A Novel Hybrid Synchronization of Two Coupled Complex Networks

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Abstract—This paper deals with the hybrid synchronization problem of two coupled complex networks. By using the linear feedback controller, several useful hybrid synchronization criteria of two coupled networks are obtained based on the Lyapunov stability theory and Lasalle’s invariant principle. Analytical results also show that two coupled complex networks can realize hybrid synchronization under suitable conditions. Numerical simulations are then given to verify the effectiveness of the proposed hybrid synchronization scheme.

I. INTRODUCTION

In the 17th century, Huygens discovered that two pendulum clocks hanging at the same beams were able to synchronize their phase oscillations. Since then, various synchronization phenomena have been observed in coupled chaotic oscillators. Among them are complete synchronization (CS), phase synchronization (PS), lag synchronization (LS), and generalized synchronization (GS) [1-11].

Complex networks are shown to exist in various fields of real world [5,6]. In general, complex networks consist of a large number of nodes and the links among them. Since their flexibility and generality for representing virtually any natural and man-made structure, complex networks have been received much attention as an interdisciplinary subject. Among various complex dynamical behaviors, complete synchronization (CS) is a significant and interesting phenomena, which can be understood as the adjustment of rhythms or coherence of states by interaction. Due to its theoretical importance and practical applications, complete synchronization (CS) of complex network has attracted a great deal of attentions of all kinds of sciences, ranging from physical to chemical, biological, information technology, mathematical, and even to social sciences. Hitherto, many different methods have been used to study complete synchronization (CS) of complex networks. Some authors utilized a distance from the collective spatial states of the coupled system to the synchronization manifold to investigate global convergence for complete regular coupling configuration. The “master stability function” is used to study the local stability of the synchronization manifold. The left eigenvector corresponding to the zero eigenvalue of the diffusive coupling matrix is utilized to investigate the global synchronization of an array of linearly coupled dynamical systems. I. V. Belykh et al applied connection graph stability method to study the synchronization properties of the small-world blinking model of coupled periodic or chaotic oscillators. The authors utilized pinning control to obtain sufficient conditions for guaranteeing to synchronize a dynamical network to a homogenous solution, etc [1-5].

Other synchronization phenomena especially anti-synchronization (AS) is also considered. As a special case of generalized synchronization, chaos anti-synchronization (AS) can be characterized by the vanishing of the sum of relevant variables. AS phenomenon has been observed experimentally and numerically in the coupled chaotic systems. For example, researchers observed AS in piecewise linearly coupled Chua’s circuits and in coupled laser systems. It is a noticeable phenomenon in chaotic systems, which has important application significance. For instance, using anti-synchronization to lasers, one may generate not only drop-outs of the intensity as with ordinary low frequency fluctuations but also short pulses of high intensity, which offers some new ways for generating pulses of special shapes. Applying anti-synchronization to communication systems, one may transmit digital signals by the transform between synchronization and anti-synchronization continuously, which will strengthen the security and secrecy. Therefore, many researchers focus on the topic of anti-synchronization. An adaptive control method for anti-synchronization of uncertain Chua’s chaotic system is proposed. The authors obtained sufficient condition of synchronization and anti-synchronization of Colpitts oscillators using active control. W. Li et al study the problem of anti-synchronization for two different chaotic systems by a nonlinear control technique. By using the linear feedback controllers which are simple, efficient, and easy to implement in practical applications, several sufficient conditions for determining AS and adaptive AS in the linearly coupled systems are derived. Especially, the authors investigated the anti-synchronization of a class of delayed chaotic neural networks based on Halanany inequality and Lyapunov stability theory.

Recently, coexistence of anti-synchronization and complete
synchronization (hybrid synchronization) in chaotic systems are investigated intensively. Most of these works were concerned with hybrid synchronization only in two coupled chaotic systems, however, hybrid synchronization of two coupled networks has not been explicitly considered and studied.

In view of this reason, in this paper, we consider hybrid synchronization of two coupled networks via the linear feedback control which is easy to implement in practical applications. Based on the Lyapunov’s stability theorem and Lasalle’s invariance principle, sufficient conditions of the global hybrid synchronization of two coupled networks are obtained [6-11].

The rest of the paper is then organized as follows. Section II introduces the model and preliminaries. Hybrid synchronization of two coupled networks are obtained [6-11]. Based on the Lyapunov’s stability theorem and Lasalle’s invariance principle, we rewrite the driving network in the form of

\[
\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^{N} b_{ij} \Gamma x_j(t),
\]

and the response network as

\[
\dot{y}_i(t) = f(y_i(t)) + c \sum_{j=1}^{N} b_{ij} \Gamma y_j(t) + u_i(t),
\]

where \(i = 1, 2, \cdots, N\), \(u_i(t) = -k_i \Gamma (x_i^p + y_i^p)\) are feedback controllers, \(x_i^p, y_i^p \in R^n\) are the state variables of node \(i\) in the driving network and the response network respectively, \(f : R^n \times [0, \infty) \rightarrow R^n\) is a continuous function, \(c\) is the coupling strength, \(\Gamma = \text{diag}\{\gamma_1, \cdots, \gamma_n\}\), \(B = (b_{ij}) \in R^{N \times N}\) (B is not necessary symmetric) is irreducible matrix with zero-sum rows and \(b_{ij} \geq 0\) for \(i \neq j\), \(b_{ij}\) is defined as follows: if there is a connection from node \(j\) to node \(i (i \neq j)\), \(b_{ij} > 0\), otherwise \(b_{ij} = 0 (i \neq j)\) for \(i, j = 1, 2, \cdots, N\).

In this paper, we have the following definitions, hypothesis and Lemma:

**Definition 1.** The driving network (2.3) and the response network (2.4) are said to achieve hybrid synchronization if the driving network and the response network achieve outer anti-synchronization, while the driving network and the response network achieve inner synchronization respectively, i.e.,

\[
\lim_{t \rightarrow \infty} \|x_i^p(t) - x_i^p(t)\| = 0,
\]

\[
\lim_{t \rightarrow \infty} \|y_i^p(t) - y_i^p(t)\| = 0,
\]

\[
\lim_{t \rightarrow \infty} \|x_i(t) + y_i(t)\| = 0, i, j = 1, 2, \cdots, N.
\]

**Definition 2.** Matrix \(B = (b_{ij}) \in R^{N \times N}\) is said to belong to \(A_1\), if

(i). \(b_{ij} \geq 0 (i \neq j), b_{ii} = -\sum_{j=1, j \neq i}^{N} b_{ij}\), for \(i = 1, 2, \cdots, N\);

(ii). Real parts of eigenvalues of \(B\) are all negative except an eigenvalue \(0\) with multiplicity 1.

It is clear that \(B\) is irreducible. Suppose \(\xi = (\xi_1, \cdots, \xi_N)(\xi_i > 0, i = 1, \cdots, N)\) is the left eigenvector corresponding to eigenvalue \(0\) of \(B, \Xi = \text{diag}\{\xi_1, \cdots, \xi_N\}\)

\(U = \Xi - \xi \xi^T\). With regard to the system functions \(f(\cdot)\), the following assumption is satisfied:

**Assumption 1 (A1).** Let \(P = \text{diag}\{p_1, \cdots, p_n\}, p_i > 0, i = 1, 2, \cdots, N, f(x, t) \in C(R^n \times [0, \infty), R^n)\) satisfies:

\((x - y)^T P[(f(x, t) - f(y, t) - \Delta(x - y))] \leq -\alpha(x - y)^T(x - y)\)

for some \(\alpha > 0\), all \(x, y \in R^n\) and \(t > 0\), where \(\Delta = \text{diag}\{\Delta_1, \cdots, \Delta_n\}\).

Many chaotic systems satisfy Assumption H1, for example, Chua’s oscillator, Rössler system, Lorenz system, Chen system, and Lu system.

**Lemma 1.** ([1-11]) Suppose \(B\) is symmetrical, then for any two vectors \(u = (u_1, \cdots, u_N)^T, v = (v_1, \cdots, v_N)^T\), we have

\[u^T B v = -\sum_{j > i}^{N} (u_j - u_i)(v_j - v_i).\]

**III. Hybrid Synchronization of Two Coupled Networks**

In order to achieve hybrid synchronization of the driving network and the response network, we only need realize inner synchronization of the driving network, outer anti-synchronization between the driving network and the response network. The driving network can be rewritten in compact form as

\[
\frac{dX(t)}{dt} = F(X(t)) + (cB \otimes \Gamma)X(t), i = 1, 2, \cdots, N,
\]

where \(X(t) = (x_1(t)^T, \cdots, x_N(t)^T)^T, F(X(t)) = (f(x_1(t))^T, \cdots, f(x_N(t))^T)^T\), \(\otimes\) is the Kronecker product. Letting \(e_i = x_i^p + y_i^p\), then the anti-synchronization error systems between the driving network (2.3) and the response network (2.4) can be written as

\[
\frac{de_i(t)}{dt} = f(x_i^p(t)) + f(y_i^p(t)) + c \sum_{j=1}^{N} b_{ij} \Gamma e_j(t) - k_i \Gamma e_i(t),
\]

where \(i = 1, 2, \cdots, N\). Then we may prove the main result:
Theorem 1. Suppose A1 holds, $f(x, t)$ is odd with regard to $x$, if there exists constants $c$, $\Delta_j, \gamma_j, K (j = 1, \cdots, n)$ such that
\[
\begin{cases}
\Delta_j U + (c \gamma_j \Xi B)^* \leq 0, \\
\Delta_j I_N - \gamma_j K + c \gamma_j B^* < 0, j = 1, \cdots, n,
\end{cases}
\]
where $K = \text{diag}\{k_1, \cdots, k_n\}$, then the driving network (2.3) and the response network (2.4) achieve hybrid synchronization.

The gains $k_i$ of the controller $u_i(t)$ derived above requires that the system parameters must be known a priori. However, in many real applications, it can be difficult to determine exactly the values of the system parameters. Consequently, the gains $k_i$ cannot be appropriately chosen to guarantee the stability of the error systems (3.1). Even we know the values of the system parameters exactly, the gains $k_i$ derived may be so large that it is of no significance in some real applications. To overcome these drawbacks, adaptive gains $k_i$ are adapted. In this case the controllers $u_i(t)$ are chosen as
\[
\begin{align*}
u_i(t) &= -k_i(t)\gamma_i x^i(t) + \beta_i(x^i(t) + y^i(t)), i = 1, 2, \cdots, N \\
\end{align*}
\]
where the time-varying gains $k_i(t)$ are updated according to the following adaptive algorithm:
\[
\dot{k}_i(t) = \beta_i(x^i(t) + y^i(t)) \Gamma(x^i(t) + y^i(t)), i = 1, 2, \cdots, N
\]
and $\beta_i > 0$. Then two coupled networks can be rewritten as
\[
\begin{align*}
\frac{dx^i(t)}{dt} &= f(x^i(t)) + c \sum_{j=1}^{N} b_{ij} \Gamma x^j(t), \\
\frac{dy^i(t)}{dt} &= f(y^i(t)) + c \sum_{j=1}^{N} b_{ij} \Gamma y^j(t) + u_i(t),
\end{align*}
\]
where $i = 1, 2, \cdots, N$. Letting $e^i = x^i - y^i$, then the anti-synchronization errors between the driving network (3.4) and the response network (3.5) can be written as:
\[
\begin{align*}
\frac{de^i(t)}{dt} &= f(x^i(t)) + f(y^i(t)) + c \sum_{j=1}^{N} b_{ij} \Gamma e^j(t) - k_i \Gamma e^i(t), \\
\dot{k}_i(t) &= \beta_i e^i(t)^T \Gamma e^i(t), i = 1, 2, \cdots, N,
\end{align*}
\]
Then we can get the following results:

Theorem 2. Suppose that A1 holds, $f(x, t)$ is odd with regard to $x$. Under the adaptive controllers (3.3) and the adaptive gains (3.4), two coupled complex networks (3.5) and (3.6) will achieve hybrid synchronization globally, where $k_i(0)$ and $\beta_i(i = 1, 2, \cdots, N)$ are arbitrary positive constants.

IV. NUMERICAL RESULTS

In this section, numerical simulations are presented to verify the theorems obtained in the previous section. We consider chaotic Chua’s circuits as the node dynamic system. Chua’s circuit is described by
\[
\begin{align*}
\dot{x}_1 &= p(x_2 - x_1 - g(x_1)), \\
\dot{x}_2 &= x_1 - x_2 + x_3, \\
\dot{x}_3 &= -q x_2,
\end{align*}
\]
where $g(x) = m_0 x + \frac{1}{2}(m_1 - m_0)(|x + 1| - |x - 1|)$. Chua’s circuit is chaotic for parameters $p = 10$, $q = 14.87$, $m_0 = -0.68$, $m_1 = -1.27$.

We consider two coupled networks with 5 nodes described by
\[
\begin{aligned}
\frac{dx^i(t)}{dt} &= f(x^i(t)) + c \sum_{j=1}^{5} b_{ij} \Gamma x^j(t), i = 1, 2, \cdots, 5, \\
\frac{dy^i(t)}{dt} &= f(y^i(t)) + c \sum_{j=1}^{5} b_{ij} \Gamma x^j(t) - k_i \Gamma x^i(t) + y^i(t), i = 1, 2, \cdots, 5,
\end{aligned}
\]
where $\Gamma = \text{diag}(1, 1, 1)$. The coupling matrix $B$ is
\[
B = \begin{bmatrix}
-1 & 1/4 & 1/4 & 1/4 & 1/4 \\
1/3 & -1 & 0 & 1/3 & 1/3 \\
1/4 & 1/4 & 1/4 & 1/4 & -1
\end{bmatrix}
\]
The left eigenvector of $B$ associated with eigenvalue 0 is $\xi = [0.2222, 0.1667, 0.1667, 0.2222, 0.2222]$, and $\lambda_1 = 0$, $\lambda_2 = -1.2500$, $\lambda_3 = -1.5000$, $\lambda_4 = -1.0000$, $\lambda_5 = -1.2500$ are the eigenvalues of $B$, $\Xi = F = \text{diag}\{0.2222, 0.1667, 0.1667, 0.2222, 0.2222\}$. Under the adaptive controllers (3.3) and the adaptive linear feedback controllers. What's more, even the time-varying gains $k_i(t)$ derived may be such odd with regard to $x$, so
\[(x+y)^T P [f(x, t) + f(y, t) - \Delta(x+y)] \leq -\alpha(x+y)^T (x+y),
\]
for some $\alpha > 0$, all $x, y \in \mathbb{R}^n$ and $t > 0$. If $c = 5$, $K = 4I_5$, then the eigenvalues of $\Delta_j U + (c \gamma_j \Xi B)^*(j = 1, \cdots, n)$ are $-0.8334, -0.7221, -0.0334, 0.0000$, and the eigenvalues of $\Delta_j I_N - \gamma_j K + c \gamma_j B^*(j = 1, \cdots, n)$ are $-8.5346, -7.2500, -7.2500, -6.0000, 0.9654$, the conditions of Theorem 1 are satisfied, therefore, two coupled networks (4.1) can achieve hybrid synchronization.

In order to illustrate how to apply Theorem 2 to judge whether two coupled networks can achieve hybrid synchronization, taking suitable positive parameters $P = \Gamma = \text{diag}\{1, 1, 1\}$, $c = 5$, $(k_1(0), k_2(0), k_3(0), k_4(0), k_5(0)) = (0.1, 0.1, 0.1, 0.1, 0.1)$, the time-varying gains $k_i(t)(i = 1, 2, \cdots, 5)$ are small compared with the constant gains $k_i(t)(i = 1, 2, \cdots, 5)$, two coupled complex networks can realize hybrid synchronization fast, which is of great significance in some real applications.

V. CONCLUSION

We have further investigated the hybrid synchronization of two coupled complex networks. By using the linear feedback controllers which are simple, efficient, and easy to implement in practical applications, some criteria of hybrid synchronization of two coupled networks are obtained. Analytical results
show that under suitable conditions, two complex networks can realize hybrid synchronization: outer anti-synchronization between the driving network and the response network, and inner synchronization in the driving network and the response network, respectively. Numerical simulations are also given to show the effectiveness of the proposed hybrid synchronization scheme. It sheds light on the future real-world application.

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REFERENCES


