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Dynamic optimization for multi-agent systems with external disturbances

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Abstract:

This paper studies the dynamic optimization problem for multi-agent systems in the presence of external disturbances. Different from the existing distributed optimization results, we formulate an optimization problem of continuous-time multi-agent systems with time-varying disturbance generated by an exosystem. Based on internal model and Lyapunov-based method, a distributed design is proposed to achieve the optimization. Finally, an example is given to illustrate the proposed optimization design.

Keywords: Distributed optimization; Multi-agent systems; Disturbance rejection; Internal model

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1 Introduction

The coordination of multi-agent systems has been studied increasingly with many significant results on consensus and formation for the past decade. In recent years, distributed optimization has attracted much attention to seek an optimal solution for multi-agent systems [1, 2]. Although most results on distributed optimization were based on discrete-time models, one of emerging topics on distributed optimization is how to design continuous-time systems to achieve the optimization [3–5]. The convergence analysis was provided for a continuous-time algorithm with fixed undirected graph in [4], while a continuous-time dynamics was proposed for optimal computation of convex intersection with uniformly jointly strongly connected communication graph in [3]. Moreover, a continuous-time system for the positive constrained optimization problem was investigated in [5], and a continuous optimization problem was studied with discrete-time communication in [6]. The results show the advantages to employ continuous-time models for optimization: i) It is easy to study the case when practical systems such as robots are the agents to search the optimal solution in real time. ii) Many advanced control techniques can be used

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to facilitate the analysis of convergence rate and disturbance rejection, and moreover, overcome the diminishing step-size problem in discrete-time algorithm.

In fact, more and more attention has been paid to multi-agent systems with disturbances. In reality, agents have to face various (environmental) disturbances. One of the effective methods was developed based on internal model principle from the viewpoint of output regulation [7]. Also, distributed output regulation was studied [8–10] for multi-agent systems to track an active leader and/or reject a modeled disturbance.

The objective of this paper is to study the distributed optimization of continuous-time multi-agent systems with rejecting external disturbances. The motivation is as follows: i) the considered agents need to be equipped with disturbance rejection control scheme when they achieve their optimization in a region with disturbance; ii) the research is a combination of the new results of distributed optimization and distributed output regulation, which may provide a unified way to treat consensus, disturbance rejection, and distributed optimization in some sense. The contribution of the paper can be summarized as follows:

- We first give a problem formulation for the distributed optimization with rejecting the external disturbances during the optimization process;
- We employ the internal model technique to provide effective distributed protocol to achieve the exact optimization in the presence of external disturbances.

This paper is organized as follows. In Section 2, we formulate the dynamic optimization problem with rejecting external disturbances. Then, in Section 3, we present the main result of the paper, along with an illustrative example. In Section 4, some concluding remarks are given.

2 Preliminaries and formulation

In this section, we recall some preliminaries of graph theory and convex optimization, and then formulate our problem.

Denote I_n as the $n \times n$ identity matrix, $\mathbf{1}_N$ as the vector of N entries equal to 1, and $\otimes$ as the matrix Kronecker product. $\mathbb{R}^n$ denotes n-dimensional Euclidean space. Moreover, for vectors $x_1, \dots, x_m$, denote $(x_1, \dots, x_m) = [x_1^{\mathrm{T}} \ \cdots \ x_m^{\mathrm{T}}]^{\mathrm{T}}$.

Let us introduce some concepts related to convex functions [11]. A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if $(y - x)^{\mathrm{T}}(\nabla f(y) - \nabla f(x)) > 0$ for $x \neq y \in \mathbb{R}^n$, and f is a m-strongly convex ($m > 0$) if $(y-x)^{\mathrm{T}}(\nabla f(y)-\nabla f(x)) > m\|y-x\|^2$ for $x \neq y$. A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz with constant $M > 0$, or simply M-Lipschitz if $\|g(x) - g(y)\| \leqslant M\|x - y\|$, $\forall x, y \in \mathbb{R}^n$.

Graph theory has been widely used for multi-agent control [12]. A weighted digraph is described by a triplet $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ where $\mathcal{V} = \{1, 2, \dots, N\}$ is the node set, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set (without self-loops), and $\mathcal{A} = [a_{ij}]_{i,j=1,\dots,N}$ is the weighted adjacency matrix of $N \times N$. An edge of $\mathcal{G}$ is denoted by an ordered pair of nodes $(j, i) \in \mathcal{E}$ with j being a neighbor of i. A directed path of $\mathcal{G}$ is an ordered sequence of distinct nodes in $\mathcal{V}$ such that any consecutive nodes in the sequence correspond to an edge of $\mathcal{G}$. $\mathcal{G}$ is called strongly connected if there exists a directed path from i to j for any two nodes $i, j \in \mathcal{V}$. $\mathcal{A}$ is a nonnegative matrix with $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, $i, j \in \mathcal{V}$. The weighted in-degree and weighted out-degree of a node i are defined by $\deg^i_{\mathrm{in}} = \sum\limits_{j=1}^{N} a_{ij}$, $\deg^i_{\mathrm{out}} = \sum\limits_{j=1}^{N} a_{ji}$. A digraph is weight-balanced if for each node i, $\deg^i_{\mathrm{in}} = \deg^i_{\mathrm{out}}$. The Laplacian matrix is $\mathcal{L} = \mathcal{D} - \mathcal{A}$ with $\mathcal{D} = \mathrm{diag}(\deg^1_{\mathrm{in}}, \dots, \deg^N_{\mathrm{in}})$. Note that $\mathcal{L}\mathbf{1}_N = 0$. A digraph is weight-balanced if and only if $\mathbf{1}_N^{\mathrm{T}}\mathcal{L} = 0$.

It is time to formulate our problem. Consider a network of N agents with interaction topology described by a digraph $\mathcal{G}$. Agent i is endowed with a local cost function $f_i : \mathbb{R}^n \to \mathbb{R}$ and a dynamics

$$\dot{x}_i = u_i + d_i(t), \quad i = 1, \dots, N, \tag{1}$$

where $x_i \in \mathbb{R}^n$ is its state, u_i is its optimization protocol, and $d_i(t)$ is the local disturbance governed by the following exosystem:

$$\dot{w}_i = Sw_i, \quad d_i = Cw_i(t), \tag{2}$$

where $w_i(t) \in \mathbb{R}^p$ is the exosystem state. It is assumed that all eigenvalues of $S \in \mathbb{R}^{p\times p}$ are distinct lying on the imaginary axis, which means the boundedness of the disturbances.

The global cost function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as a sum of the local cost functions as usual [1]:

$$f(x) = \sum_{i=1}^{N} f_i(x).$$

The aim of this paper is to design a protocol u_i of the following form:

$$\begin{cases} \dot{z}_i = g_{i1}(z_i, \nabla f_i(x_i), x_{mi}), \\ u_i = g_{i2}(z_i, \nabla f_i(x_i), x_{mi}), \end{cases} \tag{3}$$

where $z_i \in \mathbb{R}^{n_{z_i}}$, $\nabla f_i(x_i)$ is the gradient information of agent i, and $x_{mi} = \sum\limits_{i=1}^{N} a_{ij}(x_i - x_j)$ is the exchanged information with its neighbors, such that the multi-agent system (10) with (3) solves the following optimization problem

$$x^* = \arg\min_{x \in \mathbb{R}^n} f(x), \quad -\infty < x^* < +\infty$$

by driving x_i to x^*.

Remark 1 The above problem can be referred to as the dynamic optimization problem with external disturbances in order to achieve the exact optimization in a distributed way. When the disturbances disappear, the problem discussed in this paper becomes the traditional distributed optimization problem in the continuous-time setup (referring to [3, 4]). On the other hand, if we do not assign the cost functions to the agent network, the consensus problem with external disturbances can be solved with the help of distributed output regulation [8–10]. Therefore, this problem provides a general framework somehow for the distributed optimization and distributed output regulation.

To proceed further, we introduce two basic conditions for the solvability of the problem, which was also used in [6].

Condition 1 The digraph $\mathcal{G}$ is strongly connected and weight-balanced.

Remark 2 Define

$$\mathrm{Sym}(\mathcal{L}) = \frac{\mathcal{L} + \mathcal{L}^{\mathrm{T}}}{2}.$$

Under Condition 1, it is known [12] that 0 is the single eigenvalue of matrices $\mathcal{L}$ and $\mathrm{Sym}(\mathcal{L})$. Moreover, there is a matrix $R \in \mathbb{R}^{N\times(N-1)}$ with $\mathbf{1}_N^{\mathrm{T}} R = 0$, $R^{\mathrm{T}} R = I_{N-1}$, $RR^{\mathrm{T}} = I_N - \dfrac{1}{N}\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}$ such that

$$R^{\mathrm{T}}\mathrm{Sym}(\mathcal{L})R \geqslant \lambda_0 I_{N-1} \tag{4}$$

for a positive real number λ_0.

Condition 2 For $i = 1, \dots, N$, the local cost function f_i is m^i-strongly convex and differentiable, and its gradient is M^i-Lipschitz on $\mathbb{R}^n$. Denote $m_{\mathrm{T}} = \min\{m^1, \dots, m^N\}$ and $M_{\mathrm{T}} = \max\{M^1, \dots, M^N\}$.

3 Main result

In this section, we propose a distributed optimization design to achieve the exact optimization with disturbance rejection. Three subsections are given for the algorithm design, optimization analysis, and simulation.

3.1 Algorithm design

Due to the disturbance $d_i(t)$, existing results on distributed optimization are not applicable. To deal with the problem, we first make some transformation. Let $p(\lambda) = \lambda^s + p_1\lambda^{s-1} + \dots + p_{s-1}\lambda + p_s$ be the minimal polynomial of S and $\tau_i = (\tau_{i1}, \dots, \tau_{\mathrm{in}})$ with

$$\tau_{ij} = \Big(d_{ij}(t), \frac{\mathrm{d}d_{ij}(t)}{\mathrm{d}t}, \dots, \frac{\mathrm{d}^{s-1}d_{ij}(t)}{\mathrm{d}t^{s-1}}\Big), \quad j = 1, \dots, n.$$

Define two matrices

$$\Phi = \left[\begin{array}{c|ccc} 0 & & I_{s-1} & \\ \hline -p_s & -p_{s-1} & \cdots & -p_1 \end{array}\right], \quad \Psi = [1 \,|\, 0_{1\times(s-1)}].$$

By a direct computation, it can be seen that τ_{ij} satisfies

$$\dot{\tau}_{ij} = \Phi\tau_{ij}, \quad d_{ij}(t) = \Psi\tau_{ij}, \tag{5}$$

which implies

$$\dot{\tau}_i = (I_n \otimes \Phi)\tau_i, \quad d_i(t) = (I_n \otimes \Psi)\tau_i. \tag{6}$$

Since the pair (Ψ, Φ) is observable, there exists a matrix G such that $F = \Phi + G\Psi$ is Hurwitz. Thus, for agent i, an internal-model-based optimization protocol can be constructed as

$$\begin{cases} \dot{v}_i = \alpha\beta \sum\limits_{j=1}^{N} a_{ij}(x_i - x_j), \\ \dot{\eta}_i = (I_n \otimes F)\eta_i + (I_n \otimes G)u_i, \\ u_i = \underbrace{-\alpha\nabla f_i(x_i) - v_i}_{\text{optimal term}} \; \underbrace{-(I_n \otimes \Psi)\eta_i}_{\text{internal model term}} \\ \qquad \underbrace{-\beta \sum\limits_{j=1}^{N} a_{ij}(x_i - x_j)}_{\text{consensus term}}. \end{cases} \tag{7}$$

Obviously, the proposed optimization protocol consists of three terms: the gradient-based optimization term to drive the agents to the optimization point, the consensus term for all agents to achieve the same point, and the internal model term to compensate the disturbance $d_i(t)$ asymptotically (referring to Chapter 6 of [7] for more details on internal model design).

Remark 3 It is worth mentioning that, by summing up all v_i subsystems, we obtain $\sum_{i=1}^{N}\dot{v}_i = 0$, which implies

$$\sum_{i=1}^{N} v_i(t) = \sum_{i=1}^{N} v_i(0) = 0 \tag{8}$$

for initial conditions $v_i(0) \in \mathbb{R}^n$ with

$$\sum_{i=1}^{N} v_i(0) = 0. \tag{9}$$

The above observation is useful in analysis of the equilibrium point. Therefore, in this paper, we always set the initial conditions $v_i(0)$ satisfying (9) by simply taking $v_i(0) = 0$ for $i = 1, \dots, N$.

By adding u_i to the dynamics (1), we obtain the following closed-loop system:

$$\begin{cases} \dot{x}_i = -\alpha\nabla f_i(x_i) - v_i - \beta\sum_{j=1}^{N} a_{ij}(x_i - x_j) \\ \qquad + (I_n \otimes \Psi)(\tau_i - \eta_i), \\ \dot{v}_i = \alpha\beta\sum_{j=1}^{N} a_{ij}(x_i - x_j), \\ \dot{\eta}_i = (I_n \otimes F)\eta_i + (I_n \otimes G)(-\alpha\nabla f_i(x_i) - v_i \\ \qquad - \beta\sum_{j=1}^{N} a_{ij}(x_i - x_j) - (I_n \otimes \Psi)\eta_i). \end{cases} \tag{10}$$

To compensate the disturbances asymptotically, the term $\tau_i(t) - \eta_i(t)$ must vanish asymptotically. Performing a transformation $\bar{\eta}_i = \eta_i - \tau_i$ gives

$$\begin{aligned} \dot{\bar{\eta}}_i = {} & (I_n \otimes F)\bar{\eta}_i + (I_n \otimes G)(-\alpha\nabla f_i(x_i) - v_i \\ & - \beta\sum_{j=1}^{N} a_{ij}(x_i - x_j) - (I_n \otimes \Psi)\bar{\eta}_i). \end{aligned} \tag{11}$$

Then, system (10) with the last equation replaced by (11) can be rewritten in the following compact form:

$$\begin{aligned} & \dot{x} = -\alpha\nabla\tilde{f}(x) - v - \beta(\mathcal{L} \otimes I_n)x - (I_{Nn} \otimes \Psi)\bar{\eta}, \\ & \dot{v} = \alpha\beta(\mathcal{L} \otimes I_n)x, \\ & \dot{\bar{\eta}} = (I_{Nn} \otimes F)\bar{\eta} - (I_{Nn} \otimes G)(\alpha\nabla\tilde{f}(x) + v \\ & \qquad + \beta(\mathcal{L} \otimes I_n)x + (I_{Nn} \otimes \Psi)\bar{\eta}), \end{aligned} \tag{12}$$

where $x = (x_1, \dots, x_N)$, $v = (v_1, \dots, v_N)$, $\bar{\eta} = (\bar{\eta}_1, \dots, \bar{\eta}_N)$, $\tilde{f}(x) = \sum_{i=1}^{N} f_i(x_i)$.

Remark 4 Suppose that system (12) has an equilibrium point at $(x^o, v^o, \bar{\eta}^o)$. Then, $(x^o, v^o, \bar{\eta}^o)$ satisfies

$$\begin{aligned} & -\alpha\nabla\tilde{f}(x^o) - v^o - \beta(\mathcal{L} \otimes I_n)x^o - (I_{Nn} \otimes \Psi)\bar{\eta}^o = 0, \\ & \alpha\beta(\mathcal{L} \otimes I_n)x^o = 0, \\ & (I_{Nn} \otimes F)\bar{\eta}^o - (I_{Nn} \otimes G)(\alpha\nabla\tilde{f}(x^o) + v^o + \beta(\mathcal{L} \otimes I_n)x^o \\ & \quad + (I_{Nn} \otimes \Psi)\bar{\eta}^o) = 0. \end{aligned} \tag{13}$$

We can conclude $(I_{Nn} \otimes F)\bar{\eta}^o = 0$ from the first and the third equations in (13), which implies $\bar{\eta}^o = 0$. Following a similar analysis given in [6], we can prove that $x^o = \mathbf{1} \otimes x^*$ and $\bar{v}_i^o = -\alpha\nabla f_i(x^*), i = 1, \dots, N$. Namely, if we can prove that the state x_i of system (12) converges to its equilibrium point x_i^o, then x_i converges to the optimizer x^*.

3.2 Optimization analysis

Here, we prove the convergence of the proposed optimization design based on Lyapunov function.

To this end, we first define the following variables to obtain a standard stability problem:

$$\tilde{x} = x - x^o, \quad \tilde{v} = v - v^o, \quad \tilde{\eta} = \bar{\eta} - (I_{Nn} \otimes G)\tilde{x}. \tag{14}$$

In the new coordinate,

$$\begin{cases} \dot{\tilde{x}} = -\alpha h - \tilde{v} - \beta(\mathcal{L} \otimes I_n)\tilde{x} \\ \qquad - (I_{Nn} \otimes \Psi)(\tilde{\eta} + (I_{Nn} \otimes G)\tilde{x}), \\ \dot{\tilde{v}} = \alpha\beta(\mathcal{L} \otimes I_n)\tilde{x}, \\ \dot{\tilde{\eta}} = (I_{Nn} \otimes F)\tilde{\eta} + (I_{Nn} \otimes FG)\tilde{x}, \end{cases} \tag{15}$$

where $h = \nabla\tilde{f}(\tilde{x} + x^o) - \nabla\tilde{f}(x^o)$.

It is time to show our main result.

Theorem 1 Under Conditions 1 and 2, there exist two constants α and $\beta > 0$ such that algorithm (7) solves the distributed optimization problem in the presence of the disturbances.

Proof Recalling Remark 4, to obtain the conclusion, it is sufficient to show that there are two constants $\alpha, \beta > 0$ such that the equilibrium point $(\tilde{x}, \tilde{v}, \tilde{\eta}) = (0, 0, 0)$ of system (15) is exponentially stable.

For this purpose, we first perform the following transformation to simplify system (15)

$$\chi = (T \otimes I_n)\tilde{x}, \quad \vartheta = (T \otimes I_n)\tilde{v}, \tag{16}$$

where T is defined by

$$T^{\mathrm{T}} = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1} & R \end{bmatrix}$$

with R specified in Remark 2. Denote $\chi = (\chi_1, \chi_{2:N})$, $\vartheta = (\vartheta_1, \vartheta_{2:N})$, where $\chi_1, \vartheta_1 \in \mathbb{R}^n$ and $\chi_{2:N}, \vartheta_{2:N} \in \mathbb{R}^{(N-1)n}$. Then, from (15), we have

$$\begin{cases} \dot{\vartheta}_1 = 0, \\ \dot{\vartheta}_{2:N} = \alpha\beta(R^{\mathrm{T}}\mathcal{L}R \otimes I_n)\chi_{2:N}, \\ \dot{\chi}_1 = -\dfrac{\alpha}{\sqrt{N}}(\mathbf{1}^{\mathrm{T}} \otimes I_n)h - \dfrac{1}{\sqrt{N}}(\mathbf{1}^{\mathrm{T}} \otimes I_n)\phi, \\ \dot{\chi}_{2:N} = -\alpha(R^{\mathrm{T}} \otimes I_n)h - \beta(R^{\mathrm{T}}\mathcal{L}R \otimes I_n)\chi_{2:N} \\ \qquad\quad -\vartheta_{2:N} - (R^{\mathrm{T}} \otimes I_n)\phi \end{cases} \tag{17}$$

with $\phi = (I_{Nn} \otimes \Psi)(\tilde{\eta} + (I_{Nn} \otimes G)(T^{-1} \otimes I_n)\chi)$.

Similar to [6], we take the following Lyapunov function candidate:

$$\begin{aligned} V_1 = {} & \frac{1}{18}\alpha(\gamma+1)\chi_1^{\mathrm{T}}\chi_1 + \frac{\gamma\alpha}{2}\chi_{2:N}^{\mathrm{T}}\chi_{2:N} \\ & + \frac{1}{2\alpha}(\alpha\chi_{2:N} + \vartheta_{2:N})^{\mathrm{T}}(\alpha\chi_{2:N} + \vartheta_{2:N}) \end{aligned} \tag{18}$$

with $\gamma > 0$ to be determined. It can be verified that

$$\begin{aligned} \dot{V}_1 = {} & -\frac{1}{9\sqrt{N}}\alpha(\gamma+1)\chi_1^{\mathrm{T}}(\mathbf{1}_N^{\mathrm{T}} \otimes I_n)(\alpha h - \phi) \\ & + \gamma\alpha\chi_{2:N}^{\mathrm{T}}(-\alpha(R^{\mathrm{T}} \otimes I_n)h - \beta(R^{\mathrm{T}}\mathcal{L}R \otimes I_n)\chi_{2:N} \\ & - \vartheta_{2:N} - (R^{\mathrm{T}} \otimes I_n)\phi) \\ & + (\alpha\chi_{2:N} + \vartheta_{2:N})^{\mathrm{T}}(-(R^{\mathrm{T}} \otimes I_n)h \\ & - \vartheta_{2:N} - (R^{\mathrm{T}} \otimes I_n)\phi) \\ = {} & -\frac{1}{9}\alpha^2(\gamma+1)\tilde{x}^{\mathrm{T}}h - \frac{7}{16}\vartheta_{2:N}^{\mathrm{T}}\vartheta_{2:N} \\ & - \gamma\alpha\beta\chi_{2:N}^{\mathrm{T}}(R^{\mathrm{T}}\mathrm{Sym}(\mathcal{L})R \otimes I_n)\chi_{2:N} \\ & + \frac{4}{9}\alpha^2\|(R^{\mathrm{T}} \otimes I_d)h\|^2 + \frac{4}{9}\alpha^2(\gamma+1)^2\chi_{2:N}^{\mathrm{T}}\chi_{2:N} \\ & - \|\frac{3}{4}\vartheta_{2:N} + \frac{2\alpha}{3}(R^{\mathrm{T}} \otimes I_n)h + \frac{2\alpha}{3}(\gamma+1)\chi_{2:N}\|^2 \\ & - \frac{1}{9\sqrt{N}}\alpha(\gamma+1)\chi_1^{\mathrm{T}}(\mathbf{1}_N^{\mathrm{T}} \otimes I_n)\phi \\ & - ((\gamma+1)\alpha\chi_{2:N} + \vartheta_{2:N})^{\mathrm{T}}(R^{\mathrm{T}} \otimes I_n)\phi. \end{aligned} \tag{19}$$

Under Condition 2, we obtain

$$\begin{aligned} & \tilde{x}^{\mathrm{T}}h \geqslant m_{\mathrm{T}}\|\tilde{x}\|^2 = m_{\mathrm{T}}\|\chi\|^2, \\ & \|(R^{\mathrm{T}} \otimes I_n)h\| \leqslant M_{\mathrm{T}}\|\tilde{x}\| = M_{\mathrm{T}}\|\chi\|. \end{aligned}$$

Also, because $\|\phi\| \leqslant \ell_1\|\tilde{\eta}\| + \ell_2\|\chi\|$ for two positive numbers ℓ_1, ℓ_2, by completing the squares,

$$\begin{aligned} & -\frac{1}{9\sqrt{N}}\alpha(\gamma+1)\chi_1^{\mathrm{T}}(\mathbf{1}_N^{\mathrm{T}} \otimes I_n)\phi \\ & = \frac{1}{9}\alpha(\gamma+1)\tilde{x}_1^{\mathrm{T}}\phi \\ & \leqslant \frac{1}{18}\alpha^2\|\chi\|^2 + \frac{1}{9}\alpha(\gamma+1)\ell_2\|\chi\|^2 + \frac{1}{18}(\gamma+1)^2\ell_1^2\|\eta\| \\ & \quad - ((\gamma+1)\alpha\chi_{2:N} + \vartheta_{2:N})^{\mathrm{T}}(R^{\mathrm{T}} \otimes I_n)\phi \\ & \leqslant \frac{1}{2}(\gamma+1)^2\alpha^2\|\chi_{2:N}\|^2 + \frac{1}{8}\|\vartheta_{2:N}\|^2 \\ & \quad + 5\ell_1^2\|\tilde{\eta}\|^2 + 5\ell_2^2\|\chi\|^2. \end{aligned} \tag{20}$$

Thus, letting $\gamma \geqslant \dfrac{4M_{\mathrm{T}}^2 + 1}{m_{\mathrm{T}}}$ leads to

$$\begin{aligned} \dot{V}_1 \leqslant {} & -(\frac{1}{18}\alpha^2 m_{\mathrm{T}} - \frac{1}{9}\alpha(\gamma+1)\ell_2 - 5\ell_2^2)\|\chi\|^2 \\ & - (\gamma\alpha\beta\lambda_0 - \frac{4}{9}\alpha^2(\gamma+1) - \frac{1}{2}(\gamma+1)^2\alpha^2)\|\chi_{2:N}\|^2 \\ & - \frac{5}{16}\|\vartheta_{2:N}\|^2 + (\frac{1}{18}(\gamma+1)^2 + 5)\ell_1^2\|\tilde{\eta}\|^2. \end{aligned}$$

Then, let us check the $\tilde{\eta}$ subsystem. Because F is Hurwitz, there exists a positive definite matrix P such that $PF + F^{\mathrm{T}}P = -2I_s$. Taking $V_0 = \tilde{\eta}^{\mathrm{T}}(I_{Nn} \otimes P)\tilde{\eta}$ gives

$$\begin{aligned} \dot{V}_0 & = -2\|\tilde{\eta}\|^2 + 2\tilde{\eta}^{\mathrm{T}}(I_{Nn} \otimes P)I_{Nn} \otimes FG)\tilde{x} \\ & \leqslant -\|\tilde{\eta}\|^2 + \ell_0\|\chi\|^2 \end{aligned}$$

for a positive real number ℓ_0.

Take the following Lyapunov function candidate for the whole system

$$V = V_1 + \ell_3 V_0, \quad \ell_3 = (\frac{1}{18}(\gamma+1)^2 + 5)\ell_1^2 + 1. \tag{21}$$

Then, we have

$$\begin{aligned} \dot{V} \leqslant {} & -(\frac{1}{18}\alpha^2 m_{\mathrm{T}} - \frac{1}{9}\alpha(\gamma+1)\ell_2 - 5\ell_2^2 - \ell_3\ell_0)\|\chi\|^2 \\ & - (\gamma\alpha\beta\lambda_0 - \frac{4}{9}\alpha^2(\gamma+1) - \frac{1}{2}(\gamma+1)^2\alpha^2)\|\chi_{2:N}\|^2 \\ & - \frac{5}{16}\|\vartheta_{2:N}\|^2 - \|\tilde{\eta}\|^2. \end{aligned} \tag{22}$$

Taking α and β satisfying

$$\begin{cases} \dfrac{1}{18}\alpha^2 m_{\mathrm{T}} - \dfrac{1}{9}\alpha(\gamma+1)\ell_2 - 5\ell_2^2 - \ell_3\ell_0 \geqslant 1, \\ \gamma\alpha\beta\lambda_0 - \dfrac{4}{9}\alpha^2(\gamma+1) - \dfrac{1}{2}(\gamma+1)^2\alpha^2 \geqslant 1 \end{cases} \tag{23}$$

gives

$$\dot{V} \leqslant -\chi^2 - \frac{5}{16}\|\vartheta_{2:N}\|^2 - \|\tilde{\eta}\|^2. \tag{24}$$

By Theorem 4.10 of [13],

$$\lim_{t\to\infty}(\chi(t), \vartheta_{2:N}(t), \tilde{\eta}(t)) = (0, 0, 0). \tag{25}$$

Therefore, $\tilde{x} \to 0$ as $t \to \infty$, i.e., $x_i \to x^*$. The proof is completed.

Remark 5 Note that the constructed Lyapunov function V in (21) is independent of the interaction digraph. Hence, following a similar proof as in Theorem 1, the above result can be extended to switching case when the switching digraph $\mathcal{G}$ keeps strongly connected and weighted-balanced, with its adjacency matrix $\mathcal{A}$ piecewise constant between switchings.

3.3 Example

Consider a five-agent network to minimize $f(x) = \sum_{i=1}^{5} f_i(x)$, $x \in \mathbb{R}$ with

$$\begin{cases} f_1(x) = (x+2)^2, \ \ f_2(x) = (x-5)^2, \\ f_3(x) = x^2 \ln(1+x^2) + x^2, \\ f_4(x) = \dfrac{x^2}{\sqrt{x^2+1}} + x^2, \ \ f_5(x) = \dfrac{x^2}{\ln(2+x^2)}. \end{cases} \tag{26}$$

It is easy to see that the local cost function f_i is m^i-strongly convex and its gradient is M^i-Lipschitz on $\mathbb{R}$ with real numbers $m^i, M^i > 0$ for $i = 1, \dots, 5$, which implies Condition 2. The network topology is shown in Fig. 1. We set all edge weights to be 1, which verifies Condition 1. The disturbance in agent dynamics is given by $d_i(t) = A_i \sin(\omega t + c_i)$, which can be generated by system (2) with

$$S = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \ \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ \ w_i(0) = \begin{bmatrix} A_i \sin c_i \\ A_i \cos c_i \end{bmatrix}.$$

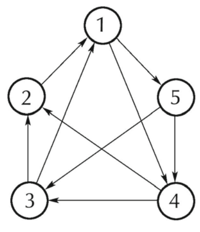

Fig. 1 Interaction topology for the network.

Here, we set $\omega = 1$. Then, system (5) is given with

$$\Phi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \ \Psi = [1 \ \ 0].$$

Let $G = [-4 \ \ -2]^{\mathrm{T}}$ be such that the matrix $F = \Phi + G\Psi$ is Hurwitz. According to Theorem 1, we can choose appropriate α and β such that the algorithm (7) solves our problem. The simulation results are shown in Figs. 2 and 3 with $(A_1, A_2, A_3, A_4, A_5) = (4\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, 3\sqrt{2}, 9\sqrt{2})$ and $c_i = \dfrac{\pi}{4}$, $i = 1, \dots, 5$.

Figs. 2 and 3 show that the state of each agent converges to the exact optimization point $x^* = 0.49$. Moreover, a larger value of β results in faster convergence, which is consistent with the disturbance-free case discussed in [6].

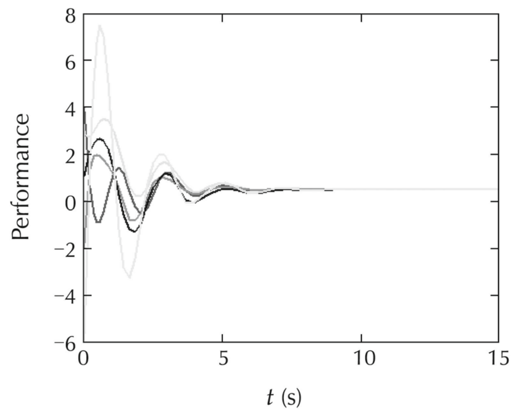

Fig. 2 Performance of (7) with $\alpha = 1, \beta = 1$.

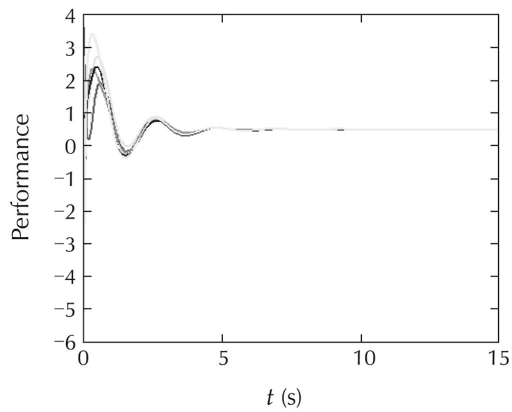

Fig. 3 Performance of (7) with $\alpha = 1, \beta = 5$.

4 Conclusions

The dynamic optimization problem with external disturbances has been studied in this paper. First, a new problem formulation was given to achieve the distributed optimization with disturbance rejection. Then, an internal-model-based algorithm was proposed to solve the problem. To our knowledge, this is the first effort to study the distributed optimization with external disturbance signals, and more complicated problems are still under investigation.

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