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Brief paper

Tracking control for multi-agent consensus with an active leader and variable topology☆

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Abstract

In this paper, we consider a multi-agent consensus problem with an active leader and variable interconnection topology. The state of the considered leader not only keeps changing but also may not be measured. To track such a leader, a neighbor-based local controller together with a neighbor-based state-estimation rule is given for each autonomous agent. Then we prove that, with the proposed control scheme, each agent can follow the leader if the (acceleration) input of the active leader is known, and the tracking error is estimated if the input of the leader is unknown.

Keywords: Multi-agent systems; Consensus; State estimation; Active leader

1. Introduction

In recent years, there has been an increasing research interest in the control design of multi-agent systems. Many results have been obtained with local rules applied to each agent in a considered multi-agent system. These neighbor rules for each agent are based on the average of its own information and that of its neighbors or its leader (Fax & Murray, 2004; Jadbabaie, Lin, & Morse, 2003; Lin, Broucke, & Francis, 2004; Olfati-Saber & Murray, 2004; Savkin, 2004). For example, Jadbabaie et al. (2003) demonstrated that a simple neighbor rule makes all agents eventually move in the same direction despite the absence of centralized coordination and each agent's set of neighbors changing with time as the system evolves under a joint connection condition. Also, with a similar technique, Lin et al. (2004) studied three formation strategies for groups of mobile autonomous agents. The stability analysis of multi-vehicle formations was given with a Nyquist-type criterion in (Fax & Murray, 2004). Moreover, by a Lyapunov-based approach, Olfati-Saber and Murray (2004) solved the average-consensus problem with directed interconnection graphs or time-delays.

In reality, some variables of the agents and/or the leader in a multi-agent system may not be able to be measured. Fax and Murray (2004) raised this important issue regarding observer design for multi-agent systems, and first tackled this problem. However, many works remain to be done for the distributed observer design of networks of multiple agents.

With this background, we consider a consensus problem with an active leader with an underlying dynamics. Here, some variables (that is, the velocity and maybe the acceleration) of an active leader cannot be measured, and each agent only gets the measured information (that is, the position) of the leader once there is a connection between them. In this paper, we propose an "observer" by inserting an integrator into the loop for each agent to estimate the leader's velocity. To analyze the problem, a Lyapunov-based approach is developed. With the proposed estimation rule and a selected Lyapunov function, the leader-following problem can be solved if the leader's input is known, while the tracking error can also be analyzed if the input is unknown.

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2. Problem formulation

To solve coordination problems, graph theory is helpful. An undirected graph $\mathscr{G}$ on vertex set $\mathscr{V}=\{1,2,\ldots,n\}$ contains $\mathscr{V}$ and a set of unordered pairs $\mathscr{E}=\{(i,j):i,j\in\mathscr{V}\}$, which are called $\mathscr{G}$'s edges. If there is an edge between two vertices, the two vertices are called adjacent. A graph is simple if it has no self-loops or repeated edges. If there is a path between any two vertices of a graph $\mathscr{G}$, then $\mathscr{G}$ is connected, otherwise disconnected. A subgraph $\mathscr{X}$ of $\mathscr{G}$ is an induced subgraph if two vertices of $\mathscr{V}(X)$ are adjacent in $\mathscr{X}$ if and only if they are adjacent in $\mathscr{G}$. An induced subgraph $\mathscr{X}$ of $\mathscr{G}$ that is maximal, subject to being connected, is called a component of $\mathscr{G}$.

Here, we consider a system consisting of n agents and a leader. In the sequel, the state of agent i is denoted by x_i for $i=1,\ldots,n$. With regarding the n agents as the vertices in $\mathscr{V}$, the relationships between n agents can be conveniently described by a simple and undirected graph $\mathscr{G}$, which is defined so that (i,j) defines one of the graph's edges in case agents i and j are neighbors. $N_i(t)$ denotes the set of labels of those agents which are neighbors of agent i $(i=1,\ldots,n)$ at time t. The weighted adjacency matrix of $\mathscr{G}$ is denoted by $A=[a_{ij}]\in R^{n\times n}$, where $a_{ii}=0$ and $a_{ij}=a_{ji}\geqslant 0$ ($a_{ij}>0$ if there is an edge between agent i and agent j). Its degree matrix $D=\mathrm{diag}\{d_1,\ldots,d_n\}\in R^{n\times n}$ is a diagonal matrix, where diagonal elements $d_i=\sum_{j=1}^{n}a_{ij}$ for $i=1,\ldots,n$. Then the Laplacian of the weighted graph is defined as

$$L=D-A, \tag{1}$$

which is symmetric. In what follows, we mainly concern a graph $\bar{\mathscr{G}}$ associated with the system consisting of n agents and one leader. In fact, $\bar{\mathscr{G}}$ contains n agents (related to graph $\mathscr{G}$) and the leader with directed edges from some agents to the leader. By "the graph, $\bar{\mathscr{G}}$, of this system is connected", we mean that at least one agent in each component of $\mathscr{G}$ is connected to the leader.

For the multi-agent system under consideration, the relationships between neighbors (and the interconnection topology) change over time. Suppose that there is an infinite sequence of bounded, non-overlapping, contiguous time-intervals $[t_i,t_{i+1}),\ i=0,1,\ldots,$ starting at $t_0=0$.

Denote $\mathscr{S}=\{\bar{\mathscr{G}}_1,\bar{\mathscr{G}}_2,\ldots,\bar{\mathscr{G}}_N\}$ as a set of the graphs with all possible topologies, which includes all possible interconnection graphs (involving n agents and a leader), and denote $\mathscr{P}=\{1,2,\ldots,N\}$ as its index set.

To describe the variable interconnection topology, we define a switching signal $\sigma:[0,\infty)\to\mathscr{P}$, which is piecewise-constant. Therefore, N_i and the connection weight a_{ij} ($i=1,\ldots,n$, $j=1,\ldots,n$) are time-varying, and moreover, Laplacian L_p ($p\in\mathscr{P}$) associated with the switching interconnection graph is also time-varying (switched at $t_i,\ i=0,1,\ldots$), though it is a time-invariant matrix in any interval $[t_i,t_{i+1})$. In our problem, we assume that there are fixed positive constants α_{ij} $(i=1,\ldots,n;\ j=1,\ldots,n)$ such that

$$a_{ij}(t)=\begin{cases}\alpha_{ij}=\alpha_{ji} & \text{if agents } i \text{ and } j \text{ are connected at } t,\\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

Meanwhile, the connection weight between agent i and the leader, denoted by b_i, is time-varying, too. We assume that there are fixed positive constants β_i $(i=1,\ldots,n)$ such that

$$b_i(t)=\begin{cases}\beta_i & \text{if agent } i \text{ is connected to the leader at } t,\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

The next lemma was given to check the positive definiteness of a matrix (Horn & Johnson, 1985).

Lemma 1. *Suppose that a symmetric matrix is partitioned as*

$$E=\begin{pmatrix}E_1 & E_2\\ E_2^{\mathrm{T}} & E_3\end{pmatrix},$$

where E_1 and E_3 are square. E is positive definite if and only if both E_1 and $E_3-E_2^{\mathrm{T}}E_1^{-1}E_2$ are positive definite.

The following result is well-known in algebraic graph theory (Godsil & Royle, 2001) and establishes a direct relationship between the graph connectivity and its Laplacian.

Lemma 2. *Let $\mathscr{G}$ be a graph on n vertices with Laplacian L. Denote the eigenvalues of L by $\lambda_1(L),\ldots,\lambda_n(L)$ satisfying $\lambda_1(L)\leqslant\cdots\leqslant\lambda_n(L)$. Then $\lambda_1(L)=0$ and $\mathbf{1}=[1,1,\ldots,1]^{\mathrm{T}}\in R^n$ is its eigenvector. Moreover, if $\mathscr{G}$ is connected, $\lambda_2>0$.*

In this paper, all the considered agents move in a plane:

$$\dot{x}_i=u_i\in R^2,\quad i=1,\ldots,n, \tag{4}$$

where u_i is the control input. The leader of this considered multi-agent system is active; that is, its state variables keep changing. Its underlying dynamics can be expressed as follows:

$$\begin{cases}\dot{x}_0=v_0,\\ \dot{v}_0=a(t)=a_0(t)+\delta(t), \qquad x_0,\ v_0,\ \delta\in R^2,\\ y=x_0,\end{cases} \tag{5}$$

where $y(t)=x_0(t)$ is the measured output and $a(t)$ is the (acceleration) input. Note that (5) is completely different from the agent dynamics (4). In other words, the agents will track a leader with a different dynamics.

In our problem formulation, the input $a(t)$ may not be completely known. We assume that $a_0(t)$ is known and $\delta(t)$ is unknown but bounded with a given upper bound $\bar{\delta}$ (that is, $\|\delta(t)\|\leqslant\bar{\delta}$). The input $a(t)$ is known if and only if $\bar{\delta}=0$. On the other hand, $y=x_0$ is the only variable that can be obtained directly by the agents when they are connected to the leader. Our aim here is to propose a decentralized control scheme for each agent to follow the leader (i.e., $x_i\to x_0$).

Since $v_0(t)$ cannot be measured even when the agents are connected to the leader, its value cannot be used in the control design. Instead, we have to estimate v_0 during the evolution. Note that, each agent has to estimate v_0 only by the information obtained from its neighbors in a decentralized way. The estimate of $v_0(t)$ by agent i is denoted by $v_i(t)$ $(i=1,\ldots,n)$.

Therefore, for each agent, the local control scheme consists of two parts:

- a neighbor-based feedback law:

$$u_i = -k\left[\sum_{j\in N_i(t)} a_{ij}(t)(x_i - x_j) + b_i(t)(x_i - x_0)\right] + v_i, \quad k>0,\ i=1,\ldots,n, \tag{6}$$

  where N_i is the set consisting of agent i's neighbor agents;
- a dynamic neighbor-based system to estimate v_0

$$\dot v_i = a_0 - \gamma k\left[\sum_{j\in N_i(t)} a_{ij}(t)(x_i - x_j) + b_i(t) \times (x_i - x_0)\right], \quad i=1,\ldots,n, \tag{7}$$

  for some positive constant $\gamma < 1$. In fact, (7), can be viewed as an "observer" in some sense.

Note that u_i in (6) is a local controller of agent i, which only depends on the information from its neighbors, and, in fact, when $v_0=0, a=0$, the proposed control law (6) is consistent with the one given by Olfati-Saber and Murray (2004). In addition, with the neighbor-based estimation rule in a form of observer (7) to estimate the leader's velocity, each agent relies only on the locally available information at every moment. In other words, each agent cannot "observe" or "estimate" the leader directly based on the measured information of the leader if it is not connected to the leader. In fact, it has to collect the information of the leader in a distributed way from its neighbor agents.

Take

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Regarding the switching interconnection graphs, the closed-loop system can be expressed as

$$\begin{cases} \dot x = u = -k(L_\sigma + B_\sigma)\otimes I_2 x + kB_\sigma \mathbf{1}\otimes x_0 + v, \\ \dot v = \mathbf{1}\otimes a_0 - \gamma k(L_\sigma + B_\sigma)\otimes I_2 x + \gamma k(B_\sigma \mathbf{1})\otimes x_0, \end{cases} \tag{8}$$

where $I_l \in R^{l\times l}$ (for any positive integer l) is the identity matrix and $\otimes$ denotes the Kronecker product, $\sigma : [0,\infty) \to \mathscr{P} = \{1,2,\ldots,N\}$ is a piecewise-constant switching signal with successive switching times, B_σ is an $n\times n$ diagonal matrix whose ith diagonal element is $b_i(t)$ at time t, L_σ is the Laplacian for the n agents. Note that, even in the case when the interconnection graph is connected, $b_i(t)$ may be always 0 for some i, and therefore, B_σ may not be of full rank.

Denote $\bar x = x - \mathbf{1}\otimes x_0$ and $\bar v = v - \mathbf{1}\otimes v_0$. Because $-k(L_\sigma + B_\sigma)\otimes I_2 x + kB_\sigma\mathbf{1}\otimes x_0 = -k(L_\sigma + B_\sigma)\otimes I_2 \bar x$ (invoking Lemma 2), we can obtain an error dynamics of (8) as follows:

$$\dot\varepsilon = F_\sigma \varepsilon + g, \quad g = \begin{pmatrix} 0 \\ -\mathbf{1}\otimes\delta \end{pmatrix} \tag{9}$$

where

$$\varepsilon = \begin{pmatrix} \bar x \\ \bar v \end{pmatrix}, \quad F_\sigma = \begin{pmatrix} -k(L_\sigma + B_\sigma) & I_n \\ -\gamma k(L_\sigma + B_\sigma) & 0 \end{pmatrix} \otimes I_2.$$

3. Main results

In this section, the convergence analysis of system (9) is given for the consensus problem of multi-agent system (8). If the information of the input $a(t)$ can be used in local control design, we can prove that all the agents can follow the leader, though the leader keeps changing. If not, we can also get some estimation of the tracking error. We first assume that the interconnection graph $\bar{\mathscr{G}}$ is always connected, though the interconnection topology keeps changing; and then we consider an extended case.

As mentioned above, $\bar{\mathscr{G}}$ is connected if at least one agent in each of its component is connected with the leader. To be specific, if there are $m \geqslant 1$ components, then the Laplacian L_p (for any $p\in\mathscr{P}$) of the graph associated with n agents have m zero eigenvalues. For simplicity, we can rearrange the indices of n agents such that L_p can be rewritten as a block diagonal matrix:

$$L_p = \begin{pmatrix} L_p^1 & & & \\ & L_p^2 & & \\ & & \ddots & \\ & & & L_p^m \end{pmatrix},$$

where each block matrix L_p^i is also a Laplacian of the corresponding component. For convenience, denote $M_p = L_p + B_p$, where L_p is the weighted Laplacian and B_p ($p\in\mathscr{P}$) is the diagonal matrix as defined in Section 2. The next lemma is given for M_p.

Lemma 3. *If graph $\bar{\mathscr{G}}_p$ is connected, then the symmetric matrix M_p associated with $\bar{\mathscr{G}}_p$ is positive definite.*

Proof. We only need to prove the case when $m=1$. Let $\lambda_1,\ldots,\lambda_n$ be the eigenvalues of Laplacian L_p in the increasing order. From Lemma 2, $\lambda_1 = 0$ and $\lambda_i > 0, i \geqslant 2$. Denote n eigenvectors of L_p by $\zeta_i,\ i=1,\ldots,n$, with $\zeta_1 = \mathbf{1}$, an eigenvector of L_p corresponding to $\lambda_1 = 0$. Then any nonzero vector $z\in R^n$ can be expressed by $z = \sum_{i=1}^n c_i\zeta_i$ for some constants $c_i, i=1,2,\ldots,n$. Moreover, $B_p \neq 0$ since there is at least one agent connected to the leader. Without loss of generality, we assume $b_j > 0$ for some j, and it is obvious $\zeta_1^{\mathrm T} B_p \zeta_1 \geqslant b_j$. Therefore, in either the case when $c_2 = \cdots = c_n = 0$ (so $c_1 \neq 0$) or the case when $c_i \neq 0$ for some $i \geqslant 2$, we always have

$$z^{\mathrm T} M_p z = z^{\mathrm T} L_p z + z^{\mathrm T} B_p z \geqslant \sum_{i=2}^n \lambda_i c_i^2 \zeta_i^{\mathrm T}\zeta_i + z^{\mathrm T} B_p z > 0$$

for $z\neq 0$, which implies the conclusion. □

Based on Lemma 3 and the fact that the set $\mathscr{P}$ is finite,

$$\bar{\lambda} = \min\{\text{eigenvalues of } M_p \in R^{n\times n},\ \forall \bar{\mathscr{G}}_p \text{ is connected}\} > 0, \tag{10}$$

is fixed and depends directly on the constants α_{ij} and β_i for $i=1,\dots,n$, $j=1,\dots,n$ given in (2) and (3). Its estimation is also related to the minimum nonzero eigenvalue of Laplacian L_p, which has been widely studied in different situations (Merris, 1994).

In some existing works (including Jadbabaie et al., 2003; Lin et al., 2004), the convergence analysis depends on theory of nonnegative matrices or stochastic matrices. However, F_p of system (9) fails to be transformed easily to a matrix with some properties related to stochastic matrices, and therefore, the effective methods used in Jadbabaie et al. (2003) may not work. Here, we propose a Lyapunov-based approach to deal with the problem.

Theorem 4. *For any fixed* $0<\gamma<1$ *and* $\bar{\lambda}$ *defined in* (10), *we take a constant*

$$k > \frac{1}{4\gamma(1-\gamma^2)\bar{\lambda}}. \tag{11}$$

If the switching interconnection graph keeps connected, then

$$\lim_{t\to\infty} \|\varepsilon(t)\| \leqslant C, \tag{12}$$

for some constant C depending on $\bar{\delta}$. *Moreover, if* $a(t)$ *is known* (*i.e.*, $a(t)=a_0(t)$ *or* $\bar{\delta}=0$),

$$\lim_{t\to\infty} \varepsilon(t) = 0. \tag{13}$$

Proof. Take a Lyapunov function $V(\varepsilon)=\varepsilon^{\mathrm{T}}(t)P\varepsilon(t)$ with symmetric positive definite matrix

$$P = \begin{pmatrix} I_n & -\gamma I_n \\ -\gamma I_n & I_n \end{pmatrix} \otimes I_2. \tag{14}$$

The interconnection graph is time-varying, but the interconnection graph associated with F_p for some $p\in\mathscr{P}$ is connected on an interval $[t_i, t_{i+1})$ with its topology unchanged. Consider the derivative of $V(\varepsilon)$:

$$\begin{aligned}\dot{V}(\varepsilon)|_{(9)} &= \varepsilon^{\mathrm{T}}(F_p^{\mathrm{T}}P + PF_p)\varepsilon + 2\varepsilon^{\mathrm{T}}Pg \\ &\leqslant -\varepsilon^{\mathrm{T}}Q_p\varepsilon + 2(1+\gamma)\bar{\delta}\|\varepsilon\|,\end{aligned} \tag{15}$$

where

$$Q_p = -(F_p^{\mathrm{T}}P + PF_p) = \begin{pmatrix} 2k(1-\gamma^2)M_p & -I_n \\ -I_n & 2\gamma I_n \end{pmatrix} \otimes I_2 \tag{16}$$

is a positive definite matrix because $2\gamma I - (1/2k(1-\gamma^2))M_p^{-1}$ and M_p are positive definite (by virtue of (11), Lemmas 1 and 3).

Let $\mu_{i,j}$, $i=1,\dots,n$, $j=1,2$ denote the (at most) $2n$ different eigenvalues of Q_p though $Q_p\in R^{4n\times 4n}$ defined in (16). Based on $\lambda_i(M_p)$, the eigenvalues of M_p, we have the $2n$ eigenvalues in the following forms:

$$\mu_{i,1} = (1-\gamma^2)k\lambda_i(M_p)+\gamma + \sqrt{[(1-\gamma^2)k\lambda_i(M_p)+\gamma]^2 - 4\gamma(1-\gamma^2)k\lambda_i(M_p)+1},$$

$$\mu_{i,2} = (1-\gamma^2)k\lambda_i(M_p)+\gamma - \sqrt{[(1-\gamma^2)k\lambda_i(M_p)+\gamma]^2 - 4\gamma(1-\gamma^2)k\lambda_i(M_p)+1},$$

for $i=1,\dots,n$. Clearly, the smallest eigenvalue of Q_p will be found in the form of $\mu_{i,2}$ for some i.

Note that (11) implies $k\lambda_i(M_p) > 1/4\gamma(1-\gamma^2)$. In this case, $\mu_{i,2}$ increases as $k\lambda_i(M_p)$ increases. Therefore, the minimum eigenvalue of Q_p will be no less than

$$\bar{\mu} = (1-\gamma^2)k\bar{\lambda}+\gamma - \sqrt{[(1-\gamma^2)k\bar{\lambda}-\gamma]^2+1} > 0, \tag{17}$$

which is obtained by taking $\lambda_i(M_p)=\bar{\lambda}$ with a given k satisfying (11). In addition, since the eigenvalues of P are either $\mu_{\min}=1-\gamma$ or $\mu_{\max}=1+\gamma$, we have

$$(1-\gamma)\|\varepsilon\|^2 \leqslant V(\varepsilon) \leqslant (1+\gamma)\|\varepsilon\|^2. \tag{18}$$

Therefore,

$$\min \frac{\varepsilon^{\mathrm{T}}Q_p\varepsilon}{\varepsilon^{\mathrm{T}}P\varepsilon} \geqslant \frac{\bar{\mu}}{\mu_{\max}} = 2\beta,$$

where $\beta = \bar{\mu}/2(1+\gamma) > 0$ with $\bar{\mu}$ defined in (17).

Due to (18),

$$\|\varepsilon\| \leqslant \frac{1}{\sqrt{1-\gamma}}\sqrt{V(\varepsilon)}.$$

Therefore, from (15),

$$\begin{aligned}\dot{V}(\varepsilon)|_{(9)} &\leqslant -2\beta V(\varepsilon) + 2\sqrt{\frac{(1+\gamma)^2V(\varepsilon)}{1-\gamma}}\bar{\delta} \\ &\leqslant -\beta V(\varepsilon) + \frac{(1+\gamma)^2\bar{\delta}^2}{(1-\gamma)\beta}\end{aligned}$$

or equivalently,

$$V(\varepsilon(t)) \leqslant V(\varepsilon(t_i))e^{-\beta(t-t_i)} + \frac{(1+\gamma)^2\bar{\delta}^2}{(1-\gamma)\beta^2}(1-e^{-\beta(t-t_i)}), \quad t\in[t_i, t_{i+1}).$$

Thus, with $t_0=0$,

$$V(\varepsilon(t)) \leqslant V(\varepsilon(0))e^{-\beta t} + \frac{(1+\gamma)^2\bar{\delta}^2}{(1-\gamma)\beta^2}(1-e^{-\beta t}), \tag{19}$$

which implies (12) with taking $C=((1+\gamma)/(1-\gamma)\beta)\bar{\delta}$. Furthermore, if $\bar{\delta}=0$, then (13) is obtained. □

Next, we consider an extended case: the interconnection graph is not always connected. Let $T>0$ be a (sufficient large) constant, and then we have a sequence of interval $[T_j, T_{j+1})$, $j=0,1,\dots$ with $T_0=t_0$, $T_{j+1}=T_j+T$. Each interval $[T_j, T_{j+1})$ consists of a number of intervals (still expressed in the form of $[t_i, t_{i+1})$, during which the interconnection graph is time-invariant), including the intervals during which the graphs are connected and those during which the graphs are not. We assume that there is a constant $\tau > 0$, often called dwell time, with $t_{i+1} - t_i \geqslant \tau,\ \forall i$.

Denote the total length of the intervals associated with the connected graphs as T_j^{c} in $[T_j, T_{j+1})$ and the total length of the intervals with the unconnected graphs as T_j^{d} in $[T_j, T_{j+1})$. In what follows, we denote an upper bound of T_j^{d} $(j = 0, 1, \ldots)$ as $T^{\mathrm{d}}(< T)$, and a lower bound of T_j^{c} $(j = 0, 1, \ldots)$ as $T^{\mathrm{c}}(= T - T^{\mathrm{d}})$.

Theorem 5. *During each time interval $[T_j, T_{j+1})$, if the total period that the interconnection graph is connected (i.e., T^{c}) is sufficient large, then (12) still holds with k given in (11). Moreover, (13) holds if $\bar{\delta} = 0$ (or equivalently $a(t) = a_0(t)$).*

Proof. Still take a Lyapunov function $V(\varepsilon) = \varepsilon^{\mathrm{T}} P \varepsilon$ with P defined in (14), and then we have (15). If the graph associated with F_p for some $p \in \mathscr{P}$ is connected during $[t_i, t_{i+1})$, then, according to Theorem 4, we have

$$V(\varepsilon(t_{i+1})) \leqslant \mathrm{e}^{-\beta(t_{i+1}-t_i)} V(\varepsilon(t_i)) + \frac{(1+\gamma)^2}{(1-\gamma)\beta^2}\bar{\delta}^2.$$

If the graph associated with F_q for some $q \in \mathscr{P}$ is not connected during $[t_l, t_{l+1})$. The minimum eigenvalue of Q_q is $\gamma - \sqrt{1+\gamma^2}(< 0)$ and, by (18), we have

$$-\varepsilon^{\mathrm{T}} Q_q \varepsilon \leqslant \left(\sqrt{1+\gamma^2} - \gamma\right) \varepsilon^{\mathrm{T}}\varepsilon \leqslant \frac{\alpha}{2} V(\varepsilon),$$

where $\alpha = (2\sqrt{1+\gamma^2} - 2\gamma)/(1-\gamma)$.

Similarly, with (18),

$$\dot{V}(\varepsilon(t))|_{(9)} \leqslant \alpha V(\varepsilon(t)) + \frac{2(1+\gamma)^2}{\alpha(1-\gamma)}\bar{\delta}^2, \quad t \in [t_l, t_{l+1}),$$

and therefore,

$$V(\varepsilon(t_{l+1})) \leqslant \mathrm{e}^{\alpha(t_{l+1}-t_l)} V(\varepsilon(t_l)) + \frac{2(1+\gamma)^2}{\alpha^2(1-\gamma)}(\mathrm{e}^{\alpha T^{\mathrm{d}}} - 1)\bar{\delta}^2. \quad (20)$$

Denote

$$\eta = \max\left\{\frac{(1+\gamma)^2}{\beta^2(1-\gamma)}, \frac{2(1+\gamma)^2}{\alpha^2(1-\gamma)}(\mathrm{e}^{\alpha T^{\mathrm{d}}} - 1)\right\}.$$

It is not hard to see that there are at most $m_{\mathrm{d}} = [T^{\mathrm{d}}/\tau] + 1$ intervals (in $[T_j, T_{j+1})$) associated with unconnected graphs. Therefore, we have

$$\begin{aligned} V(\varepsilon(T_{j+1})) &\leqslant \mathrm{e}^{-\beta T_j^{\mathrm{c}} + \alpha T_j^{\mathrm{d}}} V(\varepsilon(T_j)) + (1 + \mathrm{e}^{T^{\mathrm{d}}} + \mathrm{e}^{2T^{\mathrm{d}}} \\ &\quad + \cdots + \mathrm{e}^{m_{\mathrm{d}} T^{\mathrm{d}}})\eta\bar{\delta}^2 \\ &\leqslant \mathrm{e}^{-\beta T^{\mathrm{c}} + \alpha(T - T^{\mathrm{c}})} V(\varepsilon(T_j)) + \bar{\eta}\bar{\delta}^2 \end{aligned} \quad (21)$$

with

$$\bar{\eta} = \frac{\mathrm{e}^{(m_{\mathrm{d}}+1)T^{\mathrm{d}}} - 1}{\mathrm{e}^{T^{\mathrm{d}}} - 1}\eta > 0.$$

If $\beta T_{\mathrm{c}} > \alpha(T - T^{\mathrm{c}})$ or $T_{\mathrm{c}} > \alpha T/(\alpha+\beta)$, then $\nu = \mathrm{e}^{-\beta T^{\mathrm{c}} + \alpha(T - T^{\mathrm{c}})} < 1$. Thus,

$$\begin{aligned} V(\varepsilon(T_{j+1})) &\leqslant \nu^{j+1} V(\varepsilon(T_0)) + (\nu^j + \cdots + 1)\bar{\eta}\bar{\delta}^2 \\ &\leqslant \nu^{j+1} V(\varepsilon(T_0)) + \frac{1-\nu^{j+1}}{1-\nu}\bar{\eta}\bar{\delta}^2. \end{aligned}$$

For any $t > 0$, there is j such that $T_j < t < T_{j+1}$ with

$$V(\varepsilon(t)) \leqslant \mathrm{e}^{\alpha T^{\mathrm{d}}} V(\varepsilon(T_j)) + \bar{\eta}\bar{\delta}^2.$$

Thus, (12) is obtained with taking

$$C = \sqrt{\frac{(\mathrm{e}^{\alpha T^{\mathrm{d}}} + 1 - \nu)\bar{\eta}}{(1-\nu)(1-\gamma)}}\bar{\delta}.$$

Furthermore, if $\bar{\delta} = 0$, then $C = 0$, which implies (13), or $\varepsilon \to 0$ as $t \to \infty$. $\square$

In fact, the proposed estimation idea can be extended to the case of an active leader with the following dynamics:

$$\begin{cases} \dot{x}_0^1 = x_0^2, \\ \dot{x}_0^2 = x_0^3, \\ \vdots \\ \dot{x}_0^{\kappa} = a(t) = a_0(t) + \delta(t), \\ y = x_0 = x_0^1 \in R^2, \end{cases} \quad (22)$$

where $y(t)$ is the measured output variable of the leader and $a(t)$ is its input variable. The dynamics of each agent is still taken in the form of (4). Then we will construct an observer as we did for system (5). Here, for the space limitations, we only give the corresponding error system, which can be expressed as

$$\begin{pmatrix} \dot{\bar{x}}^1 \\ \dot{\bar{x}}^2 \\ \vdots \\ \dot{\bar{x}}^{\kappa} \end{pmatrix} = \begin{pmatrix} -kM_p & I_n & & \\ -\gamma_1 kM_p & 0 & I_n & \\ \vdots & & & \\ -\gamma_{\kappa-2} kM_p & & 0 & I_n \\ -\gamma_{\kappa-1} kM_p & & & 0 \end{pmatrix} \otimes I_2 \begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \vdots \\ \bar{x}^{\kappa} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -\mathbf{1}\otimes\delta \end{pmatrix}$$

or equivalently in a compact form:

$$\dot{\varepsilon} = F_p\varepsilon + g \in R^{2n\kappa},$$

where $k > 0$ and $0 < \gamma_j < 1$, $(j = 1, \ldots, \kappa - 1)$ are suitable real numbers and $\bar{x}^i = x^i - \mathbf{1} \otimes x_0^i \in R^{2n}$ with $x^1 = x$ and x^i $(2 \leqslant i \leqslant \kappa)$ as the vector whose components are the respective estimated values of x_0^i by n agents.

To obtain the results similar to Theorems 4 or 5, we need to find a suitable quadratic Lyapunov function; that is, to construct an appropriate positive definite matrix P such that $F_p^{\mathrm{T}} P + P F_p$ is negative definite then the corresponding graph $\mathscr{G}_p$ is connected. For example, when $\kappa = 3$, we can choose

$$F_p = \begin{pmatrix} -kM_p & I_n & 0 \\ -\dfrac{8k}{9}M_p & 0 & I_n \\ -\dfrac{4k}{9}M_p & 0 & 0 \end{pmatrix} \otimes I_2, \quad P = \begin{pmatrix} I_n & -\frac{2}{3}I & 0 \\ -\frac{2}{3}I_n & I_n & -\frac{1}{2}I_n \\ 0 & -\frac{1}{2}I_n & I_n \end{pmatrix} \otimes I_2.$$

4. Conclusions

This paper studied the consensus problem of a group of autonomous agents with an active leader, whose velocity cannot be measured. To solve the problem, a distributed feedback (i.e., (6)) along with a distributed state-estimation rule (i.e., (7)) was proposed for each continuous-time dynamical agent, and Lyapunov-based convergence analysis was given for the considered multi-agent system with a varying interconnection topology.

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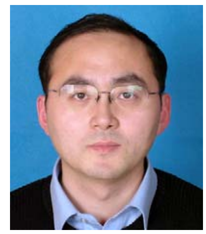

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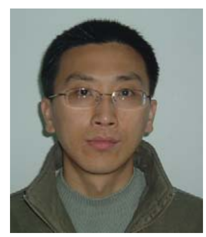

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