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Consensus of Multi-Agent Systems with an Active Leader and Asymmetric Adjacency Matrix

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Abstract— **Multi-agent systems (MAS) have received an increasing attention over the past few years. Here an MAS is a system consisted of multiple interacting intelligent agents. This paper further investigates the consensus of MAS with an active leader and asymmetric adjacency matrix. In particular, the state of the active leader is changing and unmeasured. Under some suitable conditions, we have proved that: $i)$ each agent of MAS can follow the active leader if the input of the active leader is known beforehand; $ii)$ the tracking error of MAS can be effectively estimated if the input of the active leader is unknown beforehand. Moreover, numerical simulations are then given to verify the effectiveness of the proposed theorems.**

I. INTRODUCTION

Over the past ten years, the coordinating motion of multi-agent systems (MAS) has received an increasing attention in various fields, such as mathematical, physical, engineering, and biological sciences [1]-[16]. Some potential real-world applications of MAS include formation control [1], [3], [5], flocking [4], [5], consensus [6]-[8], and network synchronization [13]-[17].

It is well known that the research of consensus has a very long history in computer science, especially in automata theory and distributed computation [7]. In fact, consensus or synchronization is one of typical collective behaviors in MAS. In detail, consensus of MAS means that all agents can reach a general agreement by using some local follow-up interactions between agents. To reveal the inherent mechanics of consensus in MAS, some typical mathematical models were introduced, such as the Vicsek model [4], Couzin-Levin model [13], and so on. Some recent advances were reported on the consensus of MAS. For example, the graph theory approach was applied to further investigate the consensus of Vicsek model [5].

Leader-follower method indicates another effective route to consensus of MAS. Recently, a neighbor-based observer was designed to estimate the unmeasurable state of an active leader in MAS [9]. Note that the interconnection between agents of MAS was described by the undirected graphs [9]. That is, the adjacency matrix is symmetric. However, in most real-world MAS, the interconnection between agents is more likely to be described by the directed graphs. And the corresponding adjacency matrix is asymmetric. This paper further explores the consensus of MAS with an active leader and asymmetric adjacency matrix. Under some soft conditions, some interesting results are attained as follows: $i)$ each agent of MAS can follow the active leader if the input of the active leader is known beforehand; $ii)$ the tracking error of MAS can be estimated if the input of the active leader is unknown beforehand. In addition, numerical simulations are also given to validate the effectiveness of the proposed criteria.

This paper is organized as follows. Section II describes the fundamental problem and gives several necessary lemmas. The main results are given in Section III. In Section IV, the numerical simulations are then used to justify the effectiveness of the deduced criteria. Some concluding remarks are drawn in Section V.

II. PROBLEM FORMULATION

Let $G = (V, \varepsilon, A)$ be a weighted directed graph of MAS, where $V = \{1, 2, \cdots, n\}$ is the set of vertices, $\varepsilon \subset V \times V = \{(i, j) : i, j \in V\}$ is the set of ordered edges, and $A = (a_{ij}) \in R^{n\times n}$ is the weighted adjacency matrix with $a_{ii} = 0$ and $a_{ij} \geq 0$ for $i \neq j$. Here, $a_{ij} > 0$ if and only if there exists an edge from i to j. If $(i, j) \in \varepsilon$, then j is called a neighbor of i. The set of all neighbors of i is denoted by $N_i = \{j \in V : (i, j) \in \varepsilon, j \neq i\}$. The degree matrix of G is given by $D = \text{diag}\{d_1, \cdots, d_n\} \in R^{n\times n}$, where $d_i = \sum_{j\in N_i} a_{ij}$ for $i = 1, 2, \cdots, n$. Thus the Laplacian of the weighted digraph G is defined by $L = D - A$. In general, the Laplacian L of the weighted digraph G is asymmetric.

Hereafter, we focus on an MAS with digraph $\bar{G}$ consisting of n agents and an active leader. In detail, $\bar{G}$ contains two independent parts: digraph G with n agents and the leader with directed edges from some agents to the leader. To describe the digraph $\bar{G}$, let $B = \text{diag}\{b_1, \cdots, b_n\}$ be the adjacency matrix of leader associated with $\bar{G}$, where $b_i > 0$ if the leader is a neighbor of agent i, otherwise, $b_i = 0$. Hereafter, assume that there at least exists one $b_i > 0$. The norm of a vector $u \in R^n$ is defined as $||u|| = \sqrt{u^T u}$. The spectral norm of $M \in R^{n\times n}$ is defined

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as $||M|| = \max_{1\leqslant i\leqslant n} \sqrt{\lambda_i}$, where λ_i are the eigenvalues of $M^T M$.

The state vectors of all agents of MAS are described by

$$\dot{x}_i = u_i \in R^m,\ i = 1, \cdots, n, \tag{1}$$

where u_i are the control inputs. Suppose that the leader of MAS is active, that is, its state variables keep updating. And its underlying dynamics is given by

$$\begin{cases} \dot{x}_0 = v_0, \\ \dot{v}_0 = a(t) = a_0(t) + \delta(t), \quad x_0,\, v_0,\, \delta \in R^m, \\ y = x_0, \end{cases} \tag{2}$$

where $y(t) = x_0(t)$ is the measured output and $a(t)$ is the (acceleration) input. It should be especially pointed out that the dynamics (2) of the leader is totally different from the agent dynamics (1). That is, all agents of MAS will track its leader with different dynamics.

In the above problem formulation, the input $a(t)$ may not be completely known. Hereafter, suppose that $a_0(t)$ is known and $\delta(t)$ is unknown but bounded by a given upper bound $\bar{\delta}$. That is, $||\delta(t)|| \leq \bar{\delta}$. Thus the input $a(t)$ is known if and only if $\bar{\delta} = 0$. In the following, the main idea is to design a suitable decentralized control scheme for each agent to follow the leader. That is, $x_i \to x_0$.

Since $v_0(t)$ cannot be measured even if all agents of MAS are connected to the leader, then its value cannot be used in the design of controller. Instead, one needs to estimate v_0 during the dynamical evolution. It should be especially pointed out that each agent of MAS can estimate v_0 only by using the information obtained from its neighbors in a decentralized way. The estimated value of $v_0(t)$ by agent i is denoted by $v_i(t)$ for $i = 1, \cdots, n$.

According to [9], for each agent i, the local control scheme includes the following two parts:

(A) A neighbor-based feedback law:

$$u_i = -k\left[\sum_{j\in N_i} a_{ij}(x_i - x_j) + b_i(x_i - x_0)\right] + v_i, \tag{3}$$

where $k > 0$ and $i = 1, \cdots, n$.

(B) A dynamic neighbor-based system to estimate v_0:

$$\dot{v}_i = a_0 - \gamma k\left[\sum_{j\in N_i} a_{ij}(x_i - x_j) + b_i(x_i - x_0)\right], \tag{4}$$

where $0 < \gamma < 1$ and $i = 1, \cdots, n$.

Using the above neighbor-based observer (3) to estimate the velocity of leader, each agent of MAS only relies on the locally available information at any time t. In other words, each agent of MAS cannot “observe” or ”estimate” the leader directly based on the measured information of the leader if it has not connection with the leader. Therefore, it is necessary for some agents to collect the information of the leader in a distributed way from its neighbor agents during the dynamical evolution.

Denote

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Then the closed-loop system can be described by

$$\begin{cases} \dot{x} = u = -k(L+B)\otimes I_m x + kB\mathbf{1}\otimes x_0 + v, \\ \dot{v} = \mathbf{1}\otimes a_0 - \gamma k(L+B)\otimes I_m x + \gamma kB\mathbf{1}\otimes x_0, \end{cases} \tag{5}$$

where $\otimes$ is Kronecker product and $\mathbf{1} = (1, 1, \cdots, 1)^T$.

Denote $\bar{x} = x - \mathbf{1}\otimes x_0$ and $\bar{v} = v - \mathbf{1}\otimes v_0$. And the error dynamics of (5) is given by

$$\begin{cases} \dot{\bar{x}} = -k(L+B)\otimes I_m \bar{x} + \bar{v}, \\ \dot{\bar{v}} = -\gamma k(L+B)\otimes I_m \bar{x} - \mathbf{1}\otimes\delta. \end{cases}$$

Also it can be rewritten as follows:

$$\dot{\omega} = F\omega + g, g = \begin{pmatrix} 0 \\ -\mathbf{1}\otimes\delta \end{pmatrix}, \tag{6}$$

where

$$\omega = \begin{pmatrix} \bar{x} \\ \bar{v} \end{pmatrix}, \quad F = \begin{pmatrix} -k(L+B) & I_n \\ -\gamma k(L+B) & 0 \end{pmatrix} \otimes I_m.$$

To begin with, one definition and several necessary lemmas are introduced in the following.

Definition 1: A digraph is called strongly connected if and only if there exists a directed path from vertex i to vertex j for any pair of vertices (i, j).

Lemma 1: [11] The graph G is strongly connected if and only if its Laplacian is irreducible.

Lemma 2: [10] Let Q and R be two symmetric matrices, and matrix S has suitable dimension. Then

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} > 0$$

if and only if $R > 0$ and $Q - SR^{-1}S^T > 0$.

Lemma 3: [12] Suppose that $L = (l_{ij}) \in R^{n\times n}$ satisfies the following conditions:

1) $l_{ij} \leq 0,\ i \neq j,\ l_{ii} = -\sum_{j=1}^{n} l_{ij},\ i = 1, 2, \cdots, n,$
2) L is irreducible,

then one has

i) All real parts of eigenvalues of L are positive except an eigenvalue 0 with multiplicity 1.
ii) L has the right eigenvector $(1, 1, \cdots, 1)^T$ corresponding to the eigenvalue 0.
iii) Let $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T$ be the left eigenvector of L corresponding to the eigenvalue 0, then $\xi_i > 0$ for $i = 1, 2, \cdots, n$.

Lemma 4: If $L = (l_{ij})_{n\times n}$ is an irreducible matrix satisfying $l_{ij} = l_{ji} \leq 0$ for $i \neq j$ and $l_{ii} = -\sum_{j=1, i\neq j}^{n} l_{ij}$ for $i = 1, 2, \cdots, n$, then all eigenvalues of the matrix

$$\tilde{L} = \begin{pmatrix} l_{11}+\varepsilon_1 & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn}+\varepsilon_n \end{pmatrix}$$

are positive, where $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ are nonnegative constants with $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n > 0$.

Proof. Denote $\Lambda = \mathrm{diag}\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$. Let λ be the eigenvalue of $\tilde{L}$ and $x = (x_1, x_2, \cdots, x_n)^T$ be the corresponding right eigenvector. Then one has

$$x^T \tilde{L} x = x^T L x + x^T \Lambda x = \lambda x^T x.$$

According to Lemma 3 and its assumptions, then $x^T L x \geq 0$ and $x^T \Lambda x \geq 0$, which implies $\lambda \geq 0$.

If $\lambda = 0$, then one gets

$$x^T \tilde{L} x = x^T L x + x^T \Lambda x = 0.$$

Thus x should satisfy $x^T L x = 0$ and $x^T \Lambda x = 0$. And $x^T L x = 0$ implies $x = \alpha(1, 1, \cdots, 1)^T$. However, one also has $x^T \Lambda x \neq 0$. Therefore, it is a contradiction and $\lambda > 0$. Thus the proof is completed here. ■

III. MAIN RESULTS

In this section, our main results are given in the following two different cases: Case I: Strongly connected digraph and Case II: Non-strongly connected digraph.

A. Case I: Strongly Connected Digraph

In this subsection, suppose that the directed graph G is strongly connected. That is, its Laplacian L is irreducible. Denote $M = (L + B)$, $\Xi = \mathrm{diag}\{\xi_1, \cdots, \xi_n\}$, and $M^s = \Xi M + M^T \Xi$, where $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T$ is the left eigenvector of L corresponding to the eigenvalue 0 with $\xi_i > 0$.

Theorem 1: For any given $0 < \gamma < 1$, select a constant

$$k > \frac{\max \xi_i}{2\gamma(1-\gamma^2)\underline{\lambda}},$$

where $\underline{\lambda}$ is the minimal eigenvalue of M^s. If the interconnection graph G keeps strongly connected or L is irreducible, then there exists some constant C satisfying

$$\lim_{t\to\infty} ||\omega(t)|| \leqslant C\bar{\delta}.$$

Moreover, if $a(t)$ is known, that is, $a(t) = a_0(t)$ or $\bar{\delta} = 0$, then one has

$$\lim_{t\to\infty} ||\omega(t)|| = 0.$$

Proof. Construct a Lyapunov candidate $V(\omega) = \omega^T P \omega$ with symmetric positive definite matrix

$$P = \begin{pmatrix} \Xi & -\gamma\Xi \\ -\gamma\Xi & \Xi \end{pmatrix} \otimes I_m.$$

Differentiating $V(t)$, one obtains

$$\begin{aligned} \dot{V}(\omega) &= \omega^T(PF + F^T P)\omega + 2\omega^T P g \\ &\leqslant \omega^T(PF + F^T P)\omega + 2||\omega||(1+\gamma)\bar{\delta}\max \xi_i. \end{aligned}$$

Since L is irreducible, then one has

$$\begin{aligned} (\Xi L + L^T \Xi)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} &= \Xi L \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + L^T \Xi \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \Xi L \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + L^T \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix} \\ &= 0 \end{aligned}$$

Thus, $\Xi L + L^T \Xi$ has zero row sum. Since L is irreducible, then ΞL is irreducible. Therefore, $\Xi L + L^T \Xi$ is irreducible. According to Lemma 4, M^s is positive definite. When $k > \frac{\max \xi_i}{2\gamma(1-\gamma^2)\lambda}$, one gets

$$k(1-\gamma^2)M^s - \Xi(2\gamma\Xi)^{-1}\Xi = k(1-\gamma^2)M^s - \frac{\Xi}{2\gamma} > 0\,.$$

Moreover, from Lemma 2, the matrix

$$Q = -(PF + F^T P) = \begin{pmatrix} k(1-\gamma^2)M^s & -\Xi \\ -\Xi & 2\gamma\Xi \end{pmatrix} \otimes I_m$$

is positive definite. Denote $\underline{\mu}$ be the minimum eigenvalue of Q.

In addition, since the maximal and minimal eigenvalues of P are $(1+\gamma)\max \xi_i$ and $(1-\gamma)\min \xi_i$, respectively, then one has

$$(1-\gamma)\min \xi_i ||\omega||^2 \leqslant V(\omega) \leqslant (1+\gamma)\max \xi_i ||\omega||^2.$$

Therefore, one gets

$$\omega^T Q \omega \geqslant \underline{\mu}\omega^T \omega \geqslant \underline{\mu}\frac{V(\omega)}{(1+\gamma)\max \xi_i}$$

and

$$||\omega|| \leqslant \sqrt{\frac{V(\omega)}{(1-\gamma)\min \xi_i}}.$$

Let $\frac{\underline{\mu}}{(1+\gamma)\max \xi_i} = 2\beta$, then one obtains

$$\dot{V}(\omega) \leqslant -2\beta V(\omega) + 2(1+\gamma)\max \xi_i \bar{\delta}\sqrt{\frac{V(\omega)}{(1-\gamma)\min \xi_i}}.$$

Thus one has

$$\frac{d\sqrt{V(\omega)}}{dt} \leqslant -\beta\sqrt{V(\omega)} + (1+\gamma)\max \xi_i \bar{\delta}\sqrt{\frac{1}{(1-\gamma)\min \xi_i}}.$$

Therefore, one gets

$$\sqrt{V(\omega)} \leqslant \sqrt{V(\omega(0))}e^{-\beta t} + \frac{(1+\gamma)\max \xi_i \bar{\delta}(1-e^{-\beta t})}{\beta\sqrt{(1-\gamma)\min \xi_i}}.$$

It implies $\lim_{t\to\infty} ||\omega(t)|| \leqslant C\bar{\delta}$, where $C = \frac{(1+\gamma)\max \xi_i}{(1-\gamma)\beta \min \xi_i}$.

Furthermore, if $\bar{\delta} = 0$, then $\lim_{t\to\infty} ||\omega(t)|| = 0$.

And the proof is thus completed. ■

Remark 1: Under the conditions of Theorem 1, we now have proved that: $i)$ each agent of MAS can follow the active leader if the input of the active leader is known beforehand; $ii)$ the tracking error of MAS can be effectively estimated if the input of the active leader is unknown beforehand.

B. Case II: Non-Strongly Connected Digraph

If the interconnection graph G is not strongly connected, then its Laplacian L is reducible. Denote

$$L =(l_{ij})_{n\times n} = \begin{pmatrix} L_{11} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & L_{qq} & 0 & \cdots & 0 \\ L_{q+1,1} & \cdots & L_{q+1,q} & L_{q+1,q+1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ L_{p1} & \cdots & L_{pq} & L_{p,q+1} & \cdots & L_{pp} \end{pmatrix},$$

where $L_{ii} \in R^{m_i,m_i}$ are irreducible or one dimensional zero matrices for $i = 1, \cdots, q$ and $L_{ii} \in R^{m_i,m_i}$ are irreducible for $i = q+1, \cdots, p$. For each $q+1 \leq i \leq p$, there exists some $k < i$ satisfying $L_{ik} \neq 0$. For simplification, denote $S_i = m_1 + \cdots + m_i$ with $S_0 = 0$ and $\xi^{q+1} = (\xi_1^{q+1}, \cdots \xi_{m_{q+1}}^{q+1})^T$ be the left eigenvector of $L_{q+1,q+1} + \mathrm{diag}\{\sum_{i=1}^{S_q} l_{S_q+1,i}, \sum_{i=1}^{S_q} l_{S_q+2,i}, \cdots, \sum_{i=1}^{S_q} l_{S_{q+1},i}\}$ corresponding to the eigenvalue 0, where $\xi_i^{q+1} > 0$ and $\Xi^{q+1} = \mathrm{diag}\{\xi_1^{q+1}, \cdots \xi_{m_{q+1}}^{q+1}\}$.

Remark 2: If $L_{ik} = 0 (i > k)$, then the Laplacian L is a block diagonal matrix and the result follows immediately. If each L_{ii} corresponds to at least one agent, then it follows from Section 3.1 that either every agent in the graph G can follow the leader or that the tracking error can be estimated.

Theorem 2: If $b_{S_i+1} + \cdots + b_{S_{i+1}} > 0$ for $i = 1, \cdots, q$, then one can select some large enough k satisfying that each agent of MAS can follow the leader or the tracking errors can be estimated.

Proof. According to Theorem 1, if there exists at least one agent in every part of graph G which corresponds to L_{ii}, $i = 1, \cdots, q$ connecting with the leader, then all agents in L_{ii}, $i = 1, \cdots, q$ can follow the leader or the tracking errors can be estimated. Also, there exists a positive constant C satisfying

$$||x_j - x_0|| \leq C\bar{\delta}, \quad j = 1, \cdots, S_q .$$

For subsystem $x_{S_q+1}, \cdots, x_{S_{q+1}}$ in $L_{q+1,q+1}$, one has

$$\begin{cases} \dot{x}_i = -k(\sum_{j=1}^{S_{q+1}} l_{ij}x_j + b_i(x_i - x_0)) + v_i, \\ \dot{v}_i = a_0 - \gamma k(\sum_{j=1}^{S_{q+1}} l_{ij}x_j + b_i(x_i - x_0)), \end{cases}$$

where $i \in [S_q + 1, S_{q+1}]$.

Denote

$$x^h = \left(x_{S_{h-1}+1}, \cdots, x_{S_h}\right)^T, \; v^h = \left(v_{S_{h-1}+1}, \cdots, v_{S_h}\right)^T,$$

where $h = 1, 2, \cdots, p$.

Thus one gets

$$\begin{cases} \dot{x}^{q+1} = - k(L_{q+1,1} \otimes I_m x^1 + \cdots + (L_{q+1,q+1} + B^{q+1}) \\ \qquad \otimes I_m x^{q+1}) + kB^{q+1}\mathbf{1} \otimes x_0 + v^{q+1}, \\ \dot{v}^{q+1} = \mathbf{1} \otimes a_0 - \gamma k(L_{q+1,1} \otimes I_m x^1 + \cdots + (L_{q+1,q+1} \\ \qquad + B^{q+1}) \otimes I_m x^{q+1}) + \gamma k B^{q+1}\mathbf{1} \otimes x_0, \end{cases}$$

where $B^{q+1} = \mathrm{diag}\{b_{S_q+1}, \cdots, b_{S_{q+1}}\}$.

Denote

$$\bar{x}^h = x^h - \mathbf{1} \otimes x_0, \; \bar{v}^h = v^h - \mathbf{1} \otimes v_0,$$

where $h = 1, 2, \cdots, p$.

Then the error dynamics is described by

$$\begin{cases} \dot{\bar{x}}^{q+1} = - k(L_{q+1,1} \otimes I_m \bar{x}^1 + \cdots + (L_{q+1,q+1} \\ \qquad + B^{q+1}) \otimes I_m \bar{x}^{q+1}) + \bar{v}^{q+1}, \\ \dot{\bar{v}}^{q+1} = - \gamma k(L_{q+1,1} \otimes I_m \bar{x}^1 + \cdots + (L_{q+1,q+1} \\ \qquad + B^{q+1}) \otimes I_m \bar{x}^{q+1}) - \mathbf{1} \otimes \delta. \end{cases}$$

Denote $\omega^h = \left(\bar{x}^h, \bar{v}^h\right)^T$ for $h = 1, 2, \cdots, p$, thus one obtains

$$\dot{\omega}^{q+1} = F^1\omega^1 + \cdots + F^q\omega^q + F^{q+1}\omega^{q+1} + g,$$

where $g = (0, -\mathbf{1} \otimes \delta)^T$ and

$$F^i = \begin{pmatrix} -kL_{q+1,i} & 0 \\ -\gamma k L_{q+1,i} & 0 \end{pmatrix} \otimes I_m, \; i = 1, \cdots, q,$$

$$F^{q+1} = \begin{pmatrix} -k(L_{q+1,q+1} + B^{q+1}) & I_n \\ -\gamma k(L_{q+1,q+1} + B^{q+1}) & 0 \end{pmatrix} \otimes I_m .$$

Construct a Lyapunov candidate

$$V(\omega^{q+1}) = (\omega^{q+1})^T P^{q+1}\omega^{q+1}$$

with the symmetric positive definite matrix

$$P^{q+1} = \begin{pmatrix} \Xi^{q+1} & -\gamma\Xi^{q+1} \\ -\gamma\Xi^{q+1} & \Xi^{q+1} \end{pmatrix} \otimes I_m,$$

whose maximal and minimal eigenvalues are $(1 + \gamma)\max \xi_i^{q+1}$ and $(1 - \gamma)\min \xi_i^{q+1}$, respectively. Then one has

$$(1-\gamma)\min \xi_i^{q+1}||\omega^{q+1}||^2 \leqslant V(\omega^{q+1}) \leqslant (1+\gamma)\max \xi_i^{q+1}||\omega^{q+1}||^2 .$$

Differentiating $V(\omega^{q+1})$ and one gets

$$\begin{aligned} \dot{V}(\omega^{q+1}) =& 2(\omega^{q+1})^T P^{q+1}(F^1\omega^1 + \cdots + F^q\omega^q \\ & + F^{q+1}\omega^{q+1} + g) \\ \leqslant& 2(||P^{q+1}F^1|| \cdot ||\omega^1|| + \cdots + ||P^{q+1}F^q|| \cdot ||\omega^q|| \\ & + \bar{\delta}||P^{q+1}||)||\omega^{q+1}|| + (\omega^{q+1})^T(P^{q+1}F^{q+1} \\ & + (F^{q+1})^T P^{q+1})\omega^{q+1} \\ \leqslant& 2(C||F^1 P^{q+1}|| + \cdots + C||F^q|| \cdot ||P^{q+1}|| \\ & + ||P^{q+1}||)\bar{\delta}||\omega^{q+1}|| + (\omega^{q+1})^T(P^{q+1}F^{q+1} \\ & + (F^{q+1})^T P^{q+1})\omega^{q+1} \\ =& - (\omega^{q+1})^T Q^{q+1}\omega^{q+1} + \mathbf{C}\bar{\delta}||\omega^{q+1}||, \end{aligned}$$

where

$$\mathbf{C} = 2(C||F^1P^{q+1}|| + \cdots + C||F^q|| \cdot ||P^{q+1}|| + ||P^{q+1}||)$$

and

$$Q^{q+1} = -(P^{q+1}F^{q+1} + (F^{q+1})^T P^{q+1}).$$

According to Theorem 1 and Lemma 4, for the large enough k, Q^{q+1} is a positive definite matrix. This is because that for each $q+1 \leq i \leq p$, there exists some $k < i$ satisfying $L_{ik} \neq 0$. Let $\underline{\mu}^{q+1}$ be the minimal eigenvalue of Q^{q+1}. Then one obtains

$$(\omega^{q+1})^T Q^{q+1}\omega^{q+1} \geqslant \underline{\mu}^{q+1}(\omega^{q+1})^T\omega^{q+1} \geqslant \underline{\mu}^{q+1}\frac{V(\omega^{q+1})}{(1+\gamma)\max \xi_i}.$$

Denote $\frac{\underline{\mu}^{q+1}}{(1+\gamma)\max \xi_i} = 2\beta^{q+1}$, then one gets

$$\dot{V}(\omega^{q+1}) \leqslant -2\beta^{q+1}V(\omega^{q+1}) + \mathbf{C}\bar{\delta}\sqrt{\frac{V(\omega^{q+1})}{(1-\gamma)\min \xi_i^{q+1}}}.$$

Thus, one has

$$\frac{d\sqrt{V(\omega^{q+1})}}{dt} \leqslant \beta^{q+1}\sqrt{V(\omega^{q+1})} + \frac{\mathbf{C}\bar{\delta}}{2}\sqrt{\frac{1}{(1-\gamma)\min \xi_i^{q+1}}}.$$

Therefore, one gets

$$\sqrt{V(\omega^{q+1})} \leqslant \sqrt{V(\omega^{q+1}(0))}e^{-\beta t} + \frac{\mathbf{C}\bar{\delta}(1-e^{-\beta t})}{2\beta^{q+1}(1-\gamma)\min \xi_i^{q+1}}.$$

Similarly, one can prove that all agents corresponding to the subsystem $L_{q+2,q+2}, \cdots, L_{pp}$ can also follow the leader or the tracking errors can be estimated. That is, each agent in the directed graph G can follow the leader or the tracking error can be estimated. Thus the proof is completed. ∎

Remark 3: In general, the interconnection topology of MAS is time-varying. Similarly, according to [9], it is easy to generalize the results of Theorems 1 and 2 to be the case of the time-varying interconnection topology.

Remark 4: It is well known that most graphs of real-world MAS are more likely to be the directed graphs. Moreover, the undirected graphs can easily be regarded as being directed by viewing each undirected edge between the vertices i and j as the union of two directed edges. Therefore, our results indeed generalize the results in [9].

IV. NUMERICAL SIMULATIONS

To validate the effectiveness of the proposed theories, several numerical simulations are then given in the following. Here, suppose that an MAS has 6 three-dimensional agents and a leader with velocity v_0.

When $\bar{\delta} = 0$, assume that v_0 satisfies the following Lorenz system

$$\begin{cases} \dot{x}_0 = v_0, \\ \dot{v}_{01} = a(v_{02} - v_{01}) + \delta_1(t), \\ \dot{v}_{02} = cv_{01} - v_{01}v_{03} - v_{02} + \delta_2(t), \\ \dot{v}_{03} = v_{01}v_{02} - bv_{03} + \delta_3(t), \end{cases}$$

where $a = 10$, $b = 8/3$, and $c = 28$.

The distance between agent x_i and leader x_0 is given by

$$E_i = \sqrt{(x_{i1} - x_{01})^2 + (x_{i2} - x_{02})^2 + (x_{i3} - x_{03})^2}.$$

A. Case I: Strongly Connected Digraph

Here, the adjacency matrices and parameters are given as follows:

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 & 0 & 1 \\ 1 & 3 & 0 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 2 & 3 \\ 1 & 2 & 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 1 & 1 & 0 \end{pmatrix},$$

$B = \text{diag}\{0, 5, 0, 0, 0, 0\}$, $\gamma = 0.8$, and $k = 5$. Fig. 1 shows the distances between agents x_i and leader x_0 in strongly connected graph for $\delta(t) = (\sin t, \cos t, \sin t)$ and $i = 1, \cdots, 6$. Fig. 2 shows the distances between agents x_i and leader x_0 in strongly connected graph for $\bar{\delta} = 0$ and $i = 1, \cdots, 6$. Here, the horizontal axis is the time t in all figures.

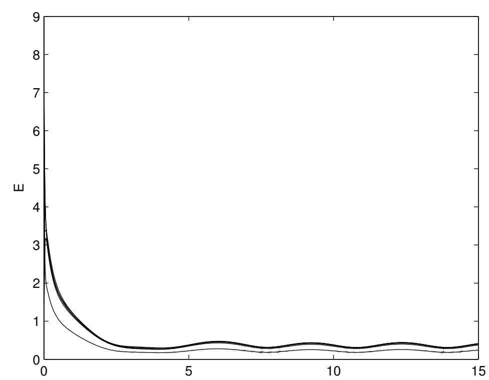

Fig. 1. The distances between agents x_i and leader x_0 in strongly connected graph, where $\delta(t) = (\sin t, \cos t, \sin t)$ and $i = 1, \cdots, 6$.

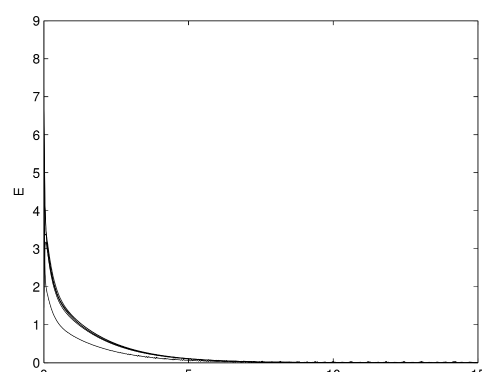

Fig. 2. The distances between agents x_i and leader x_0 in strongly connected graph, where $\bar{\delta} = 0$ and $i = 1, \cdots, 6$.

B. Case II: Non-Strongly Connected Digraph

In this subsection, the adjacency matrices and parameters are given as follows:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 3 \\ 1 & 2 & 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 1 & 1 & 0 \end{pmatrix},$$

$B = \text{diag}\{3, 5, 0, 0, 0, 0\}$, $\gamma = 0.8$, and $k = 5$. Fig. 3 shows the distances between agents x_i and leader x_0 in non-strongly connected graph for $\delta(t) = (\sin t, \cos t, \sin t)$ and $i = 1, \cdots, 6$. Fig. 2 shows the distances between agents x_i and leader x_0 in non-strongly connected graph for $\bar{\delta} = 0$ and $i = 1, \cdots, 6$.

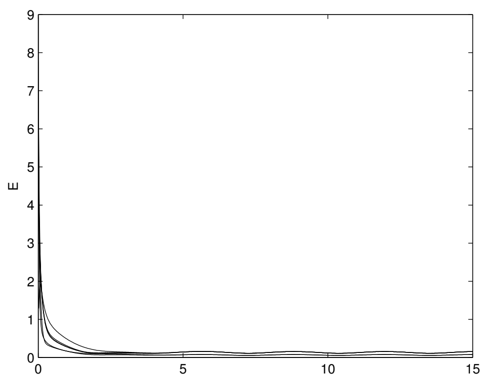

Fig. 3. The distances between agents x_i and leader x_0 in non-strongly connected graph, where $\delta(t) = (\sin t, \cos t, \sin t)$ and $i = 1, \cdots, 6$.

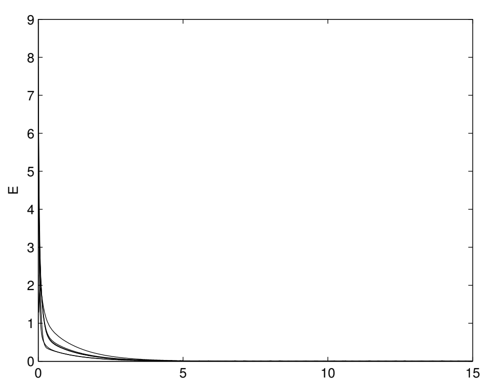

Fig. 4. The distances between agents x_i and leader x_0 in non-strongly connected graph, where $\bar{\delta} = 0$ and $i = 1, \cdots, 6$.

Figs. 1 and 3 indicate us that the tracking errors between agents x_i and leader x_0 can be effectively estimated for the connected or unconnected graph under a given disturbance. Similarly, Figs. 2 and 4 show us that each agent x_i of MAS can follow the leader x_0 for the connected or unconnected graph without disturbance. All above numerical results consist with the theoretical analysis results in Theorems 1 and 2.

V. CONCLUDING REMARKS

This paper has further investigated the consensus of MAS with an active leader and asymmetric adjacency matrix. It should be especially pointed out that the state of the active leader is changing and unmeasured. Under some suitable assumptions, some interesting results are attained as follows: $i)$ each agent of MAS can follow the active leader if the input of the active leader is known beforehand; $ii)$ the tracking error of MAS can be estimated if the input of the active leader is unknown beforehand. Finally, several numerical simulations are also given to justify the effectiveness of the proposed criteria. Some potential real-world engineering applications will be further explored in the near future.

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