Synchronization stability of three chaotic systems with linear coupling

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Abstract

This Letter introduces a new method—mode decomposition—for stability analysis of periodic orbits. Using this method, the stability of a periodic solution of an autonomous system, as well as the stability of synchronization within three chaotic systems with linear coupling, can be analyzed. As an example, a rigorous sufficient condition on the coupling coefficients for achieving chaos synchronization is obtained, for the case of three-coupled identical Lorenz systems. Numerical simulations are shown for demonstration.

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1. Introduction

Nonlinear dynamics of coupled systems have been intensively studied in the last few decades [1–3], in various fields such as biology [4,5], physics [6,7], chemistry [8,9], and engineering [10].

Dynamical behaviors of simply (e.g., linearly) coupled nonlinear oscillators can be enormously complicated. Consider for instance three identical oscillators coupled in a ring, which can be phase-locked in four basic patterns: three oscillators move synchronously; three oscillators move along a ring apart by one third of the periodic interval; two oscillators move synchronously, while the third moves in a different phase; two oscillators move half a phase out of step, while the third moves twice as rapidly as its neighbors. The situation for linearly coupled chaotic systems is even more complex, since a strange attractor has infinitely many unstable periodic orbits. Yet, chaos synchronization between two coupled systems is possible and, in fact, has been frequently observed both numerically and experimentally in the past decade.

Since the earlier discussion of chaos synchronization [11], there has been tremendous interest in studying the synchronization phenomenon of chaotic systems [12]. Over the last decade, many new types of chaos...
synchronization have appeared: stochastic synchronization [13], lag synchronization [14], adaptive synchronization [15], phase synchronization [16], and generalized synchronization [17], to name only a few.

We have also investigated chaos synchronization between two coupled identical Lorenz [18], Chen [19], and Lü [20] systems, from a rigorous mathematical approach. However, for three-coupled identical systems, the stability of periodic orbit and chaos synchronization are much more complex, and there does not seem to be much result reported in the literature about this special synchronization configuration.

In this Letter, we investigate the stability of a periodic solution of an autonomous system and then chaos synchronization in an array of chaotic systems with linear coupling. We will first formulate the problem of the stability of systems synchronization, and then transform the stability of synchronized solution into a lower-dimensional problem using a new technique called mode decomposition. Then, a specific example of three-coupled identical Lorenz systems is studied, with a rigorous sufficient condition on the coupling coefficients for chaos synchronization derived. To that end, computer simulations are reported for illustration and verification.

2. Stability of a synchronized solution

It is well-known that the stability of a periodic solution of an autonomous system depends on the multipliers of the variational equation of the system. A coupled system of three identical n-D subsystems is a 3n-D system, so is its variational equation. It turns out that, the stability of synchronization (or out-of-phase solution) of such a 3n-D coupled system only depends on a relevant n-D linear homogeneous equation, called the reduced variational equation [2].

To begin with, we briefly introduce some concepts. Let $S$ be an open domain in $\mathbb{R}^n (n \geq 2)$ and consider a $C^1$ function $f: S \to \mathbb{R}^n$, i.e., continuously differentiable on $S$. Assume that the n-D equation (linear oscillator)

$$\dot{x} = f(x) \quad (x \in S),$$

has a unique periodic solution (regardless of possible phase shifts), $\psi(t)$, with the least period $T > 0$.

Let $\rho_1 = 1$, $\rho_2, \ldots, \rho_n$ be the Floquet multipliers of the variational equation of system (1) with respect to $\psi(t)$:

$$\dot{y} = A(t)y,$$

where $A(t) = Df(\psi(t))$, and $Df$ is the Jacobian of $f$. Furthermore, assume that $\psi(t)$ is hyperbolically orbitally asymptotically stable, that is, $|\rho_j| < 1$ for $2 \leq j \leq n$.

Now, we consider the coupled system consisting of three identical oscillators described by (1), with a symmetric coupling, as follows:

$$\begin{cases}
\dot{x} = f(x) + C(x, y, z), \\
\dot{y} = f(y) + C(y, z, x), \\
\dot{z} = f(z) + C(z, x, y),
\end{cases}$$

where $C: S \times S \times S \to \mathbb{R}^n$ is a $C^1$ function, which can be nonlinear. For simplicity, let $U = (x, y, z)^T$ and $F(U)$ be the vector field defined by (3), so that (3) can be briefly written as

$$\dot{U} = F(U).$$

In the following, utilizing the symmetry of the reduced variational equation associated with (4), we will establish a structural property of its monodromy matrix, thereby in the study of synchronization of this three-coupled system only two non-unity Floquet multipliers are needed to compute.

First, note that system (4) defines a group $\Omega = \{\omega\}$, i.e., $\omega F(U) = F(\omega U)$, where $\omega$ is a circular permutation, acting on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ by $\omega U = (y, z, x)$. 
Assume that the coupling function $C(x, y, z)$ satisfies

$$C(x, x, x) = 0, \quad \forall x \in S.$$  \hspace{1cm} (5)

Then, the coupled system (4) has an in-phase periodic solution (or synchronized solution), $U(t) = \Phi(t) = (\varphi(t), \varphi(t), \varphi(t))^T$.

Let $B_1(t) = D_2C(\varphi(t), \varphi(t), \varphi(t))$ and $B_2(t) = D_3C(\varphi(t), \varphi(t), \varphi(t))$. Then, from (5), we have

$$D_1C(\varphi(t), \varphi(t), \varphi(t)) = -(B_1(t) + B_2(t)),$$  \hspace{1cm} (6)

where $D_iC(x, y, z)$ denotes the derivative of $C(x, y, z)$ with respect to its $i$th component, $i = 1, 2, 3$. The corresponding variational equation of (4) with respect to $\Phi(t)$ is

$$\begin{align*}
\dot{x} &= (A(t) - B_1(t) - B_2(t))x + B_1(t)y + B_2(t)z, \\
\dot{y} &= B_2(t)x + (A(t) - B_1(t) - B_2(t))y + B_1(t)z, \\
\dot{z} &= B_1(t)x + B_2(t)y + (A(t) - B_1(t) - B_2(t))z,
\end{align*}$$  \hspace{1cm} (7)

or, in a matrix form,

$$\dot{U} = DF(\Phi(t))U.$$  \hspace{1cm} (8)

It is easy to see that the variational equation (8) inherits the $\omega$-symmetry, namely,

$$\omega DF(\Phi(t))U = DF(\Phi(t))(\omega U), \quad U \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n.$$  \hspace{1cm} (9)

Next, introduce a transformation of variables:

$$U = PW, \quad P = \begin{pmatrix} I & I & 0 \\ I & -\frac{1}{2}I & \frac{\sqrt{3}}{2}I \\ I & \frac{1}{2}I & -\frac{\sqrt{3}}{2}I \end{pmatrix},$$  \hspace{1cm} (10)

where $W$ is the new variable of dimension $3n$, and $I$ is the $n \times n$ identity matrix. The inverse matrix of $P$ is

$$P^{-1} = \begin{pmatrix} \frac{1}{2}I & \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}I & \frac{1}{2}I & \frac{1}{2}I \\ 0 & \frac{\sqrt{3}}{2}I & -\frac{\sqrt{3}}{2}I \end{pmatrix}.$$

It is easy to verify that, under the transformation (10), the variational equation (8) is converted to

$$\frac{dW}{dt} = \begin{pmatrix} A(t) & O_{3n \times 2n} \\ O_{2n \times 2n} & R(t) \end{pmatrix}W,$$  \hspace{1cm} (11)

where

$$R(t) = \begin{pmatrix} A(t) - \frac{3}{2}(B_1(t) + B_2(t)) & \frac{\sqrt{3}}{2}(B_1(t) - B_2(t)) \\ -\frac{\sqrt{3}}{2}(B_1(t) - B_2(t)) & A(t) - \frac{3}{2}(B_1(t) + B_2(t)) \end{pmatrix}.$$

This means that the transformation (10) decomposes (7) into two independent equations, in which one is

$$\frac{dW_1}{dt} = A(t)W_1,$$  \hspace{1cm} (12)

and the other is

$$\frac{dW_2}{dt} = R(t)W_2.$$  \hspace{1cm} (13)

This decomposition is called mode decomposition.
In the case of a coupled system consisting of two identical oscillators, similar mode decomposition can be found in [3,18].

Since Eq. (13) is a linear homogeneous equation with periodic coefficients, the Floquet theory implies that its trivial solution is asymptotically stable if and only if the maximal modulus of its multipliers is strictly less than one. We have the following results.

**Theorem 2.1.** Let the periodic solution \( \varphi(t) \) of the individual oscillator (1) be hyperbolically orbitally asymptotically stable, and assume that the coupling function \( C(x, y, z) \) satisfies (5). Then, the synchronized solution \( \Phi(t) \) of the three-coupled system (3) is hyperbolically orbitally asymptotically stable if and only if the trivial solution of system (3) is asymptotically stable.

Note that the orbit \( \Phi(t) \) itself may have a reduced variational equation, other than (13). However, we do not intend to find and show this reduced variational equation because both the original and the reduced variational equations are equivalent in determining the stability of \( W(t) \). Moreover, the symmetry of (13) allows us to obtain some structural properties of its monodromy matrix, which is discussed next.

Define a linear operator \( \theta : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \) by

\[
\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}, \quad x, y \in \mathbb{R}^n.
\]

This operator, \( \theta \), has a matrix representation

\[
R_\theta = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\]

Obviously, (13) has \( \theta \)-symmetry, namely,

\[
\theta R(t)W = R(t)(\theta W), \quad \text{for } W \in \mathbb{R}^{2n}.
\]

Therefore, we conclude that if \( W_2(t) \) is a solution of Eq. (13), then so is \( \theta W_2(t) \). The \( \theta \)-symmetry of (13) comes from the \( \omega \)-symmetry of the variational equation (7) and the fact that \( R_\theta \) is the exact matrix representation of \( \omega \) on a \( 2n \)-D invariant subset.

**Theorem 2.2.** The monodromy matrix \( M \) of the reduced variational equation of (13) is of the following form:

\[
M = \begin{pmatrix} M_1 & M_2 \\ -M_2 & M_1 \end{pmatrix},
\]

where \( M_1 \) and \( M_2 \) are \( n \times n \) matrices.

**Proof.** Let \( \xi(t) \) be the principal matrix solution of (13), i.e., \( \xi(0) = I_{2n} \), where \( I_{2n} \) is the identity matrix of order \( 2n \). Then, \( M = \xi(T) \).

We first show that \( \xi(t) \) is commutative with matrix \( R_\theta \):

\[
R_\theta \xi(t) = \xi(t)R_\theta.
\]

Indeed, each column of \( R_\theta \xi(t) \) is a solution of (13), as just concluded, so that \( R_\theta \xi(t) \) is a matrix solution of (13). On the other hand, \( R_\theta \xi(t) \) is also a matrix solution of (13) according to the basic theory of linear homogeneous equations. Since \( \xi(0) = I_{2n} \), these two matrix solutions take the same value, \( I_{2n} \), at \( t = 0 \). Thus, (18) holds.

Thus, in particular, we have

\[
R_\theta M = MR_\theta,
\]
which implies (17), as desired. The proof is completed. □

It follows from (17) that the characteristic polynomial of $M$ is given by

$$p_M(\lambda) = \left| \lambda I - (M_1 + iM_2) \right| \cdot \left| \lambda I - (M_1 - iM_2) \right|, \tag{20}$$

where $| \cdot |$ denotes the determinant of a matrix, and $i^2 = -1$. Let

$$|\lambda I - (M_1 + iM_2)| = p_1(\lambda) + ip_2(\lambda), \tag{21}$$

where $p_1(\lambda)$ and $p_2(\lambda)$ are polynomials with real coefficients. Then, $p_M(\lambda)$ can be written as

$$p_M(\lambda) = p_2^2(\lambda) + p_2^2(\lambda),$$

yielding the following result.

**Corollary 2.3.** The real roots of $p_M(\lambda)$, if it has any, have even number of multiplicities.

3. Chaos synchronization of coupled systems with linear coupling

We investigated the problem of chaos synchronization for a two-coupled systems of identical chaotic subsystems with linear coupling [21]. In this Letter, we further investigate the chaos synchronization problem yet for three-coupled systems of identical chaotic subsystems, also with linear coupling. In this Letter, we mainly utilize the mode decomposition method introduced above. As a concrete example, the Lorenz system is used for illustration.

The Lorenz system [18] can be used to describe different physical systems such as laser devices, disk dynamos, and some other problems related to flow and heat convection. Recently, the Lorenz attractor has been mathematically confirmed to exist. The Lorenz system is given by

$$\begin{align*}
\dot{x} &= a(y - x), \\
\dot{y} &= cx - xz - y, \\
\dot{z} &= xy - bz,
\end{align*} \tag{22}$$

which has a chaotic attractor, when $a = 10$, $b = \frac{8}{3}, c = 28$.

Synchronization of two identical Lorenz systems has been widely studied under different conditions over the last decade. We have also derived some global asymptotic synchronization conditions for two linearly coupled identical Lorenz systems [21]. When three identical Lorenz systems are linearly coupled in a ring, the situation is much more complicated, as mentioned above, and has not been carefully investigated. Here, we want to ask if there is also chaos synchronization in such a three-coupled system. The following theoretical analysis and numerical simulations provide a positive answer to this question.

More precisely, consider the following linearly coupled system of three identical Lorenz systems:

$$\begin{align*}
\dot{X} &= F(X) + \tilde{D}(Y - X), \\
\dot{Y} &= F(Y) + \tilde{D}(Z - Y), \\
\dot{Z} &= F(Z) + \tilde{D}(X - Z),
\end{align*} \tag{23}$$

where $X, Y, Z \in \mathbb{R}^3$, $X = (x_1, x_2, x_3)^T$, $F(X) = (f_1(X), f_2(X), f_3(X))^T$, $f_1(X) = a(x_2 - x_1)$, $f_2(X) = cx_1 - x_1x_3 - x_2$, $f_3(X) = x_1x_2 - bx_3$, and $\tilde{D}$ is the linearly coupled coefficient matrix.

In general, for some matrices $\tilde{D}$, the three chaotic oscillators will operate separately and independently in their own chaotic orbits, i.e., they are not synchronizing. However, if there is a suitable coupling structure, then the three identical oscillators will run into the same chaotic orbit and then evolve together simultaneously:

$$X(t) \to Y(t) \to Z(t), \quad \text{as } t \to \infty. \tag{24}$$
Let \( X(t) = \varphi(t) \) be chaotic solution of the single Lorenz system
\[
\dot{X} = F(X),
\]
(25)

Then, \( (\varphi(t), \varphi(t), \varphi(t)) \) is a solution of the three-coupled system (23). The key problem here is whether or not the combined solution of this form is overall asymptotically stable. If it is unstable, then the three chaotic solutions, \( \varphi(t), \varphi(t) \) and \( \varphi(t) \), will not synchronize.

To proceed, we first linearize (23) at \( (\varphi(t), \varphi(t), \varphi(t)) \), to get
\[
\begin{pmatrix}
\dot{\xi} \\
\dot{\eta} \\
\dot{\zeta}
\end{pmatrix} =
\begin{pmatrix}
A(t) - \bar{D} & \bar{D} & 0 \\
0 & A(t) - \bar{D} & \bar{D} \\
\bar{D} & 0 & A(t) - \bar{D}
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta \\
\zeta
\end{pmatrix},
\]
(26)

where
\[
A(t) =
\begin{pmatrix}
-a & a & 0 \\
-c - \varphi_3(t) & -1 & -\varphi_1(t) \\
\varphi_2(t) & \varphi_1(t) & -b
\end{pmatrix}
\quad \text{and} \quad \varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))^T.
\]

According to the analysis given in Section 2, we have a mode decomposition as follows:
\[
\dot{W}_1 = A(t)W_1,
\]
(27)

and
\[
\dot{W}_2 = R(t)W_2 = \begin{pmatrix}
A(t) - \frac{3}{2}\bar{D} & \frac{\sqrt{3}}{2}\bar{D} \\
-\frac{\sqrt{3}}{2}\bar{D} & A(t) - \frac{3}{2}\bar{D}
\end{pmatrix}W_2.
\]
(28)

It is noticed that for any coupled matrix \( \bar{D} \), the variational equation (26) can be decomposed into two independent equations (27) and (28). Furthermore, when the coupled matrix \( \bar{D} \) changes, it only influences Eq. (28), but Eq. (27) is not affected.

Numerical simulation reveals that the synchronized solution \( \Phi(t) \) of the three-coupled system (23) is asymptotically stable for some coupled matrices \( \bar{D} \). Therefore, we focus on the solution of Eq. (28), and look for conditions on the coupled matrix \( \bar{D} \) for the stability of the synchronized solution \( \Phi(t) \).

Since Eq. (27) corresponds to the evolution of the synchronized solution \( \phi(t) \), all its Lyapunov exponents correspond to those of each single oscillator (25), for which the maximum Lyapunov exponent is positive when the oscillator is chaotic. However, all Lyapunov exponents of (28) are the horizontal Lyapunov exponents of the three-coupled system (23). Hence, the synchronized solution is stable if and only if all horizontal Lyapunov exponents are negative.

In the following, we derive a sufficient condition on the coupling matrix for the stability of the synchronized solution of the three-coupled system (23). It is noted that this condition is derived based on rigorous mathematical theory, and does not need to compute the Lyapunov exponents numerically.

First, recall the following inequality [22]:
\[
\left\| W_2(0) \right\|_2 e^{\int_0^t \alpha(s) \, ds} \leq \left\| W_2(t) \right\|_2 \leq \left\| W_2(0) \right\|_2 e^{\int_0^t \beta(s) \, ds}.
\]
(29)

This inequality holds for (28), when \( \alpha(t) \) and \( \beta(t) \) are the minimum and maximum characteristic roots of the symmetric matrix
\[
B(t) = \frac{1}{2}(R(t) + R^T(t)) = \begin{pmatrix}
\frac{1}{2}(A(t) + A^T(t)) - \frac{3}{2}\bar{D} & 0 \\
0 & \frac{1}{2}(A(t) + A^T(t)) - \frac{3}{2}\bar{D}
\end{pmatrix}.
\]

For simplicity, let us consider a simple coupling form, e.g., \( \bar{D} = \text{diag}(d, 0, 0) \) \( (d > 0) \). The characteristic polynomial of \( B(t) \) is
\[
P(\lambda) = (\lambda^3 + p\lambda^2 + q\lambda + r)^2.
\]
(30)
where

\[ p = 1 + a + b + \frac{3}{2}d, \]
\[ q = b + \left( a + \frac{3}{2}d \right)(1 + b) - \frac{1}{4}q^2(t) - \frac{1}{4}(\psi_3(t) - c - a)^2, \]
\[ r = b\left( a + \frac{3}{2}d \right) - \frac{1}{4}q^2(t) - \frac{b}{4}(\psi_3(t) - c - a)^2. \]

According to the Routh–Hurwitz criterion, all characteristic roots of Eq. (30) are negative, i.e., the trivial solution of (28) is stable, if and only if \( p > 0, q > 0, r > 0 \) and \( pq - r > 0 \).

It has been shown that there is a bounded region \( \Gamma \subset \mathbb{R}^3 \), containing the whole attractor, such that all orbits of (22) in it will never leave it, and the closure of the attractor is given by [23]:

\[ \Gamma = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + (z - a - c)^2 = C\}, \]  

where

\[ C = \frac{b^2(a + c)^2}{4(b - 1)}. \]

And, for the solution of (25), we have

\[ \psi_2^2(t) + (\psi_3(t) - a - c)^2 \leq \frac{b^2(a + c)^2}{4(b - 1)}. \]  

Therefore,

\[ r > b\left( a + \frac{3}{2}d \right) - \frac{b}{4}\left[ \psi_2^2(t) + (\psi_3(t) - a - c)^2 \right] = b\left( a + \frac{3d}{2} \right) - \frac{b^3(a + c)^2}{16(b - 1)}. \]

Since

\[ s = pq - r = b(1 + b) + (1 + b)\left( a + \frac{3}{2}d \right)\left( 1 + a + b + \frac{3}{2}d \right) - \frac{1}{4}\left( a + b + \frac{3}{2}d \right)\psi_2^2(t) - \frac{1}{4}\left( 1 + a + \frac{3}{2}d \right)(\psi_3(t) - a - c)^2, \]

we have

\[ s > b(1 + b) + (1 + b)\left( a + \frac{3}{2}d \right)\left( 1 + a + b + \frac{3}{2}d \right) - \frac{b^2(a + c)^2}{16(b - 1)}\left( a + b + \frac{3}{2}d \right). \]

Since \( s > 0 \), we assume that

\[ f\left( a + \frac{3}{2}d \right) = \left( a + \frac{3}{2}d \right)^2 + \left[ (1 + b) - \frac{b^2(a + c)^2}{16(b^2 - 1)} \right]\left( a + \frac{3}{2}d \right) + b - \frac{b^3(a + c)^2}{16(b^2 - 1)} > 0. \]

Thus,

\[ d > -\frac{2}{16}a + \frac{1}{3}\left( -1 - b + \frac{b^2(a + c)^2}{16(b^2 - 1)} + \sqrt{\Delta} \right) = d_1, \]

or

\[ 0 < d < -\frac{2}{16}a + \frac{1}{3}\left( -1 - b + \frac{b^2(a + c)^2}{16(b^2 - 1)} - \sqrt{\Delta} \right) = d_2, \]
where
\[
\Delta = \left[1 + b - \frac{b^2(a + c)^2}{16(b^2 - 1)}\right]^2 - 4b + \frac{b^3(a + c)^2}{4(4b^2 - 1)}.
\]

According to (30), it is clear that \( p > 0 \). If \( r > 0 \) and \( pq - r > 0 \), then \( q = \frac{r + (pq - r)}{p} > 0 \). Furthermore, since \( r > 0 \), we assume that
\[
b\left( a + \frac{3}{2}d \right) - \frac{b^3(a + c)^2}{16(b^2 - 1)} > 0,
\]
that is, \( d > -\frac{2}{3}a + \frac{b^2(a + c)^2}{24(b^2 - 1)} \). Hence, if
\[
d > \max\left( \frac{2}{3}a + \frac{b^2(a + c)^2}{24(b^2 - 1)}, d_1 \right), \tag{33}
\]
or
\[
\max\left( \frac{2}{3}a + \frac{b^2(a + c)^2}{24(b^2 - 1)}, 0 \right) < d < d_2, \tag{34}
\]
then all roots of (30) are negative with a common lower bound, denoted by \(-\varepsilon\) (\( \varepsilon > 0 \)).

The above analysis shows that if the condition (33) or (34) holds, then the synchronized solution of the three-coupled Lorenz system (23) is stable, implying that the three-coupled identical Lorenz system (23) achieves chaos synchronization.

4. Numerical simulations

In this section, numerical simulations are given to verify the mode decomposition method, using the linearly coupled three identical Lorenz systems as an example.

In these numerical simulations, the fourth-order Runge–Kutta method was used to solve the three-coupled Lorenz system (23), with time step size 0.001. The parameters were selected as follows: \( a = 10 \), \( b = \frac{8}{3} \), \( c = 28 \), with initial values \( x_1(0) = 1 \), \( x_2(0) = 0.3 \), \( x_3(0) = 0.5 \), \( y_1(0) = 0 \), \( y_2(0) = 0.5 \), \( y_3(0) = 1 \), \( z_1(0) = -1 \), \( z_2(0) = -0.5 \), \( z_3(0) = -1 \).

Fig. 1 shows the time series of \( x_1 \), \( y_1 \), \( z_1 \) for the coupled system (23). From the figure, we can see that the linearly coupled three identical Lorenz systems are indeed achieving chaos synchronization.

According to the theoretical results of (33) and (34), when \( d > 250.0444 \), the synchronized solution of the coupled system (23) is stable. However, our numerical simulations show that actually when \( d > 7.51 \) the synchronized solution is already stable. This means that the theoretical result is rather too conservative, but this is common in stability analysis, particularly for chaos synchronization.

Remarks:

1. For the three-coupled system (23), when the coupling coefficient \( d \) oversteps a critical value, such as \( d = 7.51 \) according to our numerical simulations, the synchronized solution will be stable.
2. It should be emphasize that applying the mode decomposition method for stability analysis, as suggested above, does not need to compute the horizontal Lyapunov exponents of the coupled system (23) numerically. Instead, the stability can be determined based on rigorous mathematical analysis.
3. Numerical simulations have also shown that the different coupling structures, for instance, using the coupled matrices \( \tilde{D} = \text{diag}(d, 0, 0) \) and \( \tilde{D} = \text{diag}(0, 0, d) \), may lead to different types of stability domains. Generally speaking, the more connections among the three coupled subsystems, the easier the chaos synchronization.
5. Conclusions

In this Letter, we have further investigated the stability of chaos synchronization in a linearly coupled multi-system, particularly a linearly coupling of three identical Lorenz subsystems. We have derived a sufficient condition for the stability of periodic solution of the three coupled identical oscillators, based on a new and efficient method—mode decomposition. The potential of this method for other types of linearly coupled multiple chaotic oscillators will be further investigated elsewhere in the near future.

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References