Multi-Wing Butterfly Attractors from the Modified Lorenz Systems

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Abstract— **Based on the sawtooth wave function, this paper initiates an approach for generating novel multi-wing butterfly chaotic attractors from the generalized first and second types of modified Lorenz systems. Our theoretical analysis shows that every index-2 saddle-focus equilibrium corresponds to a unique wing in the butterfly attractors. Compared with the traditional ring-shaped multiscroll Lorenz chaotic attractors, the proposed multi-wing butterfly chaotic attractors are much easier to be constructed and implemented by analog circuits. Furthermore, a module-based unified circuit diagram is designed for realizing various multi-wing attractors.**

I. INTRODUCTION

Over the last four decades, as a very interesting nonlinear phenomenon, chaos has been intensively investigated within the mathematics, science, and engineering communities [1-15]. In 1963, Lorenz found the first classical chaotic system. In 1999, Chen and Ueta introduced a dual-system of the Lorenz system. [6] In 2002, Lü and Chen discovered the critical chaotic system between the Lorenz and the Chen systems [5].

On the other hand, in 1984, Chua presented the first chaotic circuit, which builds a connection between the chaos theory and the nonlinear circuit theory. As a natural generalization of Chua's double-scroll attractors, many multiscroll chaotic attractors have been designed and implemented over the past two decades. Recently, Lü and Chen reviewed the main advances in theories, methods, implementations, and applications of multi-scroll chaotic attractors generation and circuit designs [4].

Also, in 1993, Miranda and Stone introduced a so-called proto-Lorenz system based on a nonparametric quadratic polynomial transformation, which can generate an n-scroll chaotic attractor [14]. In 2006, Yu *et al.* proposed a generalized proto-Lorenz system family by using a novel parameterized nth-order polynomial transformation, which can create various ring multiscroll chaotic attractors [10]. However, their algebraic forms are rather complex. Therefore, it is very difficult to physically realize these ring-shaped multiscroll chaotic attractors by using analog circuit. To simplify the algebraic form of the Lorenz system while keeping the butterfly structure of the Lorenz attractor, Elwakil *et al.* presented the first and second types of modified Lorenz systems recently [1-3]. A natural question to follow is whether or not one can generate various multi-wing butterfly chaotic attractors from the first and second types of modified Lorenz systems by introducing some simple nonlinear functions. In the following, we give a positive answer to this question.

More precisely, this paper proposes a novel method for creating various multi-wing butterfly chaotic attractors from the generalized first and second types of modified Lorenz systems. Compared with the traditional ring-shaped multiscroll Lorenz chaotic attractors, these multi-wing butterfly chaotic attractors are much easier to be designed and realized by analog circuits. Moreover, a module-based unified circuit diagram is constructed for implementing these multi-wing attractors. In particular, it is the first time in the literature to generate the multi-wing butterfly chaotic attractors from the modified Lorenz systems.

The rest of the paper is organized as follows. Section II briefly reviews the first and second types of modified Lorenz systems. The generalized multi-wing Lorenz systems are then introduced in Section III. Section IV designs a module-based unified circuit diagram for implementing the multi-wing butterfly chaotic attractors. Concluding remarks are drawn in Section V.

II. TWO TYPES OF MODIFIED LORENZ SYSTEMS

The first type of modified Lorenz system is given by [1-3]

$$\begin{cases} \frac{dx}{d\tau} = a\,(y\,-\,x) \\ \frac{dy}{d\tau} = (c\,-\,z)sgn(x) \\ \frac{dz}{d\tau} = f(x)\,-bz\,, \end{cases} \tag{1}$$

where $a\,=\,0.9$, $b = 0.1$, $c = 2$, sgn(x) is defined by sgn(x)= 1 for $x\,\geq\,0$ and sgn(x)= -1 for $x\,<\,0$, and $f(x) = |x|$ is a absolute-valued function. And the two-wing butterfly attractor of system (1) is shown in Fig. 1 (a).

The second type of modified Lorenz system is described by [1-3]

$$\begin{cases} \frac{dx}{d\tau} = a\,(y\,-\,x) \\ \frac{dy}{d\tau} = -\,z\,sgn(x) \\ \frac{dz}{d\tau} = f(x)\,-1\,, \end{cases} \tag{2}$$

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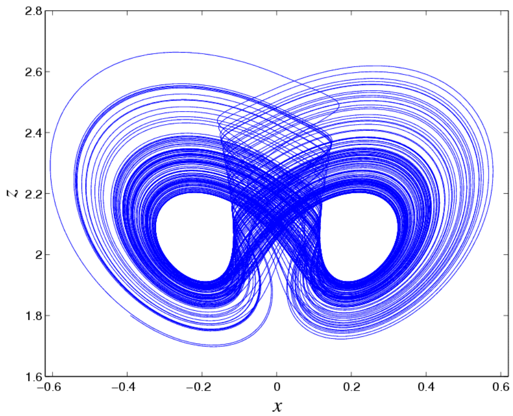
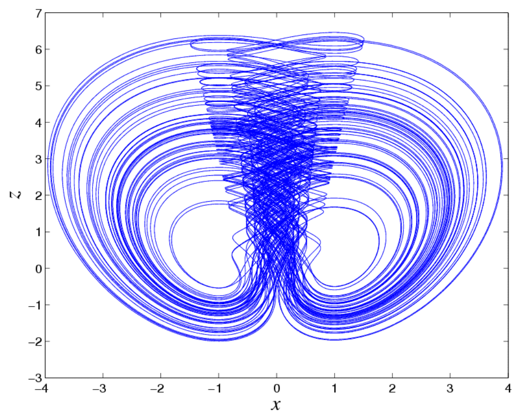
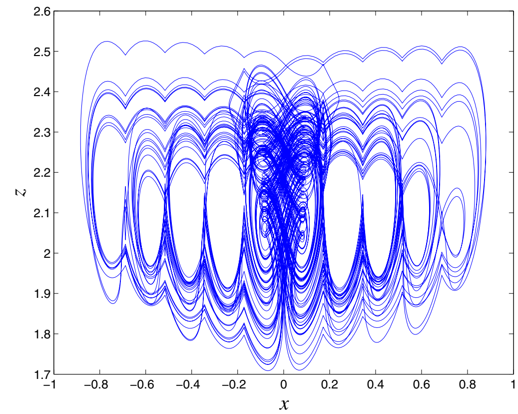


(a) The first type of modified Lorenz attractor

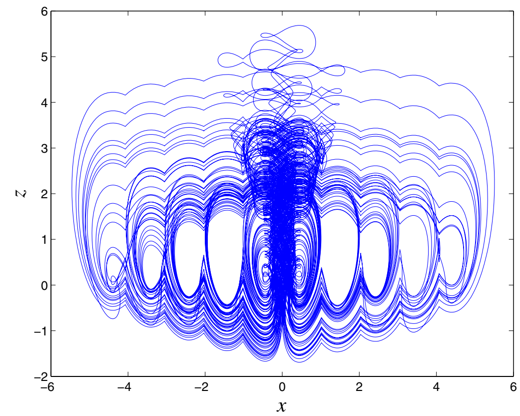


(b) The second type of modified Lorenz attractor

Fig. 1. Two types of modified Lorenz attractors.



(a) The first type of 10-wing butterfly attractor



(b) The second type of 10-wing butterfly attractor

Fig. 2. 10-wing butterfly attractors.

where $a = 0.55$, sgn(x) and $f(x) = |x|$ are defined in system (1). Figure 1 (b) shows the typical two-wing butterfly attractor of system (2).

III. Multi-Wing Butterfly Attractors

Based on the sawtooth wave function theory, the generalized first type of modified Lorenz system is described by

$$\begin{cases} \frac{dx}{d\tau} = a\,(y\,-\,x) \\ \frac{dy}{d\tau} = (c\,-\,z)sgn(x) \\ \frac{dz}{d\tau} = g(x)\,-\,bz\,, \end{cases} \tag{3}$$

where $a = 0.5$, $b = 0.1$, $c = 2$, sgn(x) is a sign function, and $g(x)$ is a sawtooth wave function given as follows:

$$g(x) = g_0(x) + \sum_{n=1}^{N} g_n(x)\,, \tag{4}$$

in which

$$\begin{cases} g_0(x) = k\,|x| \\ g_n(x) = -G_n[2 + sgn(x - E_n) - sgn(x + E_n)] \\ G_n = \frac{A}{A_n}\,, E_n = \frac{nA}{k}\,, \end{cases} \tag{5}$$

where $1 \leq n \leq N$.

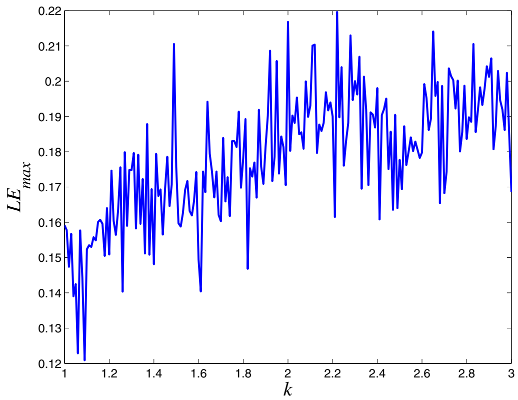
Clearly, system (3) with (4) and (5) can generate a maximum of $2N + 2$ wings in a butterfly attractor. When $A = 0.43$, $k = 2.5$, $A_1 = A_2 = A_3 = A_4 = 2.06$, $G_1 = G_2 = G_3 = G_4 = 0.2087$, $E_1 = 0.1720$, $E_2 = 0.3440$, $E_3 = 0.5160$, $E_4 = 0.6880$, system (3) with (4) and (5) can create a $10-$wing butterfly chaotic attractor as shown in Fig. 2 (a). Figure 3 (a) shows the spectrum of largest Lyapunov exponents of this system.

Similarly, the generalized second type of modified Lorenz system is given by

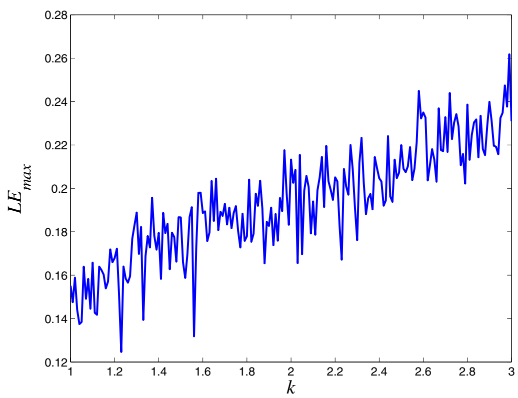
$$\begin{cases} \frac{dx}{d\tau} = a\,(y\,-\,x) \\ \frac{dy}{d\tau} = -\,z\,sgn(x) \\ \frac{dz}{d\tau} = g(x)\,-\,1\,, \end{cases} \tag{6}$$

where $a = 0.5$, sgn(x) is a sign function, and $g(x)$ is a sawtooth wave function defined by (4) and (5).

Obviously, system (6) with (4) and (5) can create a maximum of $2N + 2$ wings in a butterfly attractor. When $A = 2.3$, $k = 2.26$, $A_1 = A_2 = A_3 = A_4 = 2.06$, $G_1 = G_2 = G_3 = G_4 = 1.1165$, $E_1 = 1.0177$, $E_2 = 2.0354$, $E_3 = 3.0531$, $E_4 = 4.0708$, system (6) with (4) and (5) can generate a $10-$wing butterfly chaotic attractor as shown in Fig. 2 (b). Figure 3 (b) shows the spectrum of largest Lyapunov exponents of this system.



(a) The first type of multi-wing system



(b) The second type of multi-wing system

Fig. 3. Spectrum of largest Lyapunov exponents.

IV. Circuit Implementation of Multi-Wing Butterfly Attractors

A module-based unified circuit diagram has been constructed to implement the multi-wing butterfly chaotic attractors of system (3) or (6) with (4) and (5), as shown by Figs. 4 and 5.

The procedure consists of four steps: Step I: *proportional compression transformation of state variables*; Step II: *transformation from differential to integral operations*; Step III: *transformation of the time-scale*; Step IV: *module-based design of nonlinear functions*.

First, one designs the circuit diagram to implement the generalized first type of multi-wing butterfly chaotic attractors. Let $a = 0.5$, $b = 0.1$, $c = 2$. Carry out the transformation from differential to integral operations on system (3), giving

$$\begin{cases} x = \int [-ax + ay]d\tau \\ y = \int [-z\,sgn(x) + c\,sgn(x)]d\tau \\ z = \int [g(x) - bz]d\tau. \end{cases} \tag{7}$$

Since there are several anti-adders, let $\tau = \frac{t}{\tau_0}$ and carry out

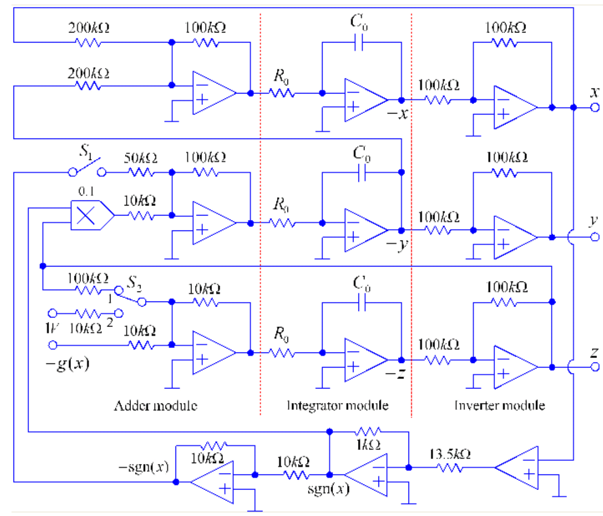


Fig. 4. Circuit diagram for realizing the first and second types of multi-wing butterfly attractors.

the transformation of the time-scale as follows:

$$\begin{cases} x = -\left(-\frac{1}{R_0C_0}\right)\int [-ax - a(-y)]dt \\ y = -\left(-\frac{1}{R_0C_0}\right)\int [-z\,sgn(x) - c(-sgn(x))]dt \\ z = -\left(-\frac{1}{R_0C_0}\right)\int (-(-g(x)) - bz)dt\,, \end{cases} \tag{8}$$

where $\tau_0 = R_0 * C_0$ is the transformation factor of the time-scale. If R_0 and C_0 are increased, then the phase portraits become more and more sparse; if R_0 and C_0 are decreased, then the phase portraits become more and more dense.

Similarly, one constructs the circuit diagram to realize the generalized second type of multi-wing butterfly chaotic attractors. Let $a = 0.5$. Carry out the transformation from differential to integral operations on system (6), giving

$$\begin{cases} x = -\left(-\frac{1}{R_0C_0}\right)\int [-ax - a(-y)]dt \\ y = -\left(-\frac{1}{R_0C_0}\right)\int [-z\,sgn(x)]dt \\ z = -\left(-\frac{1}{R_0C_0}\right)\int (-(-g(x)) - 1)dt\,. \end{cases} \tag{9}$$

All operational amplifiers shown in Figs. 4 and 5 are TL082. Assume that the supply voltages and saturated voltages of all operational amplifiers are $\pm E = \pm 15\,V$ and $\pm E_{sat} = \pm 13.5\,V$, respectively. For convenient adjustment and higher precision, all resistors are precisely adjustable resistors or potentiometers.

When S_1 is switched on and S_2 is located at 1, the circuit in Figs. 4 and 5 can generate a $10-$wing butterfly chaotic attractor as shown in Fig. 6 (a), where $R = 4.0k\Omega$, $R_1 = R_2 = R_3 = R_4 = 47.92k\Omega$, $E_1 = 0.172V, E_2 = 0.344V, E_3 = 0.516V, E_4 = 0.688$. When S_1 is switched off and S_2 is located at 2, the circuit in Figs. 4 and 5 can generate a $10-$wing butterfly chaotic attractor as shown in Fig. 6 (b), where $R = 4.42k\Omega$, $R_1 = R_2 = R_3 = R_4 = 8.96k\Omega$, $E_1 = 1.018V, E_2 = 2.036V, E_3 = 3.054V, E_4 = 4.072$.

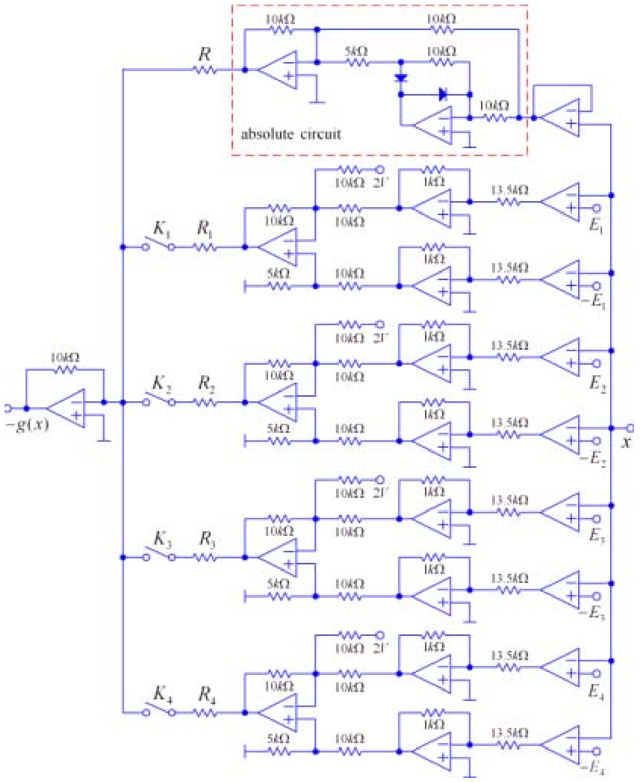


Fig. 5. Circuit diagram for realizing the nonlinearity $-g(x)$.

V. Conclusions

We have proposed a novel approach for creating various multi-wing butterfly chaotic attractors from the generalized first and second types of modified Lorenz systems. Theoretical analysis indicates that every wing in the butterfly attractors corresponds to a unique index-2 saddle-focus equilibrium. Moreover, the proposed multi-wing butterfly chaotic attractors are much easier to be constructed and implemented by analog circuits compared with the traditional ring-shaped multiscroll Lorenz chaotic attractors. Also, a module-based unified circuit diagram has been constructed to implement various multi-wing attractors. Furthermore, multi-wing butterfly chaotic attractors have the general applications in engineering, such as encryption and secret communication.

Acknowledgments

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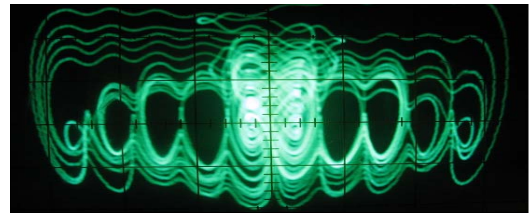
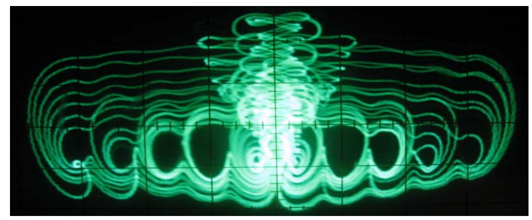


(a) The first type of 10−wing butterfly attractors



(b) The second type of 10−wing butterfly attractors

Fig. 6. Experimental observations of the 10−wing butterfly attractors.

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