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Distributed attitude synchronization control of multi-agent systems with switching topologies[☆]

Johan Thunberg[a,1], Wenjun Song[b], Eduardo Montijano[c], Yiguang Hong[b], Xiaoming Hu[a]

[a] KTH Royal Institute of Technology, S-100 44 Stockholm, Sweden

[b] Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

[c] Centro Universitario de la Defensa and Instituto de Investigación en Ingeniería de Aragón, Zaragoza, Spain

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ABSTRACT

This paper addresses the attitude synchronization problem in multi-agent systems with directed and switching interconnection topologies. Two cases for the synchronization problem are discussed under different assumptions about the measurable information. In the first case the agents can measure their rotations relative to a global reference coordinate frame, whilst in the second case they can only measure the relative rotations between each other. Two intuitive distributed control laws based on the axis–angle representations of the rotations are proposed for the two cases, respectively. The invariance of convex balls in $SO(3)$ is guaranteed. Moreover, attitude synchronization is ensured under the well-known mild switching assumptions, the joint strong connection for the first case and joint quasi-strong connection for the second case. To show the effectiveness of the proposed control schemes, illustrative examples are provided.

1. Introduction

We address the problem of attitude synchronization. In this problem a system of rigid bodies in the three-dimensional Euclidean space shall synchronize or reach consensus in their attitudes (rotations or orientations). Or equivalently formulated, they all shall have the same rotation relative to some fixed world coordinate frame. The problem is interesting since it has applications in the real world, *e.g.*, systems of satellites, UAVs or networks of cameras. It is challenging since the kinematics are nonlinear and the system evolves on the compact manifold $SO(3)$, *i.e.*, the group of orthogonal matrices in $\mathbb{R}^{3\times 3}$ with determinant equal to 1.

As we know, connectivity is key to achieving the collective behavior in a multi-agent network. In fact, the topologies for the practical multi-agent networks may change over time. In the study of variable topologies, a well-known connectivity assumption, called (uniform) joint connection without requiring connectedness of the graph at every moment, was employed to guarantee multi-agent consensus for first-order or second-order linear or nonlinear systems (Cheng, Wang, & Hu, 2008; Hong, Gao, Cheng, & Hu, 2007; Jadbabaie, Lin, & Morse, 2003; Shi & Hong, 2009). Most existing results on the attitude synchronization problem were obtained for the case of fixed topologies.

The objective of the paper is to solve the attitude synchronization problem in continuous time systems. To provide simple and intuitive control designs, we present distributed angular velocity controllers making use of the axis–angle representation to describe the rotations of the agents. In Thunberg, Montijano, and Hu (2011) solutions for two different synchronization cases were proposed in order to synchronize the rotations. Here we largely relax the assumptions that the neighborhood or connectivity graph needs to be undirected and fixed, and consider directed and switching graphs.

We provide control laws that will lead to synchronization when the interaction graph is jointly strongly connected for the first case and jointly quasi-strongly connected in the second case. In fact, almost global attitude synchronization was achieved in Sarlette,

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E-mail addresses: johan.thunberg@math.kth.se (J. Thunberg), jsongwen@amss.ac.cn (W. Song), emonti@unizar.es (E. Montijano), yghong@iss.ac.cn (Y. Hong), hu@math.kth.se (X. Hu).

[1] Tel.: +46 87906659; fax: +46 8 225320.

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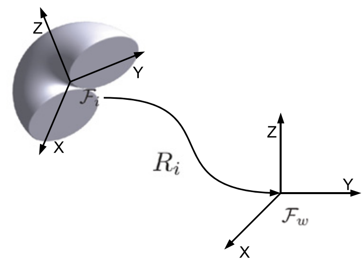

Fig. 1. A rigid body agent (agent i) here is illustrated as a half-torus. The absolute rotation or orientation of this rigid body agent is the rotation R_i of its body frame $\mathcal{F}_i$ to the fixed world frame $\mathcal{F}_W$.

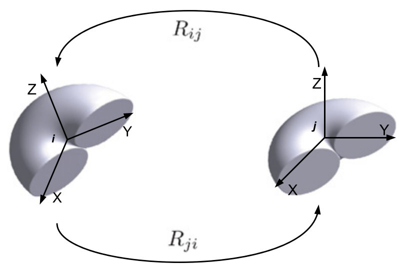

Fig. 2. Two rigid body agents, agent i and agent j, here are illustrated as half-tori. The relative rotations or orientations R_{ij} and R_{ji} between them are the rotation of $\mathcal{F}_j$ to $\mathcal{F}_i$ and the rotation from $\mathcal{F}_i$ to $\mathcal{F}_j$, respectively.

Sepulchre, and Leonard (2009) based on the joint connection, where auxiliary variables were introduced. Here the proposed controller is relatively simple in the static linear feedback form, which is very common in consensus control to make each agent move toward the average position of its neighbors.

In this work the control design is conducted on a kinematic level. A reason for this is that control laws in the robotics community often are specified on a kinematic level. The dynamic equations are platform dependent and differ between applications. We show that the well-known consensus control laws used for positional consensus of systems of agents with single integrator dynamics can be used for rotational consensus after a simple transformation, meaning that the same type of control laws can be used for rotational and positional consensus.

The paper proceeds as follows. In Section 2 we introduce the axis–angle representation and the kinematics of the agents. In Section 3 we introduce two standard cases for the general synchronization problem and propose a control scheme for each case. Sections 4 and 5 show the attitude synchronization under joint connection assumptions in the two cases, respectively. In Section 6 two illustrative examples are provided to show the synchronization when the two different controllers are used. The paper is concluded in Section 7.

2. Preliminaries

We consider a system of n agents (rigid bodies). We denote the world frame as $\mathcal{F}_W$ and the instantaneous body frame of agent i as $\mathcal{F}_i$, where $i \in \{1, 2, \dots, n\}$. For agent $i \in \{1, \dots, n\}$, let $R_i(t) \in SO(3)$ denote the rotation of $\mathcal{F}_i$ in the world frame $\mathcal{F}_W$ at time t, where $SO(3)$ represents the group of rotation matrices (Murray & Sastry, 1994). Let $R_{ij}(t) \in SO(3)$ denote the rotation of $\mathcal{F}_j$ in the frame $\mathcal{F}_i$, *i.e.*, $R_{ij}(t) = R_i^T(t)R_j(t)$, where $i, j \in \{1, 2, \dots, n\}$. We will throughout the text refer to the rotation $R_i(t)$ as the *absolute rotation* of agent i, whereas the rotation $R_{ij}(t)$ will be referred to as the *relative rotation* between agent j and agent i. The difference between the two kinds of rotations is illustrated in Figs. 1 and 2.

Let $\boldsymbol{x}_i(t)$ and $\boldsymbol{x}_{ij}(t)$ denote the axis–angle representations of the rotations $R_i(t)$ and $R_{ij}(t)$, respectively. The axis–angle representation is obtained from the logarithm as

$$\widehat{\boldsymbol{x}}_i = \log(R_i),$$
$$\widehat{\boldsymbol{x}}_{ij} = \log(R_i^T R_j),$$

where $\widehat{\boldsymbol{p}} \in so(3)$ denotes the skew symmetric matrix generated by $\boldsymbol{p} = [p_1, p_2, p_3]^T \in \mathbb{R}^3$, *i.e.*,

$$\widehat{\boldsymbol{p}} = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}. \tag{1}$$

Define the absolute state of the entire system (in terms of the axis–angle representation) as $\boldsymbol{x}(t) = [\boldsymbol{x}_1^T(t), \boldsymbol{x}_2^T(t), \dots, \boldsymbol{x}_n^T(t)]^T$. Note that

$$\boldsymbol{x}_{ij} = -\boldsymbol{x}_{ji},$$

and in general

$$\boldsymbol{x}_j - \boldsymbol{x}_i \neq \boldsymbol{x}_{ij},$$

which can be seen from the Baker–Campbell–Hausdorff formula (see Murray & Sastry, 1994). The axis–angle representation $\boldsymbol{x}_i$ of a rotation matrix R_i is unique for $\|\boldsymbol{x}_i\| < \pi$, which is almost all $SO(3)$. To be more precise, the open ball $B_\pi(I)$ in $SO(3)$ with radius π around the identity is diffeomorphic to the open ball $B_\pi(\mathbf{0}) = \{\|\boldsymbol{z}\| < \pi : \boldsymbol{z} \in \mathbb{R}^3\}$ via the logarithmic map and the mapping from skew symmetric matrices to $\mathbb{R}^3$. In this paper we restrict the rotations of the agents to be contained in this almost global region.

Provided that the rotations are contained in this restricted set, there is a close relationship between the Riemannian distance of two elements in $SO(3)$ and the axis–angle representation, given as follows

$$d_R(R_i, R_j) = \|\boldsymbol{x}_{ij}\|_2 = \theta_{ij},$$
$$d_R(I, R_i) = \|\boldsymbol{x}_i\|_2 = \theta_i,$$

where d_R denotes the Riemannian metric on $SO(3)$. The name axis–angle comes from the fact that the vector $\boldsymbol{x}_i$ can be equivalently written as $\boldsymbol{x}_i = \theta_i \boldsymbol{u}_i$, where $\boldsymbol{u}_i$ is the rotational axis and θ_i is the angle of rotation around the axis.

Denote the instantaneous angular velocity of $\mathcal{F}_i$ relative to $\mathcal{F}_W$ expressed in the frame $\mathcal{F}_i$ as $\boldsymbol{\omega}_i$. The kinematics of $\boldsymbol{x}_i$ is given by

$$\dot{\boldsymbol{x}}_i = L_{\boldsymbol{x}_i}\boldsymbol{\omega}_i, \tag{2}$$

where the transition matrix $L_{\boldsymbol{x}_i}$ is given by

$$L_{\boldsymbol{x}_i} = L_{\theta_i \boldsymbol{u}_i} = I_3 + \frac{\theta_i}{2}\widehat{\boldsymbol{u}}_i + \left(1 - \frac{\operatorname{sinc}(\theta_i)}{\operatorname{sinc}^2\left(\frac{\theta_i}{2}\right)}\right)\widehat{\boldsymbol{u}}_i^{\,2}. \tag{3}$$

The proof is found in Junkins and Schaub (2003). The function $\operatorname{sinc}(\beta)$ is defined so that $\beta\operatorname{sinc}(\beta) = \sin(\beta)$ and $\operatorname{sinc}(0) = 1$. It was shown in Malis, Chaumette, and Boudet (1999) that $L_{\theta_i \boldsymbol{u}_i}$ is invertible for $\theta \in (-2\pi, 2\pi)$. Note, however, that in this paper $\theta \in [0, \pi)$.

To represent the connectivity between the agents we introduce a directed graph (or digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The set $\mathcal{V} = \{1, \dots, n\}$ is the node set and the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. Agent i has a corresponding node $i \in \mathcal{V}$. Let $\mathcal{N}_i \in \mathcal{V}$ comprise the neighbors of agent i. The directed edge from j to i, denoted as (j, i), belongs to $\mathcal{E}$ if and only if $j \in \mathcal{N}_i$. We assume that $i \in \mathcal{N}_i$. The adjacency matrix $A = [a_{ij}]_{n\times n}$ is defined such that $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$.

A directed path of $\mathcal{G}$ is an ordered sequence of distinct nodes in $\mathcal{V}$ such that any consecutive two nodes in the sequence correspond to an edge of the digraph. An agent i is connected to an agent j if there is a directed path starting from j and ending in i. A digraph is called quasi-strongly connected if there exists a (rooted) spanning tree or a center, *i.e.*, at least one node such that all the other nodes are connected to it. A digraph is called strongly connected if there is a directed path between any pair of nodes.

Instead of using the term *communication graph* for $\mathcal{G}$, we deliberately use the terms *neighborhood graph* or *interaction graph*. This is due to the reason that explicit communication does not necessarily take place between the agents. Instead they can choose to just observe each other via cameras or other sensors.

The connectivity graph $\mathcal{G}$ may be time-varying. In the time-varying case, we denote all possible topologies of the graph $\mathcal{G}$ as $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N$. Let $\sigma(t) : [0, \infty) \to \{1, 2, \ldots, N\}$ be a right continuous piecewise constant switching signal function. We assume there is a dwell time $\tau_D > 0$ as a lower bound between any two consecutive switching times, i.e., the switching instances $\{\tau_k \mid k = 1, 2, \ldots\}$ satisfy

$$\inf_k(\tau_{k+1} - \tau_k) \geq \tau_D.$$

Define the union graph of $\mathcal{G}_{\sigma(t)}$ during the time interval $[t_1, t_2)$ as

$$\mathcal{G}([t_1, t_2)) = \bigcup_{t\in[t_1,t_2)} \mathcal{G}_{\sigma(t)} = \left(\mathcal{V}, \bigcup_{t\in[t_1,t_2)} \mathcal{E}_{\sigma(t)}\right),$$

where $t_1 < t_2 \leq +\infty$.

Definition 1. The graph $\mathcal{G}_{\sigma(t)}$ is said to be (uniformly) jointly quasi-strongly connected if there exists a constant $T > 0$ such that the union graph $\mathcal{G}([t, t + T))$ is quasi-strongly connected for any $t \geq 0$. The graph $\mathcal{G}_{\sigma(t)}$ is said to be (uniformly) jointly strongly connected if there exists a constant $T > 0$ such that the union graph $\mathcal{G}([t, t + T))$ is strongly connected for any $t \geq 0$.

If $j \in \mathcal{N}_i(t)$, agent i obtains information about agent j. This information is either the absolute rotation R_j or the relative rotation R_{ij}. In Section 3 we will provide two simple linear control laws, which are based on absolute rotations and relative rotations, respectively. If agent i obtains the absolute rotation R_j, this rotation can either be transmitted by means of communication, *i.e.*, agent j sends R_j to agent i, or measured by a camera attached to agent i, *i.e.*, agent i observes agent j together with an object of known rotation in the world frame $\mathcal{F}_W$. In the latter case, agent i can calculate R_j without the need of communication. The relative rotation R_{ij} can often be obtained without communication, if *e.g.*, a camera is used.

3. Formulation and control design

Let us now consider the attitude synchronization problem. We find a feedback control law $\boldsymbol{\omega}_i$ for each agent i using either absolute rotations or relative rotations so that the absolute rotations of all agents converge to some common rotation in the global frame $\mathcal{F}_W$ as time goes to infinity, *i.e.*,

$$R_i - R_j \to O, \quad \forall i, j, \text{ as } t \to \infty,$$

where $O \in \mathbb{R}^{3\times3}$ denotes a matrix in which all elements are zero. Equivalently this can be stated as

$$R_{ij} \to I, \quad \forall i, j, \text{ as } t \to \infty.$$

This is illustrated in Fig. 3.

In terms of the axis–angle representations of the rotations, we get the following conditions

$$\boldsymbol{x}_i - \boldsymbol{x}_j \to \boldsymbol{0} \quad \forall i, j, \text{ as } t \to \infty,$$

or equivalently

$$\boldsymbol{x}_{ij} \to \boldsymbol{0} \quad \forall i, j, \text{ as } t \to \infty.$$

A controller solving this problem will be referred to as a *synchronization controller*.

Based on the axis–angle representation, we will propose a unified method with simple controllers to study two synchronization cases, where available variables are different for each agent.

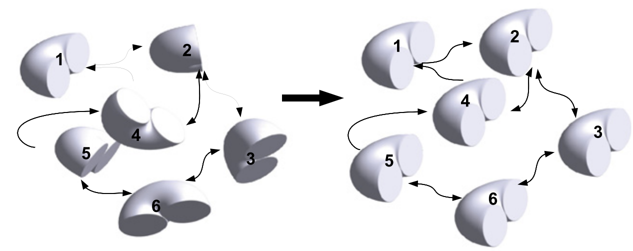

Fig. 3. The rotations of all agents in the system shall converge to some common rotation in the world frame $\mathcal{F}_W$.

3.1. Synchronization using absolute rotations

In this case we assume that each agent i can have access to the absolute rotations $\{R_j(t)$ or $\boldsymbol{x}_j(t) : j \in \mathcal{N}_i(t)\}$.

An obvious solution to this problem is to choose

$$\boldsymbol{\omega}_i = -k_i\boldsymbol{x}_i,$$

where $k_i > 0$. Due to the structure (3) of $L_{\boldsymbol{x}_i}$, the rotations would asymptotically reach synchronization at the identity of the frame $\mathcal{F}_W$. However there are reasons why this solution is not preferable. Since there is no collaboration between agents in this case, the controller does not make use of valuable information that can be gathered from the neighbors and the controller could be regarded as an open-loop controller in terms of synchronization (not in terms of state feedback), making this solution less robust to noise and erroneous measurements.

Instead we consider the following well-known consensus control law based on the absolute rotations of all neighbors of agent i,

$$\boldsymbol{\omega}_i = \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(\boldsymbol{x}_j - \boldsymbol{x}_i), \tag{4}$$

where $[a_{ij}(t)]_{n\times n}$ is the adjacency matrix of the digraph $\mathcal{G}_{\sigma(t)}$.

3.2. Synchronization using relative rotations

In this case we assume that only relative rotations $\{R_{ij}(t)$ or $\boldsymbol{x}_{ij}(t) : j \in \mathcal{N}_i(t)\}$ are available for agent i.

We propose the following control law for this problem based on relative rotations as

$$\boldsymbol{\omega}_i = \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)\boldsymbol{x}_{ij}, \tag{5}$$

where once again $[a_{ij}(t)]_{n\times n}$ is the adjacency matrix of the digraph $\mathcal{G}_{\sigma(t)}$.

3.3. The case of $SO(2)$

Let us now consider a special case, when all the rotational axes of the agents are parallel. This is the case when *e.g.*, all the agents are located in a plane and rotating around the axis normal to the plane. It is easy to show that the set of rotations defined in this manner for a fixed rotational axis is equivalent to $SO(2)$.

An interesting observation is that if the absolute rotations of all agents are contained within the open ball $B_{\pi/2}(I)$ of radius r less than $\pi/2$ centered around the identity in the frame $\mathcal{F}_W$, then controllers (4) and (5) are the same. This can be seen as follows. Assume $\boldsymbol{u}^*$ is the axis such that

$$\boldsymbol{x}_i = \theta_i\boldsymbol{u}^*, \quad \forall i,$$

and then the absolute rotations of the agents are contained in a subset of $SO(3)$ which is equivalent to $SO(2)$. The controller (4) reduces to the following form

$$\boldsymbol{\omega}_i = \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(\theta_j - \theta_i)\boldsymbol{u}^*, \tag{6}$$

and controller (5) reduces to the following form

$$\begin{aligned}\boldsymbol{\omega}_i &= \sum_{j\in\mathcal{N}_i} a_{ij}(t)\left[\log\left(\exp\left((\theta_j - \theta_i)\widehat{\boldsymbol{u}^*}\right)\right)\right]^\vee \\ &= \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(\theta_j - \theta_i)\boldsymbol{u}^*, \qquad (7)\end{aligned}$$

where $\exp(\cdot)$ is the matrix exponential and the operator $(\cdot)^\vee$ is the inverse map of the hat operator (1). We see that (6) and (7) are equal.

4. Absolute rotation case

The control law (4) is used as a consensus controller in systems of agents with single integrator dynamics. The question is if this simple control law also works for rotations expressed in the axis–angle representation. The answer is yes and is given by the following theorem.

Theorem 2. *If the rotations of the agents initially are contained in a closed ball $\bar{B}_r(I)$ of radius r less than π in $SO(3)$, where I corresponds to the identity in $\mathcal{F}_W$, and the graph $\mathcal{G}_{\sigma(t)}$ is jointly strongly connected, then controller (4) is a synchronization controller.*

Note that the assumptions on $\mathcal{G}$ are standard in the literature, and provided that the agents initially are uniformly distributed in $SO(3)$, the agents will initially be contained in a ball $\bar{B}_r(I)$ of radius r less than π with probability 1, and hence (4) is a synchronization controller.

Before we proceed with the proof of Theorem 2, we state the following lemma, guaranteeing the existence and uniqueness of solutions and continuous dependence of initial conditions when using the proposed control law.

Lemma 3. *For $i \in \{1, \ldots, n\}$, the function*

$$f_i(t, \boldsymbol{x}) = L_{\boldsymbol{x}_i} \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(\boldsymbol{x}_j - \boldsymbol{x}_i), \tag{8}$$

i.e., *the right hand side of the closed loop dynamics (2) for $\boldsymbol{x}_i$ when using controller (4) is Lipschitz continuous in $\boldsymbol{x}$ in a closed ball $(\bar{B}_r(\boldsymbol{0}))^n$ of radius r less than π in $\mathbb{R}^{3n}$.*

Proof. The function

$$\sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(\boldsymbol{x}_j - \boldsymbol{x}_i)$$

is linear, continuously differentiable and hence Lipschitz. Regarding the function $L_{\boldsymbol{x}_i}$, it can be written as a sum of three expressions by (3),

$$L_{\boldsymbol{x}_i} = I_3 + \frac{\widehat{\boldsymbol{x}}_i}{2} + \left(1 - \frac{\mathrm{sinc}(\|\boldsymbol{x}_i\|)}{\mathrm{sinc}^2\left(\frac{\|\boldsymbol{x}_i\|}{2}\right)}\right)\left(\frac{\widehat{\boldsymbol{x}}_i}{\|\boldsymbol{x}_i\|}\right)^2,$$

where the third expression in the sum is defined as the zero matrix when $\boldsymbol{x}_i = \boldsymbol{0}$. It can be shown by computation that this Jacobian matrix is continuously differentiable in $\boldsymbol{x}_i$ (or $\boldsymbol{x}$) on $(\bar{B}_r(\boldsymbol{0}))^n$ and hence Lipschitz. It follows that (8) is Lipschitz in $\boldsymbol{x}$ on $(\bar{B}_r(\boldsymbol{0}))^n$. □

Proof. We have as a requirement that $\theta_i(t) < \pi$, $\forall t \geq t_0$. In order to show that this requirement is fulfilled when the controller (4) is used, we show that if each agent initially is contained in the closed ball $\bar{B}_r(I)$ with radius $r < \pi$, then the agent will stay in the ball $\bar{B}_r(I)$. This is equivalent to show that the closed (compact) set $(\bar{B}_r(\boldsymbol{0}))^n$ is (positively) invariant for $\boldsymbol{x}(t) = \boldsymbol{\varphi}(t; t_0, \boldsymbol{x}^0)$, where $\boldsymbol{\varphi}(t; t_0, \boldsymbol{x}^0)$ is the solution at time t, starting from the state $\boldsymbol{x}^0 \in (\bar{B}_r(\boldsymbol{0}))^n$ at the initial time t_0.

Since $\boldsymbol{x} \in (\bar{B}_r(\boldsymbol{0}))^n$, we know that $\boldsymbol{x} \in (\bar{B}_{r'}(\boldsymbol{0}))^n$ for $r' \in (r, \pi)$. We know by Lemma 3 that the system is Lipschitz in $\boldsymbol{x}$ on $(\bar{B}_{r'}(\boldsymbol{0}))^n$. Thus if $\boldsymbol{x}^0 \in (\bar{B}_r(\boldsymbol{0}))^n$, it follows by using Theorem 3.1 in Khalil (2002) that there is $\delta > 0$ such that the system has a unique solution $\boldsymbol{\varphi}(t; t_0, \boldsymbol{x}^0)$ on $[t_0, t_0 + \delta]$.

Now we introduce the function

$$V(\boldsymbol{x}) = \max\{\theta_i^2 : i = 1, \ldots, n\}.$$

Let $\mathcal{I}(t) = \{i \in \mathcal{V} : V(\boldsymbol{x}(t)) = \theta_i^2(t)\}$. It is known that

$$D^+V(\boldsymbol{x}(t)) = \max_{i\in\mathcal{I}(t)} \frac{d}{dt}\theta_i^2,$$

(see Danskin, 1966; Francis, Lin, & Maggiore, 2007; Shi & Hong, 2009) where $D^+V(\boldsymbol{x}(t))$ is the upper Dini derivative of $V(\boldsymbol{x}(t))$; see Lee (2009) (also referred to as the upper right-hand derivative in Khalil, 2002).

Suppose $i \in \mathcal{I}(t)$. We investigate how $\theta_i^2(t)$ changes.

$$\begin{aligned}\frac{d(\theta_i^2/2)}{dt} &= \frac{d((\boldsymbol{x}_i^T\boldsymbol{x}_i)/2)}{dt} \\ &= \boldsymbol{x}_i^T L_{\boldsymbol{x}_i} \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(\boldsymbol{x}_j - \boldsymbol{x}_i) \qquad (9) \\ &\leq \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)(\theta_i\theta_j - \theta_i^2) \qquad (10) \\ &\leq 0,\end{aligned}$$

since $\boldsymbol{x}_i^T L_{\boldsymbol{x}_i} = \boldsymbol{x}_i^T$. Thus $D^+V(\boldsymbol{x}(t)) \leq 0$ and it follows by the comparison lemma (Lemma 3.4 in Khalil, 2002), that

$$V(\boldsymbol{x}(t)) \leq V(\boldsymbol{x}(t_0))$$

for $t \in [t_0, t_0 + \delta]$. Thus any solution that starts in the compact set $(\bar{B}_r(\boldsymbol{0}))^n$ will remain in $(\bar{B}_r(\boldsymbol{0}))^n$. Now it follows by Theorem 3.3 in Khalil (2002) that the solution will stay in $(\bar{B}_r(\boldsymbol{0}))^n$ for arbitrary time. So $(\bar{B}_r(\boldsymbol{0}))^n$ is positively invariant.

The key point of this invariance proof is the structure of the Jacobian matrix $L_{\boldsymbol{x}_i}$. The second and the third terms in (3) are orthogonal to $\boldsymbol{x}_i$; see Fig. 4.

Now, since $(\bar{B}_r(\boldsymbol{0}))^n$ is invariant, if $\boldsymbol{x}^0 \in (\bar{B}_r(\boldsymbol{0}))^n$ the solution $\boldsymbol{x}(t) = \boldsymbol{\varphi}(t; t_0, \boldsymbol{x}^0) \in (\bar{B}_r(\boldsymbol{0}))^n$ is unique, bounded and continuous. Thus, the positive limit set of $\boldsymbol{x}(t)$ denoted as Ω is nonempty and compact, and $\boldsymbol{x}(t)$ approaches Ω as $t \to \infty$. We will show that Ω consists of only synchronization points, *i.e.*, points $\boldsymbol{x}$ for which $\boldsymbol{x}_1 = \cdots = \boldsymbol{x}_n$.

We know that $V \circ \boldsymbol{x}$ is a continuous function of time. We also know that $V \circ \boldsymbol{x}$ is decreasing from the invariance proof above. Since $V \circ \boldsymbol{x}$ is bounded from below, it has a lower limit denoted by $\alpha \geq 0$. If $\alpha = 0$, the rotations of all agents will be asymptotically synchronized at the identity. Thus, the subsequent proof will only consider the case that $\alpha > 0$.

For any point $\bar{\boldsymbol{x}}^0 \in \Omega$ there exists an increasing sequence $\{t_k\}$, such that $t_k \to \infty$, $\boldsymbol{x}(t_k) \to \bar{\boldsymbol{x}}^0$ as $k \to \infty$. By continuity of V we have that $V(\bar{\boldsymbol{x}}^0) = \lim_{k\to\infty} V(\boldsymbol{x}(t_k)) = \alpha$.

Now we claim that Ω consists of only synchronization points. In order to prove this, we choose any point $\bar{\boldsymbol{x}}^0 \in \Omega$. If $\bar{\boldsymbol{x}}^0$ is a synchronization point, it corresponds to an equilibrium and will remain as a synchronization point. Hence, in the reminder of the proof we will focus on the case when $\bar{\boldsymbol{x}}^0$ is not a synchronization

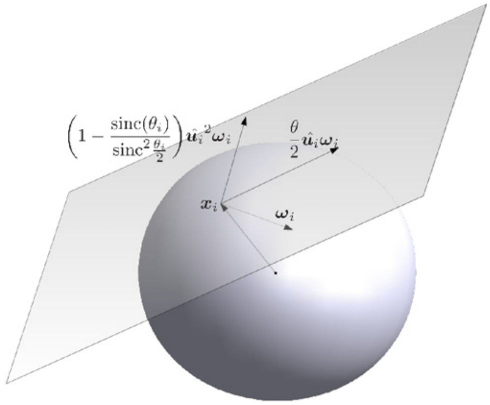

Fig. 4. The second and the third terms of (3) are orthogonal to $\boldsymbol{x}_i$ and lie in the tangent plane of the sphere with radius $\|\boldsymbol{x}_i\|_2$ at the point $\boldsymbol{x}_i$. If agent i has the largest axis–angle norm of all agents, $\boldsymbol{\omega}_i$ is inward-pointing on the boundary of the sphere, unless the rotation of agent i is synchronized with its neighbors, in which case $\boldsymbol{\omega}_i = \mathbf{0}$.

point and show that this assumption leads to a contradiction. We will study the solution $\bar{\boldsymbol{x}}(t) = \boldsymbol{\varphi}(t; \bar{t}_0, \bar{\boldsymbol{x}}^0)$ (with corresponding $\bar{\theta}(t)$). We know that $V(\bar{\boldsymbol{x}}(\bar{t}_0)) = V(\bar{\boldsymbol{x}}^0) = \alpha$.

Let us start by showing that $\bar{\boldsymbol{x}}(t)$ will leave Ω and not return and $V(\bar{\boldsymbol{x}}(t))$ will decrease, *i.e.*, there exists $t' > \bar{t}_0$ such that $V(\bar{\boldsymbol{x}}(t')) < \alpha$. Before we proceed with this, we need the following three lemmas. In the lemmas, let us consider a fixed $s \geq \bar{t}_0$ such that the switching signal function $\sigma(t)$ is constant on $[s, s+\varepsilon)$ for some $\varepsilon > 0$. Due to the assumption of a positive lower bound on the dwell time τ_D, for any $s \geq t_0$ there is ε such that this assumption holds. Let

$$K(t) = \{i : \bar{\theta}_i^2(t) = \alpha\} \quad \text{for } t \geq \bar{t}_0.$$

Lemma 4. *If agent $i \in K(t')$ for $t' \in [s, s+\varepsilon)$, but has a neighbor j such that $\bar{\boldsymbol{x}}_i(t') \neq \bar{\boldsymbol{x}}_j(t')$ (which is equivalent to the statement that $\dot{\bar{\theta}}_i^2(t') < 0$), then there is $\delta' > 0$ such that $i \notin K(t)$ for $t \in (t', t'+\delta')$, where $t'+\delta' < s+\varepsilon$.*

Proof. According to (9) and the fact that $a_{ij}(t)$ is constant on $[s, s+\epsilon)$, it follows that $\dot{\bar{\theta}}_i^2$ is a continuous function of $\bar{\boldsymbol{x}}$ during this time period and $\bar{\boldsymbol{x}}$ is continuous in t. Assume that $i \in K(t')$, then we have that $\dot{\bar{\theta}}_i^2(\bar{\boldsymbol{x}}(t')) < 0$ due to the fact that (9) is strictly smaller than (10) in this case. Then there exists $\delta' > 0$ such that $\dot{\bar{\theta}}_i^2(\bar{\boldsymbol{x}}(t)) < 0$ for $t \in [t', t'+\delta'] \subset [t', s+\varepsilon)$. The result follows. □

Lemma 5. *For an agent $i \notin K(t')$ at time $t' \in [s, s+\varepsilon)$, it holds that $i \notin K(t)$ for all times $t \in (t', s+\varepsilon)$.*

Proof. In the same manner as in the proof of Lemma 4, we note that $\dot{\bar{\theta}}_i^2$ is a continuous function of $\bar{\boldsymbol{x}}$ which is continuous in t. Now assume the statement is not true, and then there is agent i such that $\bar{\theta}_i^2(t') < \alpha$ and $\bar{\theta}_i^2(t'') = \alpha$ for some $t'' \in (t', s+\varepsilon)$. We consider two possible cases for $\dot{\bar{\theta}}_i^2(\bar{\boldsymbol{x}}(t''))$ according to (10). In the first case $\dot{\bar{\theta}}_i^2(\bar{\boldsymbol{x}}(t'')) < 0$ and in the second case $\dot{\bar{\theta}}_i^2(\bar{\boldsymbol{x}}(t'')) = 0$.

In the first case, by continuity there is ε'' such that $\dot{\bar{\theta}}_i^2(\bar{\boldsymbol{x}}(t)) < 0$ for $t \in [t''-\varepsilon'', t'']$, where $t''-\varepsilon'' \geq s$. Combining this fact with the fact that we know that $\bar{\theta}_i^2(t) \leq \alpha$ for $t \in [s, s+\epsilon)$, we reach the conclusion that $\bar{\theta}_i^2(t'') < \alpha$, which is a contradiction.

Thus we are left with the second case. If $\dot{\bar{\theta}}_i^2(\bar{\boldsymbol{x}}(t'')) = 0$, $\bar{\boldsymbol{x}}_j(t'') = \bar{\boldsymbol{x}}_i(t'')$, $\forall j \in \mathcal{N}_i(t'')$ by (9), *i.e.*, for agent $j \in \mathcal{N}_i(t'')$ we have that $j \in K(t'')$. Now we want to show that $(\dot{\bar{\theta}}_j \circ \bar{\boldsymbol{x}})(t'') = 0$ for $j \in \mathcal{N}_i(t'')$. Suppose if this is not true, then there exists $j \in \mathcal{N}_i(t'')$ such that $\dot{\bar{\theta}}_j^2(\bar{\boldsymbol{x}}(t'')) < 0$. Then, by continuity there exists $\varepsilon'' > 0$ such that $\dot{\bar{\theta}}_j^2(\bar{\boldsymbol{x}}(t'')) < 0$ for $t \in [t''-\varepsilon'', t''] \subset [s, s+\varepsilon)$. Once again, by combining this fact with the fact that $\bar{\theta}_j^2(t) \leq \alpha$ for $t \in [s, s+\epsilon)$, we reach the conclusion that $\bar{\theta}_j^2(t'') < \alpha$, which is a contradiction.

Hence $\dot{\bar{\theta}}_j^2(\bar{\boldsymbol{x}}(t'')) = 0$ for $j \in \mathcal{N}_i(t'')$. It follows, by repeating this argument for the neighbors of the neighbors of agent i and so on, that $\bar{\boldsymbol{x}}_i(t'') = \bar{\boldsymbol{x}}_j(t'')$ for all agents j that are contained in the connected component of $\mathcal{G}_{\sigma(t'')}$ containing agent i. Let us denote the subset of the states of $\bar{\boldsymbol{x}}$ contained in this connected component as $\bar{\boldsymbol{x}}_{c_i}$. The dynamics of $\bar{\boldsymbol{x}}_{c_i}$ in time interval $[s, s+\varepsilon)$ can be written as

$$\dot{\bar{\boldsymbol{x}}}_{c_i} = \boldsymbol{f}(\bar{\boldsymbol{x}}_{c_i}).$$

Then $\boldsymbol{f}(\bar{\boldsymbol{x}}_{c_i}(t'')) = \mathbf{0}$, but $\boldsymbol{f}(\bar{\boldsymbol{x}}_{c_i}(t')) \neq \mathbf{0}$ which implies that $\bar{\boldsymbol{x}}_{c_i}(t)$ has converged to an equilibrium point during the finite time interval $[t', t'']$. This is a contradiction since $\boldsymbol{f}$ is Lipschitz continuous in $\bar{\boldsymbol{x}}_{c_i}$ during $[s, s+\varepsilon)$. □

Lemma 6. *Assume that agent i in $K(s)$ is connected to agent j such that $\bar{\boldsymbol{x}}_i(s) \neq \bar{\boldsymbol{x}}_j(s)$, and then for any $t' \in (s, s+\varepsilon)$ it holds that $i \neq K(t')$.*

Proof. We will prove it by contradiction. Suppose if the conclusion is not true, then there exists $t'' \in (s, s+\varepsilon)$ such that $\bar{\theta}_i^2(t'') = \alpha$. Let n_i be the shortest distance of the directed paths in $\mathcal{G}_{\sigma(s)}$ between agent i and any agent j which is connected to i and $\bar{\boldsymbol{x}}_i(s) \neq \bar{\boldsymbol{x}}_j(s)$. If agent i has a neighbor j such that $\bar{\boldsymbol{x}}_i(s) \neq \bar{\boldsymbol{x}}_j(s)$, then $n_i = 1$. If it has no such neighbors, but there is a neighbor k to a neighbor of agent i such that $\bar{\boldsymbol{x}}_i(s) \neq \bar{\boldsymbol{x}}_k(s)$, then $n_i = 2$ and so on.

Let $t'' = s + \delta'$, where $\delta' > 0$. Clearly, there are $n_i + 1$ agents in the shortest path of all possible paths between agent i and any agent j that is connected to i and satisfies $\bar{\boldsymbol{x}}_i(s) \neq \bar{\boldsymbol{x}}_j(s)$. For convenience let us rename agent i as agent 1, the second agent in the shortest path as agent 2 and so on. The last agent is consequently renamed as agent $n_i + 1$. Then we have $\bar{\boldsymbol{x}}_1(s) = \bar{\boldsymbol{x}}_2(s) = \cdots = \bar{\boldsymbol{x}}_{n_i}(s) \neq \bar{\boldsymbol{x}}_{n_i+1}(s)$. By combining Lemmas 4 and 5 we know that since $s + \frac{\delta'}{2} > s$ it follows that $\bar{\theta}_{n_i}^2(t) < \alpha$ for $t \in [s+\frac{\delta'}{2}, s+\varepsilon)$. Since $s + \frac{3\delta'}{4} > s + \frac{\delta'}{2}$ it follows that $\bar{\theta}_{n_i-1}^2(t) < \alpha$ for $t \in [s+\frac{3\delta'}{4}, s+\varepsilon)$. Repeating this argument for the last k agents in the path, we obtain that $\bar{\theta}_{n_i-k}^2(t) < \alpha$ for $t \in [s+\delta'\sum_{l=0}^{k}(\frac{1}{2})^{1+l}, s+\varepsilon)$ when $k = 0, \ldots, n_i - 1$. Thus $\bar{\theta}_1^2(t) < \alpha$ when $t \geq s + \delta'\sum_{l=0}^{n_i}(\frac{1}{2})^{1+l} < t''$ from which it follows that $\bar{\theta}_1^2(t'') < \alpha$. But this is a contradiction. □

From Lemmas 5 and 6, it follows $K(t') \subset K(t'')$ if $s \leq t'' < t' < s+\varepsilon$. Now we show that $K(t') \subset K(t'')$ if $\tau_k \leq t'' < t' \leq \tau_{k+1}$. This can be shown as follows. Suppose if $\sigma(s)$ would have been constant on $[\tau_k, \tau_k + \varepsilon')$, where $\varepsilon' > \tau_{k+1} - \tau_k$, then by Lemmas 5 and 6, we would get that $K(t') \subset K(t'')$ if $\tau_k \leq t'' < t' < \tau_{k+1} + \varepsilon'$. But the solution subject to this alternative switching signal function is identical to our solution until the time τ_{k+1}.

As soon as agent i in K is connected to agent j such that $\boldsymbol{x}_i \neq \boldsymbol{x}_j$, it will immediately leave K according to Lemma 6 and cannot return. If $K(\bar{t}_0)$ is nonempty, since $\mathcal{G}_{\sigma(t)}$ is uniformly jointly strongly connected, at least one agent has left K at time $t' > \bar{t}_0 + T$. If this argument is repeated, for $t' > \bar{t}_0 + nT$, $K(t') = \emptyset$ and $V(\bar{x}(t')) < \alpha$, where n is the number of agents. Let us now proceed to a final important lemma.

Lemma 7. *For a given $\bar{\boldsymbol{x}}^0$ in Ω which is not a synchronization point and any initial time $\bar{t}_0 \geq t_0$, there exists $\delta_\alpha(\bar{\boldsymbol{x}}^0, t_f - \bar{t}_0) > 0$ such that $V(\bar{\boldsymbol{x}}(t_f)) \leq \alpha - \delta_\alpha(\bar{\boldsymbol{x}}^0, t_f - \bar{t}_0)$ for any $t_f > \bar{t}_0 + 2nT$, i.e., δ_α is independent of the absolute time $\bar{t}_0$.*

Proof. Without any knowledge about the switching signal function σ, due to the assumption on the bounded dwell time τ_D, there is an upper bound $M \triangleq \lceil (t_f - \bar{t}_0)/\tau_D \rceil$ on the number of possible switching times during the time interval $[\bar{t}_0, t_f]$. Between two consecutive switching times τ_i and τ_{i+1}, $\sigma(s) \in \{1, \ldots, N\}$ is constant for $s \in [\tau_i, \tau_{i+1})$. Let $\tau_k, \tau_{k+1}, \ldots, \tau_{k'}$ be all the switching times in the interval $[\bar{t}_0, t_f]$ for $k' \leq k + M$. Then the vector $[\sigma(\bar{t}_0), \sigma(\tau_k), \sigma(\tau_{k+1}), \ldots, \sigma(\tau_{k'})]^T$ describes the different topologies during the time interval $[\bar{t}_0, t_f]$. We call a specific choice of the vector $[\sigma(\bar{t}_0), \sigma(\tau_k), \sigma(\tau_{k+1}), \ldots, \sigma(\tau_{k'})]^T$ a scenario. We will only consider feasible scenarios, where feasible in this context refers to the fact that the union graph is strongly connected over a time interval of length T. The number of feasible scenarios is finite and denoted as M'.

Let k' correspond to a specific feasible scenario. Note that $V(\bar{\boldsymbol{x}}(t_f)) < \alpha$ since $t_f > \bar{t}_0 + 2nT$. Let us now consider $\tau_k, \ldots, \tau_{k'}$ as variables in $\mathbb{R}^+$. It is easy to show, using the continuous dependency theorem of initial conditions, that $V(\bar{\boldsymbol{x}}(t_f))$ is a continuous function of all the variables $\tau_k, \ldots, \tau_{k'}$; we could write this as $V(\bar{\boldsymbol{x}}(t_f))(\tau_k, \ldots, \tau_{k'})$. The vector $[\tau_k, \ldots, \tau_{k'}]^T$ is contained in a compact set C defined as

$$C = \{(\tau_k, \ldots, \tau_{k'}) : \bar{t}_0 \leq \tau_i \leq t_f \text{ for } i = k, \ldots, k', \\ \tau_{i+1} \geq \tau_i + \tau_D \text{ for } i = k, \ldots, k' - 1\}.$$

Since C is compact and $V(\bar{\boldsymbol{x}}(t_f))$ is continuous on C it obtains its maximum $V_{\max}$ on C which must be strictly less than α, *i.e.*, $V_{\max} = \alpha - \delta_{s,k'}$ where $\delta_{s,k'} > 0$. The symbol s is an index over scenarios.

Now, since the number of scenarios is finite and the number of choices of k' is finite we know that

$$\delta = \min_{s,k'}\{\delta_s\},$$

exists and is equal to $\delta_\alpha(\bar{\boldsymbol{x}}^0, t_f - \bar{t}_0)$ that we seek. $\square$

Remark 8. Lemma 7 is essential for the last part of the proof. It is important that δ_α is independent of $\bar{t}_0$, in order to show that α cannot be a lower bound of V when $\bar{\boldsymbol{x}}^0$ is not a synchronization point.

Now we are ready for the last part of the proof. Namely, we shall show that the assumption that $\bar{\boldsymbol{x}}^0$ is not a synchronization point leads to a contradiction. Since $\bar{\boldsymbol{x}}^0 \in \Omega$, for any $\zeta > 0$ there is a time t' such that $\boldsymbol{x}(t')$ is contained in a ζ-neighborhood of $\bar{\boldsymbol{x}}^0$. Since $\dot{\boldsymbol{x}}$ and $\dot{\bar{\boldsymbol{x}}}$ are piecewise continuous in t and Lipschitz in $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ respectively on $(\bar{B}_r(\boldsymbol{0}))^n$, according to the continuous dependence theorem on initial conditions, we can find a $\zeta > 0$ such that

$$\left|V(\boldsymbol{x}(t_f')) - V(\bar{\boldsymbol{x}}(t_f'))\right| < \delta_\alpha(\bar{\boldsymbol{x}}^0, t_f' - t'),$$

if

$$\|\boldsymbol{x}(t') - \bar{\boldsymbol{x}}(t')\| = \|\boldsymbol{x}(t') - \bar{\boldsymbol{x}}^0\| \leq \zeta,$$

where $t_f' > t' + 2nT$ and t' is used to substitute $\bar{t}_0$ in the foregoing part of the proof. But we also know that $V(\bar{\boldsymbol{x}}(t_f')) \leq \alpha - \delta_\alpha(\bar{\boldsymbol{x}}^0, t_f' - t')$ from Lemma 7. Thus

$$V(\boldsymbol{x}(t_f')) < \alpha.$$

Hence the statement that $\bar{\boldsymbol{x}}^0$ is not a synchronization point is false. $\square$

The following corollary is a special case of Theorem 2, where the interconnection topology is a fixed digraph.

Corollary 9. *If the absolute rotations of the agents initially are contained in a closed ball $\bar{B}_r(I)$ of radius r less than π in $SO(3)$ and the graph $\mathcal{G}$ is strongly connected, then the controller*

$$\boldsymbol{\omega}_i = \sum_{j \in \mathcal{N}_i} a_{ij}(\boldsymbol{x}_j - \boldsymbol{x}_i),$$

is a synchronization controller.

5. Relative rotation case

In this section, we study the second case when only relative rotations are available. Under some stronger assumptions on the initial rotations of the agents and some weaker assumptions on the graph $\mathcal{G}$, the following theorem ensures that (5) is a synchronization controller.

Theorem 10. *If there is $Q \in SO(3)$, such that the rotations of the agents initially are contained within a closed ball $\bar{B}_r(Q)$ of radius r less than $\pi/2$ centered around Q, then the controller (5) is a synchronization controller if and only if $\mathcal{G}_{\sigma(t)}$ is jointly quasi-strongly connected.*

Before we proceed with the proof, in the same manner as in Section 4 we state a lemma about the Lipschitz continuity.

Lemma 11. *For $i \in \{1, \ldots, n\}$, the function*

$$f_i'(t, \boldsymbol{x}) = L_{\boldsymbol{x}_i} \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)\boldsymbol{x}_{ij}, \tag{11}$$

i.e., the right hand side of the closed loop dynamics for $\boldsymbol{x}_i$ when using controller (5) is Lipschitz continuous in $\boldsymbol{x}$ in a closed ball $(\bar{B}_r(\boldsymbol{0}))^n$ of radius r less than $\pi/2$ in $\mathbb{R}^{3n}$.

Proof. The Jacobian matrix $L_{\boldsymbol{x}_i}$ is continuously differentiable on $(\bar{B}_r(\boldsymbol{0}))^n$ and hence Lipschitz. Regarding the function $\sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) \boldsymbol{x}_{ij}$, we have that

$$\widehat{\boldsymbol{x}}_{ij} = \log(\exp(\widehat{\boldsymbol{x}}_i)^T \exp(\widehat{\boldsymbol{x}}_j))$$

is continuously differentiable and hence Lipschitz. It follows that (11) is Lipschitz in $\boldsymbol{x}$ on $(\bar{B}_r(\boldsymbol{0}))^n$. $\square$

In Theorem 10, since only the relative rotations are used, we can without loss of generality assume that $Q = I$ and Lemma 11 can be applied. Thus, since the rotation matrix is a smooth function of the axis–angle representation via the exponential map (Rodrigues formula), nice properties such as existence and uniqueness of solutions and continuous dependence of initial conditions are guaranteed for the rotations.

We divide the proof of Theorem 10 into three parts. In the first part we show that for each rotation R_i that is located on the boundary of the geodesic convex hull, the time derivative of R_i is inward-pointing (relative to the convex hull) on the boundary of the convex hull at time t. A set is convex in $SO(3)$ if every point on the shortest path (geodesic curve segment) between any two points in the set is contained in the set. The convex hull of the rotations is the smallest convex set containing the rotations. If this hull is shrinking as time progresses, the question is if it asymptotically shrinks to one point (rotation in $SO(3)$), in which case the rotations would asymptotically reach consensus.

In order to show that the hull is shrinking and furthermore shrinks to one point, we use a trick. In part 2 of the proof we introduce a smooth bijection (diffeomorphism) from $B_\pi(Q)$ to $\mathbb{R}^3$, and in part 3 we actually show that the convex hull (the convex hull in $\mathbb{R}^3$ using the Euclidean metric) is shrinking and furthermore shrinks to one point for the variables in $\mathbb{R}^3$. As a consequence it follows that the rotations reach consensus.

Proof. In the case of 2 agents, the rotations will be contained in a set equivalent to $SO(2)$ and it is clear from Section 3.3 that the control law (4) coincides with the control law (5) on $\bar{B}_r(Q)$, but this control law has already been proven to be a synchronization control law. For the general case we proceed as follows.

Part 1: let $X(t) = \{R_i(t) : i = 1, \ldots, n\}$ and $X_i(t) = \{R_j(t) : j \in \mathcal{N}_i(t)\}$. The following relations hold

$$\mathrm{conv}(X_i(t)) \subset \mathrm{conv}(X(t)) \subset \bar{B}_r(Q),$$

where $\mathrm{conv}(X_i(t))$ and $\mathrm{conv}(X(t))$ denote the Riemannian convex hull of the elements in $X_i(t)$ and $X(t)$, respectively. The time derivative of R_i is given by

$$\dot{R}_i = R_i\,\widehat{\boldsymbol{\omega}}_i = R_i \sum_{j\in\mathcal{N}_i(t)} \widehat{\boldsymbol{x}}_{ij}. \tag{12}$$

Let us introduce the following function

$$\gamma_{i,t}(R) = \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t) d_R^2(R_j(t), R).$$

The function $\gamma_{i,t}$ defined on the convex set $\bar{B}_r(Q)$ is strictly convex; see *e.g.*, Moakher (2003). The covariant derivative (gradient on $SO(3)$) of $\gamma_{i,t}(R)$ at $R_i(t)$, see Moakher (2003), is given by

$$\nabla\gamma_{i,t}(R)|_{R_i(t)} = -R_i \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)\widehat{\boldsymbol{x}}_{ij}. \tag{13}$$

Since (12) and (13) are opposite in sign, this implies that $R_i(t)\widehat{\boldsymbol{\omega}}_i(t)$ points in the gradient descent direction of $\gamma_{i,t}$, a direction that is inward-pointing on the boundary of $\mathrm{conv}(X_i(t))$; see Theorem 3.11 in Afsari (2011) and the corresponding proof.

Part 2: let us introduce the Cayley transform restricted to $\bar{B}_r(Q)$. This transform $\mathrm{Cay} : B_{\pi/2}(I) \to so(3)$ is defined as

$$\mathrm{Cay}(R) = (I - R)(I + R)^{-1}$$

and it is its own inverse, *i.e.*, $\mathrm{Cay}(\mathrm{Cay}(R)) = R$. Before we proceed we state the following important lemma about the Cayley transform.

Lemma 12. *The transform Cay is a geodesic map, i.e., the geodesic curve segment between two rotations Q_1 and Q_2 in $SO(3)$ which are closer than π to each other corresponds to a straight line between $(Cay(Q_1))^\vee$ and $(Cay(Q_2))^\vee$ in $\mathbb{R}^3$. So a convex hull in $SO(3)$ corresponds to a polytope in $\mathbb{R}^3$.*

Proof. In order to show this we use the unit quaternion representation for a rotation matrix and the gnomic projection. The unit quaternion $\boldsymbol{q} \in \mathbb{R}^4$ representing a rotation matrix R is given as

$$\boldsymbol{q} = [\cos(\theta/2), \sin(\theta/2)\boldsymbol{u}^T]^T,$$

where once again θ is the rotational angle and $\boldsymbol{u}$ is the rotational axis. The mapping between rotation matrices in $B_\pi(Q)$ and unit quaternions is a smooth bijection. A unit quaternion is an element of the sphere $\mathbb{S}^3$ embedded in $\mathbb{R}^4$. We see that the identity rotation corresponds to the quaternion $[1, 0, 0, 0]^T$, since $\theta = 0$. A geodesic line segment between two rotations Q_1 and Q_2 in $\bar{B}_r(Q)$ corresponds to a great circle segment on the quaternion sphere between the corresponding quaternions $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ Hartley, Trumpf, and Da (2010). Now, if we use the gnomic projection in order to project the quaternions onto the three-dimensional tangent plane touching the point $[-1, 0, 0, 0]^T$ on the sphere, the point in the plane will be

$$\left[-1, -\frac{\sin(\theta/2)}{\cos(\theta/2)}\boldsymbol{u}^T\right]^T = [-1, -\tan(\theta/2)\boldsymbol{u}^T]^T.$$

The gnomic projection of a point on the sphere is the point in the plane where the straight line through the sphere point and the sphere center crosses the plane. It is known that the gnomic projection is a geodesic map, so great circles in the sphere correspond to straight lines in the plane.

Thus if we can show that $(\mathrm{Cay}(R))^\vee = -\boldsymbol{u}\tan\theta$ we are done. A simple calculation shows that this is true, and it follows from the Rodrigues formula. We can equivalently show that

$$\tan(\theta/2)\widehat{\boldsymbol{u}}(I + R) = R - I.$$

Using the Rodrigues formula we get that

$$\begin{aligned}\tan(\theta/2)\widehat{\boldsymbol{u}}(I + R) &= \tan(\theta/2)(2 - (1 - \cos(\theta)))\widehat{\boldsymbol{u}} \\ &\quad + \tan(\theta/2)\sin(\theta)\widehat{\boldsymbol{u}}^2 \\ &= \sin(\theta)\widehat{\boldsymbol{u}} + (1 - \cos(\theta))\widehat{\boldsymbol{u}}^2 \\ &= R - I. \quad \square\end{aligned}$$

Using Lemma 12 we map the rotations in $\bar{B}_r(Q)$ to $\mathbb{R}^3$ by the Cayley transform as

$$\boldsymbol{p}_i(t) = (\mathrm{Cay}(R_i))^\vee.$$

We introduce the variable $\boldsymbol{p} = [\boldsymbol{p}_1^T, \ldots, \boldsymbol{p}_n^T]^T$.

Part 3: it would be desirable to show that $\dot{\boldsymbol{p}}_i$ ($\dot{\boldsymbol{p}}$) is Lipschitz in $\boldsymbol{p}$ and piecewise constant in t with the dwell time τ_D as a lower bound on the difference between two consecutive switching times, and also show that $\dot{\boldsymbol{p}}_i(t)$ is inward-pointing on the boundary of $\mathrm{conv}(\{\boldsymbol{p}_j(t) : j \in \mathcal{N}_i(t)\})$, the convex hull being defined in $\mathbb{R}^3$ in this context. If we show that, then all the necessary requirements are fulfilled in order to apply Theorems 3.6 and 3.8 in Francis et al. (2007). Theorem 3.6 guarantees that any convex set in $\mathbb{R}^3$ is invariant for the $\boldsymbol{p}$-system. In particular $\mathrm{conv}(\{\boldsymbol{p}_j(t_0) : \forall j\})$ is invariant, which implies that the set

$$\mathrm{Cay}^{-1}(\mathrm{conv}(\{\boldsymbol{p}_j(t_0) : \forall j\})) \in \bar{B}_r(I)$$

is invariant for the rotations and since all the initial rotations $R_1(t_0), \ldots, R_n(t_0)$ are contained in this set, $(\bar{B}_r(I))^n$ is invariant in $(SO(3))^n$ (or $(\bar{B}_r(\mathbf{0}))^n$ is invariant in $\mathbb{R}^3$). Theorem 3.8 guarantees asymptotic consensus (or agreement) for the $\boldsymbol{p}$-system if and only if $\mathcal{G}_{\sigma(t)}$ is jointly quasi-strongly connected. This in turn guarantees asymptotic consensus for the rotations under the same premises on $\mathcal{G}_{\sigma(t)}$.

It is indeed true that $\dot{\boldsymbol{p}}_i$ fulfills these requirements. The mapping from $\boldsymbol{x}_i$ to $\boldsymbol{p}_i$, defined as

$$g = (\cdot)^\vee \circ \mathrm{Cay} \circ \exp\circ\widehat{(\cdot)} : B_\pi(\mathbf{0}) \to \mathbb{R}^3,$$

is bijective and smooth (diffeomorphism). Thus $\dot{\boldsymbol{p}}_i$ is Lipschitz in $\boldsymbol{p}$. We know that $\dot{\boldsymbol{x}}_i$ is piecewise constant in t with the dwell time τ_D as a lower bound on the difference between two consecutive switching times. Hence $\dot{\boldsymbol{p}}_i$ also has this property. Furthermore, we know that $\dot{R}_i$ is inward-pointing on the boundary of $\mathrm{conv}(X_i)$ for all i. Since the mapping Cay is a geodesic map, the tangent vector $\dot{\boldsymbol{p}}_i$ is inward-pointing on the boundary of $\{\boldsymbol{p}_j(t) : j \in \mathcal{N}_i(t)\}$ (in $\mathbb{R}^3$) at each time t. $\square$

In order to motivate the use of Theorem 5.1, one could consider a system of rigid bodies equipped with more than one camera, each of limited field of view. The initial rotations are close to each other, *i.e.*, the radius r of the convex ball is small. Each camera observes a subset of the other agents, and this subset of agents that is observed by the camera is the same for every rotation in the ball. The agents will remain in the ball due to invariance. Because of limited processing capacity, only one image can be processed from the different cameras of an agent, and relative rotations can be calculated to the neighboring agents observed in the image. If we choose the switching signal function for the change of active

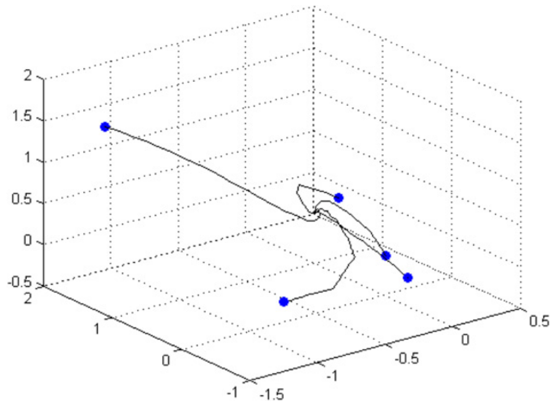

Fig. 5. The agents use controller (4) and their rotations converge to a synchronized rotation as time goes to infinity. In this example the initial rotations of the agents, marked by solid disks, are randomly distributed in the geodesic ball $B_r(I)$ with $r = 3$. Assume there are two graphs, $\mathcal{E}_1 = \{(1, 2), (3, 4), (5, 1)\}$ with nonzero weights $a_{ij} = 4$ if $(j, i) \in \mathcal{E}_1$, $\mathcal{E}_2 = \{(2, 3), (3, 5), (4, 1)\}$ with nonzero weights $a_{ij} = 6$ if $(j, i) \in \mathcal{E}_2$. We use a periodic switching law with period $T = 0.2$. The switching signal $\sigma(t) = 1$ if $t \in [kT, kT + T/2)$, $\sigma(t) = 2$ if $t \in [kT + T/2, (k + 1)T)$, $k = 0, 1, \ldots$. The time horizon was 5 s.

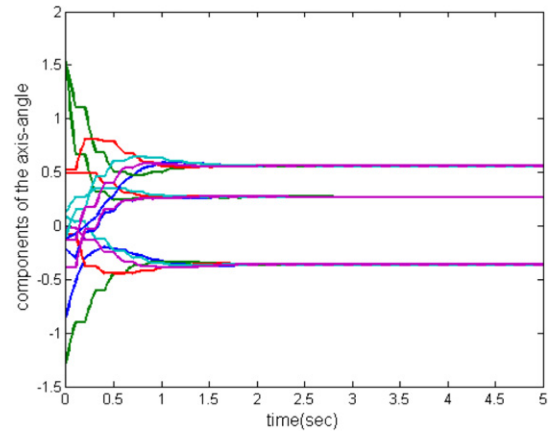

Fig. 6. The same example as Fig. 5. The three components of the axis–angle of the five agents converge to some common values as time goes to infinity.

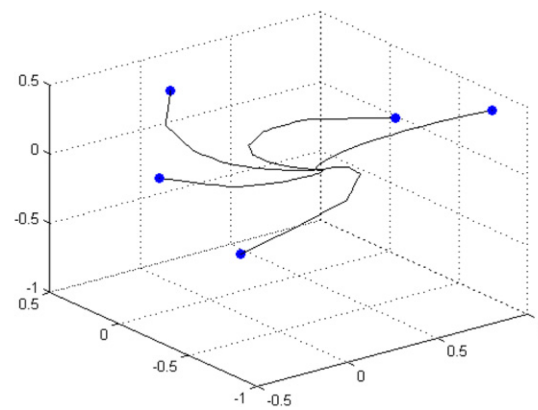

Fig. 7. The rotations of five agents converge to a synchronized rotation as time goes to infinity using controller (5). In this example the initial rotations of the agents are randomly distributed in the geodesic ball $B_r(I)$ with $r = 1.5$. Assume that there are two graphs, $\mathcal{E}_1 = \{(1, 2), (3, 4)\}$, $\mathcal{E}_2 = \{(2, 3), (3, 1), (4, 5)\}$, and $a_{ij} = 5$ if $(j, i) \in \mathcal{E}$. The switching signal is periodic with $\sigma(t) = 1$ if $t \in [kT, kT + T/2)$, $\sigma(t) = 2$ if $t \in [kT + T/2, (k + 1)T)$, $k = 0, 1, \ldots$, where the period $T = 0.2$. The time horizon was 5 s. Solid disks indicate the initial rotations of the agents.

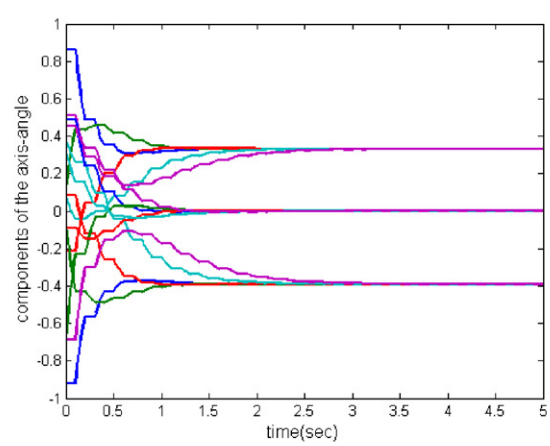

Fig. 8. The same example as Fig. 7. The three components of the axis–angle of the five agents synchronized respectively as time goes to infinity.

cameras so that there is T so that the union graph is quasi-strongly connected over any time interval $[t, t + T]$, then the rotations will reach consensus according to Theorem 5.1.

The next corollary addresses the special case when the interconnection topology in Theorem 10 is a fixed graph.

Corollary 13. *If there is $Q \in SO(3)$ such that the absolute rotations of the agents initially are contained within a closed ball $\bar{B}_r(Q)$ of radius r less than $\pi/2$ centered around Q, then the controller*

$$\boldsymbol{\omega}_i = \sum_{j \in \mathcal{N}_i} a_{ij} \boldsymbol{x}_{ij},$$

is a synchronization controller, if and only if $\mathcal{G}$ is quasi-strongly connected.

6. Illustrative examples

We will now illustrate the convergence for the two different control laws with 5 agents. The simulations were conducted in Matlab using an Euler method to solve the ordinary differentiable equations.

In Figs. 5 and 6 the control law (4) for absolute rotations is used and in Figs. 7 and 8 the control law (5) for relative rotations is used. One can see how the rotations of the agents converge from their initial rotations (blue disks in the figure) to the final rotations where they reach synchronization. Observe that the rotations are represented in the axis–angle representation as a vector in $\mathbb{R}^3$.

7. Conclusions

Using the axis–angle representation we have introduced simple and linear kinematic control laws that solve the attitude synchronization problem for systems of agents with switching and directed topologies. The control laws are either based on differences between the rotations of the agents in a global frame, referred to as *absolute rotations*, or based on the *relative rotations* between the agents.

In the case that absolute rotations are used, synchronization is achieved on almost all $SO(3)$, provided that the neighborhood graph of the agents is jointly strongly connected. When relative rotations are used, synchronization is achieved on convex balls of $SO(3)$, provided that the neighborhood graph of the agents is jointly quasi-strongly connected. In the special case when the agents are contained in a set equivalent to $SO(2)$, the two control laws are equal on convex balls of $SO(3)$.

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Thunberg, J., Montijano, E., & Hu, X. (2011). Distributed attitude synchronization control. In *50th IEEE conference on decision and control and European control conference* (pp. 1962–1967). IEEE.

Johan Thunberg received a Master of Science degree in Engineering Physics from the Royal Institute of Technology (KTH) in Stockholm, Sweden. After finishing his Master he has worked as a research assistant at the Swedish Defence Research Agency (FOI). Currently he is a graduate student in applied mathematics at the Royal Institute of Technology (KTH) in Stockholm and is funded by the Swedish National Space Technology Research Programme (NRFP) and the Swedish Research Council (VR).

Wenjun Song received her B.E. degree in Automation from Huazhong University of Science and Technology, Wuhan, China, in 2009. She is currently a Ph.D. candidate majoring in Complex Systems and Control at the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China. Her research interests include multi-agent systems and attitude control.

Eduardo Montijano received the M.Sc. and Ph.D. degrees from the Universidad de Zaragoza, Spain, in 2008 and 2012, respectively, under the supervision of Prof. Carlos Sagüés. He is currently a Professor with the Centro Universitario de la Defensa, Zaragoza, Spain. His research interests are computer vision and consensus algorithms applied to multiple robots.

Yiguang Hong (M' 99-SM' 02) received the B.Sc. and M.Sc. degrees from Peking University, Beijing, China, and the Ph.D. degree from the Chinese Academy of Sciences (CAS), Beijing. He is currently a Professor in the Institute of Systems Science, Academy of Mathematics and Systems Science, CAS, and serves as the Director of Key Lab of Systems and Control, CAS, and the Director of the Information Technology Division, National Center for Mathematics and Interdisciplinary Sciences, CAS. His research interests include nonlinear dynamics and control, multi-agent systems, robotics, and reliability of software and communication systems. Prof. Hong is also serving as the Deputy Editor-in-Chief of Acta Automatica Sinca, Editor-at-Large of the Journal of Control Theory and Applications, and Associate Editors for several journals including the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, Journal of Systems Science and Complexity, and Kybernetika. He is a recipient of the Guangzhaozhi Award at the Chinese Control Conference (in 1997), Young Author Prize of the IFAC World Congress (in 1999), Young Scientist Award of CAS (in 2001), and the Youth Award for Science and Technology of China (in 2006), and the National Natural Science Prize of China (in 2008, jointly with D. Cheng et al.).

Xiaoming Hu received the B.S. degree from the University of Science and Technology of China in 1983, and the M.S. and Ph.D. degrees from the Arizona State University in 1986 and 1989, respectively. He served as a research assistant at the Institute of Automation, the Chinese Academy of Sciences, from 1983 to 1984. From 1989 to 1990 he was a Gustafsson Postdoctoral Fellow at the Royal Institute of Technology, Stockholm, where he is currently a professor of Optimization and Systems Theory. His main research interests are in non-linear control systems, nonlinear observer design, sensing and active perception, motion planning, control of multi-agent systems, and mobile manipulation.