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Adaptive feedback synchronization of a unified chaotic system ☆

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Abstract

This Letter further improves and extends the work of Wang et al. [Phys. Lett. A 312 (2003) 34]. In detailed, the linear feedback synchronization and adaptive feedback synchronization with only one controller for a unified chaotic system are discussed here. It is noticed that this unified system contains the noted Lorenz and Chen systems. Two chaotic synchronization theorems are attained. Also, numerical simulations are given to show the effectiveness of these methods.

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1. Introduction

Chaotic behavior can be observed in many real-world physical systems, such as chemical reactors, feedback control devices, and laser systems. Recently, chaos control and synchronization attract more and more attention from various fields. Over the last decades, many methods and techniques for chaos control and synchronization had been produced [1–7], such as OGY method [6], PC method [7], feedback approach [1], adaptive method [1], time-delay feedback approach [2], backstepping design technique [3], etc.

In 1963, Lorenz found the first classical chaotic attractor [8]. In 1999, Chen found another similar but not topological equivalent chaotic attractor [9]. In 2002, Lü and Chen found the critical chaotic attractor between the Lorenz and Chen attractor [10]. It is noticed that these systems can be classified into three different types from the definition of Vaněček and Čelikovský [11]: the Lorenz system satisfies the condition $a_{12}a_{21} > 0$, the Chen system

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satisfies $a_{12}a_{21} < 0$, and the Lü system satisfies $a_{12}a_{21} = 0$, where a_{12} and a_{21} are the corresponding elements in the linear part matrix $A = (a_{ij})_{3\times 3}$ of the system. Very recently, Lü et al. unified above three chaotic systems into a new chaotic system—unified chaotic system [12], which is described by

$$\begin{cases} \dot{x} = (25\alpha + 10)(y - x), \\ \dot{y} = (28 - 35\alpha)x + (29\alpha - 1)y - xz, \\ \dot{z} = xy - \frac{8+\alpha}{3}z, \end{cases} \tag{1}$$

where $\alpha \in [0, 1]$. System (1) is chaotic for $\alpha \in [0, 1]$. When $\alpha \in [0, 0.8)$, system (1) is called the general Lorenz system; when $\alpha = 0.8$, it becomes the general Lü system; when $\alpha \in (0.8, 1]$, system (1) is called the general Chen system.

Recently, there are some results reported about this unified chaotic system [13–16]. Lu et al. further investigated the PC synchronization and its application in secure communication [13]; Lu et al. also studied its parameter identification and tracking problem [14]; Tao et al. investigated its linear feedback synchronization, non-linear feedback synchronization and generalized synchronization [15]; Wu and Lu studied the backstepping control [3]; Chen and Lü discussed the adaptive feedback synchronization with three controllers [16]. This Letter introduces the linear and adaptive feedback synchronization methods with only one controller, which contain the adaptive feedback synchronization of the Lorenz and Chen systems. In this sense, we improve and extend the results of Wang et al. [1]. Numerical simulations show the effectiveness of these techniques.

Let system (1) be the drive system, then the response system is

$$\begin{cases} \dot{u} = (25\alpha + 10)(v - u), \\ \dot{v} = (28 - 35\alpha)u + (29\alpha - 1)v - uw + u_2, \\ \dot{w} = uv - \frac{8+\alpha}{3}w. \end{cases} \tag{2}$$

2. Linear feedback synchronization

In the following, we present a theorem for the linear feedback synchronization of system (1).

Theorem 1. *Let* $u_2 = -k(v - y)$, *where* $k > k_0 = \min_{0<\beta<\beta_0} k(\alpha, \beta)$, *in which*

$$k(\alpha, \beta) = \frac{(\alpha + 8)}{12} \frac{\left[\frac{1}{\beta}(25\alpha + 10) + 63 + M_3\right]^2}{\frac{1}{3\beta}(25\alpha + 10)(\alpha + 8) - \frac{M_2^2}{4}} + 29\alpha - 1, \tag{3}$$

$$\beta_0(\alpha) = \frac{4}{3M_2^2}(25\alpha + 10)(\alpha + 8), \tag{4}$$

where M_2 *and* M_3 *are the bounds*: $|y|, |v| \leqslant M_2$, $|z|, |w| \leqslant M_3$. *Thus the response system* (2) *and the drive system* (1) *reach synchronization for all* $\alpha \in [0, 1]$.

Proof. Let the synchronous errors be $e_1 = u - x$, $e_2 = v - y$, $e_3 = w - z$. Then we have the error system

$$\begin{cases} \dot{e}_1 = (25\alpha + 10)(e_2 - e_1), \\ \dot{e}_2 = (28 - 35\alpha)e_1 + (29\alpha - 1)e_2 - xe_3 - we_1 - ke_2, \\ \dot{e}_3 = -\frac{8+\alpha}{3}e_3 + xe_2 + ve_1. \end{cases} \tag{5}$$

Consider the Lyapunov candidate

$$V = \frac{1}{2}\left(\frac{1}{\beta}e_1^2 + e_2^2 + e_3^2\right). \tag{6}$$

From system (1) and system (2), then we get

$$\begin{aligned}
\dot{V} &= \frac{1}{\beta} e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 \\
&= -\frac{25\alpha + 10}{\beta} e_1^2 - (k - 29\alpha + 1) e_2^2 - \frac{\alpha + 8}{3} e_3^2 + \left[\frac{25\alpha + 10}{\beta} + 28 - 35\alpha - w\right] e_1 e_2 + v e_1 e_3 \\
&\leqslant -\frac{25\alpha + 10}{\beta} e_1^2 - (k - 29\alpha + 1) e_2^2 - \frac{\alpha + 8}{3} e_3^2 + \left[\frac{25\alpha + 10}{\beta} + 63 + M_3\right] |e_1||e_2| + M_2 |e_1||e_3| \\
&= -e^{\mathrm{T}} P e,
\end{aligned} \tag{7}$$

where $e = (|e_1|, |e_2|, |e_3|)^{\mathrm{T}}$, and

$$P = \begin{pmatrix} \frac{25\alpha+10}{\beta} & -\frac{1}{2}\left[\frac{25\alpha+10}{\beta} + 63 + M_3\right] & -\frac{M_2}{2} \\ -\frac{1}{2}\left[\frac{25\alpha+10}{\beta} + 63 + M_3\right] & k - 29\alpha + 1 & 0 \\ -\frac{M_2}{2} & 0 & \frac{8+\alpha}{3} \end{pmatrix}.$$

To ensure that the origin of error system (5) is asymptotically stable, then the symmetrical matrix P should be positive-definite. If the symmetrical matrix P satisfies the following conditions:

$$\frac{25\alpha + 10}{\beta} > 0, \tag{8}$$

$$\frac{(25\alpha + 10)(k - 29\alpha + 1)}{\beta} - \frac{1}{4}\left[\frac{25\alpha + 10}{\beta} + 63 + M_3\right]^2 > 0, \tag{9}$$

$$\frac{(25\alpha + 10)(\alpha + 8)(k - 29\alpha + 1)}{3\beta} - \frac{\alpha + 8}{12}\left[\frac{25\alpha + 10}{\beta} + 63 + M_3\right]^2 - \frac{(k - 29\alpha + 1)M_2^2}{4} > 0, \tag{10}$$

P is a positive-definite matrix.

Obviously, condition (8) holds. If

$$\frac{(25\alpha + 10)(\alpha + 8)}{3\beta} - \frac{M_2^2}{4} > 0, \tag{11}$$

and $k > k(\alpha, \beta)$, condition (10) holds. Moreover, if condition (10) holds, condition (9) holds. Note that condition (11) is equivalent to $\beta < \beta_0(\alpha)$.

Therefore, there exists a real number k, such that, when

$$k > k_0 = \min_{0<\beta<\beta_0} k(\alpha, \beta)$$

for $\alpha \in [0, 1]$, the matrix P is positive-definite. According to the hypothesis of Theorem 1, the response system (2) and the drive system (1) realize synchronization. That is, $e_1(t), e_2(t), e_3(t) \to 0$ as $t \to \infty$.

The proof is thus completed. □

Remark. Since we do not know the detailed upper bounds M_2 and M_3, it is very difficult to determine the detailed value k_0 from the hypothesis of Theorem 1. However, we can attain the approximative upper bounds $\hat{M}_2$ and $\hat{M}_3$. Also, we have found that there is a determined relation between the feedback gain and the system parameter α and the largest conditional Lyapunov exponent λ [15].

To avoid above real difficulties, we present the adaptive feedback synchronization method in the following section.

3. Adaptive feedback synchronization

In this section, we introduce a new adaptive feedback synchronization method.

Theorem 2. *The response system (2) can synchronize the drive system (1) if $u_2 = -k(t)(v-y) = -k(t)e_2$ and*

$$\dot{k}(t) = \theta(v-y)^2 = \theta e_2^2, \tag{12}$$

where constant $\theta > 0$.

Proof. Let the synchronous errors be $e_1 = u - x$, $e_2 = v - y$, $e_3 = w - z$. Then we have the error system

$$\begin{cases} \dot{e}_1 = (25\alpha + 10)(e_2 - e_1), \\ \dot{e}_2 = (28 - 35\alpha)e_1 + (29\alpha - 1)e_2 - xe_3 - we_1 - ke_2, \\ \dot{e}_3 = -\frac{8+\alpha}{3}e_3 + xe_2 + ve_1. \end{cases} \tag{13}$$

Define a Lyapunov candidate

$$V = \frac{1}{2}\left(\frac{1}{\beta}e_1^2 + e_2^2 + e_3^2 + \frac{(k-k^*)^2}{\theta}\right), \tag{14}$$

where the constant $\beta > 0$ and k^* is a real constant.

According to system (1) and system (2), then we have

$$\begin{aligned} \dot{V} &= \frac{1}{\beta}e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + \frac{k-k^*}{\theta}\dot{k} \\ &= -\frac{25\alpha+10}{\beta}e_1^2 - (k^* - 29\alpha + 1)e_2^2 - \frac{\alpha+8}{3}e_3^2 + \left[\frac{25\alpha+10}{\beta} + 28 - 35\alpha - w\right]e_1e_2 + ve_1e_3 \\ &\leqslant -\frac{25\alpha+10}{\beta}e_1^2 - (k^* - 29\alpha + 1)e_2^2 - \frac{\alpha+8}{3}e_3^2 + \left[\frac{25\alpha+10}{\beta} + 63 + M_3\right]|e_1||e_2| + M_2|e_1||e_3| \\ &= -e^{\mathrm{T}}P^*e, \end{aligned} \tag{15}$$

where $e = (|e_1|, |e_2|, |e_3|)^{\mathrm{T}}$, and

$$P^* = \begin{pmatrix} \frac{25\alpha+10}{\beta} & -\frac{1}{2}\left[\frac{25\alpha+10}{\beta} + 63 + M_3\right] & -\frac{M_2}{2} \\ -\frac{1}{2}\left[\frac{25\alpha+10}{\beta} + 63 + M_3\right] & k^* - 29\alpha + 1 & 0 \\ -\frac{M_2}{2} & 0 & \frac{8+\alpha}{3} \end{pmatrix}.$$

Similarly, when $k^* > k_0$, where k_0 is defined in Theorem 1, the symmetrical matrix P^* is positive-definite.

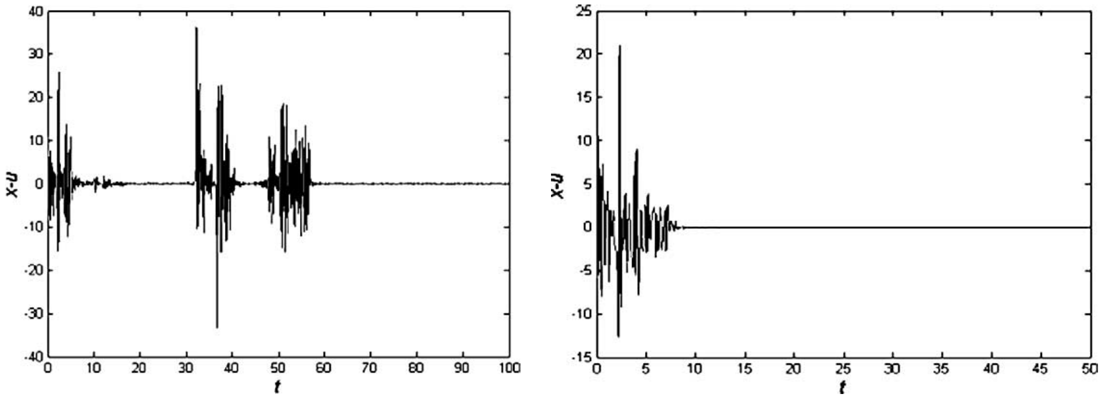
Since $\dot{V} \leqslant 0$, then $e_1, e_2, e_3, k - k^* \in L_\infty$. From the error system (13), $\dot{e}_1, \dot{e}_2, \dot{e}_3 \in L_\infty$. Since $\dot{V} \leqslant -e^{\mathrm{T}}P^*e$ and P^* is a positive-definite matrix, then we have

$$\int_0^t \lambda_{\min}(P^*)\|e\|^2\,dt \leqslant \int_0^t eP^*e^{\mathrm{T}}\,dt \leqslant \int_0^t -\dot{V}\,dt = V(0) - V(t) \leqslant V(0),$$

where $\lambda_{\min}(P^*)$ is the minimum eigenvalue of positive-definite matrix P^*. Thus, $e_1, e_2, e_3 \in L_2$. According to the Barbalat's lemma, $e_1(t), e_2(t), e_3(t) \to 0$ as $t \to \infty$. Therefore, the response system (2) synchronize the drive system (1) by using the adaptive feedback controller.

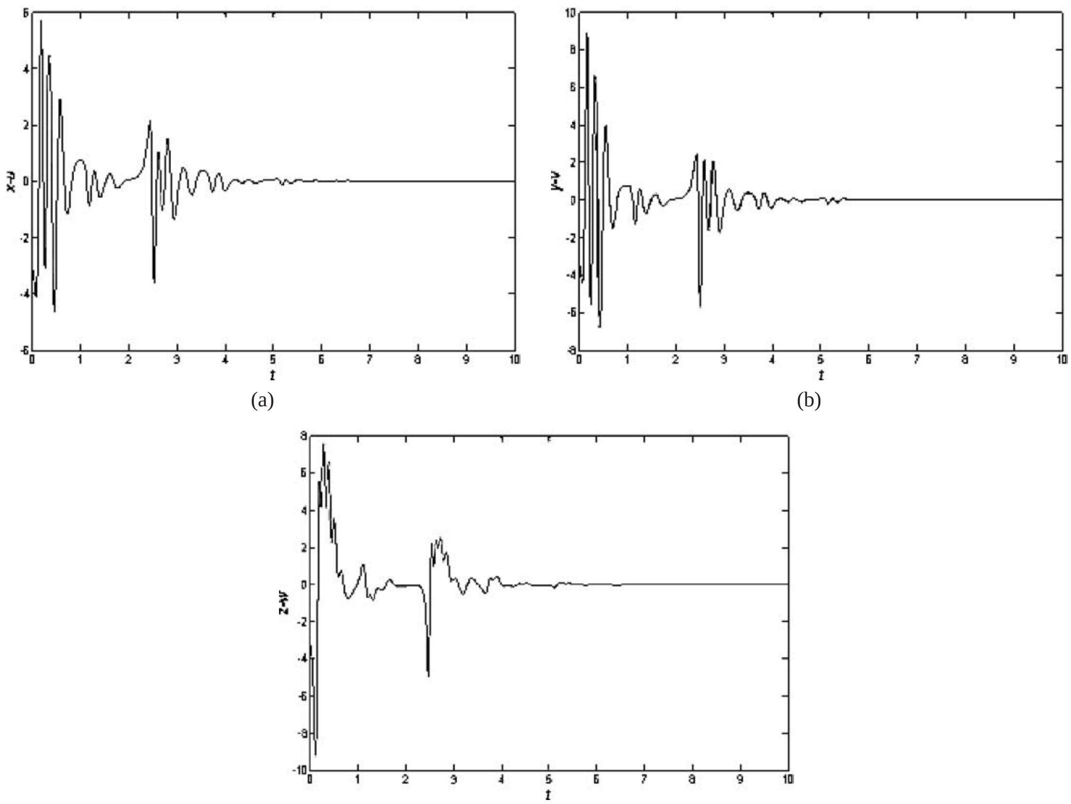
This completes the proof. □

Remark. The proof of Theorem 2 has improved the proof in Ref. [1]. Since the Lyapunov function (14) is a function of variables $e_1(t), e_2(t), e_3(t), (k - k^*)$, we cannot directly deduce $e_1(t), e_2(t), e_3(t) \to 0$ as $t \to \infty$ from the positive-definite matrix P^*.



(a) (b)

Fig. 1. The synchronous errors $e_1(t)$ for the linear feedback synchronization. (a) $k = 4$, (b) $k = 4.3$.



(c)

Fig. 2. The synchronous errors for the adaptive feedback synchronization. (a) $e_1(t)$, (b) $e_2(t)$, (c) $e_3(t)$.

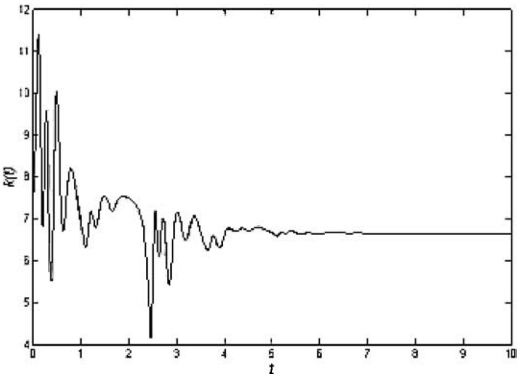


Fig. 3. The parameter $k(t)$ for the adaptive feedback synchronization.

4. Numerical simulations

In this section, some numerical simulations are given to verify the effectiveness of above two kinds of methods. In all simulations, assume that $\alpha = 0.8$, the initial conditions of drive and response systems are $(1, 2, 3)$ and $(4, 5, 6)$, respectively.

Fig. 1 shows the synchronous errors for linear feedback control method. Obviously, the response and drive systems cannot realize synchronization for $k = 4$.

Figs. 2, 3 display the effectiveness of adaptive feedback control approach, where $\theta = 1$ and $k(0) = 7$. The synchronous errors are shown in Fig. 2 and the parameter $k(t)$ is displayed in Fig. 3.

5. Conclusions

This Letter further investigates the linear feedback synchronization and adaptive feedback synchronization with only one controller for the unified chaotic system. Especially, since the unified system contains the Lorenz and Chen systems as special cases, our methods are also valid for the Lorenz and Chen systems. Moreover, our proof for the adaptive feedback synchronization theorem is rigorous, which improves the proof in Ref. [1]. Numerical simulations also show the effectiveness of above approaches. Furthermore, the adaptive feedback synchronization method with only one controller has widely applicative prospect in secure communication.

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