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Technical communique

Analysis of a class of discrete-time systems with power rule☆

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Abstract

In this communique, the properties of a class of discrete-time dynamic systems with power rule are studied and comparisons with their continuous-time counterparts as well as the linear discrete-time systems are made. It is shown that using power rule can be beneficial for improving dynamic behaviors of discrete-time systems. For example, a power control law may stabilize a discrete-time system which is not stabilizable by the linear control law.
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1. Introduction

Power rules (i.e. systems involving terms with power other than one) have shown some attractive dynamic characteristics and have been used in, for example, continuous finite-time control (Hong, Huang, & Xu, 2001), robust sliding mode control design (Levant, 2001; Yu & Man, 1998), adaptive control design (Rao, 1998; Yu & Xu, 1996), and neural network learning (Zak, 1989). Comparing to the linear control law (i.e. power is one), a power control law can substantially improve the system response. For instance, regulation control with power greater than one would produce a larger driving force than the linear control when the system state is far away from its equilibrium. On the other hand, regulation control with a fractional power would produce a larger driving force when the system state is in the neighborhood of the equilibrium.

The promising application potential of power rule in continuous-time dynamic systems motivates us to explore its use in discrete-time systems. Furthermore, a dynamic system with power rule is nonlinear in nature. Its digital implementation with an appropriate sampling mechanism is imperative for practical applications. Exploring the dynamic behaviors of the system with power rule in discrete-time domain will further the understanding of the sampled-data systems with power rule, resulting in enhanced practical implementations.

The objective of this communique is to disclose some inherent dynamic behaviors of a simple discrete-time system under a power rule, which were not known before. In order to focus on the fundamental understanding and effect of the power rule on dynamic systems, we concentrate on the simplest first order discrete-time system. We show that the dynamic behaviors of such a system are quite different from those of its continuous-time counterpart. First, the discrete-time system with power rule has non-zero equilibria and limit cycles, in addition to the original equilibrium—the origin that is the only equilibrium of its continuous-time counterpart. This induces much complicated dynamic behaviors. Second, a power regulation control may stabilize the system in certain regions which are not stabilizable under a linear regulation control. Third, by choosing an appropriate power value, the stability region can be enhanced.

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A typical first-order system will be used as an illustrative example to evaluate the effect of sampling period.

This communique is organized as follows. Section 2 gives the problem formulation. Section 3 shows the system properties with power rule. Section 4 verifies the derived properties with two numerical examples.

2. Problem formulation

Consider a simple RL circuit in series connection. Its dynamics from input voltage, u, to the circuit output current, x, is

$$L\dot{x} + Rx = u, \tag{1}$$

where L is the inductance, and R is the resistance. Many control methods have been developed for such class of control problems. A power control law, such as the terminal sliding mode control (Levant, 2001; Yu & Man, 1998)

$$u = -|x|^p \operatorname{sgn}(x), \quad 0 < p < 1,$$

where $\operatorname{sgn}(\cdot)$ is the sign function, is able to achieve finite time convergence.

If the system parameter R is unknown, the adaptive power law below can achieve an improved convergence (Rao, 1998),

$$\dot{\hat{R}}(t) = -x^p, \quad p = 3,$$

where $\hat{R}$ is an estimate of the unknown R.

Furthermore, if the circuit has a nonlinear unknown resistance $R(x)$, universal function approximation methods such as neural networks can be used to obtain $\hat{R}$. It was reported (Zak, 1989) that, incorporating a fractional power in the neural learning algorithms can significantly improve the learning speed (from exponential convergence to finite time convergence).

In digital implementation, however, a finite sampling rate is used. Assume that the circuit works under a sampling rate with interval T. The discretization with the Euler method yields

$$L\frac{x_{k+1} - x_k}{T} + Rx_k = u_k,$$

or

$$x_{k+1} = \lambda x_k + \mu u_k, \tag{2}$$

with $\lambda = 1 - TR/L$, and $\mu = T/L$. Note that Eq. (2) is the discrete-time system of concern in this paper.

Remark 1. Note that we use the first-order Euler discretization instead of the zero-order-hold, because the latter cannot be applied directly to nonlinear dynamic systems.

There are several interesting questions that need to be addressed: (1) What kind of responses can we expect if a power control law

$$u_k = -|x_k|^p \operatorname{sgn}(x_k), \quad p \neq 1 \tag{3}$$

is used in discrete-time system (2)? (2) What are the dynamic properties when power rule is involved (e.g. equilibrium, limit cycle, stability)? (3) Is there any advantage of using power rule for discrete-time systems?

We will address these questions in the following sections. The definitions of limit sets such as equilibrium and limit cycle (periodic orbit) for discrete-time systems, in the form of

$$x_{k+1} = a(x_k)x_k, \quad x_k \in R. \tag{4}$$

which we use in this paper, can be found in Robison (1999). Note also that we adopt the definitions of stability and attractiveness of discrete dynamic systems as in Agarwal (1992). For convenience and simplicity, a limit set is said to be stable in a region $\mathscr{D}$ if (1) the limit set consists of only points of $\mathscr{D}$, and (2) every point in $\mathscr{D}$ not belonging to the limit set will converge to the limit set.

3. Analysis

Substituting power rule (3) into discrete-time model (2) yields the following general model

$$x_{k+1} = \lambda x_k - \mu|x_k|^p \operatorname{sgn}(x_k), \quad \lambda \geqslant 0, \ \mu > 0, \tag{5}$$

where $p \geqslant 0$ is a real number. Obviously, if $p = p_1/p_2$ with p_i, $i = 1, 2$ being odd positive integers, $|x|^p \operatorname{sgn}(x) = x^p$, and if $p = 0$, $|x|^p \operatorname{sgn}(x) = \operatorname{sgn}(x)$.

Note that if $p = 1$, system (5) is a linear system. Obviously, the linear system is stable if and only if $|\lambda - \mu| < 1$, that is,

$$\lambda - 1 < \mu < \lambda + 1. \tag{6}$$

In other words, if (6) does not hold, the linear system ($p = 1$) is not stable. The instability of the system in this case is global.

In what follows, we focus on the scenario that $p \neq 1$ and (6) does not hold. That is, either

$$\alpha_1 = \frac{\lambda - 1}{\mu} > 1, \tag{7}$$

which corresponds to $\lambda - \mu \in (1, \infty)$; or

$$\alpha_2 = \frac{\lambda + 1}{\mu} < 1, \tag{8}$$

which corresponds to $\lambda - \mu \in (-\infty, -1)$. We show that, for system (5), there always exists a stable limit set in a region $\mathscr{D}$, even if system parameters λ and μ lie in the sets specified by either $\alpha_2 > \alpha_1 > 1$ (for (7)) or $1 > \alpha_2 > \alpha_1$ (for (8)).

Note that system (5) can be rewritten as

$$x_{k+1} = \lambda x_k - \mu|x_k|^p \operatorname{sgn}(x_k) = a(x_k)x_k,$$

where $a(x_k) = \lambda - \mu|x_k|^{p-1}$. Therefore, its stability condition can be derived by

$$|a(x_k)| < 1, \tag{9}$$

which depends on system parameters λ, μ, the power value p, and the state x_k.

It is worth pointing out that the system now has multiple equilibria and limit cycles. Indeed, letting $x_{k+1} = x_k$ in (5), or

$a(x_k)=\lambda-\mu|x_k|^{p-1}=1$, we can obtain the equilibria, denoted as x_e, as (Agarwal, 1992; Robison, 1999):

$$x_e=\pm\left(\frac{\lambda-1}{\mu}\right)^{1/(p-1)}=\pm\alpha_1^{1/(p-1)}. \tag{10}$$

It should be noted that $x_e=0$ is also an equilibrium of (5), because $x_k=0$ leads to $x_{k+1}=0=x_k$. Letting $x_{k+1}=-x_k$ in (5) yields a limit cycle with two limit points, denoted as x_c,

$$x_c=\pm\left(\frac{\lambda+1}{\mu}\right)^{1/(p-1)}=\pm\alpha_2^{1/(p-1)}. \tag{11}$$

It should be noted that there is no other periodic orbits (limit cycles) with order higher than 2. The existence of the above equilibria and limit cycle shows the complexity of the discrete-time dynamics with power rule.

Next we derive the stability conditions for two scenarios $\lambda<1$ and $\lambda\geqslant 1$, respectively.

Proposition 1. *Assume $\lambda<1$. $x_e=0$ is the only equilibrium, and $x_c=\pm\alpha_2^{1/(p-1)}$ is the limit cycle. When $p>1$, $x_e=0$ is stable in region $\mathscr{D}=(-\alpha_2^{1/(p-1)},\alpha_2^{1/(p-1)})$, and the limit cycle x_c is unstable. When $p<1$, $x_e=0$ is unstable and the limit cycle x_c is globally stable.*

Proof. Since $\lambda<1$, (7) does not hold. Therefore the non-zero equilibria (10) do not exist. We only need to consider (8) with $1>\alpha_2$ for equilibrium $x_e=0$ and limit cycle $x_c=\pm\alpha_2^{1/(p-1)}$. From stability condition (9)

$$-1<\lambda-\mu|x_k|^{p-1}<1,$$

and the fact $\lambda<1$, inequality $\lambda-\mu|x_k|^{p-1}<1$ always holds. Thus the problem is further simplified as $-1<\lambda-\mu|x_k|^{p-1}$, that is

$$|x_k|^{p-1}<\alpha_2. \tag{12}$$

Since $\alpha_2<1$, we have

- $p>1$: The stability condition (12) becomes

  $$0\leqslant|x_k|<\alpha_2^{1/(p-1)}<1. \tag{13}$$

  Thus for any initial state in region $\mathscr{D}=(-\alpha_2^{1/(p-1)},\alpha_2^{1/(p-1)})$, $x_k\to x_e=0$ as $k\to\infty$. The maximum $\mathscr{D}$ is $(-1,1)$ when $p\to\infty$. On the other hand, outside $\mathscr{D}$, the convergence condition does not hold, thus x_k is divergent. Clearly x_c is unstable, because states from either side of the limit cycle x_c will move away from it.
- $p<1$: According to (12), the stability condition is

  $$|x_k|>\alpha_2^{1/(p-1)}. \tag{14}$$

  It can be seen that the limit cycle $x_c=\pm\alpha_2^{1/(p-1)}$ is the boundary of the following three regions:

(i) $(-\alpha_2^{1/(p-1)},\alpha_2^{1/(p-1)})$, on which state x_k, according to condition (14), is unstable. Thus any state starting from this region will move towards the boundary x_c. Since $x_e=0$ is within this region, then it is unstable.

(ii) $(-\infty,-\alpha_2^{1/(p-1)}]\cup[\alpha_2^{1/(p-1)},\infty)$, which meet condition (14). Any state starting from these two regions will converge to the boundary specified by x_c.

In summary, the limit cycle $x_c=\pm\alpha_2^{1/(p-1)}$ is globally stable. □

Remark 2. Proposition 1 shows that even if the linear system cannot be stabilized by the linear control (with $p=1$ and $|\lambda-\mu|>1$), by choosing appropriate power $p\neq 1$ and with the same λ and μ, the system can be made stable.

Now consider the other case with $\lambda\geqslant 1$.

Proposition 2. *Assume $\lambda\geqslant 1$. There exist three equilibria: $x_e=0$, and $x_e=\pm\alpha_1^{1/(p-1)}$, and the limit cycle: $x_c=\pm\alpha_2^{1/(p-1)}$. When $p>1$, the equilibria $x_e=\pm\alpha_1^{1/(p-1)}$ are stable in region $\mathscr{D}=(-\alpha_2^{1/(p-1)},\alpha_2^{1/(p-1)})$, whereas the equilibrium $x_e=0$ and the limit cycle x_c are unstable. When $p<1$, all equilibria are unstable but the limit cycle x_c is stable in region $\mathscr{D}=(-\alpha_1^{1/(p-1)},\alpha_1^{1/(p-1)})$.*

Proof. Since $\lambda-1\geqslant 0$, we need to consider two cases: $\alpha_2>\alpha_1>1$ for (7) and $1>\alpha_2>\alpha_1$ for (8).

$p>1$: the stability condition (9) becomes

$$\alpha_1^{1/(p-1)}<|x_k|<\alpha_2^{1/(p-1)}. \tag{15}$$

The real axis can be divided into five regions. In region $(-\alpha_1^{1/(p-1)},\alpha_1^{1/(p-1)})$, condition (15) does not hold, thus the state will move towards the boundary $\pm\alpha_1^{1/(p-1)}$. Clearly the equilibrium $x_e=0$ is unstable. In region

$$(-\alpha_2^{1/(p-1)},-\alpha_1^{1/(p-1)})\cup(\alpha_1^{1/(p-1)},\alpha_2^{1/(p-1)}),$$

condition (15) holds, and the state will move towards the lower boundary $\pm\alpha_1^{1/(p-1)}$. Therefore, the equilibria $x_e=\pm\alpha_1^{1/(p-1)}$ are stable in region $\mathscr{D}=(-\alpha_2^{1/(p-1)},\alpha_2^{1/(p-1)})$. In region

$$(-\infty,-\alpha_2^{1/(p-1)})\cup(\alpha_2^{1/(p-1)},\infty),$$

condition (15) does not hold, thus the state is divergent. Since the state will move away from either side of the boundary points $x_c=\pm\alpha_2^{1/(p-1)}$, then x_c is an unstable limit cycle.

$p<1$: the stability condition (9) becomes

$$\alpha_1^{1/(p-1)}>|x_k|>\alpha_2^{1/(p-1)} \tag{16}$$

Analogous to the previous case, the real axis can be divided into five regions. In region $(-\alpha_2^{1/(p-1)},\alpha_2^{1/(p-1)})$, condition (16) does not hold, thus the state will move towards the boundary $x_c=\pm\alpha_2^{1/(p-1)}$. Clearly the equilibrium $x_e=0$ is unstable. In region

$$(-\alpha_1^{1/(p-1)},-\alpha_2^{1/(p-1)})\cup(\alpha_2^{1/(p-1)},\alpha_1^{1/(p-1)}),$$

condition (16) holds, and the state will move towards the lower boundary of $x_c = \pm\alpha_2^{1/(p-1)}$. Therefore the limit cycle $x_c = \pm\alpha_2^{1/(p-1)}$ is stable in region $\mathscr{D} = (-\alpha_1^{1/(p-1)}, \alpha_1^{1/(p-1)})$. In region

$$(-\infty, -\alpha_1^{1/(p-1)}) \cup (\alpha_1^{1/(p-1)}, \infty),$$

condition (16) does not hold, thus the state is divergent. Since the state will move away from either side of the boundary $x_e = \pm\alpha_1^{1/(p-1)}$, this pair of equilibria is unstable. □

Remark 3. Generally speaking, even if the first order linear system with power rule may be globally unstable, there exists a region $\mathscr{D}$ which may contain a stable limit set (either stable non-zero equilibria or limit cycle), and the size of $\mathscr{D}$ can be adjusted by choosing $p \neq 1$.

Remark 4. The convergence speed of the first order system with power rule is solely determined by the factor $a(x_k) = \lambda - \mu|x_k|^{p-1}$, while the convergence speed of the linear system is $|\lambda-\mu|$. If $|\lambda-\mu| \geqslant 1$, the linear system ($p=1$) does not converge whereas the nonlinear system ($p \neq 1$), as shown in the above derivations, will converge to a limit set (either stable non-zero equilibria or limit cycle).

Remark 5. Consider the case $|\lambda-\mu| < 1$, i.e. the linear system with the linear control law is globally stable. We can observe that the linear system with the power control law may not be globally stable, that is, $\forall x \in \mathscr{D}^c$

$$\mathscr{D}^c\colon\ |\lambda - \mu|x|^{p-1}| \geqslant 1.$$

Clearly, the above condition always holds for sufficiently large x when $p > 1$, or for sufficiently small x when $p < 1$. This reminds us that specific caution is indispensable when a power control law is designed for continuous-time systems but implemented in discrete-time (or sampled-data) form. This conclusion is consistent with the well-known fact that a high gain feedback control for continuous-time systems may not work well for their discrete-time counterparts.

4. Illustrative examples

Example 1. Consider system (2), which can represent either a discrete-time process, or a discretized continuous-time process model like the sampled continuous-time circuit (1),

$$x_{k+1} = \lambda x_k + \mu u_k, \quad x_0 > 0,$$

where u_k is the control input, $\lambda = 1.1$ and $\mu = 2.2$. In practice, we may not know the exact value of the system parameter μ. As such, it is hard to design a linear stabilizing feedback controller. For instance, a linear feedback control with gain β, i.e. $u = -\beta x_k$, would produce a closed-loop pole $\lambda - \mu\beta$. To guarantee a stable closed-loop, β must be chosen from a relatively narrow range $[(\lambda-1)/\mu, (\lambda+1)/\mu]=[0.0455, 0.9594]$. Clearly, without knowing the value of μ, β may be chosen outside of the narrow range improperly, leading to global instability of the closed-loop system. Apparently $\beta = 1$ would not stabilize the system. However, if the following controller with fractional power is chosen,

$$u_k = -|x_k|^{5/7}\,\mathrm{sgn}(x_k) = -x_k^{5/7}.$$

According to Proposition 2, since $p < 1$, there exists a stable limit cycle $x_c = \pm 1.1768$ and the state in the region $(-4994.4, 4994.4)$ converges to the limit cycle. On the other hand, all three equilibria, $x_e = 0$, $x_e = \pm 4994.4$ are unstable.

Example 2. Consider a terminal sliding mode designed in continuous-time (Yu & Man, 1998)

$$\dot{x} = -x^p, \quad p = \tfrac{1}{3}.$$

Discretizing it using the Euler method yields

$$x_{k+1} = x_k - T x_k^{1/3}$$

where T ($\ll 1$) is the sampling period. In this specific case, $\lambda = 1$ and $\mu = T < 1$. According to Proposition 2, all equilibria are unstable but the limit cycle is stable. In fact, since $\alpha_1 = 0$, there is no non-zero equilibrium. In order to make the system stable, we must have $-1 < a_k < 1$ where $a_k = 1 - T x_k^{-2/3}$, that is,

$$-1 < 1 - T x_k^{-2/3} < 1,$$

which yields the stability region

$$|x_k| > \left(\frac{T}{2}\right)^{3/2},$$

and the limit cycle is $x_c = \pm(T/2)^{3/2}$ (which becomes a lower bound). The origin $x_e = 0$ is also an equilibrium but unstable. If the sampling period is given *a priori*, the precision bound can be estimated precisely according to $|x_c| = (T/2)^{3/2}$. On the other hand, if the precision bound is specified by some particular performance requirement, we can then determine an appropriate sampling rate to meet the requirement. Finally, if the sampling period cannot be tuned arbitrarily due to hardware limitation, we can revise the original terminal sliding mode with an introduced parameter β as

$$\dot{x} = -\beta x^p, \quad p = \tfrac{1}{3}, \ \ \beta > 0.$$

Discretizing it using the Euler method yields

$$x_{k+1} = x_k - \beta T x_k^{1/3}$$

and the limit cycle now is $x_c = \pm(\beta T/2)^{3/2}$. As a result, the precision bound can be made sufficiently small with a small β. In the continuous-time domain, larger β implies a shorter reaching time. Considering the finite sampling rate, however, a trade-off has to be made to achieve a balance between the precision and the convergence speed.

5. Conclusion

The simple discrete-time system with power rule shows rather different behaviors comparing with its continuous-time counterpart as well as its discrete-time linear version ($p = 1$). We have shown that using power rule enables a much wider range of stability options even if the linear discrete-time system ($p = 1$) is unstable. This property can be used for designing special robust controllers when there exist system parametric uncertainties.

Since power rule introduces some useful nonlinearities (e.g. non-smoothness) in the discrete-time system, and hence generates additional equilibria, it is necessary to re-evaluate and re-design a continuous-time power control law implemented with finite sampling rate, which may converge to a point away from the origin. It is also worth noting that, unlike the continuous-time power control law, the discrete-time power control law may not be able to achieve global stability, hence care must be taken when a continuous-time power rule is implemented with limited sampling rate.

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