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Delay induced oscillation in predator–prey system with Beddington–DeAngelis functional response

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Abstract

The Beddington–DeAngelis predator–prey system with distributed delay is studied in this paper. At first, the positive equilibrium and its local stability are investigated. Then, with the mean delay as a bifurcation parameter, the system is found to undergo a Hopf bifurcation. The bifurcating periodic solutions are analyzed by means of the normal form and center manifold theorems. Finally, numerical simulations are also given to illustrate the results.

Keywords: Stability; Hopf bifurcation; Distributed delay; Predator–prey system; Normal form; Center manifold theorem

1. Introduction

In this paper, we consider the following Beddington–DeAngelis predator–prey system with distributed delay

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= rx\left(1-\frac{x}{k}\right)-\frac{axy}{1+bx+cy}, \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= y\left[-d+\int_{-\infty}^{t}F(t-\tau)\frac{ex(\tau)}{1+bx(\tau)+cy(\tau)}\,\mathrm{d}\tau\right], \end{aligned} \tag{1.1}$$

where $x(t)$ is the population of the prey at time t and $y(t)$ is the population of the predator at time t. The parameters r, k, a, b, c, d, e are positive constants with d representing the death rate of predator as well as r and k standing for the intrinsic rate of increase and the carrying capacity for the prey population, respectively. The predator consumes the prey with functional response of Beddington–DeAngelis type $\frac{axy}{1+bx+cy}$ and contributes to its growth with rate $\frac{exy}{1+bx+cy}$. The functional response in (1.1) was first introduced by Beddington [1] and DeAngelis et al. [2]. For the detailed biological backgrounds, see [1–3].

As usual, we assume that the distributed delay kernel $F(t)$ satisfies

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$$\int_0^\infty F(t)\,\mathrm{d}t = 1, \quad tF(t) \in L^1((0,\infty);\mathbb{R}). \tag{1.2}$$

The average delay for the distributed delay kernel $F(t)$ is defined as

$$\tau = \int_0^\infty tF(t)\,\mathrm{d}t.$$

In particular, the weak delay kernel

$$F(t) = \alpha \mathrm{e}^{-\alpha t}, \quad \alpha > 0, \tag{1.3}$$

and the strong delay kernel

$$F(t) = \alpha^2 t \mathrm{e}^{-\alpha t}, \quad \alpha > 0 \tag{1.4}$$

are widely used [4], then the average delays for the weak delay kernel and the strong delay kernel are $\tau = \frac{1}{\alpha}$ and $\tau = \frac{2}{\alpha}$, respectively.

The reason for introducing a delay into predator–prey models is that the rate of reproduction of predators should depend not just on the rate at which they are consuming prey at the present time but also on the rate at which they have consumed prey in the past time, and this idea has been well justified [5–7]. In fact, the distributed delay was originally introduced into biological modelling by Voltera in the 1930's [8] and has been regarded to be more realistic than discrete delay. For a predator–prey system with distributed delay, when the average delay τ is small, geometrical singular perturbation approach [9,10] has been applied to the studies of the existence and stability of the periodic solution [11,12]. However, when τ becomes quite large, geometrical singular perturbation approach fails.

If the delay kernel is a delta function of the following form:

$$F(t) = \delta(t-\tau), \quad \tau > 0, \tag{1.5}$$

where τ is a given constant, then system (1.1) reduces to the following predator–prey system with discrete time-delay

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= rx\left(1-\frac{x}{k}\right) - \frac{axy}{1+bx+cy}, \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= y\left[-d + \frac{ex(t-\tau)}{1+bx(t-\tau)+cy(t-\tau)}\right]. \end{aligned} \tag{1.6}$$

System (1.6) and some of its special cases, including the delayed ratio-dependent predator–prey system and the delayed predator–prey system with Michaelis–Menten functional response, have been paid much attention to in recent years [13–15]. By constructing a proper Liapunov functional, Beretta and Kuang showed the globally asymptotic stability of the positive equilibrium for the delayed ratio-dependent predator–prey system provided that the time delay τ is small enough [13]. Based on Liapunov-like functions, Razumikhin techniques and differential inequalities, the uniform persistence of the delayed predator–prey system with Michaelis–Menten functional response was obtained under suitable conditions [14], while the existence of the periodic solution was investigated by establishing a map with a nontrivial fixed point [15]. Recently, the existence of periodic solution of predator–prey system with periodic delay was extensively studied using the coincidence degree theory [16–18].

In addition, if the delay kernel is a delta function in the form of $F(t) = \delta(t)$, then system (1.1) reduces to the following predator–prey system without any delay:

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= rx\left(1-\frac{x}{k}\right) - \frac{axy}{1+bx+cy}, \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= y\left[-d + \frac{ex}{1+bx+cy}\right]. \end{aligned} \tag{1.7}$$

The uniform persistence, the stability of the positive equilibrium and the existence of the limit cycle under suitable restrictions on parameters were discussed for system (1.7) in [19], while the global stability of the positive equilibrium was analyzed using divergency criterion and the uniqueness of the limit cycle was studied by transforming system (1.7) into a Gause-type predator–prey system in [20,21].

In this paper, we studied the Hopf bifurcation of system (1.1) with a weak distributed delay kernel $F(t)$, where the average delay τ is taken as the bifurcation parameter. The direction of Hopf bifurcation and the stability of the bifurcating periodic solutions are analyzed with help of the theory of the normal form and center manifold [22].

The rest of the paper is organized as follows. In Section 2, the existence of Hopf bifurcation is verified, while the related stability analysis is carried out in Section 3. Then, numerical simulations are given to illustrate the theoretical results in Section 4. Finally, a conclusion is presented in Section 5.

2. Existence of Hopf bifurcation

From [19], we know that there are three equilibria for system (1.1); that is, $E_0 = (0,0)$, $E_1 = (k,0)$, $E_3 = (x^*, y^*)$, where

$$x^* = \frac{rkce - ak(e-bd) + \sqrt{[ak(e-bd) - rkce]^2 + 4racdek}}{2rce},$$
$$y^* = \frac{re}{ad}x^*\left(1 - \frac{x^*}{k}\right),$$

and if $e > \frac{1}{k}d(1+bk)$, then the equilibrium E_3 is a positive equilibrium. From the biological viewpoint, we are only interested in the equilibrium E_3.

Set

$$u_1 = x - x^*, \quad u_2 = y - y^*, \quad P(x,y) = \frac{x}{1+bx+cy}.$$

Then system (1.1) can be rewritten as follows:

$$\begin{aligned}\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} &= r(u_1+x^*)(1-\frac{u_1+x^*}{k}) - a(u_2+y^*)P(u_1+x^*, u_2+y^*),\\ \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} &= (u_2+y^*)\left[-d + e\int_{-\infty}^{t} F(t-\tau)P(u_1(\tau)+x^*, u_2(\tau)+y^*)\,\mathrm{d}\tau\right].\end{aligned} \tag{2.1}$$

Linearizing system (2.1) around E_3 yields

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Lu(t) + \int_{-\infty}^{0} K(\tau)u(t+\tau)\,\mathrm{d}\tau + H(u), \tag{2.2}$$

where $u = (u_1, u_2)^{\mathrm{T}}$ with T denoting the transpose, and

$$L = \begin{pmatrix} r - \frac{2rx^*}{k} - ay^*P_{10}(x^*,y^*) & -aP(x^*,y^*) - ay^*P_{01}(x^*,y^*) \\ 0 & 0 \end{pmatrix}, \tag{2.3}$$

$$K(\tau) = \begin{pmatrix} 0 & 0 \\ ey^*P_{10}(x^*,y^*)F(-\tau) & ey^*P_{01}(x^*,y^*)F(-\tau) \end{pmatrix}, \tag{2.4}$$

and

$$H(u) = \begin{pmatrix} -\frac{r}{k}u_1^2 - ay^*\sum\limits_{i+j=2}^{3}\frac{1}{i!j!}P_{ij}u_1^iu_2^j - au_2\sum\limits_{i+j=1}^{2}\frac{1}{i!j!}P_{ij}u_1^iu_2^j \\ ey^*\int_{-\infty}^{0}F(-\tau)\sum\limits_{i+j=2}^{3}\frac{1}{i!j!}P_{ij}u_1^i(t+\tau)u_2^j(t+\tau)\,\mathrm{d}\tau \\ +eu_2\int_{-\infty}^{0}F(-\tau)\sum\limits_{i+j=1}^{2}\frac{1}{i!j!}P_{ij}u_1^i(t+\tau)u_2^j(t+\tau)\,\mathrm{d}\tau \end{pmatrix} + \text{H.O.T.}, \tag{2.5}$$

with $\frac{\partial^{i+j}P(x,y)}{\partial x^i\,\partial y^j}\Big|_{(x,y)=(x^*,y^*)}$ denoted by P_{ij} and H.O.T for the shorthand of "higher order terms".

It is easy to obtain the characteristic equation of the linearized system (2.2) in the following form:

$$\begin{aligned}
D(\lambda) &= \det|\lambda I - L - \int_{-\infty}^{0} e^{\lambda\tau} K(\tau)\, d\tau| \\
&= \lambda^2 - ey^* P_{01}(x^*,y^*) \int_{-\infty}^{0} e^{\lambda\tau} F(-\tau)\, d\tau \lambda + \left(\frac{2rx^*}{k} - r + ay^* P_{10}(x^*,y^*)\right)\lambda - ey^* P_{01}(x^*,y^*) \\
&\quad \times \int_{-\infty}^{0} e^{\lambda\tau} F(-\tau)\, d\tau \left(\frac{2rx^*}{k} - r + ay^* P_{10}(x^*,y^*)\right) + aP(x^*,y^*) ey^* P_{10}(x^*,y^*) \int_{-\infty}^{0} e^{\lambda\tau} F(-\tau)\, d\tau \\
&\quad + ay^* P_{01}(x^*,y^*) ey^* P_{10}(x^*,y^*) \int_{-\infty}^{0} e^{\lambda\tau} F(-\tau)\, d\tau = 0. \qquad (2.6)
\end{aligned}$$

If $F(s)$ is a weak kernel ($i.e., F(s) = \alpha e^{-\alpha s},\ \alpha > 0$), then (2.6) becomes a third-order algebraic equation:

$$\lambda^3 + b_1(\alpha)\lambda^2 + b_2(\alpha)\lambda + b_3(\alpha) = 0, \qquad (2.7)$$

where

$$\begin{aligned}
b_1(\alpha) &= \alpha + \frac{2rx^*}{k} - r + ay^* P_{10}(x^*,y^*), \\
b_2(\alpha) &= \alpha\left[\frac{2rx^*}{k} - r + ay^* P_{10}(x^*,y^*) - ey^* P_{01}(x^*,y^*)\right], \\
b_3(\alpha) &= \alpha\left[ey^*(r - \frac{2rx^*}{k}) P_{01}(x^*,y^*) + aey^* P(x^*,y^*) P_{10}(x^*,y^*)\right].
\end{aligned}$$

In this paper, it is always assumed that $x^* > k/2$, then it is easily verified that $b_1(\alpha) > 0,\ b_2(\alpha) > 0,\ b_3(\alpha) > 0$.

Define a continuously differentiable function $\psi_1(\alpha) : (0, +\infty) \to \mathbb{R}$:

$$\psi_1(\alpha) = b_1(\alpha) b_2(\alpha) - b_3(\alpha).$$

The Routh–Hurwitz criterion [23] guarantees the locally asymptotic stability of equilibrium E_3 in case $\psi_1(\alpha) > 0$.

Obviously, if

$$\alpha_0 = \frac{ey^*\left(r - \frac{2rx^*}{k}\right) P_{01}(x^*,y^*) + aey^* P(x^*,y^*) P_{10}(x^*,y^*)}{\frac{2rx^*}{k} - r + ay^* P_{10}(x^*,y^*) - ey^* P_{01}(x^*,y^*)} - \left[\frac{2rx^*}{k} - r + ay^* P_{10}(x^*,y^*)\right],$$

then $\psi_1(\alpha_0) = 0$. Moreover, (2.7) has a pair of purely imaginary roots $\lambda_1 = \omega_0 i,\ \lambda_2 = -\omega_0 i$, where $\omega_0 = \sqrt{b_2(\alpha_0)}$ and a real root $\lambda_3 = -b_1(\alpha_0) < 0$.

By differentiating (2.7) with respect to α, it follows that

$$\frac{d}{d\alpha}[\mathrm{Re}\lambda_1]|_{\alpha=\alpha_0} = -\frac{1}{2[b_1^2(\alpha) + b_2(\alpha)]} \frac{d\psi_1(\alpha)}{d\alpha}|_{\alpha=\alpha_0},$$

where

$$\begin{aligned}
\frac{d\psi_1(\alpha)}{d\alpha}|_{\alpha=\alpha_0} &= \left[ey^*\left(r - \frac{2rx^*}{k}\right) P_{01}(x^*,y^*) + aey^* P(x^*,y^*) P_{10}(x^*,y^*)\right] \\
&\quad - \left[\frac{2rx^*}{k} - r + ay^* P_{10}(x^*,y^*)\right]\left[\frac{2rx^*}{k} - r + ay^* P_{10}(x^*,y^*) - ey^* P_{01}(x^*,y^*)\right] \\
&= \alpha_0\left[\frac{2rx^*}{k} - r + ay^* P_{10}(x^*,y^*) - ey^* P_{01}(x^*,y^*)\right].
\end{aligned}$$

Then $\frac{d\psi_1(\alpha)}{d\alpha}|_{\alpha=\alpha_0} > 0$, provided that $x^* > k/2$ and $\alpha_0 > 0$.

Therefore, the above analysis can be summarized as follows:

Theorem 2.1. *If $x^* > k/2$, then, when $\psi_1(\alpha) > 0$, the positive equilibrium $E_3 = (x^*, y^*)$ of system (1.1) is locally asymptotically stable. If $x^* > k/2$ and there exists $\alpha_0 > 0$ such that $\psi_1(\alpha_0) = 0$, then, as α crosses the critical value α_0, there is a Hopf bifurcation at E_3.*

On the other hand, if $F(s)$ is a strong kernel (*i.e.*, $F(s) = \alpha^2 s e^{-\alpha s},\ \alpha > 0$), then (2.6) becomes a fourth-order algebraic equation:

$$\lambda^4 + c_1(\alpha)\lambda^3 + c_2(\alpha)\lambda^2 + c_3(\alpha)\lambda + c_4(\alpha) = 0, \tag{2.8}$$

where

$$\begin{aligned}
c_1(\alpha) &= 2\alpha + \frac{2rx^*}{k} - r + ay^*P_{10}(x^*,y^*),\\
c_2(\alpha) &= \alpha^2 + 2\alpha\left(\frac{2rx^*}{k} - r + ay^*P_{10}(x^*,y^*)\right),\\
c_3(\alpha) &= \alpha^2\left[\frac{2rx^*}{k} - r + ay^*P_{10}(x^*,y^*) - ey^*P_{01}(x^*,y^*)\right],\\
c_4(\alpha) &= \alpha^2\left[ey^*\left(r - \frac{2rx^*}{k}\right)P_{01}(x^*,y^*) + aey^*P(x^*,y^*)P_{10}(x^*,y^*)\right].
\end{aligned}$$

With $x^* > k/2$, we also have $c_1(\alpha) > 0,\ c_2(\alpha) > 0,\ c_3(\alpha) > 0,\ c_4(\alpha) > 0$.

Define

$$\begin{aligned}
\psi_2(\alpha) &= c_1(\alpha)c_2(\alpha) - c_3(\alpha),\\
\psi_3(\alpha) &= c_3(\alpha)\psi_2(\alpha) - c_4(\alpha)c_1^2(\alpha).
\end{aligned}$$

By the Routh–Hurwitz criterion [23], the equilibrium E_3 is locally asymptotically stable if $\psi_2(\alpha) > 0$ and $\psi_3(\alpha) > 0$.

Denote λ_i $(i = 1,2,3,4)$ as the roots of the characteristic equation (2.8), and then we have

$$\begin{aligned}
&\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -c_1(\alpha),\\
&\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = c_2(\alpha),\\
&\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 = -c_3(\alpha),\\
&\lambda_1\lambda_2\lambda_3\lambda_4 = c_4(\alpha).
\end{aligned} \tag{2.9}$$

If $\psi_2(\alpha) > 0$ and there exists α_0 such that $\psi_3(\alpha_0) = 0$, then by the Routh–Hurwitz criterion [23], it follows that at least one root, say λ_1, with its real part equal to zero. From the fourth equation of (2.9), it follows that Im $\lambda_1 = \omega_0 \neq 0$, and hence there is another root, say λ_2, such that $\lambda_2 = \overline{\lambda}_1$. Since $\psi_3(\alpha)$ is a continuous function of its roots, λ_1 and λ_2 are complex conjugate for α in an open interval containing α_0. Therefore, Eqs. (2.9) have the following form at α_0:

$$\begin{aligned}
&\lambda_3 + \lambda_4 = -c_1(\alpha_0),\\
&\omega_0^2 + \lambda_3\lambda_4 = c_2(\alpha_0),\\
&\omega_0^2(\lambda_3 + \lambda_4) = -c_3(\alpha_0),\\
&\omega_0^2\lambda_3\lambda_4 = c_4(\alpha_0).
\end{aligned} \tag{2.10}$$

If λ_3 and λ_4 are complex conjugate, then $2\mathrm{Re}\lambda_3 = -c_1(\alpha_0) < 0$ can be derived from the first equation of (2.9). If λ_3 and λ_4 are real, then $\lambda_3 < 0$ and $\lambda_4 < 0$ from the first and fourth equations of (2.9). Moreover, it is not hard to see that

$$\frac{d}{d\alpha}[\mathrm{Re}\lambda_1]|_{\alpha=\alpha_0} = -\frac{c_1(\alpha)}{2[c_1^3(\alpha)c_3(\alpha) + (c_1(\alpha)c_2(\alpha) - 2c_3(\alpha))^2]}\left.\frac{d\psi_3(\alpha)}{d\alpha}\right|_{\alpha=\alpha_0}.$$

Thus, we have the following result.

Theorem 2.2. *If $x^* > k/2$, then, when $\psi_2(\alpha) > 0$ and $\psi_3(\alpha) > 0$, the positive equilibrium $E_3 = (x^*, y^*)$ of system (1.1) is locally asymptotically stable. If $x^* > k/2$, $\psi_2(\alpha) > 0$ and there exists $\alpha_0 > 0$ such that $\psi_1(\alpha_0) = 0$ and $\frac{d\psi_3(\alpha)}{d\alpha}|_{\alpha=\alpha_0} \neq 0$, then α_0 is the critical value of a Hopf bifurcation at E_3.*

3. Stability of bifurcating periodic solutions

In this section, the algorithm given by Hassard et al. [22] is employed to analyze the bifurcating periodic solutions. Here, for simplicity, we mainly consider the case when the distributed kernel $F(t)$ is a weak kernel, i.e. $F(s) = \alpha e^{-\alpha s},\ \alpha > 0$. In fact, the case of the strong kernel can be discussed similarly.

For convenience, we transform system (2.2) into an operator equation of the following form:

$$\frac{du_t}{dt} = Au_t + Fu_t, \tag{3.1}$$

where $u = (u_1, u_2)^{\mathrm{T}}$, $u_t = u(t+\theta)$, $\theta \in (-\infty, 0]$, and the operators A and F are defined as follows:

$$A\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -\infty < \theta < 0, \\ L\phi(\theta) + \int_{-\infty}^{0} K(\tau)\phi(\tau)\,d\tau, & \theta = 0, \end{cases} \tag{3.2}$$

and

$$F\phi(\theta) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & -\infty < \theta < 0, \\ \begin{pmatrix} -\frac{r}{k}\phi_1(0)^2 - ay^* \sum\limits_{i+j=2}^{3} \frac{1}{i!j!} P_{ij}\phi_1(0)^i\phi_2(0)^j \\ -a\phi_2(0) \sum\limits_{i+j=1}^{2} \frac{1}{i!j!} P_{ij}\phi_1(0)^i\phi_2(0)^j \\ ey^* \int_{-\infty}^{0} \alpha e^{\alpha\tau} \sum\limits_{i+j=2}^{3} \frac{1}{i!j!} P_{ij}\phi_1^i(\tau)\phi_2^j(\tau)\,d\tau \\ +e\phi_2(0) \int_{-\infty}^{0} \alpha e^{\alpha\tau} \sum\limits_{i+j=1}^{2} \frac{1}{i!j!} P_{ij}\phi_1^i(\tau)\phi_2^j(\tau)\,d\tau \end{pmatrix}, & \theta = 0, \end{cases} \tag{3.3}$$

where L and K are defined as in (2.3) and (2.4), $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^{\mathrm{T}}$. The H.O.T. in (2.5) is omitted here because determining the direction and the stability of Hopf bifurcation only needs up to the third order terms.

Note that the operator A depends on parameter α. By Theorem 2.1, a Hopf bifurcation occurs when α passes through α_0. Let

$$\mu = \alpha - \alpha_0.$$

Then the Hopf bifurcation occurs when $\mu = 0$.

The adjoint operator A^* of A is defined as

$$A^*\psi(\delta) = \begin{cases} -\frac{d\psi(\delta)}{d\delta}, & 0 < \delta < \infty, \\ L^{\mathrm{T}}\psi(\delta) + \int_{-\infty}^{0} K^{\mathrm{T}}(\tau)\psi(-\tau)\,d\tau, & \delta = 0, \end{cases} \tag{3.4}$$

where L^{T} and K^{T} are the transpose of the matrices L and K, respectively. Note that A and A^* can have complex eigenvectors. It is therefore suitable to assume that $\psi \in C([0, +\infty), \mathbb{C}^2)$ and $\phi \in C((-\infty, 0], \mathbb{C}^2)$. Define the bilinear form:

$$\langle \psi, \phi \rangle = \overline{\psi}^{\mathrm{T}}(0)\phi(0) - \int_{\theta=-\infty}^{0} \int_{\xi=0}^{\theta} \overline{\psi}^{\mathrm{T}}(\xi - \theta)K(\theta)\phi(\xi)\,d\xi\,d\theta.$$

To obtain the Poincare normal form of the operator A, we need to calculate the eigenvector q of A corresponding to the eigenvalue $i\omega_0$ and the eigenvector q^* of A^* to the eigenvalue $-i\omega_0$. Clearly,

$$q(\theta) = \begin{pmatrix} 1 \\ B \end{pmatrix} e^{i\omega_0\theta}, \quad -\infty < \theta < 0,$$

where

$$B = \frac{\alpha e y^* P_{10}(x^*, y^*)}{i\omega_0(\alpha + i\omega_0) - \alpha e y^* P_{01}(x^*, y^*)},$$

and

$$q^*(\delta) = E\begin{pmatrix} 1 \\ C \end{pmatrix} \mathrm{e}^{\mathrm{i}\omega_0\delta}, \quad 0 < \delta < \infty,$$

where

$$C = \frac{(\alpha - \mathrm{i}\omega_0)[aP(x^*, y^*) + ay^*P_{01}(x^*, y^*)]}{\mathrm{i}\omega_0(\alpha - \mathrm{i}\omega_0) + \alpha e y^* P_{01}(x^*, y^*)},$$

$$\overline{E} = \frac{(\alpha + \mathrm{i}\omega_0)^2}{(\alpha + \mathrm{i}\omega_0)^2(1 + \overline{C}B) + \alpha[\overline{C}ey^*P_{10}(x^*, y^*) + \overline{C}Bey^*P_{01}(x^*, y^*)]}.$$

It is straightforward to obtain that

$$\langle q^*, q\rangle = 1, \quad \langle q^*, \overline{q}\rangle = 0.$$

With the same notation as Hassard et al. did in [22], we first construct the coordinates to describe the center manifold ω_0 at $\mu = 0$ (i.e., $\alpha = \alpha_0$). Define

$$z(t) = \langle q^*, u_t\rangle, \quad w(t, \theta) = u_t - 2\mathrm{Re}\{z(t)q(\theta)\}. \tag{3.5}$$

On the center manifold ω_0, $w(t, \theta) = w(z(t), \overline{z}(t), \theta)$, where

$$w(z, \overline{z}, \theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\overline{z} + w_{02}(\theta)\frac{\overline{z}^2}{2} + w_{30}(\theta)\frac{z^3}{6} + \cdots, \tag{3.6}$$

where z and $\overline{z}$ are local coordinates for the center manifold ω_0 in the direction of q^* and $\overline{q^*}$. Note that w is real if u_t is real. We are only interested in real solutions.

For solution $u_t \in \omega_0$ of (3.1), from $\mu = 0$,

$$\dot{z}(t) = \mathrm{i}\omega_0 z(t) + \langle q^*(\theta), F(w(z, \overline{z}, \theta) + 2\mathrm{Re}\{z(t)q(\theta)\})\rangle = \mathrm{i}\omega_0 z(t) + [\overline{q^*}(0)]^{\mathrm{T}} F(w(z, \overline{z}, 0) + 2\mathrm{Re}\{z(t)q(0)\}),$$

or equivalently,

$$\dot{z}(t) = \mathrm{i}\omega_0 z(t) + g(z, \overline{z}), \tag{3.7}$$

where

$$g(z, \overline{z}) = [\overline{q^*}(0)]^{\mathrm{T}} F(w(z, \overline{z}, 0) + 2\mathrm{Re}\{z(t)q(0)\}). \tag{3.8}$$

Recalling (3.1) and (3.5), we have

$$\begin{aligned}\dot{w} = \dot{u}_t - \dot{z}q - \dot{\overline{z}}\overline{q} &= Aw - 2\mathrm{Re}\{\langle q^*(\theta), F(w(z, \overline{z}, \theta) + 2\mathrm{Re}\{z(t)q(\theta)\})\rangle q(\theta)\} + F(w(z, \overline{z}, \theta) + 2\mathrm{Re}\{z(t)q(\theta)\}) \\ &= Aw - 2\mathrm{Re}\{g(z, \overline{z})q(\theta)\} + F(w(z, \overline{z}, \theta) + 2\mathrm{Re}\{z(t)q(\theta)\}),\end{aligned}$$

which can also be expressed as

$$\dot{w} = Aw + H(z, \overline{z}, \theta), \tag{3.9}$$

where

$$H(z, \overline{z}, \theta) = -2\mathrm{Re}\{g(z, \overline{z})q(\theta)\} + F(w(z, \overline{z}, \theta) + 2\mathrm{Re}\{z(t)q(\theta)\}). \tag{3.10}$$

We expand the function $g(z, \overline{z})$ on the center manifold ω_0 in powers of z and $\overline{z}$; that is,

$$g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{3.11}$$

The coefficients of (3.11) can be fixed by comparing (3.11) with (3.8), where w is replaced by its expansion (3.6). In order to determine the coefficients $w_{ij}(\theta)$ of the expansion (3.6), we expand the function $H(z, \overline{z}, \theta)$ in powers of z and $\overline{z}$ on the center manifold ω_0; that is,

$$H(z, \overline{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{3.12}$$

The argument of F is

$$w+zq(\theta)+\overline{zq}(\theta)=\begin{pmatrix} w^{(1)}(\theta)+z\mathrm{e}^{\mathrm{i}\omega_0\theta}+\bar{z}\mathrm{e}^{-\mathrm{i}\omega_0\theta} \\ w^{(2)}(\theta)+zB\mathrm{e}^{\mathrm{i}\omega_0\theta}+\bar{z}\overline{B}\mathrm{e}^{-\mathrm{i}\omega_0\theta} \end{pmatrix}\mathrm{e}^{\mathrm{i}\omega_0\theta},\quad -\infty<\theta<0,$$

where $w=[w^{(1)}(\theta),w^{(2)}(\theta)]^{\mathrm{T}}$. Thus,

$$F(w(z,\bar{z},\theta)+2\mathrm{Re}\{z(t)q(\theta)\})=\begin{cases}\begin{pmatrix}0\\0\end{pmatrix}, & -\infty<\theta<0,\\ \begin{pmatrix}f_0^1\\ f_0^2\end{pmatrix}, & \theta=0,\end{cases}$$

where

$$f_0^1=-\frac{r}{k}[w^{(1)}(0)+z+\bar{z}]^2-ay^*\sum_{i+j=2}^{3}\frac{1}{i!j!}P_{ij}(w^{(1)}(0)+z+\bar{z})^i(w^{(2)}(0)+zB+\bar{z}\overline{B})^j-a(w^{(2)}(0)+zB+\bar{z}\overline{B})$$
$$\times\sum_{i+j=1}^{2}\frac{1}{i!j!}P_{ij}(w^{(1)}(0)+z+\bar{z})^i(w^{(2)}(0)+zB+\bar{z}\overline{B})^j,$$

$$f_0^2=ey^*\int_{-\infty}^0\alpha\mathrm{e}^{\alpha\tau}\sum_{i+j=2}^{3}\frac{1}{i!j!}P_{ij}(w^{(1)}(\tau)+z\mathrm{e}^{\mathrm{i}\omega_0\tau}+\bar{z}\mathrm{e}^{-\mathrm{i}\omega_0\tau})^i(w^{(2)}(\tau)+zB\mathrm{e}^{\mathrm{i}\omega_0\tau}+\bar{z}\overline{B}\mathrm{e}^{-\mathrm{i}\omega_0\tau})^j\,\mathrm{d}\tau$$
$$+e(w^{(2)}(0)+zB+\bar{z}\overline{B})\int_{-\infty}^0\alpha\mathrm{e}^{\alpha\tau}\sum_{i+j=1}^{2}\frac{1}{i!j!}P_{ij}(w^{(1)}(\tau)+z\mathrm{e}^{\mathrm{i}\omega_0\tau}+\bar{z}\mathrm{e}^{-\mathrm{i}\omega_0\tau})^i$$
$$\times(w^{(2)}(\tau)+zB\mathrm{e}^{\mathrm{i}\omega_0\tau}+\bar{z}\overline{B}\mathrm{e}^{-\mathrm{i}\omega_0\tau})^j\,\mathrm{d}\tau.$$

By (3.8), it follows that

$$g(z,\bar{z})=\overline{E}f_0^1(z,\bar{z})+\overline{EC}f_0^2(z,\bar{z}).$$

Therefore,

$$H(z,\bar{z},\theta)=-2\mathrm{Re}\{[\overline{E}f_0^1(z,\bar{z})+\overline{EC}f_0^2(z,\bar{z})]q(\theta)\}+\begin{cases}\begin{pmatrix}0\\0\end{pmatrix}, & -\infty<\theta<0,\\ \begin{pmatrix}f_0^1(z,\bar{z})\\ f_0^2(z,\bar{z})\end{pmatrix}, & \theta=0.\end{cases}$$

Set

$$G_1=-\frac{2r}{k}-2BaP_{10}(x^*,y^*)-2B^2aP_{01}(x^*,y^*)-ay^*(P_{20}+2BP_{11}+B^2P_{02}),$$
$$G_2=\frac{2eB\alpha[P_{10}(x^*,y^*)+BP_{01}(x^*,y^*)]}{\alpha+\mathrm{i}\omega_0}+\frac{ey^*\alpha[P_{20}+2BP_{11}+B^2P_{02}]}{\alpha+2\mathrm{i}\omega_0}.$$

Then

$$\left[\frac{\partial^2 f_0^1(z,\bar{z})}{\partial z^2}\right]_{z=\bar{z}=0}=\left[\frac{\partial^2\overline{f}_0^1(z,\bar{z})}{\partial z^2}\right]_{z=\bar{z}=0}=G_1,$$
$$\left[\frac{\partial^2 f_0^2(z,\bar{z})}{\partial z^2}\right]_{z=\bar{z}=0}=\left[\frac{\partial^2\overline{f}_0^2(z,\bar{z})}{\partial z^2}\right]_{z=\bar{z}=0}=G_2.$$

Therefore, by (3.10), we can obtain

$$H_{20} = \left[\frac{\partial^2 H(z,\bar{z},\theta)}{\partial z^2}\right]_{z=\bar{z}=0} = -\overline{E}(G_1+\overline{C}G_2)q(\theta) - E(G_1+CG_2)\bar{q}(\theta) + \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & -\infty<\theta<0, \\ \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, & \theta=0. \end{cases} \tag{3.13}$$

Similarly, set

$$G_3 = -\frac{2r}{k} - \overline{B}aP_{10}(x^*,y^*) - BaP_{10}(x^*,y^*) - 2B\overline{B}aP_{01}(x^*,y^*) - ay^*[P_{20}+(B+\overline{B})P_{11}+B\overline{B}P_{02}],$$

$$G_4 = \frac{eB\alpha[P_{10}(x^*,y^*)+\overline{B}P_{01}(x^*,y^*)]}{\alpha-\mathrm{i}\omega_0} + \frac{e\overline{B}\alpha[P_{10}(x^*,y^*)+BP_{01}(x^*,y^*)]}{\alpha+\mathrm{i}\omega_0} + ey^*[P_{20}+(B+\overline{B})P_{11}+B\overline{B}P_{02}].$$

Then

$$\left[\frac{\partial^2 f_0^1(z,\bar{z})}{\partial z\partial\bar{z}}\right]_{z=\bar{z}=0} = \left[\frac{\partial^2 \overline{f}_0^1(z,\bar{z})}{\partial z\partial\bar{z}}\right]_{z=\bar{z}=0} = G_3,$$

$$\left[\frac{\partial^2 f_0^2(z,\bar{z})}{\partial z\partial\bar{z}}\right]_{z=\bar{z}=0} = \left[\frac{\partial^2 \overline{f}_0^2(z,\bar{z})}{\partial z\partial\bar{z}}\right]_{z=\bar{z}=0} = G_4.$$

As a result,

$$H_{11} = \left[\frac{\partial^2 H(z,\bar{z},\theta)}{\partial z\partial\bar{z}}\right]_{z=\bar{z}=0} = -\overline{E}(G_3+\overline{C}G_4)q(\theta) - E(G_3+CG_4)\bar{q}(\theta) + \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & -\infty<\theta<0, \\ \begin{pmatrix} G_3 \\ G_4 \end{pmatrix}, & \theta=0. \end{cases} \tag{3.14}$$

On the other hand, on the center manifold ω_0 near the origin, we have

$$\dot{w}(z,\bar{z}) = w_z\dot{z} + w_{\bar{z}}\dot{\bar{z}}. \tag{3.15}$$

Substituting (3.6) for w_z and $w_{\bar{z}}$ and (3.7) for $\dot{z}$ and $\dot{\bar{z}}$, we obtain a second expression of $\dot{w}$. In comparison with (3.9), the equations for the coefficients $w_{ij}(\theta)$ can be derived as follows:

$$(2\mathrm{i}\omega_0 I - A)w_{20}(\theta) = H_{20}(\theta), \tag{3.16}$$

$$-Aw_{11}(\theta) = H_{11}(\theta), \tag{3.17}$$

and $w_{02} = \overline{w}_{20}$. Define

$$w_{20}(\theta) = \begin{pmatrix} w_{20}^{(1)}(\theta) \\ w_{20}^{(2)}(\theta) \end{pmatrix}, \quad -\infty<\theta<0.$$

By substituting (3.3) and (3.13) into (3.16), when $-\infty<\theta<0$, we have

$$\begin{pmatrix} 2\mathrm{i}\omega_0 - \frac{\mathrm{d}}{\mathrm{d}\theta} & 0 \\ 0 & 2\mathrm{i}\omega_0 - \frac{\mathrm{d}}{\mathrm{d}\theta} \end{pmatrix} \begin{pmatrix} w_{20}^{(1)}(\theta) \\ w_{20}^{(2)}(\theta) \end{pmatrix} = \begin{pmatrix} -\overline{E}(G_1+\overline{C}G_2)\mathrm{e}^{\mathrm{i}\omega_0\theta} - E(G_1+CG_2)\mathrm{e}^{-\mathrm{i}\omega_0\theta} \\ -\overline{E}(G_1+\overline{C}G_2)B\mathrm{e}^{\mathrm{i}\omega_0\theta} - E(G_1+CG_2)\overline{B}\mathrm{e}^{-\mathrm{i}\omega_0\theta} \end{pmatrix}. \tag{3.18}$$

If $\theta=0$, then

$$\begin{pmatrix} 2\mathrm{i}\omega_0\left[r-\frac{2rx^*}{k} - ay^*P_{10}(x^*,y^*)\right] & aP(x^*,y^*)+ay^*P_{01}(x^*,y^*) \\ 0 & 2\mathrm{i}\omega_0 \end{pmatrix} \begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0) \end{pmatrix} - \int_{-\infty}^{0} \begin{pmatrix} 0 & 0 \\ ey^*P_{10}(x^*,y^*)\alpha\mathrm{e}^{\alpha\tau} & ey^*P_{01}(x^*,y^*)\alpha\mathrm{e}^{\alpha\tau} \end{pmatrix} \begin{pmatrix} w_{20}^{(1)}(\tau) \\ w_{20}^{(2)}(\tau) \end{pmatrix}\mathrm{d}\tau = \begin{pmatrix} H_{20}^{(1)}(0) \\ H_{20}^{(2)}(0) \end{pmatrix}. \tag{3.19}$$

In order to obtain a continuous solution $w(\theta)$ on $(-\infty, 0]$, we consider the above equations associated with the following boundary conditions:

$$\lim_{\theta\to 0^-}\begin{pmatrix} w_{20}^{(1)}(\theta) \\ w_{20}^{(2)}(\theta)\end{pmatrix} = \begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0)\end{pmatrix}. \tag{3.20}$$

The general solution to (3.18) is

$$\begin{pmatrix} w_{20}^{(1)}(\theta) \\ w_{20}^{(2)}(\theta)\end{pmatrix} = \begin{pmatrix} k_0 \\ l_0\end{pmatrix}e^{2i\omega_0\theta} + \begin{pmatrix} k_1 \\ l_1\end{pmatrix}e^{i\omega_0\theta} + \begin{pmatrix} k_2 \\ l_2\end{pmatrix}e^{-i\omega_0\theta} \tag{3.21}$$

where

$$k_1 = -\frac{1}{i\omega_0}\overline{E}(G_1+\overline{C}G_2)e^{i\omega_0\theta}, \quad l_1 = Bk_1,$$
$$k_2 = -\frac{1}{3i\omega_0}E(G_1+CG_2)e^{-i\omega_0\theta}, \quad l_2 = \overline{B}k_2,$$

and l_0 and k_0 are determined by (3.20), i.e.

$$k_0 = w_{20}^{(1)}(0) - (k_1+k_2), \quad l_0 = w_{20}^{(2)}(0) - (l_1+l_2).$$

To find $w_{20}^{(1)}(0)$ and $w_{20}^{(2)}(0)$, we plug (3.21) into Eq. (3.19) to obtain

$$\begin{pmatrix} 2i\omega_0 - \left[r - \frac{2rx^*}{k} - ay^*P_{10}(x^*,y^*)\right] & aP(x^*,y^*) + ay^*P_{01}(x^*,y^*) \\ -\frac{ey^*\alpha P_{10}(x^*,y^*)}{\alpha+2i\omega_0} & 2i\omega_0 - \frac{ey^*\alpha P_{01}(x^*,y^*)}{\alpha+2i\omega_0}\end{pmatrix}\begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0)\end{pmatrix} = \begin{pmatrix} c_{20}^{(1)} \\ c_{20}^{(2)}\end{pmatrix}. \tag{3.22}$$

where

$$c_{20}^{(1)} = H_{20}^{(1)}(0),$$
$$c_{20}^{(2)} = H_{20}^{(2)}(0) - ey^*\alpha P_{10}(x^*,y^*)\left[\frac{k_1+k_2}{\alpha+2i\omega_0} - \frac{k_1}{\alpha+i\omega_0} - \frac{k_2}{\alpha-i\omega_0}\right]$$
$$- ey^*\alpha P_{01}(x^*,y^*)\left[\frac{l_1+l_2}{\alpha+2i\omega_0} - \frac{l_1}{\alpha+i\omega_0} - \frac{l_2}{\alpha-i\omega_0}\right].$$

Define

$$\Delta = \left[2i\omega_0 - \frac{ey^*\alpha P_{01}(x^*,y^*)}{\alpha+2i\omega_0}\right]\left[2i\omega_0 - \left(r - \frac{2rx^*}{k} - ay^*P_{10}(x^*,y^*)\right)\right] + \frac{ey^*\alpha P_{10}(x^*,y^*)}{\alpha+2i\omega_0}$$
$$\times [aP(x^*,y^*) + ay^*P_{01}(x^*,y^*)]$$

Then,

$$w_{20}^{(1)}(0) = \frac{c_{20}^{(1)}\left[2i\omega_0 - \frac{ey^*\alpha P_{01}(x^*,y^*)}{\alpha+2i\omega_0}\right] - c_{20}^{(2)}[aP(x^*,y^*) + ay^*P_{01}(x^*,y^*)]}{\Delta},$$
$$w_{20}^{(2)}(0) = \frac{c_{20}^{(2)}\left[2i\omega_0 - (r - \frac{2rx^*}{k} - ay^*P_{10}(x^*,y^*))\right] + c_{20}^{(1)}\frac{ey^*\alpha P_{10}(x^*,y^*)}{\alpha+2i\omega_0}}{\Delta}.$$

Similarly, by Eq. (3.19),

$$\begin{pmatrix} w_{11}^{(1)}(\theta) \\ w_{11}^{(2)}(\theta)\end{pmatrix} = \begin{pmatrix} p_0 \\ q_0\end{pmatrix} + \begin{pmatrix} p_1 \\ q_1\end{pmatrix}e^{i\omega_0\theta} + \begin{pmatrix} p_2 \\ q_2\end{pmatrix}e^{-i\omega_0\theta}, \tag{3.23}$$

where

$$p_1 = \frac{1}{i\omega_0}\overline{E}(G_3 + \overline{C}G_4)e^{i\omega_0\theta}, \quad q_1 = Bp_1,$$
$$p_2 = -\frac{1}{i\omega_0}E(G_3 + CG_4)e^{-i\omega_0\theta}, \quad q_2 = \overline{B}p_2,$$
$$p_0 = w_{11}^{(1)}(0) - (p_1 + p_2), \quad q_0 = w_{11}^{(2)}(0) - (q_1 + q_2).$$

To obtain $w_{11}^{(1)}(0)$ and $w_{11}^{(2)}(0)$, we substitute (3.23) into (3.17), which leads to

$$\begin{pmatrix} \frac{2rx^*}{k} - r + ay^*P_{10}(x^*, y^*) & aP(x^*, y^*) + ay^*P_{01}(x^*, y^*) \\ -ey^*P_{10}(x^*, y^*) & -ey^*P_{01}(x^*, y^*) \end{pmatrix}\begin{pmatrix} w_{11}^{(1)}(0) \\ w_{11}^{(2)}(0) \end{pmatrix} = \begin{pmatrix} c_{11}^{(1)} \\ c_{11}^{(2)} \end{pmatrix}. \tag{3.24}$$

where

$$c_{11}^{(1)} = H_{11}^{(1)}(0),$$
$$c_{11}^{(2)} = H_{11}^{(2)}(0) - ey^*P_{10}(x^*, y^*)\left[\frac{p_1 i\omega_0}{\alpha + i\omega_0} - \frac{-p_2 i\omega_0}{\alpha - i\omega_0}\right] - ey^*P_{01}(x^*, y^*)\left[\frac{q_1 i\omega_0}{\alpha + i\omega_0} - \frac{-q_2 i\omega_0}{\alpha - i\omega_0}\right].$$

Define

$$\varDelta_1 = -\left[\frac{2rx^*}{k} - r + ay^*P_{10}(x^*, y^*)\right][ey^*P_{01}(x^*, y^*)] + ey^*P_{10}(x^*, y^*)[aP(x^*, y^*) + ay^*P_{01}(x^*, y^*)]$$

Therefore, it is quite easy to get

$$w_{11}^{(1)}(0) = \frac{-c_{11}^{(1)}[ey^*P_{01}(x^*, y^*)] - c_{11}^{(2)}[aP(x^*, y^*) + ay^*P_{01}(x^*, y^*)]}{\varDelta_1},$$
$$w_{11}^{(2)}(0) = \frac{c_{11}^{(2)}\left[\frac{2rx^*}{k} - r + ay^*P_{10}(x^*, y^*)\right] + c_{11}^{(1)}ey^*P_{10}(x^*, y^*)}{\varDelta_1}.$$

Now it is time to consider $F(w(z, \bar{z}, 0) + 2\mathrm{Re}\{z(t)q(0)\})$. Since

$$\begin{aligned} F(w(z, \bar{z}, 0) + 2\mathrm{Re}\{z(t)q(0)\}) &= w_{20}(0)\frac{z^2}{2} + w_{11}(0)z\bar{z} + w_{02}(0)\frac{\bar{z}^2}{2} + \cdots + 2\mathrm{Re}\{z(t)q(0)\} \\ &= \begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0) \end{pmatrix}\frac{z^2}{2} + \begin{pmatrix} w_{11}^{(1)}(0) \\ w_{11}^{(2)}(0) \end{pmatrix}z\bar{z} + \begin{pmatrix} w_{02}^{(1)}(0) \\ w_{02}^{(2)}(0) \end{pmatrix}\frac{\bar{z}^2}{2} + \cdots + 2\mathrm{Re}\{z(t)q(0)\}, \end{aligned}$$

we have

$$\begin{aligned} f_0^1 = &-\frac{r}{k}\left[w_{20}^{(1)}(0)\frac{z^2}{2} + w_{11}^{(1)}(0)z\bar{z} + w_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots + z + \bar{z}\right]^2 \\ &- ay^*\sum_{i+j=2}^{3}\frac{1}{i!j!}P_{ij}\left(w_{20}^{(1)}(0)\frac{z^2}{2} + w_{11}^{(1)}(0)z\bar{z} + w_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots + z + \bar{z}\right)^i \\ &\times \left(w_{20}^{(2)}(0)\frac{z^2}{2} + w_{11}^{(2)}(0)z\bar{z} + w_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots + zB + \bar{z}\overline{B}\right)^j \\ &- a\left[w_{20}^{(2)}(0)\frac{z^2}{2} + w_{11}^{(2)}(0)z\bar{z} + w_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots + zB + \bar{z}\overline{B}\right] \\ &\times \sum_{i+j=1}^{2}\frac{1}{i!j!}P_{ij}\left(w_{20}^{(1)}(0)\frac{z^2}{2} + w_{11}^{(1)}(0)z\bar{z} + w_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots + z + \bar{z}\right)^i \\ &\times \left(w_{20}^{(2)}(0)\frac{z^2}{2} + w_{11}^{(2)}(0)z\bar{z} + w_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots + zB + \bar{z}\overline{B}\right)^j, \end{aligned}$$

and

$$
\begin{aligned}
f_0^2 &= ey^*\int_{-\infty}^{0}\alpha e^{\alpha\tau}\sum_{i+j=2}^{3}\frac{1}{i!j!}P_{ij}\left(w_{20}^{(1)}(\tau)\frac{z^2}{2}+w_{11}^{(1)}(\tau)z\bar{z}+w_{02}^{(1)}(\tau)\frac{\bar{z}^2}{2}+\cdots+ze^{i\omega_0\tau}+\bar{z}e^{-i\omega_0\tau}\right)^i \\
&\quad\times\left(w_{20}^{(2)}(\tau)\frac{z^2}{2}+w_{11}^{(2)}(\tau)z\bar{z}+w_{02}^{(2)}(\tau)\frac{\bar{z}^2}{2}+\cdots+zBe^{i\omega_0\tau}+\bar{z}\overline{B}e^{-i\omega_0\tau}\right)^j d\tau \\
&\quad+e\left[w_{20}^{(2)}(0)\frac{z^2}{2}+w_{11}^{(2)}(0)z\bar{z}+w_{02}^{(2)}(0)\frac{\bar{z}^2}{2}+\cdots+zB+\bar{z}\overline{B}\right]\times\int_{-\infty}^{0}\alpha e^{\alpha\tau}\sum_{i+j=1}^{2}\frac{1}{i!j!}P_{ij}\left(w_{20}^{(1)}(\tau)\frac{z^2}{2}+w_{11}^{(1)}(\tau)z\bar{z}\right. \\
&\quad\left.+w_{02}^{(1)}(\tau)\frac{\bar{z}^2}{2}+\cdots+ze^{i\omega_0\tau}+\bar{z}e^{-i\omega_0\tau}\right)^i \\
&\quad\times\left(w_{20}^{(2)}(\tau)\frac{z^2}{2}+w_{11}^{(2)}(\tau)z\bar{z}+w_{02}^{(2)}(\tau)\frac{\bar{z}^2}{2}+\cdots+zBe^{i\omega_0\tau}+\bar{z}\overline{B}e^{-i\omega_0\tau}\right)^j d\tau.
\end{aligned}
$$

Thus,

$$
g(z,\bar{z})=[\overline{q^*}(0)]^{\mathrm{T}}F(w(z,\bar{z},0)+2\mathrm{Re}\{z(t)q(0)\})=(\overline{E},\overline{EC})\begin{pmatrix}f_0^{(1)}\\ f_0^{(2)}\end{pmatrix}=\overline{E}(f_0^{(1)}+\overline{C}f_0^{(2)}). \tag{3.25}
$$

The comparison of the coefficients of (3.11) and (3.25) yields

$$
\begin{aligned}
g_{20} &= 2\overline{E}\left[-\frac{r}{k}-aP_{10}(x^*,y^*)B-aP_{01}(x^*,y^*)B^2-\frac{ay^*}{2}(p_{20}+2p_{11}B+p_{02}B^2)\right. \\
&\quad\left.+\overline{C}\left[\frac{eB\alpha}{\alpha+iw_0}(P_{10}(x^*,y^*)+BP_{01}(x^*,y^*))+\frac{ey^*\alpha}{2(\alpha+2iw_0)}(p_{20}+2Bp_{11}+B^2p_{02})\right]\right],
\end{aligned}
$$

$$
\begin{aligned}
g_{11} &= \overline{E}\left[-\frac{2r}{k}-aP_{10}(x^*,y^*)(B+\overline{B})-2aP_{01}(x^*,y^*)B\overline{B}-ay^*(p_{20}+p_{11}(B+\overline{B})+p_{02}B\overline{B})\right. \\
&\quad+\overline{C}\left[\frac{eB\alpha}{\alpha-iw_0}(P_{10}(x^*,y^*)+\overline{B}P_{01}(x^*,y^*))+\frac{e\overline{B}\alpha}{\alpha+iw_0}(P_{10}(x^*,y^*)+BP_{01}(x^*,y^*))\right. \\
&\quad\left.\left.+ey^*(p_{20}+p_{11}(B+\overline{B})+p_{02}B\overline{B})\right]\right],
\end{aligned}
$$

$$
\begin{aligned}
g_{02} &= 2\overline{E}\left[-\frac{r}{k}-aP_{10}(x^*,y^*)\overline{B}-aP_{01}(x^*,y^*)\overline{B}^2-\frac{ay^*}{2}(p_{20}+2p_{11}\overline{B}+p_{02}\overline{B}^2)\right. \\
&\quad\left.+\overline{C}\left[\frac{e\overline{B}\alpha}{\alpha-iw_0}(P_{10}(x^*,y^*)+\overline{B}P_{01}(x^*,y^*))+\frac{ey^*\alpha}{2(\alpha-2iw_0)}(p_{20}+2\overline{B}p_{11}+\overline{B}^2p_{02})\right]\right],
\end{aligned}
$$

and

$$
\begin{aligned}
g_{21} = 2\overline{E}\Big\{ & -\frac{r}{k}[w_{20}^{(1)}(0) + 2w_{11}^{(1)}(0)] - aP_{10}(x^*,y^*)\left[\frac{1}{2}\overline{B}w_{20}^{(1)}(0) + Bw_{11}^{(1)}(0) + \frac{1}{2}w_{20}^{(2)}(0) + w_{11}^{(2)}(0)\right] \\
& - aP_{01}(x^*,y^*)[\overline{B}w_{20}^{(2)}(0) + 2Bw_{11}^{(2)}(0)] - ay^*\left[\frac{1}{2}p_{20}(w_{20}^{(1)}(0) + 2w_{11}^{(1)}(0))\right. \\
& \left. + p_{11}\left(\frac{1}{2}\overline{B}w_{20}^{(1)}(0) + Bw_{11}^{(1)}(0) + \frac{1}{2}w_{20}^{(2)}(0) + w_{11}^{(2)}(0)\right) + \frac{1}{2}p_{02}\left(\overline{B}w_{20}^{(2)}(0) + 2Bw_{11}^{(2)}(0)\right)\right] \\
& - \frac{ay^*}{2}[p_{30} + p_{21}(\overline{B} + 2B) + p_{12}(B^2 + 2B\overline{B}) + p_{03}B^2\overline{B}] \\
& - \frac{a}{2}[p_{20}(\overline{B} + 2B) + 2p_{11}(2B\overline{B} + B^2) + 3p_{02}B^2\overline{B}] \\
& + \overline{C}\left[eP_{10}(x^*,y^*)\left[\frac{\alpha w_{20}^{(2)}(0)}{2(\alpha - \mathrm{i}w_0)} + \frac{\alpha w_{11}^{(2)}(0)}{\alpha + \mathrm{i}w_0} + B\int_{-\infty}^{0}\alpha e^{\alpha\tau}w_{11}^{(1)}(\tau)\,d\tau + \frac{\overline{B}}{2}\int_{-\infty}^{0}\alpha e^{\alpha\tau}w_{20}^{(1)}(\tau)\,d\tau\right]\right. \\
& + eP_{01}(x^*,y^*)\left[\frac{\alpha\overline{B}w_{20}^{(2)}(0)}{2(\alpha - \mathrm{i}w_0)} + \frac{\alpha Bw_{11}^{(2)}(0)}{\alpha + \mathrm{i}w_0} + B\int_{-\infty}^{0}\alpha e^{\alpha\tau}w_{11}^{(2)}(\tau)\,d\tau + \frac{\overline{B}}{2}\int_{-\infty}^{0}\alpha e^{\alpha\tau}w_{20}^{(2)}(\tau)\,d\tau\right] \\
& + eP(x^*,y^*)\left[\frac{1}{2}p_{20}\left(\int_{-\infty}^{0}\alpha e^{(\alpha - \mathrm{i}\omega_0)\tau}w_{20}^{(1)}(\tau)\,d\tau + 2\int_{-\infty}^{0}\alpha e^{(\alpha + \mathrm{i}\omega_0)\tau}w_{11}^{(1)}(\tau)\,d\tau\right)\right. \\
& + p_{11}\left(\frac{\overline{B}}{2}\int_{-\infty}^{0}\alpha e^{(\alpha - \mathrm{i}\omega_0)\tau}w_{20}^{(1)}(\tau)\,d\tau + B\int_{-\infty}^{0}\alpha e^{(\alpha + \mathrm{i}\omega_0)\tau}w_{11}^{(1)}(\tau)\,d\tau\right. \\
& \left. + \int_{-\infty}^{0}\alpha e^{(\alpha + \mathrm{i}\omega_0)\tau}w_{11}^{(2)}(\tau)\,d\tau + \frac{1}{2}\int_{-\infty}^{0}\alpha e^{(\alpha - \mathrm{i}\omega_0)\tau}w_{20}^{(2)}(\tau)\,d\tau\right) \\
& \left. + \frac{1}{2}p_{02}\left(\overline{B}\int_{-\infty}^{0}\alpha e^{(\alpha - \mathrm{i}\omega_0)\tau}w_{20}^{(2)}(\tau)\,d\tau + 2B\int_{-\infty}^{0}\alpha e^{(\alpha + \mathrm{i}\omega_0)\tau}w_{11}^{(2)}(\tau)\,d\tau\right)\right] \\
& + \frac{eP(x^*,y^*)}{2}\left[p_{30}\frac{\alpha}{\alpha + \mathrm{i}\omega_0} + p_{21}\frac{\alpha}{\alpha + \mathrm{i}\omega_0}(\overline{B} + 2B) + p_{12}\frac{\alpha}{\alpha + \mathrm{i}\omega_0}(B^2 + 2B\overline{B}) + p_{03}\frac{\alpha}{\alpha + \mathrm{i}\omega_0}B^2\overline{B}\right] \\
& + \left[\frac{1}{2}p_{20}\left(2B + \overline{B}\frac{\alpha}{\alpha + 2\mathrm{i}\omega_0}\right) + p_{11}\left(B^2 + B\overline{B} + B\overline{B}\frac{\alpha}{\alpha + 2\mathrm{i}\omega_0}\right)\right. \\
& \left.\left.\left. + \frac{1}{2}p_{02}\left(2B^2\overline{B} + B^2\overline{B}\frac{\alpha}{\alpha + 2\mathrm{i}\omega_0}\right)\right]\right]\right\}.
\end{aligned}
$$

Therefore, we can get the following parameters:

$$
\begin{aligned}
& c_1(0) = \frac{\mathrm{i}}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\
& \mu_2 = -\frac{\operatorname{Re} c_1(0)}{\operatorname{Re} \lambda_1'(\alpha_0)}, \\
& \tau_2 = -\frac{\operatorname{Im} c_1(0) + \mu_2 \operatorname{Im} \lambda_1'(\alpha_0)}{\omega_0}, \\
& \beta_2 = 2\operatorname{Re} c_1(0), \\
& T = \frac{2\pi}{\omega_0}(1 + \tau_2\varepsilon^2 + O(\varepsilon^4)), \quad \varepsilon^2 = \frac{\alpha - \alpha_0}{\mu_2} + O(\alpha - \alpha_0)^2.
\end{aligned}
$$

Thus, we obtain the main result of this section; that is,

Theorem 3.1. *The direction of the Hopf bifurcation described in Theorem 2.1 is determined by the sign of μ_2: if $\mu_2 > 0$ (< 0), then the bifurcating periodic solutions exist for $\alpha > \alpha_0$ ($\alpha < \alpha_0$). The periodic solutions are stable (unstable) if $\beta_2 < 0$ (> 0). The period of the bifurcating periodic solutions of system (1.1) increases (decrease) if $\tau_2 > 0$ (< 0).*

4. Numerical example

In this section, an example is given to illustrate our results.

Take $r = 0.21$, $k = 1$, $a = 0.86$, $b = 0.25$, $c = 1.27$, $d = 2.62$, $e = 5.25$, which satisfy the all assumptions of this paper. Moreover, the distributed delay kernel $F(t)$ is selected as a weak kernel. Therefore, system (1.1) becomes:

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= 0.21x\left(1 - \frac{x}{1}\right) - \frac{0.86xy}{1 + 0.25x + 1.27y}, \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= y\left[-2.62 + \int_{-\infty}^{t} \alpha \mathrm{e}^{-\alpha(t-\tau)} \frac{5.25x(\tau)}{1 + 0.25x(\tau) + 1.27y(\tau)}\,\mathrm{d}\tau\right], \end{aligned} \tag{4.1}$$

Obviously, the positive equilibrium E_3 of system (4.1) is

$$E_3 = (0.6507, 0.1112), \tag{4.2}$$

and

$$\alpha_0 = 0.3251, \quad \omega_0 = 0.3657, \tag{4.3}$$

after some simple calculations.

Based on the results in Section 3, it follows that

$$\mu_2 = -7.2556 \quad \beta_2 = -2.8655 \quad \tau_2 = 7.0836. \tag{4.4}$$

According to Theorem 2.1, E_3 is stable when $\alpha > \alpha_0$ (that is, $\psi_1(\alpha) > 0$), which is shown in Fig. 1 with $\alpha = 0.4$. When α decreases and passes through the critical value $\alpha_0 = 0.3251$, E_3 loses its stability and a Hopf bifurcation occurs; i.e., a family of periodic solutions bifurcate from E_3. By Theorem 3.1, the individual periodic orbit is stable since $\beta_2 < 0$. Since $\mu_2 < 0$, the bifurcating periodic solutions exist as the value of α slightly less than the critical value. With $\alpha = 0.3121$, Fig. 2 shows that there is a stable limit cycle.

Recall that the average delay is defined by $\tau = \frac{1}{\alpha}$. Therefore, the equilibrium E_3 is stable when $\tau(= \frac{1}{\alpha}) < \frac{1}{\alpha_0} = 3.0760$. As the average delay τ decreases to zero. System (4.1) can be described by

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= 0.21x\left(1 - \frac{x}{1}\right) - \frac{0.86xy}{1 + 0.25x + 1.27y}, \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= y\left[-2.62 + \frac{5.25x}{1 + 0.25x + 1.27y}\right]. \end{aligned} \tag{4.5}$$

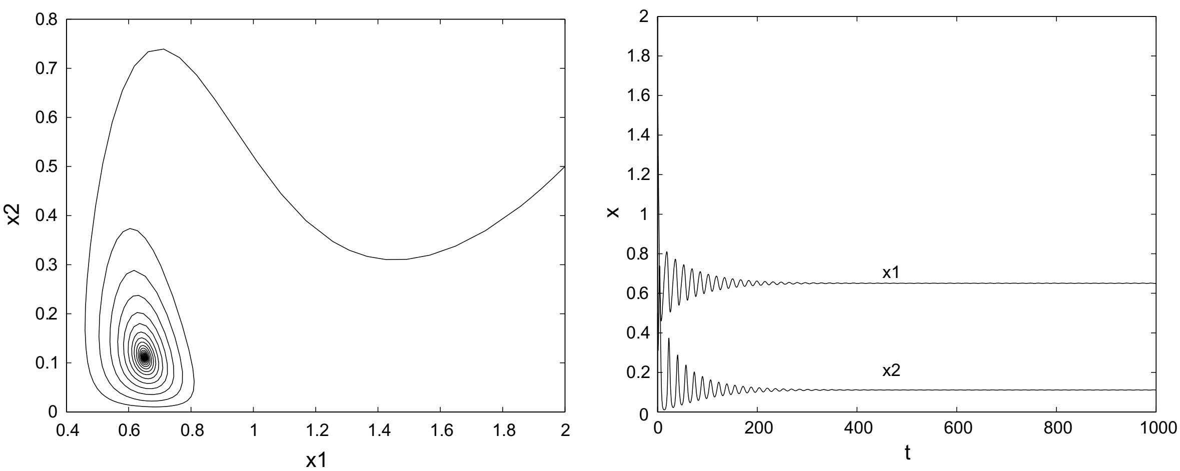

Fig. 1. E_3 is asymptotic stable, $\alpha = 0.4$.

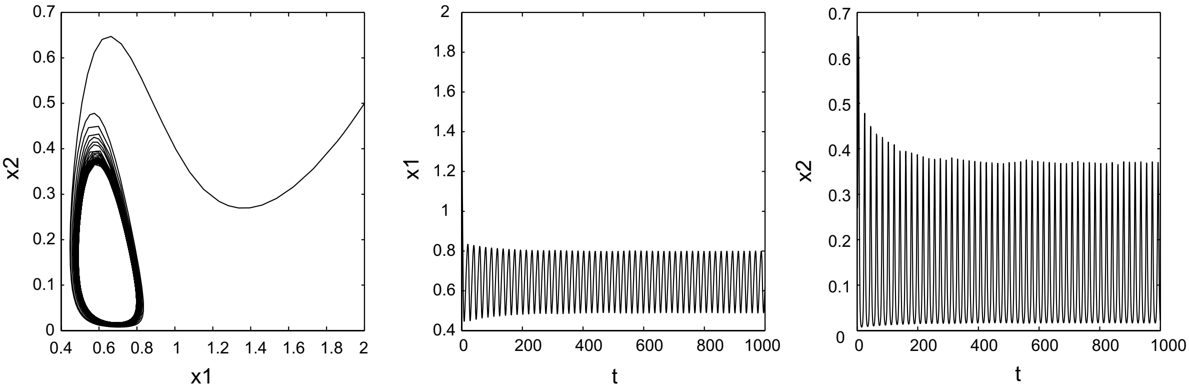

Fig. 2. There is an stable cycle, $\alpha = 0.3121$.

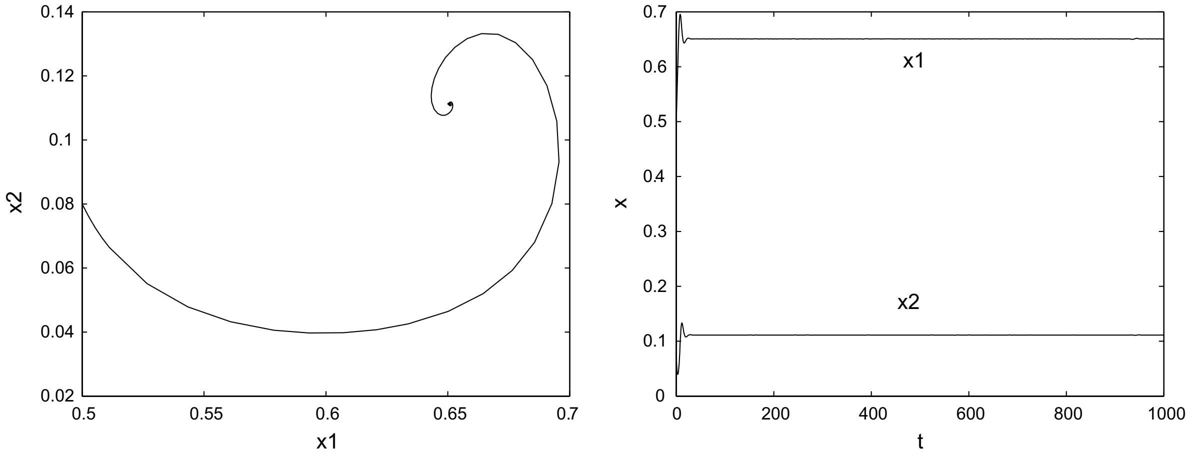

Fig. 3. E_3 is asymptotic stable, $\tau = 0$.

From [19–21], the positive equilibrium E_3 of system (4.5) is stable, which is also illustrated in Fig. 3 as $\tau = 0$. This shows that our results are consistent with those in [19–21].

5. Conclusion

In this paper, we considered the Beddington–DeAngelis predator–prey system with distributed delay (1.1). With the average delay $\tau(=\frac{1}{\alpha})$ as a bifurcation parameter, we showed that a Hopf bifurcation occurs at the critical value α_0. The bifurcating periodic solutions were analyzed in light of the normal form and center manifold. Under suitable restrictions on the parameters, when $\tau(=\frac{1}{\alpha})$ decreases to zero, our results are consistent with those in [19–21]. On the other hand, when $\tau(=\frac{1}{\alpha})$ increases and crosses a critical value, our results indicate that delays are able to destablize an otherwise stable equilibrium (that is, E_3) and generate a Hopf bifurcation.

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