$N-$Scroll Chaotic Attractors from A General Jerk Circuit

Simin Yu
College of Automation
Guangdong University of Technology
Guangzhou 510090, China

Jinhu Lü
Institute of Systems Science
Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100080, China
Email: jhlu@iss.ac.cn

Henry Leung
Department of Electrical and Computer Engineering
University of Calgary
AB, T2N 1N4, Canada
Email: leungh@ucalgary.ca

Guanrong Chen
Department of Electronic Engineering
City University of Hong Kong
Hong Kong, China
Email: gchen@ee.cityu.edu.hk

***Abstract*— This paper proposes a novel nonlinear modulating function approach for generating $n-$scroll chaotic attractors based on a general jerk circuit. The systematic nonlinear modulating function methodology developed here can arbitrarily design the swings, widths, slopes, breakpoints, equilibrium points, and shapes of the n-scroll chaotic attractors via the adjustable sawtooth wave, triangular wave, and transconductor wave functions. A block circuit diagram is designed for hardware implementation of various $3 \sim 12-$scroll attractors via switchings of the switch linkages. This is the first time to experimentally verify a $12-$scroll chaotic attractor generated by an analog circuit under the laboratory environment.**

I. Introduction

The so-called *jerk circuits* are described by $\dddot{x} = J(x, \dot{x}, \ddot{x})$. Jerk circuits have some practical applications in, for example, broadband signal generations and secure communications. This is because they are simple circuits that are easy to build, to be re-scaled (to any desired frequencies), and to analyze, predict, and control with very high accuracy [1-2]. On the other hand, multi-scroll chaotic attractors also have many practical applications [3-10]. Notice, however, that a general jerk system can only generate single- or double-scroll attractors [1-2]. Therefore, it is very interesting to ask whether or not the general jerk circuits can be slightly modified so as to generate $n-$scroll chaotic attractors. This paper gives a positive answer to this question. More precisely, this paper introduces a nonlinear modulating function approach for creating $n-$scroll chaotic attractors based on a general jerk circuit. Moreover, a block circuit diagram is designed for hardware implementations of various $3 \sim 12-$scroll attractors by using the switchings of the switch linkages.

This paper is organized as follows. In Section II, a general jerk circuit is introduced. The proposed design approach is further discussed in Section III. In Section IV, a simple circuit diagram is constructed for experimentally verifying the $n-$scroll chaotic attractors. Conclusions are finally given in Section V.

II. A General Jerk Circuit

The general jerk circuit is described by

$$\dddot{x} + \beta \ddot{x} + \gamma \dot{x} = f(x), \tag{1}$$

where β, γ are real parameters, $f(x)$ is a nonlinear function, $\dot{x} = \frac{dx}{d\tau}$ is the velocity, $\ddot{x} = \frac{d^2x}{d\tau^2}$ is the acceleration, $\dddot{x} = \frac{d^3x}{d\tau^3}$ is the jerk (or, the rate of change of the acceleration by mechanical means), $\tau = \frac{t}{R_0 C_0}$, in which $\frac{1}{R_0 C_0}$ is the transformation factor of the time scale, and also is the integral constant of the integrator. For simplicity, assume that $\beta = 0.6, \gamma = 1$. The following is a result for the stability of the equilibrium point $(x_0, 0, 0)$ of the jerk system (1).

Lemma 1: Assume that $f(x)$ is differentiable at the equilibrium point $(x_0, 0, 0)$. A necessary and sufficient condition for the stability of the equilibrium point $(x_0, 0, 0)$ is that $\beta > 0$ and $-\beta\gamma < f' < 0$, where $f' = \frac{df}{d\tau}|_{x=x_0}$.

III. Design of $n-$Scroll Chaotic Attractors

A. Modulating Function Approach

In the following, a swing modulating function of double sawtooth wave is constructed to generate multi-scroll chaotic attractors in the jerk system (1). This function is described by

$$f(x) = |F(x)| \operatorname{sgn}(x) - x, \tag{2}$$

where $F(x)$ is the swing modulating function, which controls the swings of scrolls and equilibrium points of system (1).

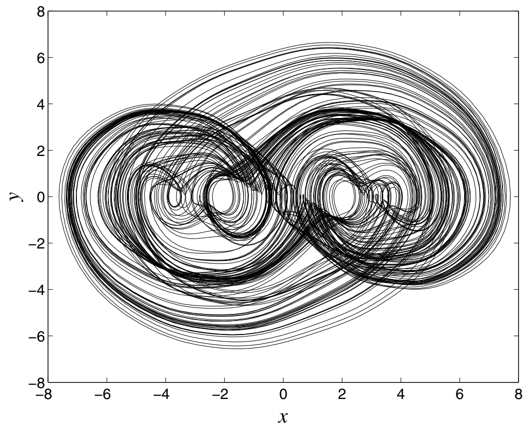
It is noticed that the modulating function $F(\cdot)$ may be an autonomous function, or a non-autonomous function produced by outer signals. Of course, $F(\cdot)$ can be a constant in the special case. To create chaos in system (1) with (2), the modulating function $F(\cdot)$ has to satisfy some conditions. Here, assume that

$$|F(\cdot)| = |A \sin(ax)|, \tag{3}$$

where $A, a > 0$ are parameters. Let $A = 5, a = 1.27$. Then, system (1) with (2) and (3) has a scroll-nesting $4-$scroll chaotic attractor, as shown in Fig. 1.

B. Adjustable Sawtooth Wave Method

To generate a chaotic attractor with an even number of scrolls, the adjustable sawtooth wave is used, which is de-



(a)

Fig. 1. Scroll-nesting 4−scroll chaotic attractor.

scribed by

$$f_1(x) = A_0\mathrm{sgn}(x) + \sum_{i=1}^{M}\left[\frac{A_{i-1}+A_i}{2}\mathrm{sgn}\left(x - \frac{2}{B}\sum_{j=0}^{i-1}A_j\right)\right] + \sum_{i=1}^{M}\left[\frac{A_{i-1}+A_i}{2}\mathrm{sgn}\left(x + \frac{2}{B}\sum_{j=0}^{i-1}A_j\right)\right] - Bx\,, \tag{4}$$

where all parameters $A_i > 0\,(i = 0, 1, 2, \cdots)$ and $B \in [0.7, 1.2]$, which can create $2M + 2\,(M = 1, 2, 3, \cdots)$ scrolls in the attractor.

Similarly, to generate a chaotic attractor with an odd number of scrolls, the adjustable sawtooth wave is used, which is described by

$$f_2(x) = \sum_{i=1}^{M}\left\{\frac{A_{i-1}+A_i}{2}\mathrm{sgn}\left[x - \frac{1}{B}(2\sum_{j=0}^{i-1}A_j - A_0)\right]\right\} + \sum_{i=1}^{M}\left\{\frac{A_{i-1}+A_i}{2}\mathrm{sgn}\left[x + \frac{1}{B}(2\sum_{j=0}^{i-1}A_j - A_0)\right]\right\} - Bx, \tag{5}$$

where all parameters $A_i > 0\,(i = 0, 1, 2, \cdots)$ and $B \in [0.7, 1.2]$, which can create $2M + 1\,(M = 1, 2, 3, \cdots)$ scrolls in the attractor.

Assume that (i) $f_1(x)$ and $f_2(x)$ are odd functions, and (ii) the zeros of $f_1(x)$ and $f_2(x)$ lie in the centers of two neighboring breakpoints. Then, one can rigorously deduce a set of recursive formulas on the parameters of sawtooth wave, as follows:

(i) The slopes of the sawtooth waves $f_1(x)$ and $f_2(x)$ are $-B$, satisfying $B \in [0.7, 1.2]$.

(ii) Denote the swings of the scrolls of the sawtooth waves $f_1(x)$ and $f_2(x)$ by $E_i\,(i = 0, 1, \cdots, M)$. Then, the recursive formulas of E_i are

$$E_0 = 2A_0, \quad E_i = A_{i-1} + A_i\,, \tag{6}$$

where $i = 1, \cdots, M$.

(iii) Denote the widths between two neighboring scrolls of the sawtooth waves $f_1(x)$ and $f_2(x)$ by $W_i\,(i = 0, 1, \cdots, M-1)$. Then, the recursive formulas of W_i (except the outside edge scroll) are

$$W_i = \frac{2A_i}{B}\,, \tag{7}$$

where $i = 0, 1, \cdots, M-1$.

(iv) Denote the breakpoints of the sawtooth waves $f_1(x)$ and $f_2(x)$ by $S_i\,(i = 0, 1, \cdots, M)$. Then, the recursive formulas of S_i are

$$\begin{cases} S_{i,\,f_1} = \frac{1}{B}\sum_{j=0}^{i-1}2A_j\,, \\ S_{i,\,f_2} = \frac{1}{B}\left(\sum_{j=0}^{i-1}2A_j - A_0\right)\,, \end{cases} \tag{8}$$

where $i = 1, 2, \cdots, M$.

(v) Denote the zeros of the sawtooth waves $f_1(x)$ and $f_2(x)$ by $P_i\,(i = 0, 1, \cdots, M)$. Then, the recursive formulas of P_i are

$$\begin{cases} P_{i,\,f_1} = \frac{1}{B}\left(\sum_{j=0}^{i-1}2A_j - A_{i-1}\right)\,, \\ P_{i,\,f_2} = \frac{1}{B}\left(\sum_{j=0}^{i-1}2A_j - A_{i-1} - A_0\right)\,, \end{cases} \tag{9}$$

where $i = 1, \cdots, M$.

In the following, all the parameters of the chaotic attractor are calculated by using the recursive formulas (6)-(9). Four kinds of 12−scroll chaotic attractors with different sizes are further investigated; that is, Type I: *multi-scroll attractors, with the sizes of the scrolls gradually increasing from the center to both sides*; Type II: *multi-scroll attractors, with the sizes of the scrolls gradually decreasing from the center to both sides*; Type III: *multi-scroll attractors, with the scrolls alternating between small and large scrolls*; Type IV: *multi-scroll attractors, with all scrolls being same in size*.

C. Adjustable Triangular Wave Approach

In most chaotic circuits, such as Chua's circuit, the four-dimensional Matsumoto-Chua-Kobayashi (MCK) chaotic circuit, and some Sprott's chaotic jerk circuits, their piecewise linear (PWL) functions have constant breakpoints and slopes. In the following, a PWL function with varying breakpoints and slopes is constructed, to create single-scroll and double-scroll attractors in the jerk system (1). The function is described by

$$\begin{aligned} f(x) &= \frac{A}{2\alpha}\left(|x+\alpha| - |x-\alpha|\right) - Bx \\ &= \begin{cases} -Bx - A & x < -\alpha \\ \frac{A-\alpha B}{\alpha}x & -\alpha \le x \le \alpha \\ -Bx + A & x > \alpha\,, \end{cases} \end{aligned} \tag{10}$$

where parameters $A > 0$, $B \in [0.8, 1.2]$, and $\alpha \in (0, \frac{A}{B}]$ represents the varying breakpoints.

Based on (10), to generate a chaotic attractor with an even number of scrolls, the adjustable triangular wave is constructed as

$$\begin{aligned} f_1(x) = \sum_{n=-M}^{M}\frac{A}{2\alpha_n}\Big[&\left|\left(x - \tfrac{2An}{B}\right) + \alpha_n\right| \\ &- \left|\left(x - \tfrac{2An}{B}\right) - \alpha_n\right|\Big] - Bx\,, \end{aligned} \tag{11}$$

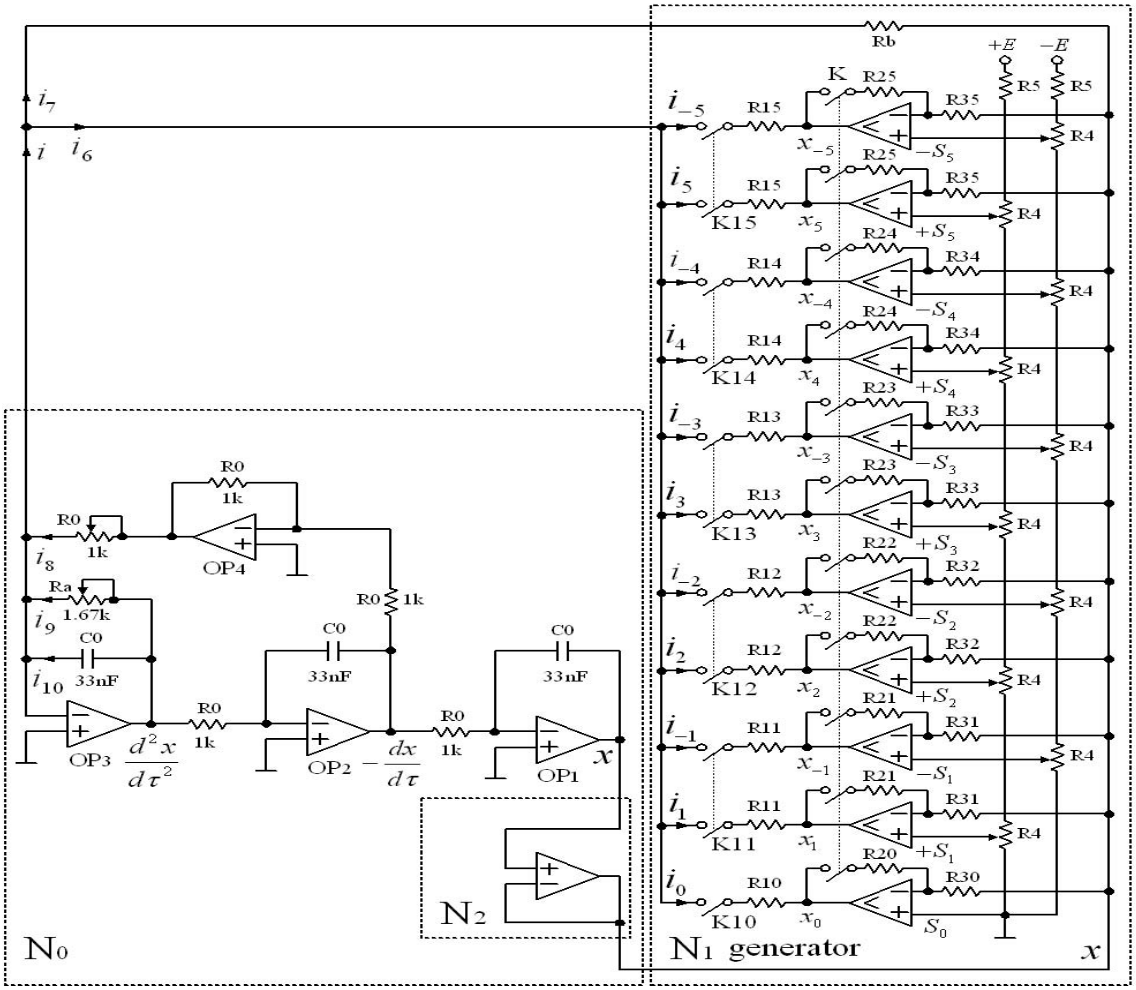


Fig. 2. Circuit diagram of $n-$scroll attractors.

where parameters $A > 0$, $0.8 \leq B \leq 1.2$, $\alpha_n \in (0, \frac{3A}{10B}]\,(n = 0, \pm 1, \cdots, \pm M)$, $M = 1, 2, \cdots$, which can create $2M+2$ scrolls in the chaotic attractor.

Similarly, to generate a chaotic attractor with an odd number of scrolls, the adjustable triangular wave is constructed as

$$\begin{aligned} f_2(x) \;=\; & \sum_{\substack{n=-M\\ n\neq 0}}^{M} \frac{A}{2\alpha_n}\left[\left|\left(x - \frac{A}{B}(2n - \frac{|n|}{n})\right) + \alpha_n\right|\right. \\ & \left. - \left|\left(x - \frac{A}{B}(2n - \frac{|n|}{n})\right) - \alpha_n\right|\right] - Bx\,, \end{aligned} \qquad (12)$$

where parameters $A > 0$, $0.8 \leq B \leq 1.2$, $\alpha_n \in (0, \frac{3A}{10B}]\,(n = \pm 1, \pm 2, \cdots, \pm M)$, $M = 1, 2, \cdots$, which can create $2M+1$ scrolls in the chaotic attractor.

D. Adjustable Transconductor Wave Method

To generate a chaotic attractor with an even number of scrolls, the adjustable transconductor wave is constructed as

$$f_1(x) = \sum_{n=-M}^{M} A \tanh\left[C_n\left(x - \frac{2nA}{B}\right)\right] - Bx\,, \qquad (13)$$

where A, B, C_n are adjustable parameters, and $M \in N$.

Similarly, to create a chaotic attractor with an odd number of scrolls, the adjustable transconductor wave is constructed as

$$f_2(x) = \sum_{\substack{n=-M\\ n\neq 0}}^{M} A \tanh\left[C_n\left(x - \left(2n - \frac{|n|}{n}\right)\frac{A}{B}\right)\right] - Bx, \qquad (14)$$

where A, B, C_n are adjustable parameters, and $M \in N$.

IV. Circuit Implementation Of $n-$Scroll Chaotic Attractors

Based on the operational principles of sawtooth waves and triangular waves, according to (1), (4), (5), (11), and (12), one can design a circuit diagram to realize various multi-scroll chaotic attractors.

Figure 2 shows such a circuit diagram. This circuit diagram includes five different parts; that is, Part I: *integrator* N_0; Part II: *sawtooth wave and triangular wave generator* N_1; Part III: *buffer* N_2; Part IV: *switch linkages, including* K, K_{10}, K_{11}, K_{12}, K_{13}, K_{14}, K_{15}; Part V: *voltage-current conversion resistors* $R10 \sim R15$. Let $R_{3j} = 1\,k\Omega$ and

$R_{2j} = 200\,k\Omega$ for $0 \le j \le 5$. When K is switched on, N_1 generates a triangular wave; when K is switched off, N_1 creates a sawtooth wave. According to (6)-(9), all experimental parameters can be closely calculated for generating Types I, II, III, and IV of chaotic attractors.

To generate an attractor with an even number of scrolls, K_{10} and K_{11} are switched on. When K_{12}, K_{13}, K_{14}, K_{15} are switched off, the circuit diagram can create a $4-$scroll attractor; when K_{12} is switched on and K_{13}, K_{14}, K_{15} are switched off, the circuit diagram can generate a $6-$scroll attractor; when K_{12}, K_{13} are switched on and K_{14}, K_{15} are switched off, the circuit diagram can create a $8-$scroll attractor; when K_{12}, K_{13}, K_{14} are switched on and K_{15} is switched off, the circuit diagram can generate a $10-$scroll attractor; when K_{12}, K_{13}, K_{14}, K_{15} are switched on, the circuit diagram can create a $12-$scroll attractor, as shown in Fig. 3.

To generate an attractor with an odd number of scrolls, K_{10} is switched off and K_{11} is switched on. When K_{12}, K_{13}, K_{14}, K_{15} are switched off, the circuit diagram can create a $3-$scroll attractor; when K_{12} is switched on and K_{13}, K_{14}, K_{15} are switched off, the circuit diagram can generate a $5-$scroll attractor; when K_{12}, K_{13} are switched on and K_{14}, K_{15} are switched off, the circuit diagram can create a $7-$scroll attractor; when K_{12}, K_{13}, K_{14} are switched on and K_{15} is switched off, the circuit diagram can generate a $9-$scroll attractor; when K_{12}, K_{13}, K_{14}, K_{15} are switched on, the circuit diagram can create a $11-$scroll attractor. Due to space limitation, figures are omitted here.

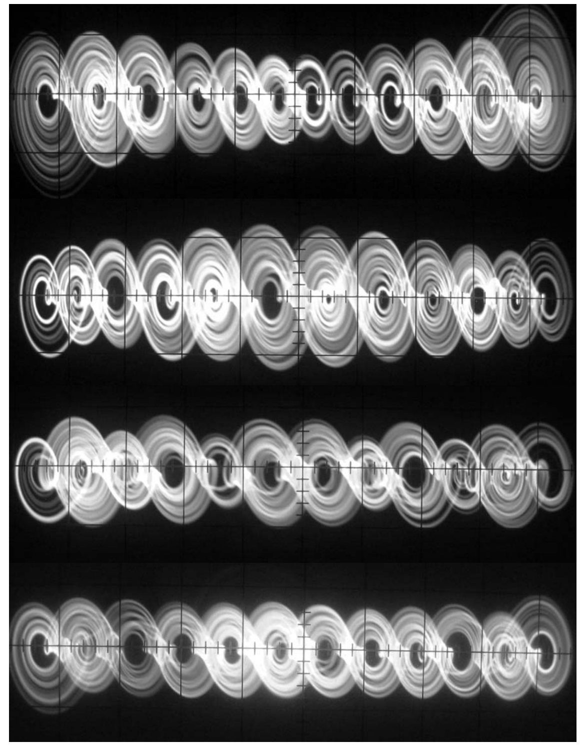


Fig. 3. Experimental observations of the $12-$scroll chaotic attractors. From up to down: (a) Type I, $x = 1.1$V/div, $y = 0.4$V/div; (b) Type II, $x = 1.05$V/div, $y = 0.5$V/div; (c) Type III, $x = 0.8$V/div, $y = 0.4$V/div; (d) Type IV, $x = 0.66$V/div, $y = 0.33$V/div.

V. Conclusion

This paper has reported a nonlinear modulating function approach for generating $n-$scroll chaotic attractors from a general jerk circuit. A simple block circuit diagram has been designed for experimental verification of various $3 \sim 12-$scroll chaotic attractors. In particular, the adjustability of the sawtooth wave and triangular wave as well as the rigorous recursive formulas provide a theoretical principle for physically realizing chaotic attractors with a large number of scrolls.

Acknowledgment

This work was supported by National Natural Science Foundation of China under Grant No.60304017 and Grant No.20336040/B06, Natural Science Foundation of Guangdong Province under Grant No.32469, Science and Technology Program of Guangzhou City under Grant No.2004J1-C0291, and the City University of Hong Kong under the SRG Grant 7001702(EE).

References

[1] J. C. Sprott, "Simple chaotic systems and circuits," *Amer. J. Phys.*, vol. 68, no. 8, pp. 758-763, Aug. 2000.

[2] J. C. Sprott, "A new class of chaotic circuit," *Phys. Lett. A*, vol. 266, no. 1, pp. 19-23, Feb. 2000.

[3] J. A. K. Suykens and J. Vandewalle, "Generation of $n-$double scrolls ($n = 1, 2, 3, 4, \cdots$)," *IEEE Trans. Circuits Syst. I*, vol. 40, no. 11, pp. 861-867, Nov. 1993.

[4] M. E. Yalcin, J. A. K. Suykens, J. Vandewalle, and S. Ozoguz, "Families of scroll grid attractors," *Int. J. Bifurcation Chaos*, vol. 12, no. 1, pp. 23-41, Jan. 2002.

[5] M. E. Yalcin, J. A. K. Suykens, and J. Vandewalle, "Experimental confirmation of $3-$ and $5-$scroll attractors from a generalized Chua's circuit," *IEEE Trans. Circuits Syst. I*, vol. 47, no. 3, pp. 425-429, Mar. 2000.

[6] J. Lü, T. Zhou, G. Chen, and X. Yang, "Generating chaos with a switching piecewise-linear controller," *Chaos*, vol. 12, no. 2, pp. 344-349, Jun. 2002.

[7] J. Lü, X. Yu, and G. Chen, "Generating chaotic attractors with multiple merged basins of attraction: A switching piecewise-linear control approach," *IEEE Trans. Circuits Syst. I*, vol. 50, no. 2, pp. 198-207, Feb. 2003.

[8] J. Lü, F. Han, X. Yu, and G. Chen, "Generating 3-D multi-scroll chaotic attractors: A hysteresis series switching method," *Automatica*, vol. 40, no. 10, pp. 1677-1687, Nov. 2004.

[9] J. Lü, G. Chen, X. Yu, and H. Leung, "Design and analysis of multi-scroll chaotic attractors from saturated function series," *IEEE Trans. Circuits Syst. I*, vol. 51, no. 12, pp. 2476-2490, Dec. 2004.

[10] S. M. Yu, S. S. Qiu, and Q. H. Lin, "New results of study on generating multiple-scroll chaotic attractors," *Science in China Series F*, vol. 46, no. 2, pp. 104-115, Feb. 2003.