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Global target aggregation and state agreement of nonlinear multi-agent systems with switching topologies[☆]

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ABSTRACT

In this paper, we discuss coordination problems of a group of autonomous agents, including the target aggregation to a convex set and the state agreement. The aggregation of the whole agent group, consisting of leaders (informed agents) and followers, to a given set is investigated with switching interconnection topologies described by the connectivity assumptions on the joint topology in the time interval $[t, +\infty)$ for any time t, and then the state agreement problem is studied in a similar way. An approach based on set stability and limit set analysis is given to study the multi-agent convergence problems. With the help of graph theory and convex analysis, coordination conditions are obtained in some important cases, and the results show that simple local rules can make the networked agents with first-order nonlinear individual dynamics achieve desired collective behaviors.

1. Introduction

Recent years have seen a large and growing literature concerned with the coordination of a group of autonomous agents, partly due to a broad application of multi-agent systems including consensus, swarming, and formation (referring to Chu, Wang, Chen, and Mu (2006), Cortés (2006), Egerstedt and Hu (2001), Fax and Murray (2004), Gazi and Passino (2004), Hu and Hong (2007) and Martinez, Cortes, and Bullo (2007)).

In the studies of multi-agent coordination, two important problems are very interesting: target aggregation, which is concerned with how a group of agents move together to a target region, and state agreement, which talks about how a group of agents reach a consensus without a given target. Sometimes, target-oriented coordination can be formulated as a leader–follower problem with multiple (virtual) leaders, while state agreement can be described as a leaderless coordination problem. In fact, a "leader" in the multi-agent systems may be a special (informed) agent, or a moving target, or a reference node to guide the whole group. Although there is usually a single leader for the leader-following formulation in many existing results, multiple (virtual) leaders can be found or needed in multi-agent coordination. In fact, a multi-leader framework may be useful in many practical problems. For example, a simple model for the fish flocking was given to simulate foraging and demonstrate that, the larger the group, the smaller the proportion of "leaders" needed to guide the group to the food source in Couzin, Krause, Franks, and Levin (2005), while moving targets can be also viewed as multiple "leaders" in pursuit-evasion operations as in Oh, Schenato, Chen, and Sastry (2007). Lin, Francis, and Maggiore (2005) also discussed an interesting model for a group of agents with straight-line formation containing two "edge leaders", where all the agents converge to the line segment specified by the two edge leaders. On the other hand, state agreement, or sometimes called consensus or synchronization, has been studied in different research areas (for example, DeGroot (1974), Lynch (1997), Lin, Francis, and Maggiore (2007) and Olfati-Saber and Murray (2004)). Without targets or leaders given in advance, all the agents achieve a consensus and their states become the same by their intra-agent interactions or communications.

Variable interconnection topologies between mobile agents pose challenging problems in the studies of multi-agent systems because of the complexity resulting from time-varying and non-smooth structures. Many efforts have been made to handle

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multi-agent controls with dynamic topologies (Hong, Hu, & Gao, 2006; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Tanner, Jadbabaie, & Pappas, 2003). Among the research results, "joint connection" or related concepts play an important role in the investigation of multi-agent coordination. For example, Tsitsiklis, Bertsekas, and Athans (1986) studied the distributed asynchronous iterations, where a team of agents achieve consensus on a common value with possibly outdated values of their neighbors. Jadbabaie, Lin, and Morse (2003) proved the consensus of a simplified Vicsek model (proposed in Vicsek, Czirok, Jacob, Cohen, and Schochet (1995)) with joint-connection assumption in a similar way. Moreover, Hong, Gao, Cheng, and Hu (2007) investigated the jointly-connected coordination for second-order agent dynamics. However, this problem becomes much more difficult if the agent dynamics are nonlinear. Moreau discussed the stability and state agreement problems for nonlinear discrete-time agents with time-varying interconnection assumption on $[t, \infty)$ in Moreau (2005). For nonlinear continuous-time agent dynamics with jointly-connected interaction graphs, results seem even harder to be obtained. Lin et al. made a good start in this research direction and provided conditions to ensure the state agreement for directed multi-agent networks under uniform joint connectivity in Lin et al. (2007).

In this paper, we consider a group of continuous-time agents with variable intra-agent (communication) connection and nonlinear agent dynamics. Also, we investigate with some connectivity assumptions given for joint topology in $[t, \infty)$, the set stability of the networked agents by virtue of graph theory, convex analysis, and stability theory. By neighborhood rules, we show that a set of agents with nonlinear individual dynamics can flock to a convex target set, or achieve the state agreement of the whole group, in some important switching jointly-connected cases. To solve the problems, we propose a limit-set-based approach, different from those adopted in Moreau (2005) and Lin et al. (2007), to deal with the convergence in either target aggregation or state agreement of the considered multi-agent networks.

The paper is organized as follows. Section 2 introduces basic concepts and preliminary results, while Section 3 formulates our problems, target aggregation and state agreement. Then Section 4 studies the set stability of the considered multi-agent system with jointly-connected topologies and the structure of its limit set with a proposed analysis technique. Furthermore, Section 5 analyzes the convergence in some important target-aggregation cases. Then, Section 6 discusses the state agreement for multi-agent systems without target sets. Finally, Section 7 gives the concluding remarks.

2. Preliminaries

In this section, we introduce some preliminary knowledge for the following discussion.

First of all, we introduce some basic concepts and notations in graph theory (referring to Godsil and Royle (2001) for details). A directed graph (or digraph) is usually denoted as $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N} = \{1, 2, \dots, n\}$ is the set of nodes and $\mathcal{E}$ is the set of arcs, each element of which is an ordered pair of distinct nodes in $\mathcal{N}$. (i, j) denotes an arc leaving from node v_i (or simply i) and entering node v_j (or j). A walk in digraph $\mathcal{G}$ is an alternating sequence $i_1e_1i_2e_2\cdots e_{k-1}i_k$ of nodes i_m and arcs $e_m = (i_m, i_{m+1}) \in \mathcal{E}$ for $m = 1, 2, \dots, k$. If there exists a walk from node i to node j then node j is said to be reachable from node i. In particular, each node is thought to be reachable by itself. A node v which is reachable from any node of $\mathcal{G}$ is called a globally reachable node of $\mathcal{G}$. $\mathcal{G}$ is said to be *quasi-strongly connected* if for every two nodes i and j there is a node k from which i and j are reachable. Given a digraph $\mathcal{G}$, its opposite graph $\mathcal{G}^*$ is the digraph formed by changing the orientation of each arc in $\mathcal{G}$. It is known that $\mathcal{G}$ is quasi-strongly connected if and only if $\mathcal{G}^*$ has a globally reachable node (Berge & Ghouila-Houri, 1965).

If $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ have the same node set, the union of the two digraphs is defined as $\mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_1 \cup \mathcal{E}_2)$. A time-varying digraph is defined as $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$ with $\sigma : t \to \mathcal{Q}$ as a piecewise constant function, where $\mathcal{Q}$ is a finite set with all the possible digraphs with node set $\mathcal{V}$.

Additionally, $\mathcal{G}([t_1, t_2))$ denotes the joint digraph in time interval $[t_1, t_2)$ with $t_1 < t_2 \le +\infty$, that is,

$$\mathcal{G}([t_1, t_2)) = \bigcup_{t\in[t_1,t_2)} \mathcal{G}(t) = (\mathcal{V}, \bigcup_{t\in[t_1,t_2)} \mathcal{E}_{\sigma(t)}). \tag{1}$$

The graph $\mathcal{G}(t)$ is called "jointly quasi-strongly connected" in $[t_1, t_2)$ if its joint digraph $\mathcal{G}([t_1, t_2))$ is quasi-strongly connected. Moreover, if there is a constant $T_0 > 0$, such that $\mathcal{G}([t, t + T_0])$ is quasi-strongly connected for any t, then $\mathcal{G}_{\sigma(t)}$ is said to be *uniformly* quasi-strongly connected (with respect to T_0).

Next, we recall some notations in convex analysis (see Rockafellar (1972) for details). A set $K \subset R^m$ is said to be convex if $(1 - \gamma)x + \gamma y \in K$ whenever $x \in K, y \in K$ and $0 < \gamma < 1$. For any set $S \subset R^m$, the intersection of all convex sets containing S is called the *convex hull* of S, denoted by $co(S)$. Particularly, the convex hull of a finite set of points $x_1, \dots, x_n \in R^m$ is a polytope, denoted by $co\{x_1, \dots, x_n\}$. We cite two lemmas on convex analysis, which can be found in Aubin and Cellina (1984).

Lemma 1 (*Best-Approximation Theorem*). *Let K be a closed convex subset of a Hilbert space X. We can associate to any $x \in X$ a unique element $\pi_K(x) \in K$ satisfying*

$$\|x - \pi_K(x)\| = \min_{y\in K} \|x - y\|,$$

where the map π_K is called the projector onto K. Moreover,

$$\langle \pi_K(x) - x, \pi_K(x) - y\rangle \le 0, \quad \forall y \in K.$$

Lemma 2. *Let K be a closed convex subset of a Hilbert space X and d_K the function defined on X by $d_K(x) \triangleq \inf\{\|x - y\| \mid y \in K\}$. Then $d_K^2(x) = \inf\{\|x - y\|^2 | y \in K\}$ is continuously differentiable and*

$$\nabla d_K^2(x) = 2(x - \pi_K(x)),$$

where $\nabla d_K^2(x)$ denotes the gradient of function d_K^2 at point x.

Then, we consider the Dini derivative for the following non-smooth analysis. Let a and b $(> a)$ be two real numbers and consider a function $h : (a, b) \to R$ and a point $t \in (a, b)$. The upper Dini derivative of h at t is defined as

$$D^+h(t) = \limsup_{s\to 0^+} \frac{h(t + s) - h(t)}{s}.$$

Obviously, when h is continuous on (a, b), h is non-increasing on (a, b) if and only if $D^+h(t) \le 0$ for any $t \in (a, b)$ (more details can be found in Rouche, Habets, and Laloy (1977)). The next result is given for the calculation of Dini derivative.

Lemma 3 (Danskin, 1966; Lin et al., 2007). *Let $V_i(t, x) : R \times R^n \to R$ be C^1 for $i = 1, 2, \dots, n$ and let $V(t, x) = \max_{i=1,2,\dots,n} V_i(t, x)$. If*

$$\mathcal{I}(t) = \{i \in \{1, 2, \dots, n\} : V(t, x(t)) = V_i(t, x(t))\}$$

is the set of indices where the maximum is reached at t, then

$$D^+V(t, x(t)) = \max_{i\in\mathcal{I}(t)} \dot{V}_i(t, x(t)).$$

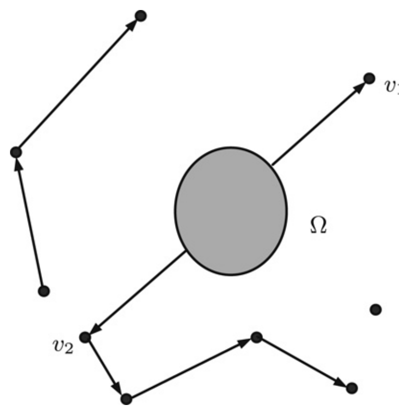

Fig. 1. v_1 and v_2 are the "leaders", which can "see" Ω.

Finally, we consider a system

$$\dot{x} = f(t, x), \tag{2}$$

where $f : R \times R^n \to R^n$ is piecewise continuous in t and continuous in x. Let $x(t) = x(t, t_0, x^0)$ be a solution of (2) with initial condition $x(t_0) = x^0$.

Definition 4. $\Omega_0 \subset R^n$ is called a positively invariant set of (2) if, for any $t_0 \in R$ and any $x^0 \in \Omega_0, x(t, t_0, x^0) \in \Omega_0$ when $t > t_0$. Then system (2) is said to be (set) stable with respect to Ω_0 if, for any t_0 and any $\varepsilon > 0$, there is $\delta > 0$ such that

$$d_{\Omega_0}(x^0) < \delta \Longrightarrow d_{\Omega_0}(x(t)) < \varepsilon, \quad \forall t \geq t_0 \tag{3}$$

with $d_{\Omega_0}(x) = \inf\{\|x - y\| | y \in \Omega_0\}$. Moreover, if δ in (3) does not depend on t_0, then system (2) is said to be uniformly stable with respect to Ω_0. System (2) is said to be (set) attractive with respect to Ω_0 in a region U_0 if $x^0 \in U_0 \Longrightarrow \lim_{t\to\infty} d_{\Omega_0}(x(t)) = 0$; and it is said to be globally attractive with respect to Ω_0 if $U_0 = R^n$.

3. Problem formulation

In this paper, we consider two kinds of coordination problems for a multi-agent system of n agents: aggregation to a target set, and state agreement.

In the target-aggregation problem, a convex region, denoted by $\Omega \subseteq R^m$, is considered as a target set (maybe a desired region or tolerance range in practical design, or a food source or nest site in animal migration (Couzin et al., 2005)) for a group of agents. In practice, there are many challenges in the target aggregation of a multi-agent group for the following reasons:

- Not all the agents can come to know the location of the desired region; in practice, only some of these n agents are "informed" agents (or called "leaders") that can get the location information, while the others (called "followers") cannot;
- Although leaders can "see" the target, they may lose their sights from time to time (due to uncertainties in the environment, for instance) and may not fully care about the target information when making tradeoffs with other agents. This makes the "connection" between the target set and "leaders" keep changing;
- Followers cannot recognize "leaders", and therefore, the local rules are applied to all the agents without any difference, while "leaders" are also affected by the information from followers if there are connections for tradeoff concerns;
- The interconnection between agents are also variable because the neighbors of the agents are time-varying due to the complex dynamics of interacted agents.

The state of agent i (that is, node v_i), is denoted as $x_i \in R^m$ ($i = 1, \ldots, n$); and set Ω is regarded as a generalized (agent) node, denoted as v_0 (see Fig. 1). At time t, if node v_i can "see" node v_j, then there is an arc (v_j, v_i) (marking the information flow) from v_j to v_i; and in this way, v_j is said to be a neighbor of v_i. Likewise, if "informed" agent v_i "sees" Ω at time t, then there is an arc (v_0, v_i) leaving from v_0 and entering v_i; and v_0 (that is, Ω) is said to be a (generalized) neighbor of v_i. Define two sets $\mathcal{N} = \{v_1, v_2, \ldots, v_n\}$ and $\bar{\mathcal{N}} = \{v_0, v_1, v_2, \ldots, v_n\}$. In what follows, when there is no confusion, we identify the index i with node v_i for convenience.

Denote $\bar{\mathcal{P}}$ as the set of all possible interconnection topologies, and $\sigma : [0, +\infty) \to \bar{\mathcal{P}}$ as a piecewise constant switching signal function to describe the switchings between the topologies. Thus, the interaction topology of the considered multi-agent network is described by $\bar{\mathcal{G}}_{\sigma(t)} = (\bar{\mathcal{N}}, \bar{\mathcal{E}}_{\sigma(t)})$. Moreover, as done in some existing works (e.g., Jadbabaie et al. (2003) and Hong et al. (2007)), we assume that there is a dwell time, denoted by a constant τ_D for $\sigma(t)$, as a lower bound between two switching times.

Let $N_i(\sigma(t))$ represent the neighbor set of node i at time t and $x_i(t) \in R^m$ denote the position vector of agent i at time t. Moreover, denote $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in R^{mn}$ and let continuous function $a_{ij}(x) > 0$ be the weight of arc (v_i, v_j), if any, for $i, j = 1, \ldots, n$. Then, the dynamics of each agent v_i with state x_i ($i = 1, \ldots, n$) is described as

$$\dot{x}_i = \sum_{j\in N_i(\sigma(t))} a_{ij}(x)(x_j - x_i) + \lambda(x_i)\chi_i(\sigma(t))f_i(x_i, t), \tag{4}$$

where $f_i(x_i, t) : R^m \times R \to R^m$ is continuous in (x_i, t), and $\lambda(x_i)$ (to mark the informed agents) and $\chi_i(\sigma(t))$ (to mark which informed agents see the target) are Boolean variables, defined respectively, as follows:

$$\lambda(x_i) = \begin{cases} 1, & \text{if } x_i \text{ is a (potential) informed agent} \\ 0, & \text{otherwise} \end{cases}$$

$$\chi_i(\sigma(t)) = \begin{cases} 1, & \text{if } x_i \text{ is connected with } \Omega \text{ at } t \\ 0, & \text{otherwise.} \end{cases}$$

Without loss of generality, we assume that the initial time of system (4) is $t_0 = 0$.

Denote

$$f_{\sigma(t)}(x, t) = \begin{pmatrix} \lambda(x_1)\chi_1(\sigma(t))f_1(x_1, t) \\ \vdots \\ \lambda(x_n)\chi_n(\sigma(t))f_n(x_n, t) \end{pmatrix} \in R^{mn}.$$

Let $A_{\sigma(t)}(x) = (\tilde{a}_{ij}(\sigma(t), x))$ be a matrix in $R^{n\times n}$ with its (i, j) entry $\tilde{a}_{ij}(\sigma(t), x) = a_{ij}(x) \cdot \chi_{ij}(\sigma(t))$, in which

$$\chi_{ij}(\sigma(t)) = \begin{cases} 1, & \text{if there is an arc from } x_i \text{ to } x_j \text{ at } t \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Define a diagonal matrix

$$D_{\sigma(t)}(x) = \operatorname{diag}\{\tilde{d}_1(\sigma(t), x), \ldots, \tilde{d}_n(\sigma(t), x)\} \in R^{n\times n}$$

with $\tilde{d}_i(\sigma(t), x) = \sum_{j\in N_i(\sigma(t))} a_{ij}(x)$, $i = 1, \ldots, n$. Then taking the (nonlinear) Laplacian matrix $L_{\sigma(t)}(x) = D_{\sigma(t)}(x) - A_{\sigma(t)}(x)$, system (4) can be rewritten in a compact form:

$$\dot{x} = -(L_{\sigma(t)}(x) \otimes I_m)x + f_{\sigma(t)}(x, t) \tag{6}$$

where $\otimes$ denotes the Kronecker product and I_m denotes the identity matrix in $R^{m\times m}$. Note that $L_{\sigma(t)}(x)$ and $f_{\sigma(t)}(x, t)$ are piecewise continuous with respect to t and continuous with respect to x, and then the solution to system (6) exists for any initial condition.

Motivated by the practical problems (referring to Couzin et al. (2005), for example), we give the following definition.

Definition 5. Suppose $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$ is the trajectory of system (6) with initial condition $x(t_0) = x^0 = (x_1^0, \ldots, x_n^0)^{\mathrm{T}}$. Then the target aggregation for initial condition $x(t_0) = x^0$ with respect to Ω is achieved if

$$\lim_{t\to+\infty} d_\Omega(x_i(t)) = 0, \quad i = 1, \ldots, n. \tag{7}$$

Moreover, if (7) holds for any initial condition $x^0 \in R^{mn}$, then the global target aggregation of system (6) is said to be achieved.

Remark 6. Note that the distance between a point v and a set Ω equals the distance between v and the closure of Ω, and Lemmas 1 and 2 still hold even if the convex set considered is boundless. Therefore, the target set in our problem, Ω, which is assumed to be convex, is not required to be bounded or closed.

State agreement, which has been studied for many years (Lin et al., 2005; Moreau, 2005), is a related problem, where there is no target given beforehand. It formulation is relatively simple: since there is no desired set Ω, $\lambda(x_i) \equiv 0$ $(i = 1, \ldots, n)$ in system (6). Therefore, the agent dynamics become

$$\dot{x}_i = \sum_{j\in N_i(\sigma(t))} a_{ij}(x)(x_j - x_i), \quad i = 1, \ldots, n, \tag{8}$$

or in a compact form,

$$\dot{x} = -(L_{\sigma(t)}(x) \otimes I_m)x. \tag{9}$$

Definition 7. Suppose $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$ is the trajectory of system (9) with initial condition $x(t_0) = x^0 = (x_1^0, \ldots, x_n^0)^{\mathrm{T}}$. Then the state agreement for initial condition x^0 is achieved if

$$\lim_{t\to+\infty} [x_i(t) - x_j(t)] = 0, \quad i, j = 1, \ldots, n. \tag{10}$$

Moreover, if (10) holds for any initial condition $x^0 \in R^{mn}$, then we say the global state agreement of system (9) is achieved.

Denote $\Xi \triangleq \{x \in R^{mn} : x_1 = \cdots = x_n\}$. Then the state agreement of system (9) is equivalent to the set attractivity of system (9) with respect to Ξ.

4. Stability analysis

In the following two sections, we focus on the target-aggregation problem. In this section, we first study the set stability and analyze the limit set of multi-agent system (6) for the following set aggregation analysis. Since we consider the system in mn-dimensional space, the convex target set $\Omega \in R^m$ is transformed to $\Omega^n = \Omega \times \cdots \times \Omega \in R^{mn}$ in the study.

Define

$$h(x_i(t)) \triangleq \frac{1}{2} d_\Omega^2(x_i(t)), \quad i \in \mathcal{N} \tag{11}$$

and

$$\hbar(x(t)) \triangleq \max_{i\in\mathcal{N}} \{h(x_i(t))\}, \tag{12}$$

where $\hbar(x(t))$ is clearly locally Lipschitz.

In the following two subsections, we will study the set stability and the properties of the limit sets, respectively.

4.1. Set stability

To study the set stability for system (6), we first give an assumption on each $f_i(x_i, t)$ in (4).

A1. There is a $\mathcal{K}$-class function κ such that

$$\langle x_i - \pi_\Omega(x_i), f_i(x_i, t)\rangle \le -\kappa(d_\Omega^2(x_i)), \quad i \in \mathcal{N}, \tag{13}$$

with d_Ω defined as in Lemma 2, where a function $\kappa : [0, +\infty) \to [0, +\infty)$ is said to be a $\mathcal{K}$-class function if it is continuous, strictly increasing, and $\kappa(0) = 0$.

Remark 8. Assumption A1 is quite mild, which makes the target set Ω "attractive", and can be easily satisfied. For example, if $x_0(t) : [0, +\infty) \to R^m$ is continuous and satisfies $x_0(t) \in \Omega$ for $t \in [0, +\infty)$, we choose $f_i(x_i, t) = x_0(t) - x_i$. Then, by Lemma 1,

$$\begin{aligned}&\langle x_i - \pi_\Omega(x_i), f_i(x_i, t)\rangle \\ &\quad = \langle x_i - \pi_\Omega(x_i), (x_0(t) - \pi_\Omega(x_i)) + (\pi_\Omega(x_i) - x_i)\rangle \\ &\quad \le \langle x_i - \pi_\Omega(x_i), \pi_\Omega(x_i) - x_i\rangle = -d_\Omega^2(x_i),\end{aligned}$$

which obviously satisfies (13).

Then we have the following lemma.

Lemma 9. *$\hbar(x(t))$ is non-increasing along any trajectory of system (6) with Assumption A1.*

Proof. According to Lemmas 1 and 2, we have

$$\nabla d_\Omega^2(x_i) = 2(x_i - \pi_\Omega(x_i)), \tag{14}$$

where π_Ω is the projector onto Ω.

With (14) and Lemma 3,

$$\begin{aligned}D^+\hbar(x(t)) &= \max_{i\in\Upsilon(t)} \frac{\mathrm{d}}{\mathrm{d}t}(h(x_i(t))) \\ &= \max_{i\in\Upsilon(t)} \langle x_i(t) - \pi_\Omega(x_i(t)), \dot{x}_i(t)\rangle \\ &= \max_{i\in\Upsilon(t)} \Bigg\langle x_i - \pi_\Omega(x_i), \sum_{j\in N_i(\sigma)} a_{ij}(x)(x_j - x_i) \\ &\qquad + \lambda(x_i)\chi_i(\sigma(t)) f_i(x_i(t), t)\Bigg\rangle\end{aligned} \tag{15}$$

where $\Upsilon(t)$ denotes the set containing all the agents that reach the maximal distance away from Ω at time t.

For any $t \in [0, +\infty)$, define $\bar{\Omega}(t) \triangleq \{p \mid \frac{1}{2} d_\Omega^2(p) \le \hbar(x(t))\}$, which is also a convex set. Suppose $x_k(t) \in \Upsilon(t)$ and $y \in \bar{\Omega}(t)$, now we claim that for any fixed $t \in [0, +\infty)$, $\langle x_k(t) - \pi_\Omega(x_k(t)), x_k(t) - y\rangle \ge 0$ for any $y \in \bar{\Omega}(t)$. If not so, based on Lemma 2, $y - x_k(t)$ is an increasing direction for d_Ω^2 at point $x_k(t)$, hence $d_\Omega^2(x_k(t)) < d_\Omega^2(y)$, which is a contradiction with $x_k(t) \in \Upsilon(t)$. Therefore,

$$\begin{aligned}&\|2x_k(t) - \pi_\Omega(x_k(t)) - y\|^2 = \|x_k(t) - \pi_\Omega(x_k(t))\|^2 \\ &\qquad + \|x_k(t) - y\|^2 + 2\langle x_k(t) - \pi_\Omega(x_k(t)), x_k(t) - y\rangle \\ &\quad \ge \|x_k(t) - \pi_\Omega(x_k(t))\|^2 + \|x_k(t) - y\|^2.\end{aligned} \tag{16}$$

Thus, as a function of y, $\|2x_k(t) - \pi_\Omega(x_k(t)) - y\|^2$ reaches the minimum when $y = x_k(t)$. Consequently, we have $x_k(t) = \pi_{\bar{\Omega}(t)}(2x_k(t) - \pi_\Omega(x_k(t)))$.

As a result, for any $x_j(t) \in \bar{\Omega}(t)$, again from Lemma 1,

$$\begin{aligned}&\langle x_k(t) - \pi_\Omega(x_k(t)), x_j(t) - x_k(t)\rangle \\ &\quad = \langle (2x_k(t) - \pi_\Omega(x_k(t))) - \pi_{\bar{\Omega}(t)}(2x_k(t) - \pi_\Omega(x_k(t))), \\ &\qquad x_j(t) - \pi_{\bar{\Omega}(t)}(2x_k(t) - \pi_\Omega(x_k(t)))\rangle \le 0.\end{aligned}$$

On the other hand, when $x_k(t)$ is an informed agent,

$$\langle x_k(t) - \pi_\Omega(x_k(t)), f_k(x_k(t), t)\rangle \le -\kappa(d_\Omega^2(x_k(t))) \le 0.$$

Therefore, with $a_{ij}(x) > 0$,

$$\left\langle x_k(t) - \pi_\Omega(x_k(t)), \sum_{j\in N_k(\sigma(t))} a_{kj}(x)(x_j(t) - x_k(t)) + \lambda(x_k)\chi_k(\sigma(t))f_k(x_k(t), t)\right\rangle \le 0. \tag{17}$$

Since $x_k(t)$ is chosen arbitrarily in $\Upsilon(t)$, $D^+\hbar(x(t)) \le 0$ by (15) and (17), which implies the conclusion. □

According to Lemma 9, Ω^n is positively invariant for (6), and $\hbar(x(t)) \le \hbar(x(t_0))$, $\forall t \ge t_0$. Therefore, it is easy to obtain the set stability of system (6):

Theorem 10. *System (6) is uniformly stable with respect to Ω^n with Assumption A1.*

4.2. Limit set

For the dynamical analysis of system (6), two more assumptions are given as follows:

A2. There exist a nonnegative constant γ and a continuous function $c_i(t)$ on $[0, +\infty)$ such that, for any $t \in [0, +\infty)$,

$$\|f_i(x_i, t)\| \le \gamma\|x_i\| + c_i(t), \quad i = 1, \ldots, n. \tag{18}$$

A3. There are $a^* > 0$ and $a_* > 0$ such that

$$a_* \le a_{ij}(x) \le a^*, \quad \forall x \in R^{mn}. \tag{19}$$

Remark 11. (18) in Assumption A2 has been widely used (see Clarke, Ledyaev, Stern, and Wolenski (1998)) in order to avoid finite escape time. Take $c(t) = \max_{i\in\mathcal{N}} c_i(t)$, which is continuous in $[t_0, +\infty)$, and clearly, $\|\dot{x}(t)\| \le (2n\sqrt{n}a^* + \sqrt{n}\gamma)\|x(t)\| + \sqrt{n}c(t)$ from Assumptions A2–A3. Suppose $x(t)$ of system (6) exists only on $[t_0, T)$ for a finite constant $T > 0$, and then, it is boundless on $[t_0, T)$. Set $c^* = \sup_{t\in[t_0,T)} c(t) < \infty$. Thus, by Gronwall's Lemma (see page 179 of Clarke et al. (1998)),

$$\|x(t) - x(t_0)\| \le (e^{(2n\sqrt{n}a^*+\sqrt{n}\gamma)t} - 1)\left(\|x(t_0)\| + \frac{c^*}{2na^* + \gamma}\right)$$

which yields a contradiction. Therefore, the solution of system (6) exists in $[t_0, +\infty)$ with Assumptions A2–A3.

Remark 12. If Ω is bounded, then, for any initial state x_0, according to Theorem 10, there is a compact set $\Omega_0 \supseteq \Omega$ such that Ω_0 is an invariant set of system (6). With the continuity of $a_{ij}(x) > 0$, $i, j = 1, 2, \ldots, n$, there are $a_* > 0$ and $a^* > 0$ such that $a_{ij}(x) \in [a_*, a^*]$, $\forall x \in \Omega_0$, $i, j \in \mathcal{N}$, which plays the same role of A3. Moreover, Theorem 10 also avoids the finite-time escape phenomena of system (6) since its solutions are bounded. Therefore, we can take the following assumption

A4. Ω is bounded

to replace Assumptions A2–A3 in the following analysis.

According to Lemma 9,

$$\lim_{t\to+\infty} \hbar(x(t)) = \hbar^*,$$

for a constant $\hbar^* \ge 0$. Note that system (6) is globally attractive with respect to Ω^n if and only if $\hbar^* = 0$.

To analyze the characteristics of the limit set of system (6), set

$$\ell(x(t)) = \min_{i\in\mathcal{N}}\{h(x_i(t))\}, \qquad \ell^* \triangleq \liminf_{t\to+\infty} \ell(x(t)).$$

Obviously, $0 \le \ell^* \le \hbar^*$.

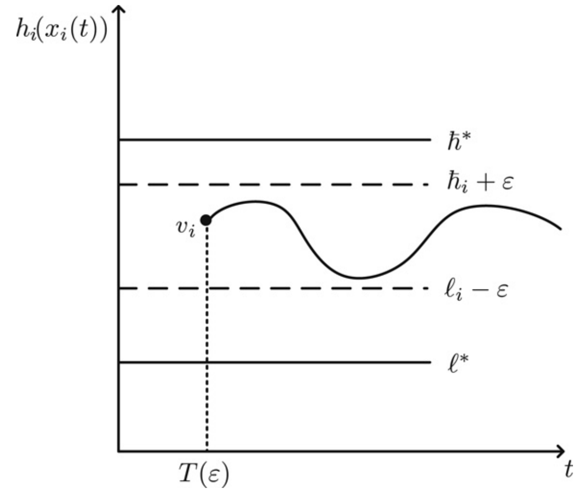

Fig. 2. $h(x_i(t)) \in (\ell_i - \varepsilon, \hbar_i + \varepsilon)$ for $t > T(\varepsilon)$.

Define

$$\hbar_i \triangleq \limsup_{t\to+\infty} h(x_i(t)), \qquad \ell_i \triangleq \liminf_{t\to+\infty} h(x_i(t)),$$

for $i = 1, \ldots, n$. With Lemma 9, $\ell^* \le \ell_i \le \hbar_i \le \hbar^*$, $i = 1, \ldots, n$. Notice that for sufficiently large t, $x_i(t)$ is almost within $(\ell_i, \hbar_i)$. To be strict, for $\forall\varepsilon > 0$, there is $T(\varepsilon) > 0$ such that, when $t > T(\varepsilon)$, (see Fig. 2)

$$h(x_i(t)) \in (\ell_i - \varepsilon, \hbar_i + \varepsilon). \tag{20}$$

Moreover, if $\ell_i < \hbar_i$, for any $\zeta \in (\ell_i, \hbar_i)$, we can find a time sequence $t_1 < t_2 < \cdots < t_k < +\infty$, $\lim_{k\to+\infty} t_k = +\infty$, such that $h(x_i(t_k)) = \zeta$. In other words, $h(x_i(t))$ fills up $(\ell_i, \hbar_i)$ along the trajectory of (6). Here we propose a method to analyze the limit set in order to study the set stability and attractivity. This idea will be used throughout the whole paper, and an important lemma is given first.

Lemma 13. *Suppose $\Omega \subset R^m$ is a convex set and $x_i, x_j \in R^m$. Then*

$$\langle x_i - \pi_\Omega(x_i), x_j - x_i\rangle \le d_\Omega(x_i)|d_\Omega(x_i) - d_\Omega(x_j)|. \tag{21}$$

On the other hand, if $d_\Omega(x_i) > d_\Omega(x_j)$, then

$$\langle x_i - \pi_\Omega(x_i), x_j - x_i\rangle \le -d_\Omega(x_i)(d_\Omega(x_i) - d_\Omega(x_j)). \tag{22}$$

Proof. If $d_\Omega(x_i) = 0$, then $x_i - \pi_\Omega(x_i) = 0$. As a result, both (21) and (22) become trivially satisfied. Hence we only suppose $d_\Omega(x_i) > 0$ in the following.

Define

$$\Omega_i \triangleq \{v | d_\Omega(v) \le d_\Omega(x_i)\}, \qquad \Omega_j \triangleq \{v | d_\Omega(v) \le d_\Omega(x_j)\},$$
$$H_1 \triangleq \{v | \langle x_i - \pi_\Omega(x_i), v - x_i\rangle > 0\}.$$

According to Lemma 2, for any $v \in H_1$, $d_\Omega(x_i) < d_\Omega(v)$. Thus, $H_1 \cap \Omega_i = \emptyset$.

Set

$$\tilde{x}_i \triangleq \pi_\Omega(x_i) + \frac{d_\Omega(x_j)}{d_\Omega(x_i)} \cdot (x_i - \pi_\Omega(x_i))$$
$$H_2 \triangleq \{v | \langle x_i - \pi_\Omega(x_i), v - \tilde{x}_i\rangle > 0\}.$$

Clearly, $\pi_\Omega(\tilde{x}_i) = \pi_\Omega(x_i)$ and we can get $H_2 \cap \Omega_j = \emptyset$ through similar analysis.

If $x_j \in \Omega_j \setminus H_1$, then $\langle x_i - \pi_\Omega(x_i), x_j - x_i\rangle \le 0$ and (21) follows immediately. Moreover, if $x_j \in \Omega_j \cap H_1$, then $x_j \notin H_2$ because $H_2 \cap \Omega_j = \emptyset$. Hence,

$$\langle x_i - \pi_\Omega(x_i), x_j - \tilde{x}_i\rangle \le 0. \tag{23}$$

Therefore, by (23) and the Cauchy–Schwarz Inequality,

$$\begin{aligned}\langle x_i - \pi_\Omega(x_i), x_j - x_i\rangle &\le \langle x_i - \pi_\Omega(x_i), \tilde{x}_i - x_i\rangle \\ &\le \|x_i - \pi_\Omega(x_i)\|\|\tilde{x}_i - x_i\| \\ &= d_\Omega(x_i)|d_\Omega(x_j) - d_\Omega(x_i)|.\end{aligned}$$

On the other hand, if $d_\Omega(x_i) > d_\Omega(x_j)$, it is not hard to see $\tilde{x}_i = \pi_{\Omega_j}(x_i)$, and

$$x_i - \tilde{x}_i = x_i - \pi_{\Omega_j}(x_i) = \frac{d_\Omega(x_i) - d_\Omega(x_j)}{d_\Omega(x_i)}(x_i - \pi_\Omega(x_i)).$$

Then, by Lemma 1,

$$\begin{aligned}&\langle x_i - \pi_\Omega(x_i), x_j - \tilde{x}_i\rangle \\ &= \frac{d_\Omega(x_i)}{d_\Omega(x_i) - d_\Omega(x_j)}\left\langle x_i - \pi_{\Omega_j}(x_i), x_j - \pi_{\Omega_j}(x_i)\right\rangle \le 0 \\ &\langle x_i - \pi_\Omega(x_i), \tilde{x}_i - x_i\rangle = -d_\Omega(x_i)(d_\Omega(x_i) - d_\Omega(x_j)),\end{aligned}$$

which completes the proof. □

Recalling the system topology $\bar{\mathcal{G}}_{\sigma(t)} = (\bar{\mathcal{N}}, \bar{\mathcal{E}}_{\sigma(t)})$ and (1), we denote $\bar{\mathcal{G}}([t, +\infty)) = (\bar{\mathcal{N}}, \cup_{s\in[t,+\infty)} \bar{\mathcal{E}}_{\sigma(s)})$ as the joint topology from t to $+\infty$.

In what follows, we assume either Assumptions A1–A3 or Assumptions A1 and A4 hold (referring to Remark 12).

Theorem 14. *If the joint topology $\bar{\mathcal{G}}([t, +\infty))$ is quasi-strongly connected for any t and $\hbar^* > 0$, then $\ell^* < \hbar^*$.*

Proof. We will prove $\ell^* < \hbar^*$ by contradiction. Suppose $\ell^* = \hbar^*$. Then $\lim_{t\to+\infty} h(x_i(t)) = \hbar^*$, $i = 1, \dots, n$. Therefore, for any $\varepsilon > 0$, there is $T(\varepsilon) > 0$ such that, when $t > T(\varepsilon)$,

$$h(x_i(t)) \in (\hbar^* - \varepsilon, \hbar^* + \varepsilon), \quad i = 1, \dots, n. \tag{24}$$

Since $\bar{\mathcal{G}}([t, +\infty))$ is quasi-strongly connected for any t, there is a sequence

$$T < t_1 < t_2 < \dots < t_p < \cdots, \quad t_{p+1} > t_p + \tau_D, \tag{25}$$

such that, at each time t_p, there is an arc from Ω to an agent node. Since the total number of the agents is finite, we can find a subsequence of (25) as follows

$$T < t_{p_1} < \dots < t_{p_k} < \cdots, \tag{26}$$

such that there is an arc from Ω pointing to a fixed agent (supposed to be v_i) at each time t_{p_k}. With the dwell time assumption, we can also assume that the system topology does not change at time intervals $(t_{p_k}, t_{p_k} + \tau_D)$ for any k. Therefore, $\chi_i(\sigma(t)) = 1$ in each time interval $(t_{p_k}, t_{p_k} + \tau_D)$. Thus, when $t \in (t_{p_k}, t_{p_k} + \tau_D)$, $\forall k$,

$$\begin{aligned}\frac{\mathrm{d}h(x_i)}{\mathrm{d}t} &= \langle x_i - \pi_\Omega(x_i), \dot{x}_i\rangle \\ &= \left\langle x_i - \pi_\Omega(x_i), \sum_{j\in N_i(\sigma)} a_{ij}(x_j - x_i) + f_i(x_i, t)\right\rangle.\end{aligned}$$

Based on Lemma 13,

$$\begin{aligned}\langle x_i - \pi_\Omega(x_i), x_j - x_i\rangle &\le (\sqrt{2(\hbar^* + \varepsilon)} - \sqrt{2(\hbar^* - \varepsilon)})\sqrt{2(\hbar^* + \varepsilon)} \\ &\le 4\varepsilon.\end{aligned} \tag{27}$$

Also, from (13), there is a $\mathcal{K}$-class function κ with $\langle x_i - \pi_\Omega(x_i), f_i(x_i, t)\rangle \le -\kappa(d^2_\Omega(x_i))$, and then

$$\langle x_i - \pi_\Omega(x_i), f_i(x_i, t)\rangle \le -\kappa(2(\hbar^* - \varepsilon)). \tag{28}$$

By (27) and (28),

$$\begin{aligned}\frac{\mathrm{d}h(x_i)}{\mathrm{d}t} &\le -\kappa(2(\hbar^* - \varepsilon)) + 4\varepsilon \cdot \sum_{j\in N_i(\sigma(t))} a_{ij}(x) \\ &\le -\kappa(2(\hbar^* - \varepsilon)) + 4(n-1)a^*\varepsilon.\end{aligned}$$

Therefore, we can choose ε sufficiently small to render

$$-\kappa(2(\hbar^* - \varepsilon)) + 4(n-1)a^*\varepsilon < -\kappa(\hbar^*),$$

which implies

$$\frac{\mathrm{d}h(x_i(t))}{\mathrm{d}t} \le -\kappa(\hbar^*), \quad \forall t \in (t_{p_k}, t_{p_k} + \tau_D), \forall k.$$

Then, for any t_{p_k},

$$h(x_i(t_{p_k} + \tau_D)) \le h(x_i(t_{p_k})) - \kappa(\hbar^*)\tau_D. \tag{29}$$

Choose ε even smaller, if necessary, to make $\kappa(\hbar^*)\tau_D > 2\varepsilon$, and then (29) contradicts (24). Thus, the conclusion holds. □

Theorem 14 will be used in the set attractivity of multi-agent analysis.

5. Aggregation convergence

In this section, we will consider the set aggregation to the target Ω^n based on the analysis technique proposed in the last section, though the dynamical behavior of system (6) is complicated with variable topologies. Based on Remark 12, the Assumption A1, along with either A4 or A2–A3, is assumed to hold throughout this section.

If $\bar{\mathcal{G}}([t, +\infty))$ is not quasi-strongly connected for some t, then (6) is not globally attractive with respect to Ω^n. In fact, if we can find a $T > 0$ such that $\bar{\mathcal{G}}([T, +\infty))$ is not quasi-strongly connected, then it is easy to find a subgraph $\mathcal{G}_1 \neq \emptyset$ of $\bar{\mathcal{G}}([T, +\infty))$ such that Ω is not in $\mathcal{G}_1$ and no arc entering $\mathcal{G}_1$. Therefore, agents in $\mathcal{G}_1$ will not be influenced by Ω when $t > T$, which implies that agents in $\mathcal{G}_{i_0}$ with some initial conditions may not converge to Ω. Therefore, the joint connection on $[t, \infty)$ is necessary to secure the convergence to the target set.

In the following, we will give sufficient convergence conditions for some important cases.

5.1. Uniform connection

In this subsection, we consider the condition when $\bar{\mathcal{G}}(\sigma(t))$ is uniformly quasi-strongly connected. In fact, the condition, "uniformly quasi-strongly connected", has been widely studied in multi-agent networks (for example, see Jadbabaie et al. (2003), Hong et al. (2007) and Lin et al. (2007)).

The next result shows that the aggregation to a given target can also be achieved under this condition.

Theorem 15. *System (6) is globally attractive with respect to Ω^n if $\bar{\mathcal{G}}(\sigma(t))$ is uniformly quasi-strongly connected.*

Proof. For any $\varepsilon > 0$ and any $\mathcal{K}$-class function κ_0, there exists $T(\varepsilon) > 0$ such that, when $t > T(\varepsilon)$,

$$h(x_i(t)) \in [0, \hbar^* + \kappa_0(\varepsilon)), \quad \forall i \in \mathcal{N}. \tag{30}$$

Note that for any $\mathcal{K}$-class function κ_0, if, for any $\varepsilon > 0$, there is $T^* > 0$ such that

$$h(x_i(t)) \in (\hbar^* - \kappa_0(\varepsilon), \hbar^* + \kappa_0(\varepsilon)), \quad \forall t > T^*, i \in \mathcal{N}, \tag{31}$$

then $\lim_{t\to+\infty} h(x_i(t)) = \hbar^*$, $i = 1, \dots, n$. Thus, $\hbar^* = 0$ holds from Theorem 14.

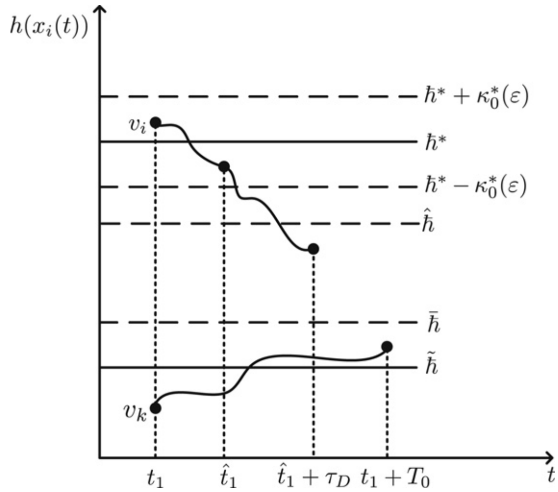

Fig. 3. $\mathcal{N}_1^\varepsilon(t_1) \supseteq \mathcal{N}_1^\varepsilon(\hat{t}_1 + \tau_D)$ and $v_i \in \mathcal{N}_1^\varepsilon(t_1) \setminus \mathcal{N}_1^\varepsilon(\hat{t}_1 + \tau_D)$.

On the other hand, there is a $\mathcal{K}$-class function κ_0^* such that, for

$\mathcal{N}_1^\varepsilon(t) \triangleq \{i \in \mathcal{N} \mid h(x_i(t)) \in (\hbar^* - \kappa_0^*(\varepsilon), \hbar^* + \kappa_0^*(\varepsilon))\}$,

we have

$$\forall \varepsilon > 0,\ \forall \tilde{T} > 0, \exists t_1 > \tilde{T} \quad s.t. \quad \mathcal{N}_1^\varepsilon(t_1) \neq \mathcal{N},$$

$$\text{and } h(x_i(t_1)) \le \tilde{\hbar}, \forall i \in \mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t_1) \tag{32}$$

for a constant $\tilde{\hbar} < \hbar^*$. Due to (30), $\mathcal{N}_1^\varepsilon(t)$ will not be empty for any $t > T$. Then we claim, if $\hbar^* > 0$, we can find a finite time sequence such that $\mathcal{N}_1^\varepsilon(t)$ is strictly decreasing when ε is sufficiently small (see Fig. 3).

Without loss of generality, suppose $t_1 > T$. Then, based on Lemma 13, $\forall v_k \in \mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t_1)$, when $t \in (t_1, t_1 + T_0)$,

$$\begin{aligned} \frac{\mathrm{d}h(x_k(t))}{\mathrm{d}t} &= \Big\langle x_k - \pi_\Omega(x_k), \sum_{j \in N_k(\sigma(t))} a_{kj}(x_j - x_k) \\ &\quad + \lambda(x_k)\chi_k(\sigma(t))f_k(x_k, t) \Big\rangle \\ &\le -\lambda(x_k)\chi_k(\sigma(t))\kappa(2h(x_k(t))) + (n-1)a^*\sqrt{2h(x_k(t))} \\ &\quad \times \left(\sqrt{2(\hbar^* + \kappa_0^*(\varepsilon))} - \sqrt{2h(x_k(t))}\right) \\ &\le -2(n-1)a^* h(x_k(t)) \\ &\quad + 2(n-1)a^*\sqrt{h(x_k(t))(\hbar^* + \kappa_0^*(\varepsilon))}, \end{aligned}$$

or equivalently,

$$\frac{\mathrm{d}\sqrt{h(x_k(t))}}{\mathrm{d}t} \le (n-1)a^*\left(\sqrt{\hbar^* + \kappa_0^*(\varepsilon)} - \sqrt{h(x_k(t))}\right)$$

which implies that, for sufficiently small ε,

$$h(x_k(t_1 + T_0)) \le \bar{\hbar} \tag{33}$$

with a constant $\bar{\hbar} < \hbar^* - \kappa_0^*(\varepsilon)$. Since $\bar{\mathcal{G}}([t_1, t_1 + T_0])$ is quasi-strongly connected, there has to be a globally reachable node in $\bar{\mathcal{G}}^*([t_1, t_1 + T_0])$, which could only be Ω. Thus, in $\bar{\mathcal{G}}([t_1, t_1 + T_0])$, there has to be an arc from a node in $\mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t_1)$ or Ω and entering a node in $\mathcal{N}_1^\varepsilon(t_1)$ at some time $\hat{t}_1 \in [t_1, t_1 + T_0]$. Therefore, there are two cases:

(1) If $v_i \in \mathcal{N}_1^\varepsilon(t_1)$ has a neighbor in $\mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t_1)$ at time $\hat{t}_1$, by $a_* \le a_{ij}(x(t)), \forall i, j \in \mathcal{N}$, during $t \in (\hat{t}_1, \hat{t}_1 + \tau_D)$, we have

$$\begin{aligned} \frac{\mathrm{d}h(x_i(t))}{\mathrm{d}t} &\le \Big\langle x_i - \pi_\Omega(x_i), \sum_{j \in N_i(\sigma(t)) \cap \mathcal{N}_1^\varepsilon(t_1)} a_{ij}(x)(x_j - x_i) \\ &\quad + \sum_{j \in N_i(\sigma(t)) \setminus \mathcal{N}_1^\varepsilon(t_1)} a_{ij}(x)(x_j - x_i) \Big\rangle \\ &\le (n-2)a^*\sqrt{2h(x_i(t))}\left(\sqrt{2(\hbar^* + \kappa_0^*(\varepsilon))} - \sqrt{2h(x_i(t))}\right) \\ &\quad - a_*\sqrt{2h(x_i(t))} \cdot \left(\sqrt{2h(x_i(t))} - \sqrt{2\bar{\hbar}}\right) \end{aligned}$$

which yields that, for sufficiently small ε,

$$h(x_i(\hat{t}_1 + T_0)) \le \hat{\hbar} \tag{34}$$

with some constant $\hat{\hbar} \in [\bar{\hbar}, \hbar^* - \kappa_0^*(\varepsilon))$.

(2) If Ω is a neighbor of $v_i \in \mathcal{N}_1^\varepsilon(t_1)$ at $\hat{t}_1$, when $t \in (\hat{t}_1, \hat{t}_1 + \tau_D)$, we obtain

$$\begin{aligned} \frac{\mathrm{d}h(x_i(t))}{\mathrm{d}t} &\le (n-1)a^*\sqrt{2h(x_i(t))}\Big(\sqrt{2(\hbar^* + \kappa_0^*(\varepsilon))} \\ &\quad - \sqrt{2h(x_i(t))}\Big) - \kappa(h(x_i(t))). \end{aligned} \tag{35}$$

With (35), there is a constant $\hat{\hbar}'$ in $(0, \hbar^* - \kappa_0^*(\varepsilon))$ (still denote $\hat{\hbar}'$ as $\hat{\hbar}$ for simplicity in what follows), for sufficiently small ε, we also have

$$h(x_i(\hat{t}_1 + \tau_D)) \le \hat{\hbar}. \tag{36}$$

Denote $t_2 = \hat{t}_1 + \tau_D$. Based on (33), (34) and (36), we obtain

$$\mathcal{N}_1^\varepsilon(t_1) \supseteq \mathcal{N}_1^\varepsilon(t_2) \quad \text{and} \quad v_i \in \mathcal{N}_1^\varepsilon(t_1) \setminus \mathcal{N}_1^\varepsilon(t_2). \tag{37}$$

Regarding t_2 as t_1 and through similar analysis, we can find $t_3 > t_2 + \tau_D$ such that

$$\mathcal{N}_1^\varepsilon(t_2) \supseteq \mathcal{N}_1^\varepsilon(t_3) \text{ and } \mathcal{N}_1^\varepsilon(t_2) \setminus \mathcal{N}_1^\varepsilon(t_3) \neq \emptyset. \tag{38}$$

Repeating the upper process yields a time sequence

$$\max\{T, \tilde{T}\} < t_1 < t_2 < \cdots < t_k$$

such that

$$\mathcal{N}_1^\varepsilon(t_i) \supseteq \mathcal{N}_1^\varepsilon(t_{i+1}) \quad \text{and} \quad \mathcal{N}_1^\varepsilon(t_i) \setminus \mathcal{N}_1^\varepsilon(t_{i+1}) \neq \emptyset, \tag{39}$$

until $\mathcal{N}_1^\varepsilon(t_k) = \emptyset$, which leads to a contradiction. Thus, $\hbar^* = 0$ and the proof is completed. □

5.2. Discussions

Uniformly quasi-strong connectedness is important in the multi-agent coordination, but it is not necessary for the global attractivity with respect to Ω for system (6). In fact, there are other cases to guarantee the target aggregation without requiring uniformly weak connectedness. Here we study the attractivity from a different viewpoint.

Denote $\mathcal{N}_1 \triangleq \{i \in \mathcal{N} \mid \lim_{t \to +\infty} h(x_i(t)) = \hbar^*\}$. $\mathcal{N}_1 = \emptyset$ can be found in some cases with switching interaction digraphs. These time-varying digraphs, with well-designed switching strategies, may make the several agents attract one another in turn (not at the same time) and lead to the oscillating motion of the agents, which fails target aggregation.

However, the multi-agent system may also fail to converge to the target if $\mathcal{N}_1 \neq \emptyset$. Here is a simple example.

Example 1. Consider a multi-agent system in the form of (4), composed of two agents $x_1, x_2 \in R$ and a target region $\Omega = \{x_0\}$ with $x_0 \equiv 0 \in R$:

$$\begin{cases} \dot{x}_1(t) = -\chi_{12}(\sigma(t))(x_2 - x_1) - \chi_{10}(\sigma(t))x_1 \\ \dot{x}_2(t) = -\chi_{21}(\sigma(t))(x_2 - x_1) - \chi_{20}(\sigma(t))x_2. \end{cases} \tag{40}$$

$\bar{\mathcal{G}}(\sigma(t)) = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$ is the topology of this multi-agent system, with $\mathcal{V} = \{x_0, x_1, x_2\}$ and $\chi_{ij}(\sigma)$ defined in (5).

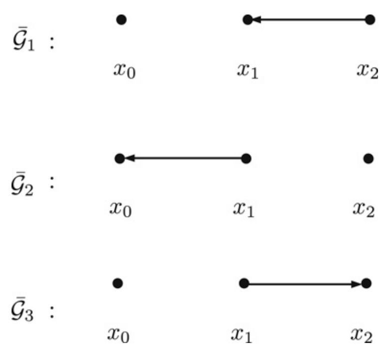

Fig. 4. Three possible topologies.

Denote $\bar{\mathcal{P}}$ as the set of all the possible interaction graphs, where $\bar{\mathcal{P}} = \{\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \bar{\mathcal{G}}_3\}$ with $\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \bar{\mathcal{G}}_3$ as shown in Fig. 4.

Set initial conditions $x_1(t_0) = \frac{1}{2}, x_2(t_0) = 1$, and $\bar{\mathcal{G}}(\sigma(t_0)) = \bar{\mathcal{G}}_1$. Define the signal $\sigma(t)$ by induction as follows: Set $m = 0$ in the beginning (that is $t = t_0$), and then we repeat the following steps:

S1. Once $x_2(t) = 1 - \sum_{k=0}^{k=m} (\frac{1}{2})^{k+2}$ at some time denoted as t_{3m+1}, we change the interconnection by setting $\bar{\mathcal{G}}(\sigma(t)) = \bar{\mathcal{G}}_2$ at $t = t_{3m+1}$. Then go to S2.

S2. Once $x_1(t) = \frac{1}{4}$ at some moment denoted as t_{3m+2}, we change the topology again by re-setting $\bar{\mathcal{G}}(\sigma(t)) = \bar{\mathcal{G}}_3$ at $t = t_{3m+2}$. Then go to S3.

S3. Once $x_1(t) = \frac{1}{2}$ at some moment denoted as t_{3m+3}, we change the interaction structure by setting $\bar{\mathcal{G}}(\sigma(t)) = \bar{\mathcal{G}}_1$ at $t = t_{3m+3}$. Let $m = m + 1$ and return to S1.

With the above procedure, we get a time sequence $t_0 < t_1 < \cdots < t_k \cdots$, and this switching signal has a dwell time no shorter than $\frac{1}{2}$. Clearly, Ω (that is, x_0) is just not the neighbor of x_2 in the joint topology $\bar{\mathcal{G}}([t, +\infty))$ for any $t > 0$. Then, $\limsup_{t\to+\infty} x_1(t) = \lim_{t\to+\infty} x_2(t) = \frac{1}{2}$ and $\liminf_{t\to+\infty} x_1(t) = \frac{1}{4}$. Thus, $\mathcal{N}_1 = \{x_2\} \neq \emptyset$, and system (40) is not attractive to Ω. $\square$

The following shows a case when the target aggregation can be achieved with $\mathcal{N}_1 \neq \emptyset$.

Theorem 16. *Suppose $\mathcal{N}_1 \neq \emptyset$. Then system (6) is globally attractive with respect to Ω^n if Ω is a neighbor of every agent in the joint topology $\bar{\mathcal{G}}([t, +\infty))$ for any t.*

Proof. As it was shown before, the attractivity of system (6) is equivalent to $\hbar^* = 0$. Suppose $\hbar^* > 0$. Then, for any $\varepsilon > 0$, there is $T(\varepsilon) > 0$ such that, if $t > T(\varepsilon)$,

$$h(x_i(t)) \in (\hbar^* - \varepsilon, \hbar^* + \varepsilon), \quad \forall i \in \mathcal{N}_1, \tag{41}$$

and

$$h(x_i(t)) \in [0, \hbar^* + \varepsilon), \quad \forall i \in \mathcal{N} \setminus \mathcal{N}_1. \tag{42}$$

Moreover, if Ω is a neighbor of every agent in the joint topology $\bar{\mathcal{G}}([t, +\infty))$ for any t, then we can find a time sequence

$$T < \tilde{t}_1 < \cdots < \tilde{t}_p < \cdots, \quad \tilde{t}_{p+1} > \tilde{t}_p + \tau_D$$

such that, for any $\tilde{t}_p$, there is an arc from Ω to some agent in $\mathcal{N}_1$ at $t = \tilde{t}_p$. Note that the number of agents in $\mathcal{N}_1$ is finite, and then we can select a subsequence

$$T < \tilde{t}_{p_1} < \cdots < \tilde{t}_{p_l} < \cdots$$

of $\{\tilde{t}_p\}$ such that there is an arc (v_0, v_k), from Ω (that is, v_0) to a fixed node $v_k \in \mathcal{N}_1$, at each moment $\tilde{t}_{p_l}$. Without loss of generality, we assume that the system topology does not change in time interval $(\tilde{t}_{p_l}, \tilde{t}_{p_l} + \tau_D)$. When $t \in (\tilde{t}_{p_l}, \tilde{t}_{p_l} + \tau_D)$,

$$\begin{aligned}\langle x_k(t) - \pi_\Omega(x_k(t)), f_k(x_k(t), t)\rangle &\leq -\kappa(d_\Omega^2(x_k(t))) \\ &\leq -\kappa(2(\hbar^* - \varepsilon))\end{aligned} \tag{43}$$

and, by Lemma 13,

$$\begin{aligned}&\left\langle x_k - \pi_\Omega(x_k), \sum_{j\in N_k(\sigma(t))} a_{kj}(x)(x_j - x_k)\right\rangle \\ &\leq \sum_{j\in N_k(\sigma(t))} 2a_{kj}(x)\sqrt{\hbar^* + \varepsilon}(\sqrt{\hbar^* + \varepsilon} - \sqrt{\hbar^* - \varepsilon}) \\ &\leq 4(n-1)a^*\varepsilon.\end{aligned}$$

As ε is sufficiently small,

$$-\kappa(2(\hbar^* - \varepsilon)) + 4(n-1)a^*\varepsilon \leq -\kappa(\hbar^*).$$

Thus, for any $\tilde{t}_{p_l}$,

$$\frac{\mathrm{d}h(x_k(t))}{\mathrm{d}t} \leq -\kappa(\hbar^*), t \in (\tilde{t}_{p_l}, \quad \tilde{t}_{p_l} + \tau_D).$$

Namely, $h(x_k(\tilde{t}_{p_l} + \tau_D)) \leq h(x_k(\tilde{t}_{p_l})) - \kappa(\hbar^*)\tau_D$, which contradicts (41) when $\kappa(\hbar^*)\tau_D > 2\varepsilon$. As a result, the assumption $\hbar^* > 0$ is not true, and then the conclusion follows. $\square$

5.3. Bidirectional topology

Here we will consider a special digraph, bidirectional graph, to describe the switching communication topology. In this case, x_i is a neighbor of x_j if and only if x_j is a neighbor of x_i, but the weight of arc (x_i, x_j) may not be equal to that of arc (x_j, x_i) (Moreau, 2005). Conventionally, if the weight of arc (x_i, x_j) equals that of arc (x_j, x_i), then the considered graph is undirected and its Laplacian ($L_{\sigma(t)}(x)$ in (6)) becomes symmetric. Obviously, an undirected topology is a special case of bidirectional topologies, but a bidirectional communication topology may not be accurately described by an undirected graph.

Here is the main result of this subsection.

Theorem 17. *System (6) with switching bidirectional topologies is globally attractive with respect to Ω^n if and only if its joint topology $\bar{\mathcal{G}}([t, +\infty))$ is connected for any t.*

Proof (*Sufficient Part*). For any $\varepsilon > 0$ and any $\mathcal{K}$-class function κ_0, there is $T(\varepsilon) > 0$ such that, when $t > T(\varepsilon)$,

$$h(x_i(t)) \in [0, \hbar^* + \kappa_0(\varepsilon)), \quad i = 1, \ldots, n. \tag{44}$$

Moreover, if, for any $\mathcal{K}$-class function κ_0 and $\varepsilon > 0$, there is $T^* > 0$ such that

$$h(x_i(t)) \in (\hbar^* - \kappa_0(\varepsilon), \hbar^* + \kappa_0(\varepsilon)), \quad \forall t > T^*, \tag{45}$$

then $\lim_{t\mapsto+\infty} h(x_i(t)) = \hbar^*$, $\forall i \in \mathcal{N}$. Furthermore, a connected bidirectional graph is certainly quasi-strongly connected. Thus, $\hbar^* = 0$ holds from Theorem 14.

Additionally, there exists a $\mathcal{K}$-class function κ_0^* such that, for

$$\mathcal{N}_1^\varepsilon(t) \triangleq \{i \in \mathcal{N} \mid h(x_i(t)) \in (\hbar^* - \kappa_0^*(\varepsilon), \hbar^* + \kappa_0^*(\varepsilon))\},$$

we have

$$\forall \varepsilon > 0, \forall \tilde{T} > 0, \exists t_1 > \tilde{T} \quad s.t. \quad \mathcal{N}_1^\varepsilon(t_1) \neq \mathcal{N}$$

$$\text{and } h(x_i(t_1)) \leq \tilde{\hbar}, \forall i \in \mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t_1) \tag{46}$$

for some $\tilde{\hbar} < \hbar^*$. Due to (44), $\mathcal{N}_1^\varepsilon(t)$ will not be empty for any $t > T$. Then we claim, if $\hbar^* > 0$, $\mathcal{N}_1^\varepsilon(t)$ is non-increasing when ε is sufficiently small and t is sufficiently large (see Fig. 5).

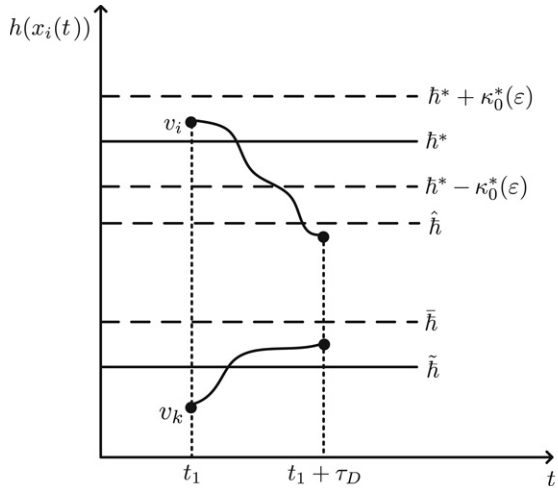

Fig. 5. $\mathcal{N}_1^\varepsilon(t)$ is non-increasing for small ε and large t.

Without loss of generality, suppose the interaction topology does not change in $(t_1, t_1 + \tau_D)$. If there is no edge between $\mathcal{N}_1^\varepsilon(t_1)$ and $\mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t_1)$, applying Lemma 9 to the subsystem formed by agents in $\mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t_1)$ gives

$$h(x_i(t)) \le \tilde{\hbar}, \quad \forall i \in \mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t_1), \quad \forall t \in (t_1, t_1 + \tau_D).$$

Thus, $\mathcal{N}_1^\varepsilon(t_1) \supseteq \mathcal{N}_1^\varepsilon(t), \quad \forall t \in (t_1, t_1 + \tau_D)$.

Also, there is an edge between $\mathcal{N}_1^\varepsilon(t_1)$ and $\mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t_1)$ when $t \in (t_1, t_1 + \tau_D)$. With $a^* \ge a_{ij}(x(t)), \forall i, j \in \mathcal{N}$ and Lemma 13, for any $v_k \in \mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t_1)$, when $t \in (t_1, t_1 + \tau_D)$,

$$\begin{aligned}\frac{\mathrm{d}h(x_k(t))}{\mathrm{d}t} &\le -\lambda(x_k)\chi_k(\sigma(t))\kappa(2h(x_k(t))) + (n-1)a^*\sqrt{2h(x_k(t))} \\ &\quad \times \left(\sqrt{2(\hbar^* + \kappa_0^*(\varepsilon))} - \sqrt{2h(x_k(t))}\right) \\ &\le -2(n-1)a^* h(x_k(t)) \\ &\quad + 2(n-1)a^*\sqrt{h(x_k(t))(\hbar^* + \kappa_0^*(\varepsilon))}. \end{aligned} \tag{47}$$

Therefore, for sufficiently small ε,

$$h(x_k(t)) \le \bar{\hbar}, \quad \forall t \in (t_1, t_1 + \tau_D), \tag{48}$$

with a constant $\bar{\hbar} < \hbar^* - \kappa_0^*(\varepsilon)$.

Furthermore, assuming $v_i \in \mathcal{N}_1^\varepsilon(t_1)$ is connected with some agent in $\mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t_1)$,

$$\begin{aligned}\frac{\mathrm{d}h(x_i(t))}{\mathrm{d}t} &\le \Bigg\langle x_i - \pi_\Omega(x_i), \sum_{j \in N_i(\sigma(t)) \cap \mathcal{N}_1^\varepsilon(t_1)} a_{ij}(x) \\ &\quad \times (x_j - x_i) + \sum_{j \in N_i(\sigma(t)) \setminus \mathcal{N}_1^\varepsilon(t_1)} a_{ij}(x)(x_j - x_i) \Bigg\rangle \\ &\le 2(n-2)a^*\sqrt{h(x_i(t))}\left(\sqrt{\hbar^* + \kappa_0^*(\varepsilon)} - \sqrt{h(x_i(t))}\right) \\ &\quad - 2a_*\sqrt{h(x_i(t))}\left(\sqrt{h(x_i(t))} - \sqrt{\bar{\hbar}}\right)\end{aligned}$$

which implies that, for sufficiently small ε,

$$h(x_i(t_1 + \tau_D)) \le \hat{\hbar} \tag{49}$$

with a constant $\hat{\hbar} \in [\bar{\hbar}, \hbar^* - \kappa_0^*(\varepsilon))$.

Based on (48) and (49), for any $t \in (t_1, t_1 + \tau_D)$,

$$\mathcal{N}_1^\varepsilon(t_1) \supseteq \mathcal{N}_1^\varepsilon(t), \quad v_i \in \mathcal{N}_1^\varepsilon(t_1) \setminus \mathcal{N}_1^\varepsilon(t_1 + \tau_D). \tag{50}$$

Therefore, $\mathcal{N}_1^\varepsilon(t)$ is non-increasing for sufficiently small ε when $t > t_1$. Moreover, (50) yields that $\mathcal{N}_1^\varepsilon(t) \supseteq \mathcal{N}_1^\varepsilon(t + \tau_D)$ and $\mathcal{N}_1^\varepsilon(t) \setminus \mathcal{N}_1^\varepsilon(t + \tau_D) \ne \emptyset$ once there is an edge between $\mathcal{N}_1^\varepsilon(t)$ and $\mathcal{N} \setminus \mathcal{N}_1^\varepsilon(t)$ for a sufficiently small ε when $t > t_1$.

Furthermore, since the joint topology $\bar{\mathcal{G}}([t, +\infty))$ is connected for any t and $\mathcal{N}_1^\varepsilon(t)$ is non-increasing, there is a sequence

$$t_1 < \bar{t}_1 < \bar{t}_2 < \cdots < \bar{t}_k < \cdots;$$

such that there exists an edge between $\mathcal{N}_1^\varepsilon(\bar{t}_k)$ and $\mathcal{N} \setminus \mathcal{N}_1^\varepsilon(\bar{t}_k)$ for any k, which will lead to $\mathcal{N}_1^\varepsilon(\bar{t}_k) = \emptyset$ for $k > n$. However, from (44), $\mathcal{N}_1^\varepsilon(t)$ cannot be empty for $t > T$, which is a contradiction. As a result, the assumption $\hbar^* > 0$ is not true and the conclusion follows.

(Necessary part): Suppose $\bar{\mathcal{G}}([T_0, +\infty))$ is not connected for some T_0. Then $\bar{\mathcal{G}}([T_0, +\infty))$ will have at least two connected components, and at least one component does not contain Ω. Consequently, the agents in the component without connecting Ω after $t > T_0$ will not converge to Ω for some initial conditions. □

In fact, if the initial position of each agent located in Ω, then the agent will never leave Ω^n, or in other words, it always "converges" to Ω without requiring any assumptions. Therefore, the necessary condition on the interconnection topology of Theorem 17 is required to ensure the "global" attractivity for all initial conditions.

6. State agreement

In the preceding sections, we showed that each agent of multi-agent system (4) can reach its target Ω based on some connectivity assumptions. In this section, we consider the case when there is no target set Ω (and therefore, there is no leader). Then the considered multi-agent dynamics becomes (9). Similar to the above sections on target aggregation, we will also investigate the stability and convergence of the multi-agent coordination, including the general digraph case and bidirectional graph case. The analysis method developed in the last sections will be also used similarly in the state-agreement analysis, and therefore, some analysis details are omitted.

First of all, we consider the set stability of system (9). Suppose K is a convex set in R^m, and $x(t)$ is the trajectory of system (9) with initial condition $x(t_0) = x^0 = (x_1^0, \ldots, x_n^0)^{\mathrm{T}}$. Denote

$$h_K(x_i(t)) \triangleq \inf\{\|x_i(t) - y\|^2, y \in K\}, \quad i \in \mathcal{N},$$

and $\hbar_K(x(t)) \triangleq \max_{i \in \mathcal{N}} h_K(x_i(t))$. Obviously, $\hbar_K(x(t)) = 0$ if and only if all the agents are in the closure of K.

The next result can be proved by the same method in the proof of Lemma 9 (by taking $f_i \equiv 0$).

Lemma 18. *For any convex set K, $\hbar_K(x(t))$ is non-increasing along any trajectory of system (9). Moreover,*

$$\lim_{t \to +\infty} \hbar_K(x(t)) = \hbar_K^*$$

for some constant $\hbar_K^* \ge 0$.

Consider a convex set $co\{x_1^0, \ldots, x_n^0\}$, a polytope formed by $x_1^0, \ldots, x_n^0$. According to Lemma 18,

$$\begin{aligned} x(t) \in (co\{x_1^0, \ldots, x_n^0\})^n &\triangleq co\{x_1^0, \ldots, x_n^0\} \\ &\quad \times \cdots \times co\{x_1^0, \ldots, x_n^0\}, \quad \forall t > t_0. \end{aligned} \tag{51}$$

Therefore, for any convex set $K \subseteq R^m$, $x(t) \in K^n \subseteq R^{mn}$ once $x(t_0) \in K^n$, which implies K^n is a positively invariant set of system (9). Based on Lemma 18, we can easily obtain the following result.

Corollary 19. *System (9) is uniformly stable with respect to set $\Xi = \{x \in R^{mn} : x_1 = \cdots = x_n\}$.*

Remark 20. Note that $(co\{x_1^0, \ldots, x_n^0\})^n$ is always compact for any initial condition $x(t_0) = x^0 = (x_1^0, \ldots, x_n^0)^{\mathrm{T}}$. Therefore, (51) ensures that any trajectory of system (9) cannot tend to infinity in finite time. Moreover, from (51), we can obtain that, for any trajectory $x(t)$ of system (9), there are $a^* > 0$ and $a_* > 0$ such that

$$a_* \le a_{ij}(x(t)) \le a^*, \quad \forall t > t_0, i, j \in \mathcal{N}. \tag{52}$$

Therefore, Assumption A3 given in Section 4 is not needed in the study of state agreement (referring to Lin et al. (2007)).

To study the state agreement of system (9), we give a definition to facilitate the following analysis.

Definition 21. Let K be a convex set in R^m, and $x(t)$ be the trajectory of system (9) with initial condition $x(t_0) = x^0$. The set agreement for $x(t_0) = x^0$ with respect to K is achieved if

$$\lim_{t\to+\infty} [h_K(x_i(t)) - h_K(x_j(t))] = 0, \quad \forall i, j \in \mathcal{N}. \tag{53}$$

Moreover, the global set agreement of system (9) with respect to K is achieved if (53) holds for any initial condition $x^0 \in R^{mn}$.

Note that the set agreement with respect to a convex set K can also be viewed as an extension of the target aggregation with respect to K (where $\lim_{t\to+\infty} h_K(x_i(t)) = 0$). In fact, we will apply the approach developed for target aggregation to the analysis of set agreement.

Let us consider the set agreement with the joint-connection assumptions. Denote the switching system topology of system (9) as $\mathcal{G}(\sigma(t)) = (\mathcal{N}, \mathcal{E}_{\sigma(t)})$. Take $\ell_K(x(t)) \triangleq \min_{i\in\mathcal{N}} h_K(x_i(t)) \ge 0$. Then, $\liminf_{t\to+\infty} \ell_K(x(t)) = \ell_K^*$ for a nonnegative constant ℓ_K^*. Clearly, the set agreement of system (9) is equivalent to $\hbar_K^* = \ell_K^*$.

Then we consider the set agreement for two important cases in the following two lemmas.

Lemma 22. *For any convex set $K \subseteq R^m$, system (9) will achieve the global set agreement with respect to K if the switching topology $\mathcal{G}(\sigma(t))$ is uniformly quasi-strongly connected.*

Proof. We only need to prove $\hbar_K^* = \ell_K^*$. Suppose $\hbar_K^* > \ell_K^*$. Similar to the proof of Theorem 15, if the switching topology $\mathcal{G}(\sigma(t))$ is uniformly quasi-strongly connected, then there is a set

$$\mathcal{N}_1^{\varepsilon}(t) \triangleq \{i \in \mathcal{N} \mid h_K(x_i(t)) \in (\hbar_K^* - \kappa_0^*(\varepsilon), \hbar_K^* + \kappa_0^*(\varepsilon))\}$$

for some $\mathcal{K}$-class function κ_0^*, such that a finite time sequence $\{t_i, i = 1, \ldots, k\}$ can be found and $\mathcal{N}_1^{\varepsilon}(t_i)$ is a strictly decreasing sequence when ε is sufficiently small. This will lead to $\mathcal{N}_1^{\varepsilon}(t_k) = \emptyset$, which yields a contradiction. Thus, the conclusion holds. □

Lemma 23. *For any convex set $K \subsetneqq R^m$, system (9) with bidirectional topology will achieve the global set agreement with respect to K if and only if the joint topology $\mathcal{G}([t, +\infty))$ is connected for any t.*

Its proof can be easily done by recalling the proofs of Theorem 17 and Lemma 22.

It is easy to see that the state agreement of system (9) implies the set agreement of system (9) with respect to any convex set. The following result shows the relationship between the state agreement and the set agreement.

Lemma 24. *The global state agreement of system (9) is achieved if and only if the global state agreement is achieved with respect to any convex set.*

Our idea to study the state agreement problems is as follows: at first, we investigate the set agreement problems using the proposed approach given in Sections 4 and 5. Then, with Lemma 24, we can easily extend the results obtained for set agreement to state agreement. In other words, the proposed approach in the preceding sections can be directly applied to the state agreement.

Combining Lemmas 24 and 22 gives a result with our approach for uniformly connected case, which was obtained in Lin et al. (2007).

Theorem 25. *System (9) will achieve the global state agreement if the switching topology $\mathcal{G}(\sigma(t))$ is uniformly quasi-strongly connected.*

Similarly, based on Lemmas 24 and 23, we obtain

Theorem 26. *System (9) with bidirectional topology will achieve the global state agreement if and only if the joint topology $\mathcal{G}([t, +\infty))$ is connected for any t.*

In Moreau (2005), the state agreement problem for multi-agent system described by discrete-time dynamics was studied. It is not hard to find that Corollary 19, Theorems 25 and 26 are consistent with Theorems 1–3 of Moreau (2005), respectively.

7. Conclusions

This paper addressed multi-agent coordination problems with the intra-agent communication topologies described by switching jointly-connected digraphs in $[t, \infty)$. The group of agents of first-order dynamics was shown to achieve target aggregation or state agreement in important cases based on non-smooth analysis. The results on state agreement obtained using the proposed approach are consistent with related existing results (e.g., in Jadbabaie et al. (2003), Lin et al. (2007) and Moreau (2005)), and the target-aggregation results may further provide strict analysis for some practical problems (e.g., in Couzin et al. (2005)). In the study of the (set) stability and convergence of these networked agents, an approach based on limit-set analysis was given. With this approach, some other coordination problems are under investigation and expected to be solved.

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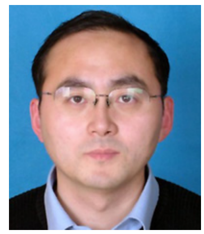

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