MAXIMUM SYNCHRONIZABILITY OF GENERAL TIME-IN Variant
DYNAMICAL NETWORKS

Jinhu Lü¹, Guanrong Chen², Daizhan Cheng¹
(¹Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080
²Department of Electronic Engineering, City University of Hong Kong)
Corresponding author E-mail: jhlu@mail.iss.ac.cn

Abstract: This paper shows that the maximum synchronizability of a general time-invariant dynamical network is completely determined by its associated internal feedback dynamics, which has a precise physical meaning in terms of synchronous communication. Also, a concept of synchronizability matrix is introduced to characterize the robustness of synchronization of the network. Based on the knowledge of synchronizability, we can purposefully increase the robustness of the network synchronization and better prevent it from attacks.

Key words: Time-invariant network, associated feedback, synchronizability matrix

1. INTRODUCTION

Recently, complex networks have attracted increasing attention from various fields of science and engineering. A complex network is a large set of interconnected nodes, in which a node can be a dynamical system. Examples of complex networks in real-world life include the World Wide Web, the Internet, electric power grids, food webs, metabolic networks, and so on. [1-4].

Today, collective motions of complex networks have been a subject of considerable interest within the science and technology communities. Especially, one of the interesting and significant phenomena in complex dynamical networks is the synchronization of all its dynamical nodes [5-12]. Although many papers discuss the network synchronization problem [5-8, 10-12], there are few results reported in the literature on how to rigorously characterize the network synchronizability. In fact, so far we are still not able to answer the basic question “What is the maximum synchronizability of a complex network?” In this paper, we show that the maximum synchronizability of a network is completely determined by its associated internal feedback dynamics, which has a precise physical meaning in terms of synchronous communication. Our analysis is based on the rigorous mathematical methods. We introduce a new concept of synchronizability matrix to characterize the robustness of synchronization of the network. Using the knowledge of synchronizability, we can purposefully increase the robustness of synchronization and better prevent the network from attacks.

This paper is organized as follows: A general dynamical network model is proposed in Section 2. In Section 3, the concepts of associated internal feedback dynamics and synchronizability matrix of a network are introduced to characterize the synchronizability of a network. Conclusions are presented in Section 4.

2. A GENERAL TIME-IN Variant
DYNAMICAL NETWORK MODEL

To better characterize the real-world complex networks, a general time-invariant dynamical network model is introduced

\[ \dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij} A x_j \]  \hspace{1cm} (2.1)

where \( i = 1, 2, \cdots, N \), \( x_i(t) = (x_{i1}(t), \cdots, x_{in}(t))^T \in \mathbb{R}^n \) is the state variable of node \( i \), \( A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n} \) is a constant inner coupling matrix between nodes, \( C = (c_{ij})_{N \times N} \) is the coupling configuration matrix of the network, where \( c_{ij} \) is defined as follows: if there is a connection from node \( i \) to node \( j \) \( (j \neq i) \), then the coupling
strength \( c_{ij} \neq 0 \); otherwise, \( c_{ij} = 0 (j \neq i) \), and the diagonal elements of \( C \) are defined by

\[
C_{ii} = -\sum_{j \neq i} C_{ij}, \quad (2.2)
\]

where \( i = 1, 2, \ldots, N \). Obviously, the uniform network model in [5] is a special case of network (2.1), where \( C \) is a 0-1 symmetric matrix and \( A \) is a 0-1 diagonal matrix. Since real-world complex networks may be directed networks, such as the WWW, whose coupling configuration matrix \( C \) is not symmetric, here it is not assumed that \( C \) is symmetric and its off-diagonal elements are nonnegative.

According to Lemma 2 in [12], the real parts of all eigenvalues of \( C \) are less than or equal to 0 and all eigenvalues with zero real part are the real eigenvalue 0. If \( C \) is an irreducible matrix, therefore 0 is an eigenvalue of multiplicity 1 [12]. If all the eigenvalues of \( C \) are real numbers satisfying \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \), then

\[
\lambda_i = 0 \quad \text{and} \quad 0 > \lambda_2 \geq \cdots \geq \lambda_N. \quad (2.3)
\]

In the following, a rigorous mathematical definition is introduced for the concept of network synchronization.

**Definition 1:** Let \( x_i(t, X_0) (i = 1, 2, \ldots, n) \) be a solution of the dynamical network

\[
\dot{x}_i = f(x_i) + g_i(x_1, x_2, \ldots, x_n), \quad (2.4)
\]

where \( i = 1, 2, \ldots, N \), \( X_0 = (x_1^0, \ldots, x_N^0)^T \in \mathbb{R}^n \), \( f: \mathbb{R}^n \to \mathbb{R}^n \) and \( g_i: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) are continuously differentiable with \( D \subseteq \mathbb{R}^n \), and \( g_i(x, x, \cdots, x) = 0 \). If there is a nonempty open subset \( E \subseteq D \), with \( x_i^0 \in E \) (\( i = 1, 2, \ldots, N \)), such that \( x_i(t, X_0) \in E \) for all \( t \geq 0 \), \( i = 1, 2, \ldots, N \), and

\[
\lim_{t \to \infty} \| x_i(t, X_0) - s(t, x_0) \|_2 = 0 \quad (2.5)
\]

where \( i = 1, 2, \ldots, N \), \( s(t, x_0) \) is a solution of the system \( \dot{x} = f(x) \) with \( x_0 \in D \), then the dynamical network (2.4) is said to realize synchronization and \( E \times \cdots \times E \) is called the region of synchrony for network (2.4).

Note that the diffusively coupled condition (2.2) of network (2.1) ensures that the synchronous solution \( x_1(t; t_0, x_1^0, \ldots, x_N^0) = x_2(t; t_0, x_1^0, \ldots, x_N^0) = \cdots = x_N(t; t_0, x_1^0, \ldots, x_N^0) \) is a solution of an individual node described by \( \dot{x}(t) = f(x(t)) \), denoted as \( s(t) \), namely,

\[
\dot{s}(t) = f(s(t)). \quad (2.6)
\]

Obviously, synchronization in network (2.1) corresponds to the motion in the invariant manifold: \( x_i(t) = x_s(t) = \cdots = x_N(t) \). Here, \( s(t) \) can be either an equilibrium point, or a periodic orbit, or an orbit of a chaotic attractor.

It is very important to point out that since a chaotic attractor is an attracting invariant set, the stability of the chaotic synchronous state \( x_i(t) = x_s(t) = \cdots = x_N(t) \) is equivalent to the stability of the zero transverse errors of the synchronous manifold for the dynamical network (2.1) [8]. However, it is quite different from the non-chaotic case. Since it is not assumed the stability of \( s(t) \), the stability of the synchronous solution \( (s^T(t), s^T(t), \cdots, s^T(t))^T \) of network (2.1) is equivalent to the stability of the error vector \((\eta_1^T(t), \eta_2^T(t), \cdots, \eta_N^T(t))^T \) about its zero solution, where \( \eta_i(t) = x_i(t) - s(t) \) \((i = 1, 2, \ldots, N) \).

### 3. MAXIMUM SYNCHRONIZABILITY OF GENERAL TIME-ININVARIANT DYNAMICAL NETWORKS

#### 3.1. Associate Internal Feedback Dynamics

Network synchronizability is an important property of complex dynamical networks. Networks with different topological structures have different degrees of network synchronizability. It has been demonstrated that, for any given coupling strength, if the number of nodes is sufficiently large, then the global coupled dynamical network will synchronize.
even if the original nearest-neighbor coupled network cannot realize synchronization under the same condition [5]. However, how to characterize the synchronizability of a network is an open problem. In the following, a new concept—associated internal feedback dynamics—is introduced for characterizing the synchronizability of network (2.1).

**Definition 2:** The self-feedback nonlinear system

\[
\dot{x}(t) = f(x(t)) + dA(x(t) - s(t)), \quad (3.1)
\]

where \(x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n\) and \(d\) is a constant, is called the associated internal feedback dynamics of the time-invariant dynamical network (2.1).

Let \(x(t) = y(t) + s(t)\), and substituting it into (2.11), yields

\[
\dot{y}(t) = f(y(t) + s(t)) - f(s(t)) + dAy(t), \quad (3.2)
\]

and its corresponding linear system is

\[
\dot{y}(t) = [Df(s(t)) + dA]y(t). \quad (3.3)
\]

For the given dynamical network (2.1), one can obtain the exponentially stable region, denoted as \(\Gamma\), of the solution \(s(t)\) of the associated internal feedback dynamics (3.1) (or the zero solution of system (3.2)) in terms of feedback parameter \(d\).

**Theorem 1:** Let \(s(t)\) be an orbit of a chaotic attractor of the given chaotic system \(\dot{x}(t) = f(x(t))\). Suppose that (H1) in [10] holds. The chaotic synchronous state \(x_1(t) = \cdots = x_N(t) = s(t)\) of dynamical network (2.1) is exponentially stable if and only if the eigenvalues \(\lambda_i \in \Gamma, \ i = 2, 3, \cdots, N\). The proof is thus completed.

**Theorem 2:** Let \(x(t) = s(t)\) be an exponentially stable solution of the individual node \(\dot{x}(t) = f(x(t))\). Suppose that (H2) in [10] holds. The synchronous solution \(\tilde{S}(t) = (\tilde{S}(t), \tilde{S}(t), \cdots, \tilde{S}(t))^T\) of dynamical network (2.1) is exponentially stable if and only if all the eigenvalues \(\lambda_i \in \Gamma, \ i = 2, 3, \cdots, N\).

**Proof.** According to (H2) in [10], the Jacobian matrix \(Df(s(t))\) is bounded and Lipschitz on \(\Omega = \{y \in \mathbb{R}^n \mid \|y\|_2 < r\}\), uniformly in \(t\). From the Lyapunov converse theorem [13], the origin is an exponentially stable solution for the nonlinear system (3.2) if and only if it is an exponentially stable equilibrium point for the linear time-varying system (3.3). Therefore, \(\Gamma\) is also the exponentially stable region of the zero solution of the linear time-varying system (3.3) in terms of feedback parameter \(d\). From Theorem 3 in [9], the synchronous solution \(S(t)\) of dynamical network (2.1) is exponentially stable if and only if all the eigenvalues \(\lambda_i \in \Gamma, \ i = 2, 3, \cdots, N\). This thus completes the proof.

**Remarks:** Theorems 1 and 2 give sufficient and necessary conditions for the exponential stability of the synchronous solution of network (2.1). It is noticed that the stable region \(\Gamma\) is completely determined by the individual node \(\dot{x}(t) = f(x(t))\) and the inner coupled matrix A of network (2.1), and that the eigenvalues of the coupled configuration matrix C determine the stability of synchronous solution of network (2.1).

Note that for a given individual node \(\dot{x}(t) = f(x(t))\), the network synchronizability is completely determined by the inner coupling matrix A and the coupling configuration matrix C. It is noticed that the unique hypothesis for matrix C is the diagonalized condition, which is a rather general condition for complex networks.

**Definition 3:** The ability that the structure of network (2.1) can ensure network (2.1) achieve synchronization is called the network synchronizability. The maximum possible set

\[\{(A, C) \mid \text{network (2.1) realizes synchronization}\}\]

is called the maximum synchronizability set, which characterizes the maximum synchronizability of network.
Obviously, for a given individual node \( \dot{x}(t) = f(x(t)) \) and an inner coupling matrix \( A \), the maximum synchronizability set of network (2.1) is completely determined by its associated internal feedback dynamics (3.1). In fact, the maximum synchronizability set of network (2.1) is

\[ \{ C \mid \lambda_\gamma \in \Gamma \text{ for } \gamma = 2,3,\ldots, N \}, \]

where \( \lambda_\gamma (\gamma = 2,3,\ldots, N) \) are the nonzero eigenvalues of \( C \).

Consider the following unidirectional coupled system:

\[
\begin{align*}
\dot{x}(t) &= f(s(t)), \\
\dot{\xi}(t) &= f(x(t)) + dA(x(t) - s(t)),
\end{align*}
\]

where \( A \) is a constant coupled matrix, and \( d \) is a coupling strength or feedback coefficient. Let the error vector be \( \xi(t) = x(t) - s(t) \). According to (3.6), its variational equation is

\[
\dot{\xi}(t) = [Df(s(t)) + dA]\xi(t). \tag{3.5}
\]

Obviously, the associated internal feedback dynamics (3.1) of network (2.1) is the response system in (3.4), and the individual node \( \dot{x}(t) = f(x(t)) \) is the drive system of (3.4). Moreover, the variational equation (3.5) (or (3.3)) is the corresponding linear system of the associated internal feedback dynamics (3.1). If the origin is an exponentially stable equilibrium of system (3.5), then the unidirectional coupled system (3.4) is synchronous. Therefore, the associated internal feedback dynamics (3.1) and the individual node of network (3.1) have their precise meaning in terms of synchronous communication.

### 3.2. Characterizing the Robustness of Synchronization

The robustness of synchronization is a key characteristic quantity of complex dynamical networks. Networks with different topological structures are likely to have different degrees of robustness of synchronization. Moreover, in a network, different edges may have different degrees of robustness of synchronization. It has been demonstrated that a scale-free network is robust against random removal of nodes, but is fragile to specific removal of the most highly connected nodes [5]. Therefore, it is very important to give a mathematical characterization for the robustness of synchronization. In the following, a new concept of synchronizability matrix is introduced to characterize the robustness of synchronization for dynamical network (2.1).

**Definition 4**: Suppose that network (2.1) can achieve synchronization for a given inner coupled matrix \( A \) and a coupled configuration matrix \( C \). If network (2.1) remains synchronized after removing one edge, \( l \), from the network, then network (2.1) is said to be robust in synchronization against the removal of edge \( l \); otherwise, network (2.1) is said to be fragile in synchronization with respect to the removal of edge \( l \).

**Definition 5**: Consider the dynamical network (2.1), and let \( \Gamma \) be the exponentially stable region of the zero solution of system (3.3) with respect to feedback parameter \( d \). The real matrix

\[
S = (s_{ij})_{N \times N} \tag{3.6}
\]

is called the synchronizability matrix for one-edge removal of network (2.1), where

\[
s_{ij} = \alpha(\lambda_2^j, \lambda_3^j, \ldots, \lambda_N^j) \tag{3.7}
\]

if \( j \neq i \) or \( c_{ij} = 0 \), then \( s_{ij} = 0 \); otherwise

\[
s_{ij} = \begin{cases} 
1, & \text{if } \lambda_k^j \in \Gamma \text{ for } k = 2, \ldots, N \\
-1, & \text{otherwise}
\end{cases}
\]

in which \( \lambda_k^j (2 \leq k \leq N) \) are the nonzero eigenvalues of transformation matrix \( C^j = (c_{lm}^j)_{N \times N} \) satisfying: if \( l=i \) and \( m=j \), then \( c_{ij}^i = 0 \) and \( c_{ij}^j = c_{ij}^i + c_{ij}^j \); otherwise,

\[
c_{lm}^j = c_{lm}^i. \]

If \( S \geq 0 \), then the dynamical network (2.1) is said to be globally robust in synchronization against one-edge removal; if \( S \leq 0 \), then the dynamical network (2.1) is said to be globally fragile in synchronization with respect to one-edge removal. If \( s_{ij} = 1 \), then the edge from node \( i \) to node \( j \) is robust in synchronization against removal and this edge is called a robust edge of network (2.1); if \( s_{ij} = -1 \), then the edge from node \( i \) to node \( j \) is fragile in synchronization with respect to removal and this edge is called a sensitive edge of network (2.1).

Note that the synchronizability matrix \( S \) characterizes the robustness of synchronization against one-edge removal of
network (2.1). Moreover, the synchronizability matrix S concretizes the robustness and fragility of synchronization against one-edge removal of network (2.1). Similarly, one can introduce the synchronizability matrices for multi-edge removals of network (2.1).

For the case of adding edges, one can similarly define the synchronizability matrix of one-edge addition (or multi-edge addition) for network (2.1).

4. CONCLUSIONS

It has been shown that the maximum synchronizability of a general time-invariant network is completely determined by its associated internal feedback dynamics. Moreover, we introduce a new concept --- synchronizability matrix --- to characterize the robustness of synchronization of a network. Our results strongly indicate that for a general time-invariant dynamical network, the associated internal feedback dynamics, the synchronizability matrix, robust and sensitive edges are the key elements responsible for the special network properties. By using the knowledge of synchronizability, one can purposefully increase the robustness of synchronization thereby better protecting against attacks to the real-world complex networks.

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