb{R}^n$ and $(\tilde{A}, \tilde{h}) = (\tilde{A}, \tilde{h})(\mathbf{a}_1) \in \mathbb{R}^{n \times (n+1)}$ (with re-designed weight $\mathbf{a}_1$) is the controllable MP of $\mathcal{T}_n$. Define $n_0 = \min \left\{ i | (\tilde{A}^i \tilde{h})_n \neq 0, 0 \leq i \leq n-1 \right\}$. Since $(\tilde{A}^i \tilde{h})_n = 0$ for $0 \leq i \leq n-1$ contradicts with that $(\tilde{A}, \tilde{h})$ is a controllable MP, n_0 is well defined. It is not hard to verify that the degree of variable δ in the $(n_0 + j)$-th component of the last row of C_{n+1} is $j-1$ for $j = 2, \ldots, n - n_0 + 1$. When $\delta > 0$ is sufficiently large, $\det C_{n+1}$ is dominated by the term $(-1)^{n-n_0+1} \delta^{n-n_0} (\tilde{A}^{n_0} \tilde{h})_n \det C_n$. Without loss of generality, suppose $(-1)^{n-n_0+1} (\tilde{A}^{n_0} \tilde{h})_n > 0$. Thus, there is sufficiently large δ such that $\det C_{n+1}(\mathbf{a}_1, \delta) > 0$ and $\det C_{n+1}(\mathbf{a}_2, \delta) < 0$, respectively. Thus, we complete the induction proof.

Given a SSC graph $\mathcal{G}$ with n follower nodes, we consider the distance-based partition $\hat{\mathcal{P}}_1, \hat{\mathcal{P}}_2, \ldots$, where the distance from the nodes in $\hat{\mathcal{P}}_r$ to the single leader is r, $r \geq 1$. For every $i_r \in \hat{\mathcal{P}}_r$, $r \geq 1$, we select arbitrarily one node $\hat{i}_r \in \hat{\mathcal{P}}_{r-1}$, where $\hat{\mathcal{P}}_0$ contains only the single leader node. Denote the spanning subgraph consisting of all nodes and arcs $\left\{ (i_r, \hat{i}_r), i_r \in \hat{\mathcal{P}}_r, r \geq 1 \right\}$ as $\hat{\mathcal{T}}$, which is an inverse tree covering all the nodes of $\mathcal{G}$. By what we have proven for inverse trees and setting all weights of arcs that belong to $\mathcal{G}$ but not $\hat{\mathcal{T}}$ are positive and sufficiently small, we conclude that there exist two positive weights, still denoted as $\mathbf{a}_1$ and $\mathbf{a}_2$, such that the determinants of CMs of $\mathcal{G}$ are positive and negative, respectively.

Define function $f(\theta) = \det C(\theta \mathbf{a}_1 + (1-\theta)\mathbf{a}_2)$, which is continuous on $[0, 1]$ satisfying $f(0) < 0$ and $f(1) > 0$. Thus, by the intermediate value theorem, there is

$0<\theta_0<1$ such that $f(\theta_0)=0$. Namely, there exists weight $\theta_0\mathbf{a}_1+(1-\theta_0)\mathbf{a}_2$ with positive components such that $\mathcal{G}$ is uncontrollable. This contradiction implies $|\hat{\mathcal{P}}_1|=1$.

Suppose $|\hat{\mathcal{P}}_1|=1$ in graph $\mathcal{G}$. Without loss of generality, let the node connected with leader $n+1$ be 1. Denote the subgraph consisting of the n follower nodes as $\mathcal{G}^1$, where node 1 is viewed as the only leader of $\mathcal{G}^1$. Then the MP $(A,b)\in\mathbb{R}^{n\times(n+1)}$ of $\mathcal{G}$ is in the form of

$$\begin{pmatrix} -\eta-\mathbf{c}_2^T\mathbf{1} & \mathbf{c}_2^T & \eta \\ h^* & A^* & \mathbf{0}\end{pmatrix}$$

where $(A^*,h^*)\in\mathbb{R}^{(n-1)\times n}$ is the MP of $\mathcal{G}^1$ and η is the weight of arc $(1,n+1)$. Since graph $\mathcal{G}$ is SSC, $\operatorname{rank}(\lambda I_n-A,b)=n$ and then $\operatorname{rank}(\lambda I_{n-1}-A^*,h^*)=n-1$ for any non-zero weights and any $\lambda\in\mathbb{C}$, which implies that the subgraph $\mathcal{G}^1$ is SSC. Consequently, the number of nodes that connect directly with node 1 is one, which means that $|\hat{\mathcal{P}}_2|=1$. Similarly, we can prove that $|\hat{\mathcal{P}}_r|=1$ for all $r\geq 3$. Thus, the proof is complete. □

Theorem 5.2 provides a necessary and sufficient graph condition of strong structural controllability, which is easy to be checked. However, it is very difficult to characterise the graph condition for the multi-leader case. Then we give three methods to 'simplify' directed graph $\mathcal{G}$ with multiple leaders.

- If there exist follower nodes $i_0,i_1,\ldots,i_p$ and leader node $n+i$ such that the induced subgraph consisting of $\{i_r,0\leq r\leq p,n+i\}$ has a chain-like partition and there is no arc from $\{i_r,0\leq r\leq p,n+i\}^c$ (^{c} denotes the complement of a set) to $\{i_r,1\leq r\leq p,n+i\}$, we delete all the arcs from $\{i_r,0\leq r\leq p\}$ to $\{i_r,0\leq r\leq p,n+i\}^c$ and the induced subgraph except node i_0, and then denote node i_0 as the new leader node '$n+i$'.
- If there are follower nodes $j_0,j_1,\ldots,j_q$ such that the induced subgraph consisting of $\{j_r,0\leq r\leq q\}$ is a directed chain (from j_0 to j_q) and there is no arc from $\{j_r,0\leq r\leq q\}^c$ to $\{j_r,0\leq r\leq q-1\}$ and no arc from $\{j_r,0\leq r\leq q-1\}$ to $\{j_r,0\leq r\leq q\}^c$, then we delete the chain except node j_q.
- Delete all the arcs starting from leader nodes to follower nodes.

Given a directed graph $\mathcal{G}$, we can apply any one of the above three methods for $\mathcal{G}$ to obtain a new digraph. Apply any one of the above three methods repeatedly until it cannot be further simplified. Denote the final graph obtained by the above procedures as $\mathcal{G}_s$. Based on the equivalent condition (4), we have the following result, whose proof is omitted here.

Theorem 5.3: *$\mathcal{G}$ is SSC if and only if $\mathcal{G}_s$ is SSC.*

6. Zero row-sum restrictions

Structural controllability was first proposed for LTI systems in Lin (1974), without restrictions on the free parameters. However, different from LTI systems in Lin (1974), the system MP of system (2) must satisfy zero row-sum restrictions. In Section 4, we gave a necessary and sufficient graph condition on the structural controllability of system (2) (satisfying zero row-sum restrictions) and provided a self-contained proof. In this section, we shall discuss the effect of the zero row-sum restrictions on structural controllability of system (2).

To make a comparison, we introduce a multi-agent system without zero row-sum restrictions:

$$\dot{y}_i(t)=\sum_{j\in(\mathcal{N}_i\cup i)\cap\mathcal{I}}\bar{a}_{ij}y_j(t)+\sum_{j\in(\mathcal{N}_i\cup i)\cap\mathcal{L}}\bar{a}_{ij}u_j(t),\quad i\in\mathcal{I}. \tag{10}$$

The interconnection graph of system (10) is denoted as $\bar{\mathcal{G}}=(\mathcal{V},\bar{\mathcal{E}})$. Different from system (2), here $\{\bar{a}_{ii},i\in\mathcal{I}\}$ are free parameters, that is, $(i,i)\in\bar{\mathcal{E}}$ for all i. Thus, the following restriction condition

$$\sum_{j=1}^{n+l}\bar{a}_{ij}=0,\quad 1\leq i\leq n$$

may not hold any more. Structural controllability subject to links failure has been studied for system (10) in Jafari et al. (2011). Theorem 2 in Jafari et al. (2011) showed that system (10) is SC if and only if $\bar{\mathcal{G}}$ has a connected partition. Next, we give a stronger conclusion for single-leader systems.

Theorem 6.1: *Consider system* (10) *with a single leader. If $\bar{\mathcal{G}}$ has a connected partition, then for any given non-zero weights $\{\bar{a}_{ij},(i,j)\in\bar{\mathcal{E}},j\neq i,i\in\mathcal{I}\}$, there exist weights $\{\bar{a}_{ii},i\in\mathcal{I}\}$ such that system* (10) *is controllable.*

Proof: Denote the MP of system (10) as $(\bar{A},\bar{b})$. We shall prove the theorem by induction on the number of follower nodes. The case of $n=1$ is straightforward. For $n=2$, let

$$\begin{pmatrix}\bar{a}_{11} & \bar{a}_{12}\\ \bar{a}_{21} & \bar{a}_{22}\end{pmatrix},\ \begin{pmatrix}\bar{a}_{13}\\ \bar{a}_{23}\end{pmatrix}$$

be the MP (here we assume $\bar{a}_{12}$ and $\bar{a}_{21}$ are free parameters, and the conclusion also holds by similar analysis if $\bar{a}_{12}$ or $\bar{a}_{21}$ or both $\bar{a}_{12}$ and $\bar{a}_{21}$ are fixed

zeros). Since $\bar{\mathcal{G}}$ has a connected partition, we have: (i) Both $\bar{a}_{13}$ and $\bar{a}_{23}$ are free or (ii) only one of $\bar{a}_{13}$ and $\bar{a}_{23}$ is free. In case (i), it is easy to find that the determinant of the CM is

$$\bar{a}_{13}\bar{a}_{23}(-\bar{a}_{11}+\bar{a}_{22})+\bar{a}_{21}\bar{a}_{13}^2-\bar{a}_{12}\bar{a}_{23}^2. \tag{11}$$

Then for given non-zero values $\bar{a}_{12}$, $\bar{a}_{21}$, $\bar{a}_{13}$ and $\bar{a}_{23}$, we can design weights for $\bar{a}_{11}$, $\bar{a}_{22}$ such that (11) does not equal zero. In case (ii), without loss of generality, suppose $\bar{a}_{13}$ is a free parameter and $\bar{a}_{23}\equiv 0$ (fixed zero). Since $\bar{\mathcal{G}}$ has a connected partition, $\bar{a}_{21}$ is certainly a free parameter. Thus, the determinant of the CM equals to $\bar{a}_{21}\bar{a}_{13}^2$, which is not zero.

Suppose this conclusion holds for n. Then we prove it for graph $\bar{\mathcal{G}}_{n+1}$ with $n+1$ follower nodes. Denote the MP of $\bar{\mathcal{G}}_{n+1}$ as $(\bar{A}_{n+1},\bar{b}_{n+1})$. Take a node from follower nodes $\{1,2,\ldots,n+1\}$, say $n+1$, which has the farthest distance with leader $n+2$, and denote the induced subgraph consisting of nodes $\{1,2,\ldots,n\}$ and leader $n+2$ as $\bar{\mathcal{G}}_n$, which still has a connected partition. The MP of $\bar{\mathcal{G}}_n$ is denoted as $(\bar{A}_n,\bar{b}_n)$.

Given $\bar{a}_{ij},(i,j)\in\bar{\mathcal{E}}(\bar{\mathcal{G}}_{n+1}),j\neq i,1\le i\le n+1$. By induction assumption, for weights $\{\bar{a}_{ij},(i,j)\in\bar{\mathcal{E}}(\bar{\mathcal{G}}_n),j\neq i,1\le i\le n\}$, there exist $\{\bar{a}_{ii},\ 1\le i\le n\}$ such that the MP $(\bar{A}_n,\bar{b}_n)$ (with re-designed weights) is controllable. Therefore, there is an invertible matrix $\Gamma\in\mathbb{R}^{n\times n}$ such that $(\Gamma\bar{A}_n\Gamma^{-1},\Gamma\bar{b}_n)$ takes the canonical form of controllable MP. An equivalent transform of $(\bar{A}_{n+1},\bar{b}_{n+1})$ takes the following form:

$$\begin{aligned}
&\begin{pmatrix}\Gamma & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix}\begin{pmatrix}\bar{A}_n & \bar{\mathbf{c}}_1\\ \bar{\mathbf{c}}_2^T & \bar{a}_{(n+1)(n+1)}\end{pmatrix}\begin{pmatrix}\Gamma^{-1} & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix},\ \begin{pmatrix}\Gamma & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix}\begin{pmatrix}\bar{b}_n\\ \gamma\end{pmatrix}\\
&=\begin{pmatrix}\Gamma\bar{A}_n\Gamma^{-1} & \Gamma\bar{\mathbf{c}}_1\\ \bar{\mathbf{c}}_2^T\Gamma^{-1} & \bar{a}_{(n+1)(n+1)}\end{pmatrix},\qquad \begin{pmatrix}\Gamma\bar{b}_n\\ \gamma\end{pmatrix}\\
&=\begin{pmatrix}0 & 1 & 0 & \cdots & 0 & \hat{c}_{11}\\ 0 & 0 & 1 & \cdots & 0 & \hat{c}_{12}\\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 1 & \hat{c}_{1(n-1)}\\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n & \hat{c}_{1n}\\ \hat{c}_{21} & \hat{c}_{22} & \cdots & \hat{c}_{2(n-1)} & \hat{c}_{2n} & \bar{a}_{(n+1)(n+1)}\end{pmatrix},\begin{pmatrix}0\\ 0\\ \vdots\\ 0\\ 1\\ \gamma\end{pmatrix}\\
&:=\bar{A}_{n+1}^e,\qquad\qquad \bar{b}_{n+1}^e
\end{aligned}$$

Since $(\bar{A}_{n+1},\bar{b}_{n+1})$ is controllable if and only if $(\bar{A}_{n+1}^e,\bar{b}_{n+1}^e)$ is controllable, we only need to prove that there exists positive weight $\bar{a}_{(n+1)(n+1)}$ such that $(\bar{A}_{n+1}^e,\bar{b}_{n+1}^e)$ is controllable. Consider $(\lambda I_{n+1}-\bar{A}_{n+1}^e,\bar{b}_{n+1}^e)$ for $\lambda\in\mathbb{C}$. We first assume $\gamma=0$. Denote the matrix by deleting the n-th row of $\bar{A}_{n+1}^e$ as $\bar{A}_{n+1}^{ef}$. Note that $\operatorname{rank}(\lambda I_{n+1}-\bar{A}_{n+1}^e,\bar{b}_{n+1}^e)=\operatorname{rank}(\lambda I_n-\bar{A}_{n+1}^{ef})+1$ for any $\lambda\in\mathbb{C}$.

Define $\beta_1=-\hat{c}_{11}$ and $\beta_{i+1}=\lambda\beta_i-\hat{c}_{1(i+1)}$ for $1\le i\le n-2$. To make the entries $\hat{c}_{1i},1\le i\le n-1$ zeros, we perform the following elementary column transformations for matrix $\lambda I_n-\bar{A}_{n+1}^{ef}$: adding the $(i+1)$-th column multiplied by β_i to the last column for $1\le i\le n-1$. The new matrix takes the following form:

$$\begin{pmatrix}\lambda & -1 & 0 & \cdots & 0 & 0\\ 0 & \lambda & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \lambda & -1 & 0\\ -\hat{c}_{21} & -\hat{c}_{22} & \cdots & -\hat{c}_{2(n-1)} & -\hat{c}_{2n} & g(\bar{a}_{(n+1)(n+1)})\end{pmatrix} \tag{12}$$

where $g(\bar{a}_{(n+1)(n+1)})=-\bar{a}_{(n+1)(n+1)}+\lambda-\sum_{i=1}^{n-1}\hat{c}_{2(i+1)}\beta_i$, which is a polynomial of variable λ with coefficients being polynomial functions of variables $\hat{c}_{ij}$, $i=1,2,1\le j\le n$. Denote the matrices by deleting the first column and the last column of the matrix in (12) as $\bar{A}^1$ and $\bar{A}^2$, respectively. It is not hard to find that $\det\bar{A}^2=(-1)^n\sum_{i=0}^{n-1}\hat{c}_{2(i+1)}\lambda^i$ and $\det\bar{A}^1=(-1)^{n-1}g(\bar{a}_{(n+1)(n+1)})$. Since there is a path from $n+1$ to some leader in $\mathcal{L}$, $\bar{\mathbf{c}}_2\neq\mathbf{0}$ and then $\bar{\mathbf{c}}_2^T\Gamma^{-1}\neq\mathbf{0}$, that is, at least one of $\{\hat{c}_{2i},1\le i\le n\}$ is not zero. Therefore, there exists sufficiently large $\varphi>0$ such that $\det\bar{A}^2\neq 0$ for all $|\lambda|\ge\varphi$. Take

$$\bar{a}_{(n+1)(n+1)}>\max_{|\lambda|\le\varphi}\left|\lambda-\sum_{i=1}^{n-1}\hat{c}_{2(i+1)}\beta_i\right|. \tag{13}$$

It is easy to find that, for any $\lambda\in\mathbb{C}$, either $\det\bar{A}^2\neq 0$ or $\det\bar{A}^1\neq 0$. Thus, $\operatorname{rank}(\lambda I_n-\bar{A}_{n+1}^{ef})=n$ and then $\operatorname{rank}(\lambda I_{n+1}-\bar{A}_{n+1}^e,\bar{b}_{n+1}^e)=n+1$ for any $\lambda\in\mathbb{C}$, which implies that the MP $(\bar{A}_{n+1}^e,\bar{b}_{n+1}^e)$ is controllable with given weights $\bar{a}_{ij},(i,j)\in\bar{\mathcal{E}}(\bar{\mathcal{G}}_{n+1})$, $j\neq i$, $1\le i\le n+1$ and re-designed weights $\bar{a}_{ii}$, $1\le i\le n+1$, where $\bar{a}_{(n+1)(n+1)}$ satisfies (13).

If $\gamma\neq 0$, we can add the $(n-1)$-th row multiplied by $-\gamma$ to the last row in matrix $(\lambda I_{n+1}-\bar{A}_{n+1}^e,\bar{b}_{n+1}^e)$ and then make similar analysis with the above arguments by substituting $\hat{c}_{2n}$ and $\bar{a}_{(n+1)(n+1)}$ with $\hat{c}_{2n}+\lambda\gamma$ and $\bar{a}_{(n+1)(n+1)}+\lambda\hat{c}_{1n}$, respectively. Thus, we complete the proof. □

Theorem 6.1 shows that for any non-zero weights of all the arcs except the n self-loop arcs (i,i), $1\le i\le n$, we always could find weights $\bar{a}_{ii}$, $1\le i\le n$ for the n self-loop arcs such that system (10) is controllable. In other words, the diagonal elements of the system matrix play

very important roles in the structural controllability of system (10).

7. Conclusions

In this article, we studied the controllability of multi-agent systems with their leaders as control inputs. Different from most existing results in the literature, the interconnection graph is directed and weighted. First, we employed weight-balanced partition to prove that the system is uncontrollable if there exists a nontrivial weight-balanced partition on the interconnection graph. Based on the distance-based and weight-balanced partitions, we provided two necessary and sufficient graph conditions of structural controllability and strong structural controllability. Moreover, we also discussed the effect of the zero row-sum restrictions of the system matrices on structural controllability.

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References

Borůvka, O. (1976), *Foundations of the Theory of Groupoids and Groups*, Basel and Stuttgart: Birkhauser Verlag.

Cao, Y., and Ren, W. (2010), 'Distributed Containment Control for Multiple Autonomous Vehicles with Double-integrator Dynamics: Algorithms and Experiments', *IEEE Transactions Control Systems Technology*, 19, 929–938.

Fax, J., and Murray, R. (2004), 'Information Flow and Cooperative Control of Vehicle Formation', *IEEE Transactions on Automatic Control*, 49, 1465–1476.

Godsil, C., and Royle, G. (2001), *Algebraic Graph Theory*, New York: Springer-Verlag.

Hong, Y., Hu, J., and Gao, L. (2006), 'Tracking Control for Multi-agent Consensus with an Active Leader and Variable Topology', *Automatica*, 42, 1177–1182.

Hu, J., and Hong, Y. (2007), 'Leader-following Coordination of Multi-agent Systems with Coupling Time Delays', *Physica A*, 374, 853–863.

Jadbabaie, A., Lin, J., and Morse, A.S. (2003), 'Coordination of Groups of Mobile Autonomous Agents using Nearest Neighbour Rules', *IEEE Transactions on Automatic Control*, 48, 988–1001.

Jafari, S., Ajorlou, A., and Aghdam, A.G. (2011), 'Leader Localization in Multi-agent Systems Subject to Failure: A Graph-theoretic Approach', *Automatica*, 47, 1744–1750.

Ji, Z., Wang, Z., Lin, H., and Wang, Z. (2009), 'Interconnection Topologies for Multi-agent Coordination under Leader-follower Framework', *Automatica*, 45, 2857–2863.

Ji, Z., Wang, Z., Lin, H., and Wang, Z. (2010), 'Controllability of Multi-agent Systems with Time-delay in State and Switching Topology', *International Journal of Control*, 83, 371–386.

Kailath, T. (1980), *Linear Systems*, Englewood Cliffs, NJ: Prentice-Hall.

Lin, C.-T. (1974), 'Structural Controllability', *IEEE Transactions on Automatic Control*, 19, 201–208.

Lin, P., and Jia, Y. (2010), 'Consensus of a Class of Second-order Multi-agent Systems with Time-delay and Jointly-connected Topologies', *IEEE Transactions on Automatic Control*, 55, 778–784.

Liu, B., Chu, T., Wang, L., and Xie, G. (2008), 'Controllability of a Leader-follower Dynamic Network with Switching Topology', *IEEE Transactions on Automatic Control*, 53, 1009–1013.

Liu, Y., Slotine, J.-J., and Barabasi, A.-L. (2011), 'Controllability of Complex Networks', *Nature*, 473, 167–173.

Lou, Y., and Hong, Y. (2012), 'Target Containment Control of Multi-agent Systems with Random Switching Interconnection Topologies', *Automatica*, 48, 879–885.

Martini, S., Egerstedt, M., and Bicchi, A. (2010), 'Controllability Analysis of Multi-agent Systems using Relaxed Equitable Partitions', *International Journal of Systems, Control and Communications*, 2, 100–121.

Mayeda, H. (1981), 'On Structural Controllability Theorem', *IEEE Transactions on Automatic Control*, 26, 795–798.

Mayeda, H., and Yamada, T. (1979), 'Strong Structural Controllability', *SIAM Journal on Control and Optimization*, 17, 123–138.

Partovi, A., Lin, H., and Ji, Z. (2010), 'Structural Controllability of High-Order Dynamic Multi-agent Systems', in *Proceedings of IEEE Conference on Robotics Automation and Mechatronics*, pp. 327–332.

Rahmani, A., Ji, M., Mesbahi, M., and Egerstedt, M. (2009), 'Controllability of Multi-agent Systems from a Graph-theoretic Perspective', *SIAM Journal on Control and Optimization*, 48, 162–186.

Ren, W., and Beard, R.W. (2005), 'Consensus Seeking in Multiagent Systems under Dynamically Changing Interaction Topologies', *IEEE Transactions on Automatic Control*, 50, 655–661.

Shi, G., and Hong, Y. (2009), 'Global Target Aggregation and State Agreement of Nonlinear Multi-agent Systems with Switching Topologies', *Automatica*, 45, 1165–1175.

Tanner, H. (2004), 'On the Controllability of Nearest Neighbour Interconnections', in *Proceedings of the 43rd IEEE Conference on Decision and Control*, pp. 2467–2472.

Twu, P., Egerstedt, M., and Martini, S. (2010), 'Controllability of Homogeneous Single-leader Networks', in *Proceedings of 49th IEEE Conference on Decision and Control*, pp. 5869–5874.

Zamani, M., and Lin, H. (2009), 'Structural Controllability of Multi-agent Systems', in *Proceedings of the American Control Conference*, pp. 5743–5748.