Upper and lower solution method for fourth-order four-point boundary value problems

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Abstract

This paper is concerned with the fourth-order ordinary differential equation

\[ u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1 \]

with the four-point boundary conditions

\[ u(0) = u(1) = 0, \quad au''(\xi_1) - bu'''(\xi_1) = 0, \quad cu''(\xi_2) + du'''(\xi_2) = 0, \]

where \( \xi_i \in [0, 1] \) (\( i = 1, 2 \)) and \( a, b, c, d \) are nonnegative constants satisfying \( ad + bc + ac(\xi_2 - \xi_1) > 0 \). Some new existence results are obtained by developing the upper and lower solution method and the monotone iterative technique.

Keywords: Lower and upper solution method; Fourth-order ordinary differential equation; Four-point boundary conditions

1. Introduction

It is well known that the upper and lower solution method is a powerful tool for proving existence results for boundary value problems. In many cases it is possible to find a minimal solution and a maximal solution between the lower solution and the upper solution by the monotone iterative technique. The upper and lower solution method has been used to deal with the multi-point boundary value problem for second-order ordinary differential equations and the two-point boundary value problem for higher-order ordinary differential equations \[6,9,8,13\]. There are fewer results on multi-point boundary value problems for higher-order equations in the literature of ordinary differential equations. For this reason, we consider the fourth-order nonlinear ordinary differential equation

\[ u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1 \]

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together with the four-point boundary conditions

\begin{align*}
    u(0) &= u(1) = 0, \\
    au''(\xi_1) - bu''(\xi_1) &= 0, \\
    cu''(\xi_2) + du''(\xi_2) &= 0,
\end{align*}

(1.2)

where \( a, b, c, d \) are nonnegative constants satisfying \( ad + bc + ac(\xi_2 - \xi_1) > 0, 0 \leq \xi_1 < \xi_2 \leq 1 \) and \( f \in C([0, 1] \times R \times R) \).

Eq. (1.1) is often referred to as a generalized beam equation, which has been studied by several authors. A brief discussion, which is easily accessible to the nonexpert reader, of the physical interpretation of the beam equation associated with some boundary conditions can be found in Zill and Cullen [14, pp. 237–243]. Several authors have studied the equation when it takes a simple form such as

\( f(t, u(t), u''(t)) = a(t)g(u(t)) \) or \( f(t, u(t), u''(t)) = f(t, u(t)) \)

under a variety of boundary conditions [7,10,3,12,1]. The type of multi-point boundary conditions considered in (1.2) are also somewhat different from the conjugate [11], focal [2], and Lidstone [4] conditions that are commonly encountered in the literature.

We will develop the upper and lower solution method for system (1.1) and (1.2) and establish some new existence results.

2. Preliminaries and lemmas

In this section, we will give some preliminary considerations and some lemmas which are essential to our main results.

**Lemma 2.1.** If \( \alpha = ad + bc + ac(t_2 - t_1) \neq 0 \) and \( h(t) \in C[t_1, t_2] \), then the boundary value problem

\begin{align*}
    u''(t) &= h(t), \\
    au(t_1) - bu'(t_1) &= 0, \\
    cu(t_2) + du'(t_2) &= 0
\end{align*}

(2.1)

has a unique solution

\[ u(t) = -\int_{t_1}^{t_2} G(t, s)h(s) \, ds, \]

where

\[ G(t, s) = \begin{cases} 
    \frac{1}{\alpha} (a(s - t_1) + b)(d + c(t_2 - t)), & t_1 \leq s < t \leq t_2, \\
    \frac{1}{\alpha} (a(t - t_1) + b)(d + c(t_2 - s)), & t_1 \leq t \leq s \leq t_2.
\end{cases} \] (2.2)

**Proof.** Integrating the first equation of (2.1) over the interval \([t_1, t]\) for \( t \in [t_1, t_2] \), we have

\[ u(t) = u(t_1) + u'(t_1)(t - t_1) + \int_{t_1}^{t} (t - s)h(s) \, ds, \] (2.3)

which implies

\[ u(t_2) = u(t_1) + u'(t_1)(t_2 - t_1) + \int_{t_1}^{t_2} (t_2 - s)h(s) \, ds, \]

\[ u'(t_2) = u'(t_1) + \int_{t_1}^{t_2} h(s) \, ds. \] (2.4)
Therefore, this equation together with the second equation of (2.1) leads to
\[
    u(t_1) = -\int_{t_1}^{t_2} \frac{1}{x} (a(t_2 - t_1) + b)(d + c(t_2 - s))h(s) \, ds,
\]
\[
    u'(t_1) = -\int_{t_1}^{t_2} \frac{a}{x} (d + c(t_2 - s))h(s) \, ds.
\]
Substituting this equation into Eq. (2.3), we have
\[
    u(t) = -\frac{1}{x} \left( \int_{t_1}^{t} (a(s - t_1) + b)(d + c(t_2 - s))h(s) \, ds \right.
    \]
\[
    \left. + \int_{t}^{t_2} (a(t - t_1) + b)(d + c(t_2 - s))h(s) \, ds \right),
\]
which implies the Lemma. □

**Lemma 2.2.** Suppose \( a, b, c, d, \xi_1, \xi_2 \) are nonnegative constants satisfying \( 0 \leq \xi_1 < \xi_2 \leq 1, b - a \xi_1 \geq 0, d - c + c \xi_2 \geq 0 \) and \( \delta = ad + bc + ac(\xi_2 - \xi_1) > 0 \). If \( u(t) \in C^4[0, 1] \) satisfies
\[
    u^{(4)}(t) \geq 0, \quad t \in (0, 1),
\]
\[
    u(0) \geq 0, \quad u(1) \geq 0,
\]
\[
    au''(\xi_1) - bu''(\xi_1) \leq 0, \quad cu''(\xi_2) + du''(\xi_2) \leq 0
\]
then \( u(t) \geq 0 \) and \( u''(t) \leq 0 \) for \( t \in [0, 1] \).

**Proof.** Let
\[
    u^{(4)}(t) = h(t), \quad t \in (0, 1),
\]
\[
    u(0) = x_0, \quad u(1) = x_1,
\]
\[
    au''(\xi_1) - bu''(\xi_1) = x_2, \quad cu''(\xi_2) + du''(\xi_2) = x_3,
\]
where \( x_0 \geq 0, x_1 \geq 0, x_2 \leq 0, x_3 \leq 0, h(t) \in C[0, 1] \) and \( h(t) \geq 0 \). In virtue of Lemma 2.1, we have
\[
    u(t) = tx_1 + (1 - t)x_0 - \int_0^1 G_1(t, \xi) R(\xi) \, d\xi + \int_0^1 G_1(t, \xi) \int_{\xi_1}^{\xi_2} G_2(\xi, s)h(s) \, ds \, d\xi,
\]
\[
    u''(t) = R(t) - \int_{\xi_1}^{\xi_2} G_2(t, s)h(s) \, ds,
\]
where
\[
    R(t) = \frac{1}{\delta} ((a(t - \xi_1) + b)x_3 + (c(\xi_2 - t) + d)x_2),
\]
\[
    G_1(t, s) = \begin{cases} 
        s(1 - t), & 0 \leq s < t \leq 1, \\
        t(1 - s), & 0 \leq t \leq s \leq 1. 
    \end{cases}
\]
\[
    G_2(t, s) = \begin{cases} 
        \frac{1}{\delta} (a(s - \xi_1) + b)(d + c(\xi_2 - t)), & \xi_1 \leq s < t \leq \xi_2, \\
        \frac{1}{\delta} (a(t - \xi_1) + b)(d + c(\xi_2 - s)), & \xi_1 \leq t \leq s \leq \xi_2. 
    \end{cases}
\]
On the other hand, the assumptions of the lemma imply \( R(t) \leq 0 \) for \( t \in [0, 1] \), and \( G_1(t, s) \geq 0 \) and \( G_2(t, s) \geq 0 \) for \( (t, s) \in [0, 1] \times [0, 1] \). Thus, \( u(t) \geq 0 \) and \( u''(t) \leq 0 \) for \( t \in [0, 1] \), which complete the proof of the lemma. □
Theorem 3.1. If there exist \( A \) function \( f(t, z(t), z''(t)) \), \( t \in (0, 1) \), \( f(0) \geq 0 \), \( f(1) \geq 0 \), \( a\alpha''(\xi_1) - b\alpha''(\xi_1) \leq 0 \), \( c\alpha''(\xi_2) + d\alpha''(\xi_2) \leq 0 \). (3.9)

Definition 1. A function \( \alpha(t) \) is said to be an upper solution of the boundary problem (1.1)–(1.2), if it belongs to \( C^4[0, 1] \) and satisfies
\[
\alpha^{(4)}(t) \geq f(t, \alpha(t), \alpha''(t)), \quad t \in (0, 1), \quad \alpha(0) \geq 0, \quad \alpha(1) \geq 0, \quad a\alpha''(\xi_1) - b\alpha''(\xi_1) \leq 0, \quad c\alpha''(\xi_2) + d\alpha''(\xi_2) \leq 0.
\]

Definition 2. A function \( \beta(t) \) is said to be a lower solution of the boundary problem (1.1)–(1.2), if it belongs to \( C^4[0, 1] \) and satisfies
\[
\beta^{(4)}(t) \leq f(t, \beta(t), \beta''(t)), \quad t \in (0, 1), \quad \beta(0) \leq 0, \quad \beta(1) \leq 0, \quad a\beta''(\xi_1) - b\beta''(\xi_1) \geq 0, \quad c\beta''(\xi_2) + d\beta''(\xi_2) \geq 0.
\]

3. Main results

We are now in a position to present and prove our main results. In what follows, we will always assume that the nonnegative constants \( a, b, c, d, \xi_1, \xi_2 \) satisfy \( 0 \leq \xi_1 < \xi_2 \leq 1, b - a\xi_1 \geq 0, d - c + c\xi_2 \geq 0 \) and \( ad + bc + ac(\xi_2 - \xi_1) > 0 \).

Theorem 3.1. If there exist \( \alpha(t) \) and \( \beta(t) \), upper and lower solutions, respectively, for the problem (1.1)–(1.2) satisfying
\[
\beta(t) \leq \alpha(t) \quad \text{and} \quad \beta''(t) \geq \alpha''(t), \quad (3.1)
\]
and if \( f : [0, 1] \times R \times R \rightarrow R \) is continuous and satisfies
\[
\begin{align*}
&f(t, u_1, v) - f(t, u_2, v) \leq 0 \quad \text{for} \quad \beta(t) \leq u_1 \leq u_2 \leq \alpha(t), \quad v \in R, \quad t \in [0, 1], \\
&f(t, u, v_1) - f(t, u, v_2) \geq 0 \quad \text{for} \quad \alpha''(t) \leq v_1 \leq v_2 \leq \beta''(t), \quad u \in R, \quad t \in [0, 1].
\end{align*}
\]
then, there exist two function sequences \( \{\alpha_n(t)\} \) and \( \{\beta_n(t)\} \) that converge uniformly to the solutions of the boundary value problem (1.1)–(1.2).

Proof. We consider the operator \( T : C^2[0, 1] \rightarrow C^4[0, 1] \) defined by
\[
Tu(t) = \int_0^1 G_1(t, \xi) \int_{\xi_1}^{\xi_2} G_2(\xi, s) f(s, u(s), u''(s)) \, ds \, d\xi,
\]
where \( G_1(t, s) \) and \( G_2(t, s) \) as in Eqs. (2.7) and (2.8), respectively, and define the set \( S \) by
\[
S = \{ u \in C^2[0, 1] | \beta(t) \leq u(t) \leq \alpha(t), \quad \alpha''(t) \leq u''(t) \leq \beta''(t) \}. \quad (3.4)
\]
The operator \( T \) has the following two properties:

(i) \( TS \subseteq S \).
(ii) \( \omega_1(t) \leq \omega_2(t), \omega_1''(t) \geq \omega_2''(t) \), where \( \omega_1(t) = Tu_1(t) \) and \( \omega_2(t) = Tu_2(t) \) for any \( u_1(t) \in S, u_2(t) \in S \) and \( u_1(t) \leq u_2(t), u_1''(t) \geq u_2''(t) \).

In fact, for any \( u(t) \in S \), let \( \omega(t) = Tu(t) \), then,
\[
\begin{align*}
&\omega^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1, \\
&\omega(0) = \omega(1) = 0, \\
&a\omega''(\xi_1) - b\omega''(\xi_1) = 0, \quad c\omega''(\xi_2) + d\omega''(\xi_2) = 0.
\end{align*}
\]
Let $\zeta(t) = x(t) - \omega(t)$, from the definition of $x(t)$ and conditions (3.2) of the theorem, we have

$$\begin{align*}
\zeta^{(4)}(t) &\geq f(t, x(t), x''(t)) - f(t, u(t), u''(t)) \geq 0, \quad t \in (0, 1), \\
\zeta(0) &\geq 0, \quad \zeta(1) \geq 0, \\
a\zeta''(\xi_1) - b\zeta''(\xi_1) \leq 0, \quad c\zeta''(\zeta_2) + d\zeta''(\zeta_2) \leq 0.
\end{align*}$$

(3.5)

In virtue of Lemma 2.2, we obtain that, $\zeta(t) \geq 0$ and $\zeta''(t) \leq 0$, i.e., $\omega(t) \leq x(t)$ and $\omega''(t) \geq x''(t)$. Similarly, we can prove that, $\beta(t) \leq \omega(t)$ and $\omega''(t) \leq \beta''(t)$. Thus, $T S \subseteq S$.

Furthermore, if we let $\omega(t) = \omega_2(t) - \omega_1(t)$, then,

$$\begin{align*}
\omega^{(4)}(t) &= f(t, u_2(t), u''_2(t)) - f(t, u_1(t), u''_1(t)) \geq 0, \quad t \in (0, 1), \\
\omega(0) &= 0, \quad \omega(1) = 0, \\
a\omega''(\xi_1) - b\omega''(\xi_1) &= 0, \quad c\omega''(\xi_2) + d\omega''(\xi_2) = 0.
\end{align*}$$

So Lemma 2.2 implies that $\omega(t) \geq 0$ and $\omega''(t) \leq 0$, i.e., $\omega_1(t) \leq \omega_2(t)$ and $\omega''_1(t) \geq \omega''_2(t)$. Thus, the second property of $T$ is true. Now, we define the sequences $\{x_n(t)\}$ and $\{\beta_n(t)\}$ by

$$\begin{align*}
x_n(t) &= T x_{n-1}(t), \quad x_0(t) = x(t), \\
\beta_n(t) &= T \beta_{n-1}(t), \quad \beta_0(t) = \beta(t).
\end{align*}$$

From the properties of $T$, we have

$$\begin{align*}
x(t) &\geq x_1(t) \geq x_2(t) \geq \cdots \geq x_n(t) \geq \cdots \geq \beta(t), \\
x''(t) &\leq x''_1(t) \leq x''_2(t) \leq \cdots \leq x''_n(t) \leq \cdots \leq \beta''(t), \\
\beta(t) &\leq \beta_1(t) \leq \beta_2(t) \leq \cdots \leq \beta_n(t) \leq \cdots \leq x(t), \\
\beta''(t) &\geq \beta''_1(t) \geq \beta''_2(t) \geq \cdots \geq \beta''_n(t) \geq \cdots \geq x''(t),
\end{align*}$$

(3.6)

which together with the conditions (3.2) of the theorem imply that

$$\begin{align*}
f(t, x(t), x''(t)) &\geq f(t, x_1(t), x''_1(t)) \geq \cdots \geq f(t, x_n(t), x''_n(t)) \geq \cdots \geq f(t, \beta(t), \beta''(t)), \\
f(t, x(t), x''(t)) &\geq \cdots \geq f(t, \beta_n(t), \beta''_n(t)) \geq \cdots \geq f(t, \beta_1(t), \beta''_1(t)) \geq f(t, \beta(t), \beta''(t)).
\end{align*}$$

So we have

$$\begin{align*}
x^{(4)}(t) &\geq x^{(4)}_1(t) \geq x^{(4)}_2(t) \geq \cdots \geq x^{(4)}_n(t) \geq \cdots \geq \beta^{(4)}(t), \\
\beta^{(4)}(t) &\leq \beta^{(4)}_1(t) \leq \beta^{(4)}_2(t) \leq \cdots \leq \beta^{(4)}_n(t) \leq \cdots \leq x^{(4)}(t). \tag{3.7}
\end{align*}$$

Hence, from Eqs. (3.6) and (3.7) we know there exists a constant $M$ such that

$$\begin{align*}
|x_n(t)| &\leq M, \quad |\beta_n(t)| \leq M, \quad t \in [0, 1], \quad n = 1, 2, \ldots, n, \\
|x^{(2)}_n(t)| &\leq M, \quad |\beta^{(2)}_n(t)| \leq M, \quad t \in [0, 1], \quad n = 1, 2, \ldots, n, \\
|x^{(4)}_n(t)| &\leq M, \quad |\beta^{(4)}(t)| \leq M, \quad t \in [0, 1], \quad n = 1, 2, \ldots, n. \tag{3.8}
\end{align*}$$

In addition, from the boundary condition we know that for each $n \in N$, there exist $\xi_n \in [0, 1], \eta_n \in [0, 1]$ and a constant $M_1$, such that $|x''_n(\xi_n)| \leq M_1$, $|\beta''_n(\eta_n)| \leq M_1$. Thus,

$$\begin{align*}
|x''_n(t)| &\leq \left| \int_{\xi_n}^{t} x^{(4)}_n(t) \, dt \right| + |x''_n(\xi_n)| \leq M_1 + M, \\
|\beta''_n(t)| &\leq \left| \int_{\eta_n}^{t} \beta^{(4)}(t) \, dt \right| + |\beta''_n(\eta_n)| \leq M_1 + M, \quad t \in [0, 1], \quad n = 1, 2, \ldots, n. \tag{3.9}
\end{align*}$$
Using the boundary condition we get that for each \( n \in N \), there exist \( \gamma_n \in [0, 1] \), \( \tau_n \in [0, 1] \), such that \( u_n'(\gamma_n) = 0 \) and \( \beta_n'(\tau_n) = 0 \). Thus,

\[
|\alpha'_n(t)| = \left| \int_{\gamma_n}^{t} \alpha''_n(t) \, dt \right| \leq M, \\
|\beta'_n(t)| = \left| \int_{\tau_n}^{t} \beta''_n(t) \, dt \right| \leq M, \quad t \in [0, 1], \quad n = 1, 2, \ldots, n, \ldots.
\]  

(3.10)

Hence, from Eqs. (3.8)–(3.10), we know that \( \{\alpha_n\}, \{\beta_n\} \) are uniformly bounded in \( C^4[0, 1] \). On the other hand, for any \( x_1, x_2 \in [0, 1] \),

\[
|\alpha_n(x_1) - \alpha_n(x_2)| \leq M|x_1 - x_2|, \quad |\beta_n(x_1) - \beta_n(x_2)| \leq M|x_1 - x_2|,
\]

\[
|\alpha''_n(x_1) - \alpha''_n(x_2)| \leq (M_1 + M)|x_1 - x_2|, \quad |\beta''_n(x_1) - \beta''_n(x_2)| \leq (M_1 + M)|x_1 - x_2|.
\]

So \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\alpha''_n\}, \{\beta''_n\} \) are equicontinuous. Thus, from the Arzela–Ascoli theorem [5] and the Eqs. (3.6), (3.7) we know that \( \{\alpha_n\}, \{\beta_n\} \) converge uniformly to the solutions of the problem (1.1)–(1.2). The proof is completed as desired. \( \Box \)

4. Applications

To illustrate our results, we present the following examples.

Example 1. Consider the boundary value problem

\[
u^{(4)}(t) = u^2 - \frac{1}{\pi^2} (u^2)^5 + \frac{1}{2} \sin \pi t, \quad 0 < t < 1,
\]

\[u(0) = u(1) = 0,
\]

\[au''(\frac{1}{2}) - bu''(\frac{1}{2}) = 0, \quad cu''(\frac{1}{3}) + du'''(\frac{1}{2}) = 0,
\]  

(4.1)

where \( a, b, c, d \) are nonnegative constants satisfying \( 3b - a > 0 \) and \( 2d - c > 0 \). It is easy to check that \( \alpha(t) = \sin \pi t \), \( \beta(t) = 0 \) are upper and lower solutions of (4.1), respectively, and that all assumptions of Theorem 3.1 are fulfilled. So the boundary problem (4.1) has at least one solution \( u(t) \) satisfying

\[0 \leq u(t) \leq \sin \pi t, \quad -\pi^2 \sin \pi t \leq u''(t) \leq 0, \quad t \in [0, 1].
\]

Example 2. Consider the fourth-order nonlinear ordinary differential equation

\[u^{(4)}(t) = f(t, u(t)), \quad 0 < t < 1,
\]  

(4.2)

together with the four-point boundary conditions

\[u(0) = u(1) = 0,
\]

\[au''(\xi_1) - bu''(\xi_1) = 0, \quad cu''(\xi_2) + du''(\xi_2) = 0,
\]  

(4.3)

where \( a, b, c, d, \xi_1, \xi_2 \) are nonnegative constants satisfying \( 0 \leq \xi_1 < \xi_2 \leq 1, b - a \xi_1 \geq 0, d - c \xi_2 \geq 0 \) and \( \delta = ad + bc + ac(\xi_2 - \xi_1) > 0, f(t, u) \in C([0, 1] \times [0, +\infty), \mathbb{R}^+) \) is nondecreasing in \( u \) and there exists a positive constant \( \mu \) such that

\[k^\mu f(t, u) \leq f(t, ku) \quad \text{for any } 0 \leq k \leq 1.
\]
It is easy to verify that the functions $\alpha(t) = k_1 g(t)$, $\beta(t) = k_2 g(t)$ are lower and upper solutions of (4.2)–(4.3), respectively, where $k_1 \leq \min\{1/a_2, (a_1)^{\mu/(1-\mu)}\}$, $k_2 \geq \max\{1/a_1, (a_2)^{\mu/(1-\mu)}\}$, and

$$a_1 = \min \left\{ 1, \frac{1}{2} \min_{t \in [0,1]} \int_{\xi_1}^{\xi_2} G_2(t, s) f(s, s(1 - s)) \, ds \right\},$$

$$a_2 = \max \left\{ 1, \frac{1}{2} \max_{t \in [0,1]} \int_{\xi_1}^{\xi_2} G_2(t, s) f(s, s(1 - s)) \, ds \right\},$$

$$g(t) = \int_{0}^{1} G_1(t, \xi) \int_{\xi_1}^{\xi_2} G_2(\xi, s) f(s, s(1 - s)) \, ds \, d\xi. \quad (4.4)$$

Thus, this boundary value problem has at least one solution.

5. Conclusion

In this paper, we study a four-point boundary value problem for a fourth-order ordinary differential equation. Some new existence results are obtained by developing the upper and lower solution method and the monotone iterative technique. Furthermore, some applications are included in the paper.

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