Synchronization: A Fundamental Phenomenon in Complex Dynamical Networks

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***Abstract*— It is well known that complex networks have become a very important part of our daily lives and have been intensively studied in all fields over the last few years. This paper proposes and reviews several network synchronization criteria for the time-invariant, time-varying, delayed, and discrete complex dynamical network models. Moreover, the maximum synchronizability of time-invariant complex dynamical networks is further investigated.**

I. Introduction

Over the last decades, complex networks occur in all fields of sciences and humanities, such as the World Wide Web, computer networks, biological neural networks, electrical power grids, food webs, and so on [1-12]. Since many real-world complex networks, such as the Internet and various e-bank networks, are very important and closely relative to our daily lives, it is extremely important to maintain the security against failures and attacks for these complex networks. Therefore, it is very necessary for us to further understand the essential nature and fundamental behaviors of network topological structures as well as their synchronization properties, towards better design and management of real-world complex networks.

Synchronization is a basic phenomenon in a wide range of real systems, such as neural networks, physiological process, biology, and so on [9]. It has been demonstrated that many real-world complex networks display various synchronization phenomena [3-12]. In this paper, we introduce and review several typical network synchronization criteria for the time-invariant [5], time-varying [3], delayed [10], and discrete [9] network models. Moreover, the maximum synchronizability of time-invariant complex dynamical networks is further explored. Our main results are: (i) the synchronization of a general time-varying complex network is completely determined by its inner-coupled matrix and its coupled configuration matrix – specifically the eigenvalues and the corresponding eigenvectors of this coupled configuration matrix, rather than the sole eigenvalues of the coupled configuration matrix for a uniform network; (ii) the maximum synchronizability of a network is completely determined by its associated feedback system with a real meaning in synchronous communication.

The rest of this paper is organized as follows: In Section II, a general time-varying complex network model is introduced and several synchronization criteria are given. The synchronization criteria for a delayed and a discrete complex network models are reviewed in Section III and Section IV, respectively. In Section V, the maximum synchronizability of time-invariant complex networks is then further investigated. Conclusions are finally drawn in Section VI.

II. Synchronization Criteria of Time-Varying Complex Dynamical Networks

In this section, we introduce a general time-varying complex dynamical network model and further investigate its synchronization criteria.

In 2002, Wang and Chen [12] proposed a simple uniform dynamical network model, which is described by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + c\sum_{j=1}^{N} c_{ij}\mathbf{A}\mathbf{x}_j,\ i = 1, 2, \cdots, N, \tag{1}$$

where $\mathbf{C} = (c_{ij})_{N\times N}$ is a $0-1$ matrix and $\mathbf{A}$ is a $0-1$ diagonal matrix.

Lü *et al.* [3-4] generalized this uniform model and introduced a general time-varying dynamical network as follows:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}(t)\,\mathbf{A}(t)\,(\mathbf{x}_j - \mathbf{x}_i),\ i = 1, 2, \cdots, N, \tag{2}$$

where $\mathbf{x}_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in \mathbf{R}^n$ is the state variable of node i, $\mathbf{A}(t) = (a_{kl}(t))_{n\times n} \in \mathbf{R}^{n\times n}$ is the inner-coupling matrix of the network at time t, $\mathbf{C}(t) = (c_{ij}(t))_{N\times N}$ is the coupling configuration matrix representing the coupling strength and the topological structure of the network at time t, in which $c_{ij}(t)$ is defined as follows: if there is a connection from node i to node j $(j \neq i)$ at time t, then $c_{ij}(t) \neq 0$; otherwise, $c_{ij}(t) = 0$ $(j \neq i)$, and the diagonal elements of matrix $\mathbf{C}(t)$ satisfy the diffusively coupled conditions as follows:

$$c_{ii}(t) = -\sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}(t)\,, \quad i = 1, 2, \cdots, N\,. \tag{3}$$

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Thus network (1) can be recasted as

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \sum_{j=1}^{N} c_{ij}(t)\mathbf{A}(t)\,\mathbf{x}_j, \quad i = 1, 2, \cdots, N. \tag{4}$$

In this paper, assume that network (4) is connected in the sense that there are no isolate clusters, that is, $\mathbf{C}(t)$ is irreducible.

If $\mathbf{A}(t)$, $\mathbf{C}(t)$ are constant matrices, network (4) becomes a time-invariant network [5]:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \sum_{j=1}^{N} c_{ij}\mathbf{A}\mathbf{x}_j, \quad i = 1, 2, \cdots, N\,. \tag{5}$$

In the following, we firstly present a rigorous definition for network synchronization, then give several network synchronization criteria [3-4,6-7].

Definition 1: *Suppose $\boldsymbol{x}_i(t\,,\boldsymbol{X}_0)$ $(i = 1, 2, \cdots, N)$ is a solution of the nonautonomous dynamical network*

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i) + \boldsymbol{g}_i(t, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N), \quad i = 1, 2, \cdots, N, \tag{6}$$

where $\boldsymbol{X}_0 = \left((\boldsymbol{x}_1^0)^T, \cdots, (\boldsymbol{x}_N^0)^T\right)^T \in \boldsymbol{R}^{nN}$, $\boldsymbol{f} : D \to \boldsymbol{R}^n$ and $\boldsymbol{g}_i : D\times\cdots\times D \to \boldsymbol{R}^n$ $(i = 1, 2, \cdots, N)$ are continuously differentiable with $D \subseteq \boldsymbol{R}^n$, and $\boldsymbol{g}_i(t, \boldsymbol{x}, \boldsymbol{x}, \cdots, \boldsymbol{x}) = 0$ for all t. If there exists a nonempty open subset $E \subseteq D$, with $\boldsymbol{x}_i^0 \in E\,(i = 1, 2, \cdots, N)$, such that $\boldsymbol{x}_i(t\,,\boldsymbol{X}_0) \in D$ for all $t \geq 0$, $i = 1, 2, \cdots, N$, and

$$\lim_{t\to\infty} \|\boldsymbol{x}_i(t,\ \boldsymbol{X}_0) - \boldsymbol{s}(t, \boldsymbol{x}_0)\|_2 = 0 \quad \text{for } 1 \leq i \leq N, \tag{7}$$

where $\boldsymbol{s}(t\,,\boldsymbol{x}_0)$ is a solution of the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ with $\boldsymbol{x}_0 \in D$, then the network (6) is said to realize synchronization and $E \times \cdots \times E$ is called the region of synchrony for network (6). Here, $\boldsymbol{X}(t, X_0) = \left(\boldsymbol{x}_1^T(t, \boldsymbol{X}_0), \boldsymbol{x}_2^T(t, \boldsymbol{X}_0), \cdots, \boldsymbol{x}_N^T(t, \boldsymbol{X}_0)\right)^T$ is called the synchronous solution of network (6), if $\boldsymbol{x}_i(t, \boldsymbol{X}_0) = \boldsymbol{x}_j(t, \boldsymbol{X}_0)$ for all $t \geq 0$ and $1 \leq i, j \leq N$.

Definition 2: [3] *Suppose $\Gamma = \{\boldsymbol{s}(t) \mid 0 \leq t < T\}$ denotes the set of $T-$periodic solutions of system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ in $\boldsymbol{R}^n$. A $T-$periodic solution $\boldsymbol{s}(t)$ is said to be orbitally stable, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that every solution $\boldsymbol{x}(t)$ of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, whose distance from Γ is less than δ at $t = 0$, will remain within a distance less than ε from Γ for all $t \geq 0$. Such an $\boldsymbol{s}(t)$ is said to be orbitally asymptotically stable if, in addition, the distance of $\boldsymbol{x}(t)$ from Γ tends to zero as $t \to \infty$. Furthermore, if there exist positive constants α, β and a real constant h such that $\|\boldsymbol{x}(t-h) - \boldsymbol{s}(t)\| \leq \alpha\, e^{-\beta t}$ for $t \geq 0$, then $\boldsymbol{s}(t)$ is said to be orbitally asymptotically stable with an asymptotic phase.*

Definition 3: *Let $\boldsymbol{s}(t)$ be a periodic solution of system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Suppose $\gamma_1 = 1, \gamma_2, \cdots, \gamma_n$ are the Floquet multipliers of the variational equation of $\boldsymbol{s}(t)$, $\dot{\boldsymbol{y}} = \boldsymbol{A}(t)\,\boldsymbol{y}$, where $\boldsymbol{A}(t) = D\boldsymbol{f}(\boldsymbol{s}(t))$ is the Jacobian of $\boldsymbol{f}$ evaluated at $\boldsymbol{s}(t)$. Then the periodic solution $\boldsymbol{s}(t)$ is said to be a hyperbolic periodic solution if $|\gamma_j| \neq 1$ for $2 \leq j \leq n$. Furthermore, $\boldsymbol{S}(t) = (\boldsymbol{s}^T(t), \cdots, \boldsymbol{s}^T(t))^T$ is said to be a hyperbolic synchronous periodic solution of network (4) if all Floquet multipliers of the variational equation of $\boldsymbol{S}(t)$ have absolute values less than 1 except one multiplier which equals 1.*

Assumption 1: *Let $\lambda_1(t), \lambda_2(t), \cdots, \lambda_N(t)$ be the eigenvalues of $\boldsymbol{C}(t)$. $\exists\, t_0 \geq 0$, for any $\lambda_i(t)$ $(1 \leq i \leq N)$, either $\lambda_i(t) \neq 0$ for all $t > t_0$, or $\lambda_i(t) \equiv 0$ for all $t > t_0$.*

Theorem 1: *Let $\boldsymbol{s}(t)$ be a hyperbolic periodic solution of an individual node $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, and be orbitally asymptotically stable with an asymptotic phase. Suppose that the coupling configuration matrix $\boldsymbol{C} = (c_{ij})_{N\times N}$ can be diagonalized. Then, $\boldsymbol{S}(t) = \left(\boldsymbol{s}^T(t), \boldsymbol{s}^T(t), \cdots, \boldsymbol{s}^T(t)\right)^T$ is a hyperbolic synchronous periodic solution of network (5), and is orbitally asymptotically stable with an asymptotic phase, if and only if the linear time-varying systems*

$$\dot{\boldsymbol{w}} = [D\boldsymbol{f}(\boldsymbol{s}(t)) + \lambda_k \boldsymbol{A}]\,\boldsymbol{w}, \quad k = 2, \cdots, N, \tag{8}$$

are asymptotic stable about their zero solutions.

Theorem 2: *Assume that $\boldsymbol{x} = \boldsymbol{s}(t)$ is an exponentially stable solution of nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, where $\boldsymbol{f} : \Omega \to \boldsymbol{R}^n$ is continuously differentiable, $\Omega = \{\boldsymbol{x} \in \boldsymbol{R}^n \,|\, \|\boldsymbol{x} - \boldsymbol{s}(t)\|_2 < r\}$. Suppose that the Jacobian $D\bar{\boldsymbol{F}}(t, \bar{\eta})$ is bounded and Lipschitz on $\bar{\Omega} = \{\bar{\eta} \in \boldsymbol{R}^{nN} \,|\, \|\bar{\eta}\|_2 < r\}$, uniformly in t. Suppose also that Assumption 1 holds and there exists a real matrix, $\boldsymbol{\Phi}(t)$, nonsingular for all t, such that $\boldsymbol{\Phi}^{-1}(t)\ (\boldsymbol{C}(t))^T\ \boldsymbol{\Phi}(t) = diag\{\lambda_1(t), \lambda_2(t), \cdots, \lambda_N(t)\}$ and $\dot{\boldsymbol{\Phi}}^{-1}(t)\,\boldsymbol{\Phi}(t) = diag\{\beta_1(t), \beta_2(t), \cdots, \beta_N(t)\}$. Then, the synchronous solution $\boldsymbol{S}(t)$ is exponentially stable in network (4) if and only if the linear time-varying systems*

$$\dot{\boldsymbol{w}} = [D\boldsymbol{f}(\boldsymbol{s}(t)) + \lambda_k(t)\ \boldsymbol{A}(t) - \beta_k(t)\boldsymbol{I}_n]\ \boldsymbol{w},\ k = 2, \cdots, N, \tag{9}$$

are exponentially stable about their zero solutions.

Theorem 3: *Suppose that $\boldsymbol{F} : \Omega \to \boldsymbol{R}^{n(N-1)}$ is continuously differentiable on $\Omega = \{\boldsymbol{x} \in \boldsymbol{R}^{n(N-1)} \,|\, \|\boldsymbol{x}\|_2 < r\}$, with $\boldsymbol{F}(t, \boldsymbol{0}) = \boldsymbol{0}$ for all t, and the Jacobian $D\boldsymbol{F}(t, \boldsymbol{x})$ is bounded and Lipschitz on Ω, uniformly in t. Suppose also that there exists a bounded nonsingular real matrix $\boldsymbol{\Phi}(t)$, such that $\boldsymbol{\Phi}^{-1}(t)\ (\boldsymbol{C}(t))^T\ \boldsymbol{\Phi}(t) = diag\{\lambda_1(t), \lambda_2(t), \cdots, \lambda_N(t)\}$ and $\dot{\boldsymbol{\Phi}}^{-1}(t)\,\boldsymbol{\Phi}(t) = diag\{\beta_1(t), \beta_2(t), \cdots, \beta_N(t)\}$. Then, the chaotic synchronous state $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \cdots = \boldsymbol{x}_N(t) = \boldsymbol{s}(t)$ is exponentially stable for dynamical network (4) if and only if the linear time-varying systems*

$$\dot{\boldsymbol{w}} = [D\boldsymbol{f}(\boldsymbol{s}(t)) + \lambda_k(t)\boldsymbol{A}(t) - \beta_k(t)\boldsymbol{I}_n]\boldsymbol{w},\ k = 2, \cdots, N \tag{10}$$

are exponentially stable about the zero solution.

Remark 1: For simplicity, all notes and proofs are omitted here and detailed proofs can refer to [3-4]. Theorems 2 and 3 show that synchronization of the time-varying network (4) is completely determined by its inner-coupling matrix $\mathbf{A}(t)$, and the eigenvalues $\lambda_k(t)$ $(2 \leq k \leq N)$ and the corresponding eigenvectors $\phi_k(t)$ ($\beta_k(t)$ are functions of $\phi_k(t)$, $2 \leq k \leq N$) of the coupling configuration matrix $\mathbf{C}(t)$. However, the synchronization of the time-invariant network (5) is completely determined only by its inner-coupling matrix $\mathbf{A}$ and the eigenvalues of the coupling configuration matrix $\mathbf{C}$ [12].

III. Synchronization Criteria of Delayed Complex Dynamical Networks

The multi time delays occur often in complex networks, however, they are rather complex. For simplification, this section only discusses the single time delay case. Li & Chen [10] introduced a delayed complex dynamical network model as follows:

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t)) + c\sum_{j=1}^{N} c_{ij}\mathbf{A}\mathbf{x}_j(t-\tau),\ i = 1, 2, \cdots, N, \quad (11)$$

where $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ is a continuously differentiable function, $\mathbf{x}_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in \mathbf{R}^n$ is the state variable of node i, τ is the time delay, $c > 0$ is the coupling strength, $\mathbf{A} = (a_{ij})_{n\times n} \in \mathbf{R}^{n\times n}$ is the inner-coupling matrix of the network, $\mathbf{C} = (c_{ij})_{N\times N}$ is the coupling configuration matrix of the network, in which c_{ij} is defined as follows: if there exists a connection between node i and node $j (j \neq i)$, then $c_{ij} = c_{ji} = 1$; otherwise, $c_{ij} = c_{ji} = 0 (j \neq i)$, and the diagonal elements of matrix $\mathbf{C}$ satisfy the diffusively coupled conditions as follows:

$$c_{ii} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} c_{ij},\ i = 1, 2, \cdots, N.$$

Moreover, assume that network (11) is also connected in the sense that there are no isolated clusters, that is, $\mathbf{C}$ is irreducible.

Lemma 1: [10] *Suppose that the eigenvalues of the matrix $\boldsymbol{C}$ satisfy the condition $0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N$. If the following $N-1$ $n-$dimensional linear time-varying delayed differential equations*

$$\dot{\boldsymbol{w}}(t) = D\boldsymbol{f}(\boldsymbol{s}(t))\boldsymbol{w}(t) + c\lambda_k \boldsymbol{A}\boldsymbol{w}(t-\tau),\ i = 2, 3, \cdots, N, \quad (12)$$

are asymptotically stable about their zero solutions, then the synchronized states $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \cdots = \boldsymbol{x}_N(t) = \boldsymbol{s}(t)$ are asymptotically stable for the delayed dynamical network (11).

IV. Synchronization Criteria of Discrete Complex Dynamical Networks

Recently, Lu and Chen [9] proposed a general discrete complex dynamical network model, which is described by

$$\mathbf{x}_i(t+1) = \mathbf{f}(\mathbf{x}_i(t)) + \sum_{j=1}^{N} c_{ij}\mathbf{f}(\mathbf{x}_j(t)),\ \ i = 1, 2, \cdots, N, \quad (13)$$

where $c_{ii} = -\sum_{j\neq i} c_{ij}$, $\mathbf{x}_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in \mathbf{R}^n$ is the state variable of the ith node, $t \in N$ is the discrete time, $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function, $\mathbf{C} = (c_{ij})_{N\times N} \in \mathbf{R}^{N\times N}$ is the coupling configuration matrix of the network, and its entries satisfy $c_{ij} \geq 0$ for all $i \neq j$.

Assumption 2: *For matrix $\boldsymbol{C} = (c_{ij})_{N\times N} \in \boldsymbol{R}^{N\times N}$, assume that $c_{ij} \geq 0$ for $i \neq j$, $c_{ii} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} c_{ij}$ for $i = 1, 2, \cdots, N$, and the real part of eigenvalues of $\boldsymbol{C}$ are all negative except an eigenvalue 0 with multiplicity one.*

Lemma 2: [9] *Suppose that Assumption 2 holds. Let $\lambda_1 = 0$, $\lambda_k = \alpha_k + j\beta_k$, where j is the imaginary unit, $k = 2, \cdots, N$. If there exist constants $0 < \gamma_0 < \gamma < 1$ and an integer $t_0 > 0$ such that*

$$\|D\boldsymbol{f}(\boldsymbol{s}(t))\|_2\,|1 + \lambda_k| \leq \gamma_0,\ \ k = 2, \cdots, N,\ t \geq t_0, \quad (14)$$

then the synchronization solution $\boldsymbol{S}(t)$ is locally exponential stable for the discrete network (13).

Lemma 3: [9] *Suppose that Assumption 2 holds and $\boldsymbol{f} \in \boldsymbol{F}(k)$ with constant $k > 0$. If there exist a positive number $c > k$ and an irreducible symmetric matrix $\boldsymbol{P} = (p_{ij})_{N\times N} \in \boldsymbol{R}^{N\times N}$ satisfying $p_{ij} = p_{ji} \geq 0$ and $p_{ii} = -\sum_{j\neq i} p_{ij}$ for $i = 1, \cdots, N$, such that*

$$(\boldsymbol{I} + \boldsymbol{C})^T \boldsymbol{P} (\boldsymbol{I} + \boldsymbol{C}) - \frac{1}{c^2}\boldsymbol{P} \geq 0, \quad (15)$$

then the synchronization solution $\boldsymbol{S}(t)$ is globally exponential stable for the discrete network (13). Moreover, the convergence rate is $O((\frac{k}{c})^t)$.

V. Maximum Synchronizability of Time-Invariant Complex Dynamical Networks

It is well know that network synchronizability is an important property of complex dynamical networks [5]. Networks with different topological structures and node dynamics have different degrees of network synchronizability. It has been discovered that, for any given coupling strength, if the number of nodes is sufficiently large, then the small-world network will synchronize, even if the original nearest-neighbor coupled network cannot realize synchronization under the same condition [12]. However, how to characterize the synchronizability of a network is an open problem. In this paper, a new concept — *associated feedback system* — is proposed for characterizing synchronizability of the time-invariant network (5) [5].

Definition 4: *The self-feedback nonlinear system*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + d\boldsymbol{A}\,(\boldsymbol{x}(t) - \boldsymbol{s}(t)), \quad (16)$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T \in \boldsymbol{R}^n$ and d is a constant, is called the associated feedback system of the time-invariant network (5).

Let $\mathbf{x}(t) = \mathbf{y}(t) + \mathbf{s}(t)$. Substituting it into (16) yields

$$\dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t) + \mathbf{s}(t)) - \mathbf{f}(\mathbf{s}(t)) + d\,\mathbf{A}\,\mathbf{y}(t). \quad (17)$$

Linearizing system (17) gets

$$\dot{\mathbf{y}}(t) = [D\mathbf{f}(\mathbf{s}(t)) + d\,\mathbf{A}]\,\mathbf{y}(t). \quad (18)$$

For the given time-invariant network (5), one can obtain the exponentially stable region, denoted as Γ, of the solution $\mathbf{s}(t)$ of the associated feedback system (16) (or the zero solution of system (17)) in terms of feedback parameter d.

Assumption 3: *Suppose that* $\boldsymbol{F} : \Omega \to \boldsymbol{R}^{n(N-1)}$ *is continuously differentiable,* $\Omega = \{\boldsymbol{x} \in \boldsymbol{R}^{n(N-1)} \,|\, \|\boldsymbol{x}\|_2 < r\}$*, the Jacobian matrix* $D\boldsymbol{F}(t, \boldsymbol{x}(t))$ *is bounded and Lipschitz on* Ω*, uniformly in t, and the coupling configuration matrix* $\boldsymbol{C}$ *can be diagonalized.*

Assumption 4: *Suppose that* $\bar{\boldsymbol{F}} : \bar{\Omega} \to \boldsymbol{R}^{nN}$ *is continuously differentiable,* $\bar{\Omega} = \{\boldsymbol{x} \in \boldsymbol{R}^{nN} \,|\, \|\boldsymbol{x}\|_2 < r\}$*, the Jacobian matrix* $D\bar{\boldsymbol{F}}(t, \boldsymbol{x}(t))$ *is bounded and Lipschitz on* $\bar{\Omega}$*, uniformly in t, and the coupling configuration matrix* $\boldsymbol{C}$ *can be diagonalized.*

Theorem 4: *Let* $\boldsymbol{s}(t)$ *be an orbit of a chaotic attractor of the given chaotic system* $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t))$*. Assumption* 3 *holds. The chaotic synchronous state* $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \cdots = \boldsymbol{x}_N(t) = \boldsymbol{s}(t)$ *of network (5) is exponentially stable if and only if the eigenvalues* $\lambda_i \in \Gamma$*,* $i = 2, 3, \cdots N$*.*

Theorem 5: *Let* $\boldsymbol{x}(t) = \boldsymbol{s}(t)$ *be an exponentially stable solution of the individual node* $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t))$*. Assumption* 4 *holds. The synchronous solution* $\boldsymbol{S}(t)$ *of network (5) is exponentially stable if and only if all the eigenvalues* $\lambda_i \in \Gamma$*,* $i = 2, 3, \cdots N$*.*

Remark 2: Note that the stable region Γ is completely determined by the individual node $\dot{\mathbf{x}}(t) = \mathbf{f}(x(t))$ and the inner coupled matrix $\mathbf{A}$ of network (5), and that the eigenvalues of the coupled configuration matrix $\mathbf{C}$ determine the stability of synchronous solution of network (5). Γ is the maximum region of the eigenvalues of the coupled configuration matrix $\mathbf{C}$.

Definition 5: *The ability that the structure of network (5) can ensure network (5) achieve synchronization is called the network synchronizability. The maximum possible set*

$$\{(\boldsymbol{A}, \boldsymbol{C}) \,|\, \textit{network (5) realizes synchronization}\}$$

is called the maximum synchronizability set, which characterizes the maximum synchronizability of network (5).

For a given individual node $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ and a inner coupling matrix $\mathbf{A}$, the maximum synchronizability set of network (5) is completely determined by its associated feedback system (16). In fact, the maximum synchronizability set of network (5) is

$$\{\mathbf{C} \,|\, \lambda_i \in \Gamma \quad \text{for} \quad i = 2, 3, \cdots N\},$$

where λ_i $(i = 2, 3, \cdots N)$ are the nonzero eigenvalues of $\mathbf{C}$.

Consider the unidirectional coupled system:

$$\begin{cases} \dot{\mathbf{s}}(t) = \mathbf{f}(\mathbf{s}(t)), \\ \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + d\,\mathbf{A}\,(\mathbf{x}(t) - \mathbf{s}(t)), \end{cases} \tag{19}$$

where $\mathbf{A}$ is a constant coupled matrix, and d is a coupling strength or feedback coefficient. Let the error vector be $\xi(t) = \mathbf{x}(t) - \mathbf{s}(t)$. From (17), its variational equation is

$$\dot{\xi}(t) = [D\mathbf{f}(\mathbf{s}(t)) + d\,\mathbf{A}]\,\xi(t). \tag{20}$$

Therefore, the associated feedback system (16) of network (5) is the response system in (19), and the individual node $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ is the drive system of (19). Furthermore, the variational equation (20) (or (18)) is the corresponding linear system of the associated feedback system (16). If the origin is an exponentially stable equilibrium of system (20), then the unidirectional coupled system (19) is synchronous. Thus the associated feedback system (16) and the individual node of network (5) have their definitely physical meaning in terms of synchronous communication.

VI. Conclusion

This paper introduces several network synchronization criteria for time-invariant, time-varying, delayed, and discrete complex dynamical network models. Moreover, the maximum synchronizability of time-invariant networks is also explored. Synchronization is a fundamental nonlinear phenomenon in complex dynamical networks. Furthermore, the network synchronization can be classified as identical synchronization, phase synchronization, partial synchronization, and so on. Since the space limitation, this paper only deals with identical synchronization case. It is very important to better understand the essential nature and mechanism of network synchronization in the future.

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