Adaptive Pinning Synchronization of A General Complex Dynamical Network

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Abstract— **This paper further investigates and answers two fundamental questions in the complex dynamical networks: i) how many nodes should a general complex dynamical network with fixed network structure and coupling strength be pinned to reach network synchronization? ii) how much coupling strength should a general complex dynamical network with fixed network structure and pinning nodes be employed to reach network synchronization? In the above framework, the coupling-configuration matrix and the inner-coupling matrix are not necessarily symmetric. Also, the pinning nodes can be randomly selected. Furthermore, our adaptive pinning controllers are rather simple compared with some traditional controllers. Finally, a BA network example is then given to show the effectiveness of the proposed synchronization criteria.**

I. Introduction

Today, complex networks lie in everywhere in our daily life, such as the Internet, World Wide Web, communication networks, power grid networks, social networks, genetic regulatory networks, and so on [1-8]. Over the past two decades, complex networks have been intensively investigated in various disciplines, such as mathematical, physical, biological, engineering, and social sciences [1-8].

Synchronization is a kind of typical collective behaviors and basic motions in nature [1-8]. Recently, one of the interesting and significant phenomena in complex dynamical networks is the synchronization of all dynamical nodes in a network. More recently, adaptive synchronization in networks or coupled oscillators has received an increasing attention. For example, Zhou and her colleagues introduced several adaptive synchronization criteria for an uncertain complex dynamical network [4].

Since a complex network normally has a large number of nodes, it is usually difficult to control a complex network by adding the controllers to all nodes. To reduce the number of the controllers, a natural approach is to control a complex network by pinning part of nodes. Wang and Chen proposed an effective measure to pin a complex dynamical network to its equilibrium [7]. However, we do not know how many nodes a complex network should be pinned to achieve network synchronization. Therefore, it is very interesting to ask the following two fundamental questions in complex dynamical networks: i) how many nodes should a general complex dynamical network with fixed network structure and coupling strength be pinned to achieve network synchronization? ii) how much coupling strength should a general complex dynamical network with fixed network structure and pinning nodes be employed to realize network synchronization? In this paper, one gives a complete answer to these two basic questions as above. In brief, one provides a simple formula for calculating the number of pinning nodes and the value of the coupling strength. Here, the coupling configuration matrix and the inner-coupling matrix are not necessary symmetric and the pinning nodes can be randomly selected. It is very useful for the future practical engineering design.

The left paper is organized as follows. A general complex dynamical network model is introduced in Section II. In Section III, several locally and globally adaptive pinning synchronization criteria for the general complex dynamical networks are deduced. A BA network example is then given to show the effectiveness of the pinning control method as above in Section IV. Conclusions are finally drawn in Section V.

II. Preliminaries

A. A generally complex dynamical network model

Consider a generally controlled complex dynamical network consisting of N identical nodes with linearly diffusive couplings, which is described by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j=1}^{N} c_{ij}\,\mathbf{A}\,\mathbf{x}_j + \mathbf{u}_i(\mathbf{x}_1, \cdots, \mathbf{x}_N), \quad (1)$$

where $1 \le i \le N$, $\mathbf{x}_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in \mathbf{R}^n$ is the state vector of the ith node, $\mathbf{f} : \Omega \times \mathbf{R}^+ \to \mathbf{R}^n$ is a smooth nonlinear vector field, node dynamics is $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, $\mathbf{u}_i \in \mathbf{R}^n$ are the control inputs satisfying $\mathbf{u}_i(\mathbf{x}, \cdots, \mathbf{x}) = \mathbf{0}$. Here, $\mathbf{A} \in \mathbf{R}^{n\times n}$ is the inner-coupling matrix and $\mathbf{C} = (c_{ij})_{N\times N} \in \mathbf{R}^{N\times N}$ is the coupling configuration matrix. If there is a link from node i to node j $(j \ne i)$, then $c_{ij} > 0$ and c_{ij} is the coupling strength; otherwise, $c_{ij} = 0$. Assume that $\mathbf{C}$ is a diffusive matrix satisfying $\sum_{j=1}^{N} c_{ij} = 0$.

Suppose that the coupling matrix $\mathbf{C}$ is irreducible but needs not to be symmetric, and that the inner coupling matrix $\mathbf{A}$ also needs not to be symmetric. Hereafter, assume that $\mathbf{x} = \mathbf{s}(t; t_0, \mathbf{x}_0)$, denoted as $\mathbf{s}(t)$, is a solution of the node system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$. Then $\mathbf{S}(t) = (\mathbf{s}^T(t), \mathbf{s}^T(t), \cdots, \mathbf{s}^T(t))^T$ is a synchronous solution of the controlled complex dynamical network (1) since it is a diffusive coupling network. Note that $\mathbf{s}(t)$ can be an equilibrium point, a periodic orbit, an aperiodic orbit, or a chaotic orbit in the phase space.

B. Mathematical preliminaries

Define error vectors as $\mathbf{e}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t)$, where $1 \le i \le N$. Then the error system of network (1) is given by

$$\dot{\mathbf{e}}_i = \mathbf{f}(\mathbf{x}_i, t) - \mathbf{f}(\mathbf{s}, t) + \sum_{j=1}^{N} c_{ij}\,\mathbf{A}\,\mathbf{e}_j + \mathbf{u}_i(\mathbf{x}_1, \cdots, \mathbf{x}_N), \quad (2)$$

where $1 \le i \le N$.

To realize synchronization, the controllers $\mathbf{u}_i$ should guide the error system (1) to approach zero as t goes to infinity. That is, $\lim_{t \to +\infty} \|\mathbf{e}_i(t)\|_2 = 0$, where $1 \le i \le N$.

Hypothesis 1: (**H1**) Suppose that $\|\mathbf{Df}(\mathbf{s})\|_2$ is bounded. That is, there exists a nonnegative constant α satisfying $\|\mathbf{Df}(\mathbf{s})\|_2 \le \alpha$.

Hereafter, assume that $\mathbf{A} \ne 0$ and $\|\mathbf{A}\|_2 = \gamma > 0$. Denote $\rho_{\min}$ as the minimum eigenvalue of the matrix $\frac{\mathbf{A}+\mathbf{A}^T}{2}$. Suppose also that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ are the eigenvalues of the matrix $\frac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^T}{2}$, where $\hat{\mathbf{C}}$ is a modifying matrix of $\mathbf{C}$ by replacing the diagonal elements c_{ii} by $\frac{\rho_{\min}}{\gamma} c_{ii}$.

Lemma 1: [9] Assume that $\mathbf{E}$, $\mathbf{F}$ are the $n \times n$ Hermitian matrices. Suppose that $\xi_1 \ge \xi_2 \ge \cdots \ge \xi_n$, $\zeta_1 \ge \zeta_2 \ge \cdots \ge \zeta_n$, and $\varsigma_1 \ge \varsigma_2 \ge \cdots \ge \varsigma_n$ are the eigenvalues of $\mathbf{E}$, $\mathbf{F}$, $\mathbf{E} + \mathbf{F}$, respectively. Then one has $\xi_i + \zeta_n \le \varsigma_i \le \xi_i + \zeta_1, \quad 1 \le i \le n$.

III. Adaptive Pinning Synchronization of a Complex Dynamical Network

This section further investigates the locally and globally adaptive pinning synchronization of a complex dynamical network. Several network synchronization criteria are drawn.

A. Locally adaptive pinning synchronization

Assume that l nodes $i_1, i_2, \cdots, i_l$ are selected and pinned with the adaptive controllers, which is described by

$$\begin{cases} \mathbf{u}_{i_k} = -d_{i_k}\mathbf{e}_{i_k},\ \dot{d}_{i_k} = h_{i_k}\|\mathbf{e}_{i_k}\|_2^2, & 1 \le k \le l \\ \mathbf{u}_{i_k} = 0, & \text{otherwise}, \end{cases} \quad (3)$$

where $h_{i_k}\,(k = 1, \cdots, l)$ are any positive constants. Thus network (1) can be rewritten as follows:

$$\begin{cases} \dot{\mathbf{x}}_{i_k} = \mathbf{f}(\mathbf{x}_{i_k}, t) + \sum_{j=1}^{N} c_{i_k j}\mathbf{A}\mathbf{x}_j - d_{i_k}\mathbf{e}_{i_k}, & 1 \le k \le l \\ \dot{d}_{i_k} = h_{i_k}\|\mathbf{e}_{i_k}\|_2^2, & 1 \le k \le l \\ \dot{\mathbf{x}}_{i_k} = \mathbf{f}(\mathbf{x}_{i_k}, t) + \sum_{j=1}^{N} c_{i_k j}\mathbf{A}\mathbf{x}_j, & \text{otherwise}. \end{cases} \quad (4)$$

Without loss of generality, one selects the first l nodes as the pinning nodes. From (2) and (3), linearizing system (2) at zero yields

$$\begin{cases} \dot{\mathbf{e}}_i = \mathbf{Df}(\mathbf{s})\mathbf{e}_i + \sum_{j=1}^{N} c_{ij}\mathbf{A}\mathbf{e}_j - d_i\mathbf{e}_i, & 1 \le i \le l \\ \dot{d}_i = h_i\|\mathbf{e}_i\|_2^2, & 1 \le i \le l \\ \dot{\mathbf{e}}_i = \mathbf{Df}(\mathbf{s})\mathbf{e}_i + \sum_{j=1}^{N} c_{ij}\mathbf{A}\mathbf{x}_j, & (l+1) \le i \le N, \end{cases} \quad (5)$$

where $\mathbf{Df}(\mathbf{s})$ is the Jacobian of $\mathbf{f}$ evaluated at $\mathbf{x} = \mathbf{s}$.

Theorem 1: Suppose that **H1** holds. If there exists a natural number $1 \le l \le N$ satisfying $\lambda_{l+1} < -\frac{\alpha}{\gamma}$, the synchronous solution $\mathbf{S}(t)$ of network (1) is locally asymptotically stable under the pinning adaptive controller

$$\begin{cases} \mathbf{u}_i = -d_i\mathbf{e}_i,\ \dot{d}_i = h_i\|\mathbf{e}_i\|_2^2, & 1 \le i \le l \\ \mathbf{u}_i = 0, & (l+1) \le i \le N, \end{cases} \quad (6)$$

where h_i are positive constants for $1 \le i \le l$.

Proof. Construct a Lyapunov candidate as follows:

$$V = \frac{1}{2}\sum_{i=1}^{N} \mathbf{e}_i^T\mathbf{e}_i + \frac{1}{2}\sum_{i=1}^{l} \frac{(d_i - d)^2}{h_i}, \quad (7)$$

where d are positive constants satisfying $d > \alpha + \gamma\lambda_1$.

Thus the differential coefficient of V is described by

$$\begin{aligned} \dot{V} &= \tfrac{1}{2}\sum_{i=1}^{N}(\dot{\mathbf{e}}_i^T\mathbf{e}_i + \mathbf{e}_i^T\dot{\mathbf{e}}_i) + \sum_{i=1}^{l}\tfrac{(d_i - d)\dot{d}_i}{h_i} \\ &\le \mathbf{e}^T(\alpha\mathbf{I}_N + \gamma\hat{\mathbf{C}} - \mathbf{D})\mathbf{e} \\ &= \mathbf{e}^T(\alpha\mathbf{I}_N + \gamma\tfrac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^T}{2} - \mathbf{D})\mathbf{e} \end{aligned}$$

where $\mathbf{e} = (\|\mathbf{e}_1\|_2, \|\mathbf{e}_2\|_2, \cdots, \|\mathbf{e}_N\|_2)^T$, $\mathbf{D} = \operatorname{diag}\{\underbrace{d, \cdots, d}_{l}, \underbrace{0, \cdots, 0}_{N-l}\}$.

Assume that $\hat{\lambda}_1 \ge \hat{\lambda}_2 \ge \cdots \ge \hat{\lambda}_n$ are the eigenvalues of the matrix $\left(\frac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^T}{2} - \frac{\mathbf{D}}{\gamma}\right)$. According to Lemma 1, since $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ are the eigenvalues of the matrix $\frac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^T}{2}$, one has

$$\begin{cases} -\frac{d}{\gamma} + \lambda_N \le \hat{\lambda}_i \le -\frac{d}{\gamma} + \lambda_1, & 1 \le i \le l \\ \lambda_N \le \hat{\lambda}_i \le \lambda_1, & l+1 \le i \le N \end{cases}$$

and $-\frac{d}{\gamma} + \lambda_i \le \hat{\lambda}_i \le \lambda_i$ for $1 \le i \le N$.

Since $\left(\frac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^T}{2} - \frac{\mathbf{D}}{\gamma}\right)$ is a real symmetric matrix, there exists an orthogonal matrix $\mathbf{P}$ satisfying $\left(\frac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^T}{2} - \frac{\mathbf{D}}{\gamma}\right) = \mathbf{P}^T \operatorname{diag}\{\hat{\lambda}_1, \cdots, \hat{\lambda}_N\}\,\mathbf{P}$.

Therefore, one gets

$$\dot{V} \le \mathbf{e}^T(\alpha\mathbf{I}_N + \gamma\tfrac{\hat{\mathbf{C}}+\hat{\mathbf{C}}^T}{2} - \mathbf{D})\mathbf{e} \le (\mathbf{Pe})^T\mathbf{Q}(\mathbf{Pe})$$

where $\mathbf{Q} = \operatorname{diag}\{(\alpha - d + \gamma\lambda_1), \cdots, (\alpha - d + \gamma\lambda_1), (\alpha + \gamma\lambda_{l+1}), \cdots, (\alpha + \gamma\lambda_N)\}$.

Since $d > \alpha + \gamma\lambda_1$, one has $\alpha + \gamma\lambda_1 - d < 0$. From the assumption, then $\alpha + \gamma\lambda_i < 0$ for $(l+1) \le i \le N$.

Therefore, $\mathbf{Q}$ is a negative definite matrix. It follows that $\mathbf{Pe} \to 0$ as $t \to +\infty$. Since $\mathbf{P}$ is an orthogonal matrix, the error vector $\eta = (\mathbf{e_1^T}, \mathbf{e_2^T}, \cdots, \mathbf{e_N^T})^\mathbf{T} \to \mathbf{0}$ as $t \to +\infty$. That is, the synchronous solution $\mathbf{S}(t)$ of network (1) is locally asymptotically stable under the adaptive pinning controller (6). Thus the proof is completed.

Theorem 1 indicates that the network synchronization depends on three basic elements: node dynamics (α), network structure (λ_{l+1}), and inner coupling means (γ, $\rho_{\min}$). In detail, the inequality $\lambda_{l+1} < -\frac{\alpha}{\gamma}$ gives a sufficient condition of the three basic elements as above for the network synchronization. According to Theorem 1, for network (1) with fixed network structure and coupling strength, one can easily determine the number of the pinning nodes to achieve network synchronization. Moreover, for network (1) with fixed network structure and pinning nodes, one can easily determine the value of the coupling strength to realize network synchronization.

Consider a more generally complex dynamical network, which is given by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j=1}^{N} c_{ij}\, \mathbf{W}(\mathbf{x}_j, t) + \mathbf{u}_i(\mathbf{x}_1, \cdots, \mathbf{x}_N), \quad (8)$$

where $1 \le i \le N$ and $\mathbf{W} : \mathbf{R}^n \times \mathbf{R}^+ \to \mathbf{R}^n$ is an inner coupling function satisfying the diffusive condition $\sum\limits_{j=1}^{N} c_{ij}\, \mathbf{W}(\mathbf{x}_j, t) = \mathbf{0}$.

Hypothesis 2: (**H2**) Suppose that $\mathbf{W}(\mathbf{x}_j, t)$ is Lipschitz continuous. That is, there exists a positive constant δ satisfying $\|\mathbf{W}(\mathbf{x}_j, t) - \mathbf{W}(\mathbf{s}, t)\|_2 \le \delta \|\mathbf{e}_j\|_2$ for $1 \le j \le N$.

Theorem 2: Suppose **H1** and **H2** hold. If there exists a natural number $1 \le l \le N$ satisfying $\lambda_{l+1} < -\frac{\alpha}{\delta}$, the synchronous solution $\mathbf{S}(t)$ of network (8) is locally asymptotically stable under the adaptive pinning controller (6).

B. Globally adaptive pinning synchronization

Rewrite the node system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ as $\dot{\mathbf{x}} = \mathbf{B}\,\mathbf{x} + \mathbf{g}(\mathbf{x}, t)$, where $\mathbf{B}\,\mathbf{x}$ is the linear part of the node dynamics with $\mathbf{B} \in \mathbf{R}^{n\times n}$ and $\mathbf{g} : \Omega \times \mathbf{R}^+ \to \mathbf{R}^n$ is a continuously differentially nonlinear function. Thus the network (1) is recasted as follows:

$$\dot{\mathbf{x}}_i = \mathbf{B}\mathbf{x}_i + \mathbf{g}(\mathbf{x}_i, t) + \sum_{j=1}^{N} c_{ij}\, \mathbf{A}\, \mathbf{x}_j + \mathbf{u}_i(\mathbf{x}_1, \cdots, \mathbf{x}_N), \quad (9)$$

where $1 \le i \le N$.

Since $\mathbf{B}$ is a given constant matrix, there exists a nonnegative constant β satisfying $\|\mathbf{B}\|_2 \le \beta$.

Hypothesis 3: (**H3**) Suppose that $\mathbf{g}(\mathbf{x}, t)$ is Lipschitz continuous. That is, there exists a Lipschitz constant μ satisfying $\|\mathbf{g}(\mathbf{x}_i, t) - \mathbf{g}(\mathbf{s}, t)\|_2 \le \mu \|\mathbf{e}_i\|_2$ for $1 \le i \le N$.

Similarly, one can select the first l nodes as the pinning nodes. From (3) and (9), the error system is described by

$$\begin{cases} \dot{\mathbf{e}}_i &= \mathbf{B}\,\mathbf{e}_i + \mathbf{g}(\mathbf{x}_i, t) - \mathbf{g}(\mathbf{s}, t) + \sum\limits_{j=1}^{N} c_{ij}\mathbf{A}\mathbf{e}_j - d_i \mathbf{e}_i, \\ & \quad 1 \le i \le l \\ \dot{d}_i &= h_i \|\mathbf{e}_i\|_2^2, \ 1 \le i \le l \\ \dot{\mathbf{e}}_i &= \mathbf{B}\,\mathbf{e}_i + \mathbf{g}(\mathbf{x}_i, t) - \mathbf{g}(\mathbf{s}, t) + \sum\limits_{j=1}^{N} c_{ij}\mathbf{A}\mathbf{x}_j, \\ & \quad (l+1) \le i \le N\,. \end{cases} \quad (10)$$

Similarly, one obtains the following two globally adaptive pinning synchronization criteria.

Theorem 3: Suppose that **H3** holds. If there exists a natural number $1 \le l \le N$ satisfying $\lambda_{l+1} < -\frac{\beta+\mu}{\gamma}$, the synchronous solution $\mathbf{S}(t)$ of network (9) is globally asymptotically stable under the adaptive pinning controller

$$\begin{cases} \mathbf{u}_i = -d_i \mathbf{e}_i, \ \dot{d}_i = h_i \|\mathbf{e}_i\|_2^2, & 1 \le i \le l \\ \mathbf{u}_i = 0, & (l+1) \le i \le N\,, \end{cases} \quad (11)$$

where h_i are positive constants for $1 \le i \le l$.

Theorem 4: Assume that **H2** and **H3** hold. If there exists a natural number $1 \le l \le N$ satisfying $\lambda_{l+1} < -\frac{\beta+\mu}{\delta}$, the synchronous solution $\mathbf{S}(t)$ of network (8) is globally asymptotically stable under adaptive pinning controller (11).

IV. Numerical Simulation

Suppose that the controlled network (1) consists of 50 identical Lorenz systems, which is described by

$$\dot{\mathbf{x}}_i = \mathbf{B}\mathbf{x}_i + \mathbf{g}(\mathbf{x}_i) + c \sum_{j=1}^{50} \bar{c}_{ij}\, \mathbf{A}\, \mathbf{x}_j + \mathbf{u}_i, \quad (12)$$

where $1 \le i \le 50$, $\mathbf{A} = \mathrm{diag}\{1, 1.3, 1\}$, $c = 40$, $\bar{\mathbf{C}} = (\bar{c}_{ij})_{50\times 50}$ is a symmetrically diffusive coupling matrix with $\bar{c}_{ij} = 0$ or $1\,(j \ne i)$. The node dynamics is given by

$$\begin{aligned} \dot{\mathbf{x}}_i &= \begin{pmatrix} -r_1 & r_1 & 0 \\ r_3 & -1 & 0 \\ 0 & 0 & -r_2 \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix} \\ &= \mathbf{B}\mathbf{x}_i + \mathbf{g}(\mathbf{x}_i)\,, \end{aligned}$$

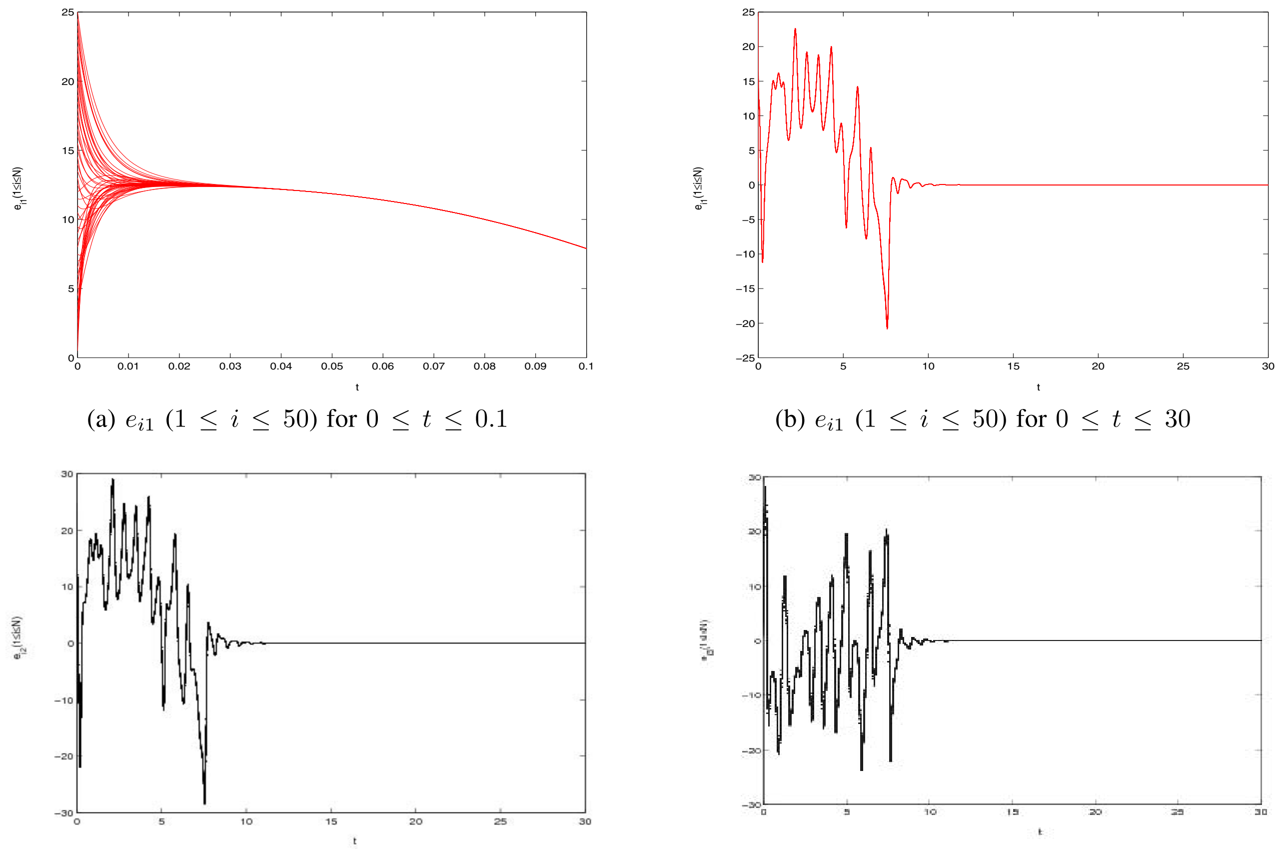
where $1 \le i \le 50$ and $r_1 = 10$, $r_2 = \frac{8}{3}$, $r_3 = 28$.

Obviously, $\gamma = \|\mathbf{A}\|_2 = 1.3$, $\beta = \|\mathbf{B}\|_2 \approx 30.0731$. From [10], there exist constants $M_1 = M_2 = 28.9180$ and $M_3 = 56.9180$ satisfying $\|x_{ij}\|, \|s_j\| \le M_j$ for $1 \le i \le 50$ and $1 \le j \le 3$. Therefore, one has

$$\begin{aligned} \|\mathbf{g}(\mathbf{x}_i) - \mathbf{g}(\mathbf{s})\|_2 &\le \sqrt{2M_1^2 + M_2^2 + M_3^2}\, \|\mathbf{e}_i\|_2 \\ &\approx 75.8183\, \|\mathbf{e}_i\|_2\,. \end{aligned}$$

Let $\mu = 75.8183$. Then one gets $-\frac{\beta+\mu}{c\gamma} = -2.0364$.

One generates network (12) via the known BA scale-free model with $m_0 = m = 5$, $N = 50$ [6]. Without loss of generality, one randomly chooses the pinning nodes. Since $\bar{\lambda}_2 = -1.9463$ and $\bar{\lambda}_3 = -2.1096$, there exists a natural number $l = 2$ satisfying $\bar{\lambda}_{2+1} = -2.1096 < -2.0364$. From Theorem 3, the synchronous solution $\mathbf{S}(t)$ of network (12) is globally asymptotically stable under controller (11).



(a) e_{i1} $(1 \leq i \leq 50)$ for $0 \leq t \leq 0.1$

(b) e_{i1} $(1 \leq i \leq 50)$ for $0 \leq t \leq 30$

(c) e_{i2} $(1 \leq i \leq 50)$ for $0 \leq t \leq 30$

(d) e_{i3} $(1 \leq i \leq 50)$ for $0 \leq t \leq 30$

Fig. 1. Synchronization errors e_{ij} $(1 \leq i \leq 50,\ 1 \leq t \leq 30)$ of network (12).

In the numerical simulation, all parameters are given as follows: $l = 2$, $h_k = 0.01$, $d_k(0) = 1$, $x_i(0) = (4 + 0.5i, 5 + 0.5i, 6 + 0.5i)^T$, $s(0) = (4, 5, 6)^T$, where $1 \leq k \leq 3$ and $1 \leq i \leq 50$. The synchronous errors e_i $(1 \leq i \leq 50)$ are shown in Fig. 1. Thus network (12) is globally asymptotically stable at zero under controller (11) with $l = 2$.

V. Conclusions

We have further studied and answered the following two fundamental questions in complex dynamical networks: i) how many nodes should a general complex dynamical network with fixed network structure and coupling strength be pinned to realize network synchronization? ii) how much coupling strength should a general complex dynamical network with fixed network structure and pinning nodes be employed to achieve network synchronization? Moreover, several novelly adaptive synchronization criteria are proposed. Here, the coupling configuration matrix and the inner-coupling matrix are not necessarily symmetric. It is very useful for the future practical engineering design.

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