International Journal of Bifurcation and Chaos, Vol. 12, No. 10 (2002) 2257–2270

LOCAL BIFURCATIONS OF THE CHEN SYSTEM

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Received September 4, 2001; Revised October 5, 2001

This paper introduces a new practical method for distinguishing chaotic, periodic and quasi-periodic orbits based on a new criterion, and apply it to investigate the local bifurcations of the Chen system. Conditions for supercritical and subcritical bifurcations are obtained, with their parameter domains specified. The analytic results are also verified by numerical simulation studies.

Keywords: Chen system; local bifurcation; supercritical; subcritical.

1. Introduction

Chaos has been extensively studied by scientists, physicists and mathematicians for more than three decades. Recently, this study has evolved from the traditional trend of understanding and analyzing chaos to the new intention of controlling and utilizing it, especially within the engineering community [Chen & Dong, 1998; Wang & Chen, 2000; Lü *et al.*, 2002c].

Lorenz found the first canonical chaotic attractor in 1963 [Lorenz, 1963; Stewart, 2000], and Chen found another similar but topologically nonequivalent chaotic attractor in 1999 [Chen & Ueta, 1999; Ueta & Chen, 2000]. It has recently been proven that the Chen system is *dual* to the Lorenz system, in a sense defined by Vaněček and Čelikovský [1996]: The Lorenz system satisfies the condition $a_{12}a_{21} > 0$ while the Chen system satisfies $a_{12}a_{21} < 0$, where a_{ij} are elements of the constant system matrix of their linear parts, $A = [a_{ij}]_{3\times 3}$. Very recently, Lü and Chen found a new chaotic system [Lü & Chen, 2002], which satisfies the condition $a_{12}a_{21} = 0$ and represents the transition between the Lorenz and the Chen attractors [Lü *et al.*, 2002a].

Over the last two years, there are some detailed investigations and studies of the Chen system [Agiza & Yassen, 2001; Čelikovský & Chen, 2002; Chang *et al.*, 2000; Lü & Zhang, 2001;

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Lü *et al.*, 2002b, 2002d; Sanchez *et al.*, 2000; Ueta & Chen, 2000a, 2000b; Wang, 1999; Yang & Yang, 2000; Yu & Xia, 2001; Zhong & Tang, 2002]. It has been found that the Chen attractor has a compound structure by merging together two simple attractors after performing a mirror operation [Lü *et al.*, 2002d]. This paper further studies the local bifurcation of the Chen system. Conditions of supercritical and subcritical bifurcations are derived and their parameter domains are specified. Finally, numerical simulations are carried out, which verify the theoretical results.

2. Dynamical Analysis of the Chen System Using a New Method

In this section, we introduce a new practical method for distinguishing chaotic, periodic and quasi-periodic orbits based on the so-called complementary-cluster energy-barrier criterion (CCEBC) [Xue, 1999], and then apply it to investigate the local bifurcative behaviors of the Chen system.

The Chen system is described by

$$\begin{cases} \dot{x} = a(y - x)\,, \\ \dot{y} = (c - a)x - xz + cy\,, \\ \dot{z} = xy - bz\,, \end{cases} \tag{1}$$

which has a chaotic attractor as shown in Fig. 1 when $a = 35$, $b = 3$, $c = 28$.

Consider the first and second equations of the Chen system:

$$\begin{cases} \dot{x} = a(y - x)\,, \\ \dot{y} = (c - a)x - xz + cy\,, \end{cases} \tag{2}$$

where z is considered as a known function of the time variable t.

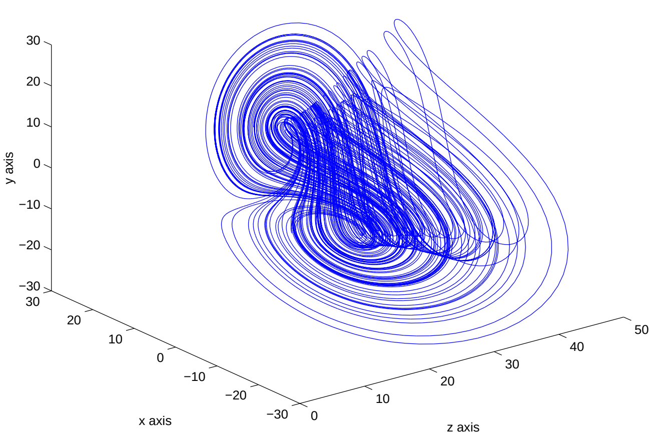


Fig. 1. The Chen chaotic attractor ($a = 35$, $b = 3$, $c = 28$).

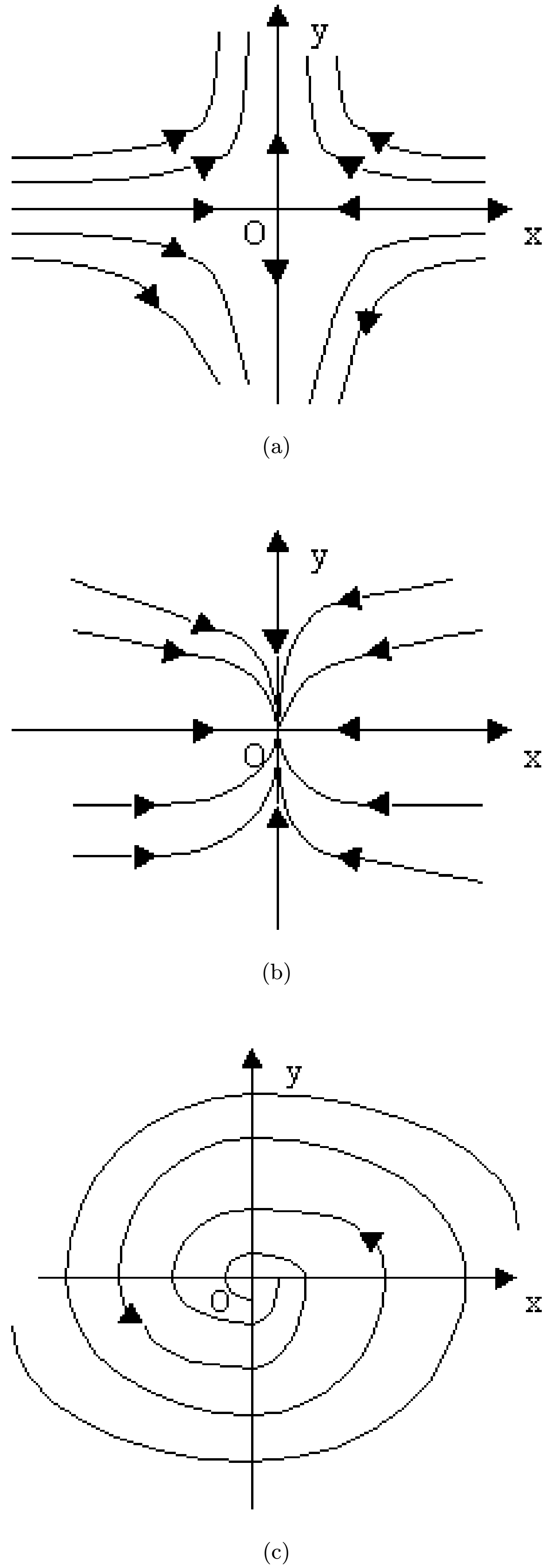


Fig. 2. The phase portrait of system (3). (a) $z < 2c - a$; (b) $2c - a < z < 2c - a + [(a - c)^2/4a]$; (c) $z > 2c - a + [(a - c)^2/4a]$.

When $t = t_0$, system (2) is a two-dimensional linear system with constant coefficients. So its dynamical behavior is very simple and, in fact, is global.

Obviously, for $z \neq 2c-a$, the origin $(0,0)$ is the only equilibrium of system (2). Linearizing this system about its equilibrium $(0,0)$ gives the following characteristic equation:

$$f(\lambda) = \lambda^2 + (a-c)\lambda + a(z - 2c + a) = 0\,. \qquad (3)$$

It can be observed that

(1) For $a > 0$, when $z < 2c - a$, the two eigenvalues satisfy $\lambda_1 > 0 > \lambda_2$, so the only equilibrium $(0,0)$ is a saddle point in the two-dimensional plane. The solution curve in the x–y plane is shown in Fig. 2(a), where the direction of arrow is the direction of the orbit as t increases. When t goes to infinity, only two orbits go to the origin, and the other orbits go to infinity along two different directions.

(2) When $2c-a < z < 2c-a+[(a-c)^2/4a]$, Eq. (3) has two different negative real roots. The only equilibrium $(0,0)$ is a node. The solution curve in the x–y plane is shown in Fig. 2(b), where the direction of arrow is the direction of the orbit as t increases. When t goes to infinity, all but two orbits go to infinity along two different directions.

(3) When $z > 2c-a+[(a-c)^2/4a]$, Eq. (3) has two complex conjugate eigenvalues with a negative real part. The only equilibrium $(0,0)$ is a focus. The solution curve in the x–y plane is shown in Fig. 2(c), where the direction of arrow is the direction of the orbit as t increases. When t goes to infinity, all orbits spiral into the origin.

Now, let us take a closer look at the dynamical behaviors of the Chen system (1). In all the simulations here, for any given initial conditions $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$, the time step size $h = 0.001$, parameter $a = 35$, $b = 3$, $c = 28$, and the time series $x(t)$, $y(t)$, $z(t)$ are generated by the fourth-order Runge–Kutta algorithm. Figure 3

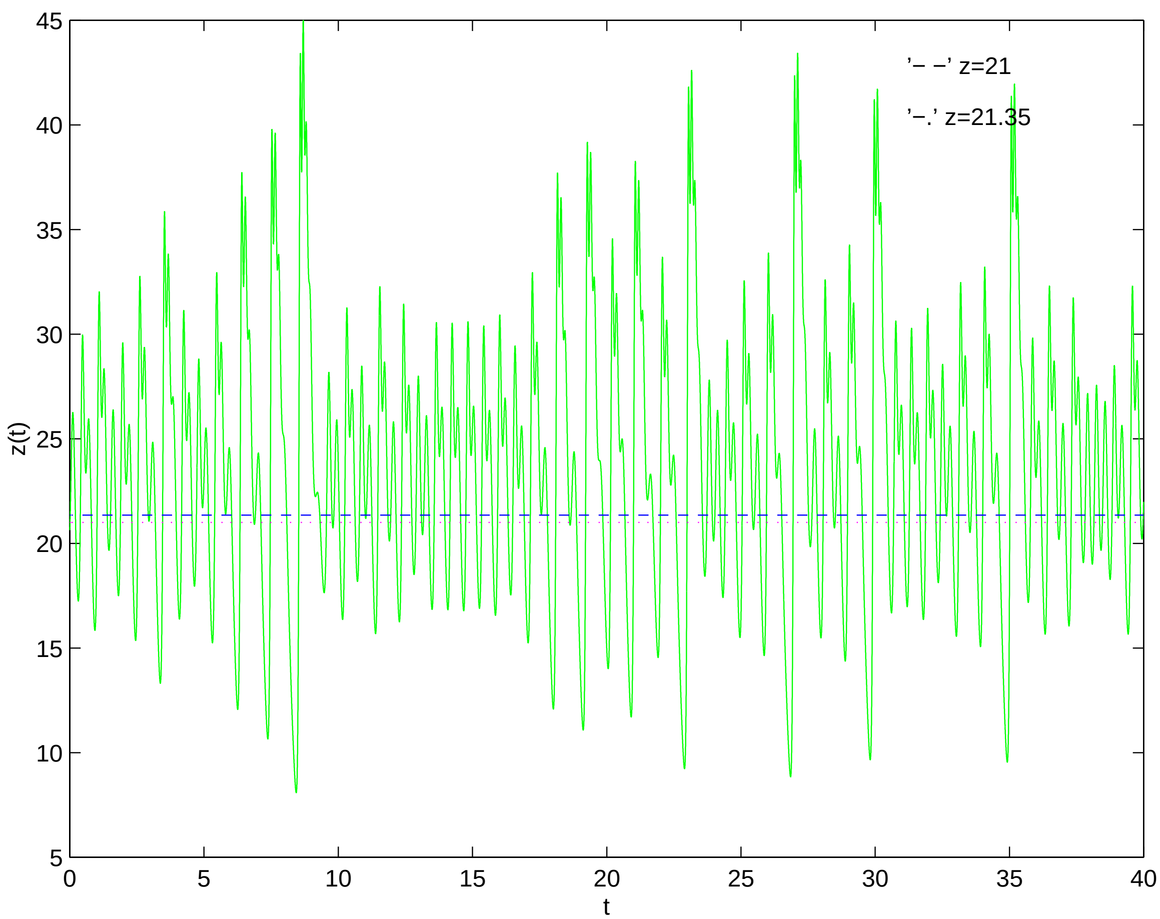


Fig. 3. The chaotic time series of $z(t)$ ($a = 35$, $b = 3$, $c = 28$).

shows the correlation between the time variable t and the function $z(t)$.

It can be seen that when $t \to \infty$, the orbit $z(t)$ goes through the straight lines $z = 2c - a$ and $z = 2c - a + [(a-c)^2/4a]$ alternatively, and repeatedly for many times. The z-axis is partitioned into three disjoint domains: $(-\infty, 2c-a)$, $(2c-a, 2c-a+[(a-c)^2/4a])$ and $(2c-a+[(a-c)^2/4a], +\infty)$, by the two straight lines $z = 2c - a$ and $z = 2c - a + [(a-c)^2/4a]$.

System (2) has different dynamical behaviors in the above three different domains. When $t \to \infty$, system (1) changes the dynamical behavior and $z(t)$ goes through these domains repeatedly, leading to complex dynamics such as the appearance of bifurcations and chaos.

It is notable that system (2) is a time-dependent system when $z(t)$ is varying in time. From Fig. 3, we can see that system (1) (or system (2)) is chaotic, where the function $z(t)$ goes through the straight lines $z = 2c - a$ and $z = 2c - a + [(a-c)^2/4a]$, alternatively. Furthermore, the time-delay when $z(t)$ goes through the straight line $z = 2c - a$ or the straight line $z = 2c - a + [(a-c)^2/4a]$ are not the same and are not distinctly related to one another, but it is different from a periodic orbit. However, the time-delay when $z(t)$ goes through the straight line $z = 2c - a$ itself (or $z = 2c - a + [(a-c)^2/4a]$ itself) is always the same. These can be seen from Fig. 4. For quasi-period, the time-delay when $z(t)$ goes through the straight line $z = 2c - a$ itself (or $z = 2c - a + [(a-c)^2/4a]$ itself) is not the same but distinctly related to one another, that is, there is a multiple correlation to one another.

In summary, we have presented a new practical method for distinguishing chaotic, periodic and quasi-periodic orbits in this section. In this method, we first select a subsystem of the original system, then analyze the dynamical behaviors of this

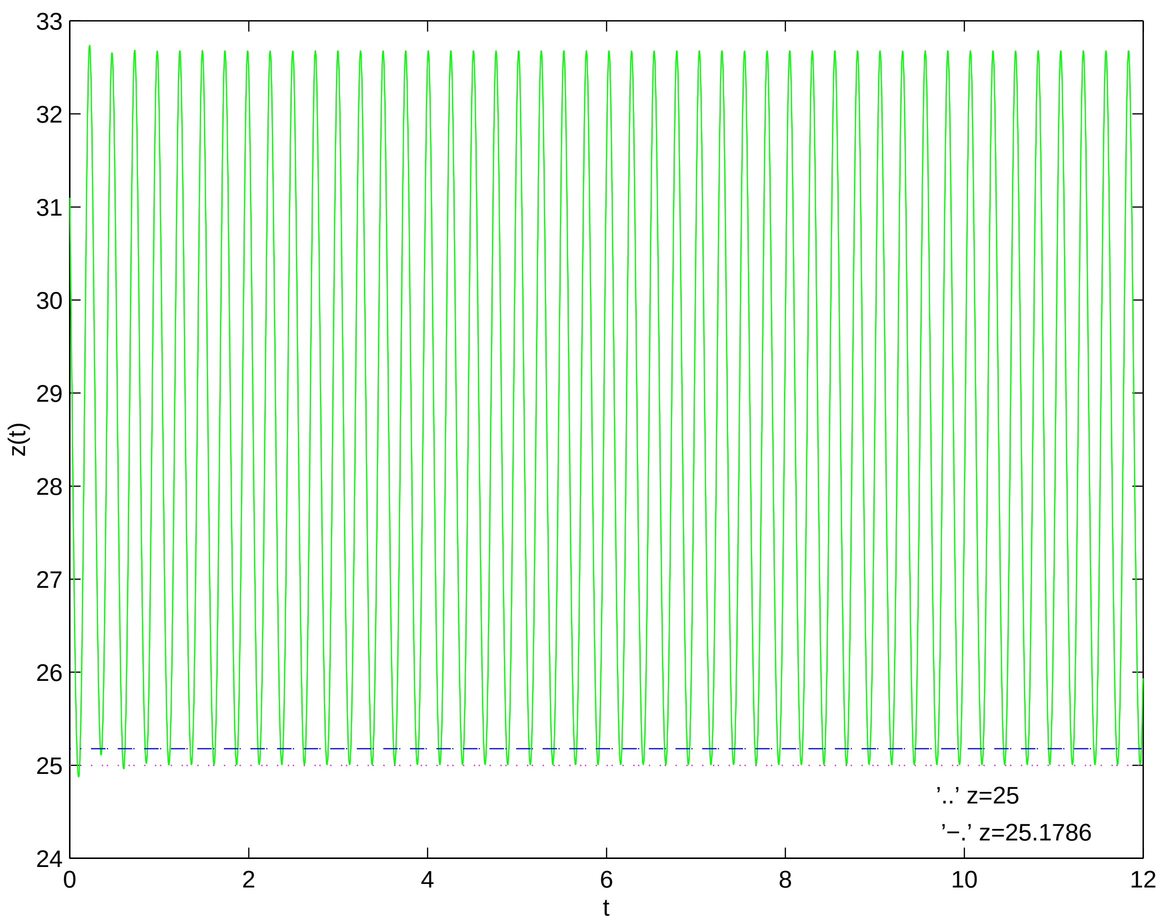


Fig. 4. The periodic time series of $z(t)$ ($a = 35$, $b = 3$, $c = 30$).

subsystem in a lower-dimensional space, and finally analyze the dynamical behaviors of the whole system based on this subsystem.

3. Linear Stability Analysis

Regarding the basic dynamical behaviors of the Chen system (1), studied in [Ueta & Chen, 2000], we have the following observations.

Lemma 1

(i) *If* $a > 2c$, *then system (1) has only one real steady state*, $S_0(0, 0, 0)$;
(ii) *If* $a < 2c$, *then system (1) has three real steady states*: $S_-(-x_0, -y_0, z_0)$, $S_0(0, 0, 0)$, $S_+(x_0, y_0, z_0)$, *where* $x_0 = y_0 = \sqrt{b(2c-a)}$ *and* $z_0 = 2c - a$.

Lemma 2

(i) *If* $a > 2c$, *then the steady state* $S_0(0, 0, 0)$ *of system (1) is asymptotic stable*;
(ii) *If* $a < 2c$, *then the steady state* $S_0(0, 0, 0)$ *of system (1) is a saddle point*;
(iii) *If* $a < 2c$ *and* $(a + b - c)c - 2a(2c - a) > 0$, *then the system steady states* S_- *and* S_+ *are all stable.*

In fact, linearizing system (1) about the equilibria S_- or S_+ yields the following characteristic equation:

$$f(\lambda) = \lambda^3 + (a+b-c)\lambda^2 + bc\lambda + 2ab(2c-a) = 0\,. \quad (4)$$

Obviously, the two equilibria $S_\pm$ have the same stability. Let

$$\begin{aligned} A &= a + b - c\,,\\ B &= bc\,,\\ C &= 2ab(2c-a)\,. \end{aligned} \quad (5)$$

Then the Routh–Hurwitz conditions lead to the conclusion that the real parts of the roots λ are negative if and only if

$$a + b - c > 0\,, \quad 2ab(2c-a) > 0$$

and $(a + b - c)c - 2a(2c - a) > 0$.

Note that the coefficients of the cubic polynomial (4) are all positive. Therefore, $f(\lambda) > 0$ for all $\lambda > 0$. Consequently, there is instability ($\mathrm{Re}(\lambda) > 0$) only if there are two complex conjugate zeros of f. Let these two zeros be $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$ for some real ω. Since the sum of the three zeros of the cubic f is

$$\lambda_1 + \lambda_2 + \lambda_3 = -(a + b - c)\,,$$

we have $\lambda_3 = -(a + b - c)$, which is on the margin of stability, where $\lambda_{1,2} = \pm i\omega$. On this margin,

$$0 = f(-(a + b - c)) = b[c^2 + (3a - b)c - 2a^2]\,,$$

that is,

$$c_{1,2} = \frac{(b - 3a) \pm \sqrt{17a^2 - 6ab + b^2}}{2}\,.$$

Figure 5 shows the maximum Lyapunov exponent of system (1), where parameters a, b are fixed. When $a = 2c$, system (1) has a simple zero eigenvalue while the other two eigenvalues have a nonzero real part. This leads to a pitchfork bifurcation, which happens on the $a = 0$ line of the c–a plane, as shown in Fig. 6.

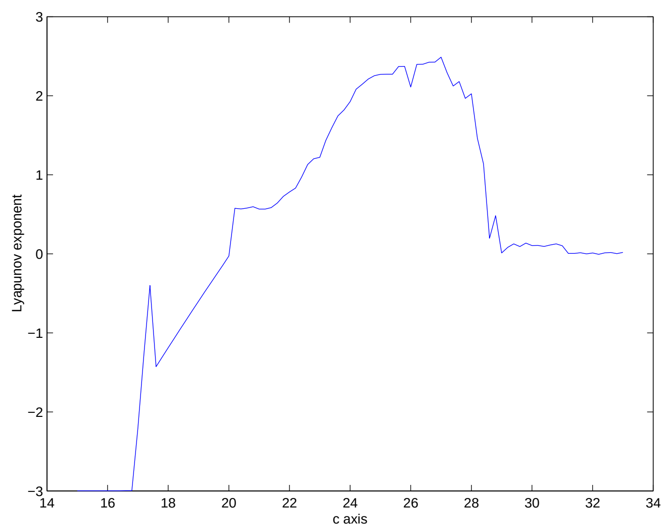


Fig. 5. Maximum Lyapunov exponent of system (1).

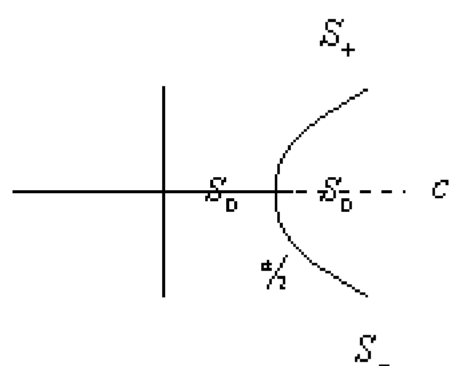


Fig. 6. Sketch of the pitchfork bifurcation diagram.

Hopf bifurcation may appear only at the steady states S_- or S_+. Due to the symmetry of S_- and S_+, it suffices to analyze the stability of S_+. According to Lemma 2, S_+ loses stability when $(a+b-c)c - 2a(2c-a) = 0$, that is,

$$b = \frac{c^2 + 3ac - 2a^2}{c} = b_0 \,. \tag{6}$$

Obviously, if $(a/2) < c < (\sqrt{17}-3)a/2$, then $b_0 < 0$, and Hopf bifurcation does not appear. Hopf bifurcation may appear only under the condition of $c > [(\sqrt{17}-3)a/2]$.

When $b = b_0$, the stability of the steady state S_+ is analyzed by linearizing the system (1) at S_+. Under the linear transform

$$\begin{cases} \xi = x - x_0 \,, \\ \eta = y - y_0 \,, \\ \zeta = z - z_0 \,, \end{cases} \tag{7}$$

system (1) becomes

$$\begin{cases} \dfrac{d\xi}{dt} = a(\eta - \xi) \,, \\ \dfrac{d\eta}{dt} = -c\xi + c\eta - x_0\zeta - \xi\zeta \,, \\ \dfrac{d\zeta}{dt} = x_0\xi + x_0\eta - b\zeta + \xi\eta \,. \end{cases} \tag{8}$$

Its Jacobian matrix $A(S_+)$ is

$$A_+ \equiv A(S_+) = \begin{pmatrix} -a & a & 0 \\ -c & c & -x_0 \\ x_0 & x_0 & -b \end{pmatrix} , \tag{9}$$

and its corresponding characteristic equation is

$$\lambda^3 + (a+b-c)\lambda^2 + bc\lambda + 2ab(2c-a) = 0 \,. \tag{10}$$

Lemma 3. *If $a < 2c$ and $b = b_0$, then the matrix A_+ has three eigenvalues: one negative real and one pair of purely imaginary conjugate, satisfying* $\mathrm{Re}\lambda'(b_0) < 0$.

In fact, letting $b = b_0$, Eq. (10) gives

$$\left(\lambda + \frac{2a(2c-a)}{c}\right)(\lambda^2 + b_0 c) = 0 \,. \tag{11}$$

Obviously, Eq. (11) not only has one negative real root, $\lambda_1 = 2a(a-2c)/c$, but also one pair of purely imaginary conjugate roots, $\lambda_{2,3} = \pm\sqrt{b_0 c}\, i = \pm\sqrt{c^2 + 3ac - 2a^2}\, i = \pm di$.

According to Eq. (10), we have

$$\lambda'(b) = -\frac{\lambda^2 + c\lambda + 2a(2c-a)}{3\lambda^2 + 2(a+b-c)\lambda + bc} \,. \tag{12}$$

Hence,

$$\alpha'(0) = \mathrm{Re}\lambda'(b_0) = \frac{-c^2 d^2}{2[c^2 d^2 + 4a^2(2c-a)^2]} < 0 \,, \tag{13}$$

and

$$\omega'(0) = \mathrm{Im}\lambda'(b_0) = \frac{c^2(c^3 - 2a^3 + 4a^2 c - ac^2)}{2d[c^2 d^2 + 4a^2(2c-a)^2]} \,. \tag{14}$$

Therefore, when $b = b_0$ and $c > (\sqrt{17}-3)a/2$, system (1) has Hopf bifurcation at S_+ and S_-.

4. Supercritical and Subcritical Bifurcations

Assume that

$$(C) : a > 0,\ b > 0,\ c > 0 \,.$$

For the Chen system (1), let the characteristic vectors of λ_1 and λ_2 be α_1 and $\beta = \alpha_2 + \alpha_3 i$, respectively, where α_1, α_2, α_3 are all real vectors.

Through algebraic calculations, we obtain

$$\alpha_1 = \begin{pmatrix} 1 \\ \dfrac{2a-3c}{c} \\ -\dfrac{2x_0}{c} \end{pmatrix} , \quad \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ \dfrac{b_0 c}{a x_0} \end{pmatrix} , \quad \alpha_3 = \begin{pmatrix} 0 \\ \dfrac{\sqrt{b_0 c}}{a} \\ \dfrac{\sqrt{b_0 c}(c-a)}{a x_0} \end{pmatrix} . \tag{15}$$

Next, perform the following transform on system (8):

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ \dfrac{\sqrt{b_0 c}}{a} & 1 & \dfrac{2a-3c}{c} \\ \dfrac{\sqrt{b_0 c}(c-a)}{a x_0} & \dfrac{b_0 c}{a x_0} & -\dfrac{2x_0}{c} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} , \tag{16}$$

so as to obtain

$$\begin{cases} \dfrac{du_1}{dt} = -du_2 + P(u_1,\, u_2,\, u_3) \\ \dfrac{du_2}{dt} = du_1 + Q(u_1,\, u_2,\, u_3) \\ \dfrac{du_3}{dt} = \lambda_1 u_3 + R(u_1,\, u_2,\, u_3)\,, \end{cases} \tag{17}$$

where

$$\begin{aligned} &P(u_1,\, u_2,\, u_3) \\ &\quad = -\frac{a}{d} f(u_1,\, u_2,\, u_3) + \frac{\lambda_1}{d} Q(u_1,\, u_2,\, u_3) \\ &Q(u_1,\, u_2,\, u_3) \\ &\quad = \frac{1}{k}\left[g(u_1,\, u_2,\, u_3) + \frac{c-a}{x_0} f(u_1,\, u_2,\, u_3)\right] \\ &\quad R(u_1,\, u_2,\, u_3) = -Q(u_1,\, u_2,\, u_3)\,, \end{aligned} \tag{18}$$

in which

$$\begin{aligned} &f(u_1,\, u_2,\, u_3) \\ &= (u_2 + u_3)\left(\frac{(c-a)d}{ax_0} u_1 + \frac{d^2}{ax_0} u_2 - \frac{2x_0}{c} u_3\right), \end{aligned} \tag{19}$$

$$\begin{aligned} &g(u_1,\, u_2,\, u_3) \\ &= (u_2 + u_3)\left(\frac{d}{a} u_1 + u_2 + \frac{2a-3c}{c} u_3\right), \end{aligned} \tag{20}$$

$$k = \frac{2ax_0^2 + b_0 c^2 + 2a(c-a)(a-2c)}{acx_0}. \tag{21}$$

Apply the method of Auchmuty and Nicolis [Hassard *et al.*, 1981; Zhang, 1991], from system (17) we can calculate the following quantities at $b = b_0$ and $O(0,\, 0,\, 0)$:

$$\begin{aligned} g_{11} &= \frac{1}{4}\left[\frac{\partial^2 P}{\partial u_1^2} + \frac{\partial^2 P}{\partial u_2^2} + i\left(\frac{\partial^2 Q}{\partial u_1^2} + \frac{\partial^2 Q}{\partial u_2^2}\right)\right] \\ &= \frac{\lambda_1}{2kd} + \frac{d((c-a)\lambda_1 - akx_0)}{2kax_0^2} + i\frac{d^2(c-a) + ax_0^2}{2kax_0^2}\,, \end{aligned} \tag{22}$$

$$\begin{aligned} g_{02} &= \frac{1}{4}\left[\frac{\partial^2 P}{\partial u_1^2} - \frac{\partial^2 P}{\partial u_2^2} - 2\frac{\partial^2 Q}{\partial u_1 \partial u_2} + i\left(\frac{\partial^2 Q}{\partial u_1^2} - \frac{\partial^2 Q}{\partial u_2^2} + 2\frac{\partial^2 P}{\partial u_1 \partial u_2}\right)\right] \\ &= \frac{-\lambda_1}{2kd} - \frac{d((c-a)\lambda_1 - akx_0 + (c-a)^2 + x_0 2)}{2kax_0^2} \\ &\quad + i\left(\frac{\lambda_1}{2ka} + \frac{(c-a)^2\lambda_1 - akx_0(c-a) - d^2(c-a) - ax_0^2}{2kax_0^2}\right)\,, \end{aligned} \tag{23}$$

$$\begin{aligned} g_{20} &= \frac{1}{4}\left[\frac{\partial^2 P}{\partial u_1^2} - \frac{\partial^2 P}{\partial u_2^2} + 2\frac{\partial^2 Q}{\partial u_1 \partial u_2} + i\left(\frac{\partial^2 Q}{\partial u_1^2} - \frac{\partial^2 Q}{\partial u_2^2} - 2\frac{\partial^2 P}{\partial u_1 \partial u_2}\right)\right] \\ &= \frac{-\lambda_1}{2kd} - \frac{d((c-a)\lambda_1 - akx_0 + (c-a)^2 + x_0 2)}{2kax_0^2} \\ &\quad + i\left(\frac{\lambda_1}{2ka} + \frac{(c-a)^2\lambda_1 - akx_0(c-a) - d^2(c-a) - ax_0^2}{2kax_0^2}\right)\,, \end{aligned} \tag{24}$$

$$\begin{aligned} G_{21} &= \frac{1}{8}\left[\frac{\partial^3 P}{\partial u_1^3} + \frac{\partial^3 P}{\partial u_1 \partial^2 u_2} + \frac{\partial^3 Q}{\partial u_1^2 \partial u_2} + \frac{\partial^3 Q}{\partial u_2^3} + i\left(\frac{\partial^3 Q}{\partial u_1^3} + \frac{\partial^3 Q}{\partial u_1 \partial^2 u_2} - \frac{\partial^3 P}{\partial u_1^2 \partial u_2} - \frac{\partial^3 P}{\partial u_2^3}\right)\right] \\ &= 0\,. \end{aligned} \tag{25}$$

Since $n = 3 > 2$, we obtain via calculations the following quantities:

$$h_{11} = \frac{1}{4}\left[\frac{\partial^2 R}{\partial u_1^2} + \frac{\partial^2 R}{\partial u_2^2}\right] = -\frac{d^2(c-a) + ax_0^2}{2kax_0^2}\,, \tag{26}$$

$$\begin{aligned} h_{20} &= \frac{1}{4}\left[\frac{\partial^2 R}{\partial u_1^2} - \frac{\partial^2 R}{\partial u_2^2} - 2i\frac{\partial^2 R}{\partial u_1 \partial u_2}\right] \\ &= \frac{d^2(c-a) + ax_0^2}{2kax_0^2} + di\frac{(c-a)^2 + x_0^2}{2kax_0^2}\,. \end{aligned} \tag{27}$$

Then we obtain the following equations:

$$\begin{cases} \lambda_1 w_{11} = -h_{11}\,, \\ (\lambda_1 - 2di)w_{20} = -h_{20}\,. \end{cases} \tag{28}$$

The solution of the above system of equations is

$$\begin{cases} w_{11} = \dfrac{d^2(c-a) + ax_0^2}{2kax_0^2\lambda_1}\,, \\ w_{20} = -\dfrac{\lambda_1(c-a)d^2 - 2d^2(c-a)^2 + x_0^2(a\lambda_1 - 2d^2) + i(\lambda_1 d(c-a)^2 + 2d^3(c-a) + x_0^2(d\lambda_1 + 2ad))}{(\lambda_1^2 + 4d^2)2kax_0^2}\,. \end{cases} \tag{29}$$

Now, let

$$\begin{aligned} G_{110} &= \frac{1}{2}\left[\frac{\partial^2 P}{\partial u_1 \partial u_3} + \frac{\partial^2 Q}{\partial u_2 \partial u_3} + i\left(\frac{\partial^2 Q}{\partial u_1 \partial u_3} - \frac{\partial^2 P}{\partial u_2 \partial u_3}\right)\right] \\ &= \frac{\lambda_1}{2ka} + \frac{(c-a)(cd^2 - 4ax_0^2 + c\lambda_1(c-a) - ackx_0)}{2kacx_0^2} \\ &\quad + i\frac{cd^2\left[(c-a)^2 + x_0^2 + akx_0 - (c-a)\lambda_1\right] - 4a(a-c)x_0^2\lambda_1 - 2ka^2x_0^3}{2kacdx_0^2}\,, \end{aligned} \tag{30}$$

$$\begin{aligned} G_{101} &= \frac{1}{2}\left[\frac{\partial^2 P}{\partial u_1 \partial u_3} - \frac{\partial^2 Q}{\partial u_2 \partial u_3} + i\left(\frac{\partial^2 Q}{\partial u_1 \partial u_3} + \frac{\partial^2 P}{\partial u_2 \partial u_3}\right)\right] \\ &= \frac{\lambda_1}{2ka} + \frac{(c-a)(-cd^2 + 4ax_0^2 + c\lambda_1(c-a) - ackx_0)}{2kacx_0^2} \\ &\quad + i\frac{cd^2[(c-a)^2 + x_0^2 - akx_0 + (c-a)\lambda_1] + 4a(a-c)x_0^2\lambda_1 + 2ka^2x_0^3}{2kacdx_0^2}\,. \end{aligned} \tag{31}$$

Then we have

$$
\begin{aligned}
g_{21} &= G_{21} + (2G_{110}w_{11} + G_{101}w_{20}) \\
&= \frac{d^2(c-a)+ax_0^2}{2k^2a^2x_0^2} + \frac{cd^2-4ax_0^2+c(c-a)\lambda_1-akcx_0}{2ck^2a^2x_0^4\lambda_1}[d^2(c-a)^2+a(c-a)x_0^2] \\
&\quad - \frac{\lambda_1(c-a)d^2-2d^2(c-a)^2+x_0^2(a\lambda_1-2d^2)}{(\lambda_1^2+4d^2)2kax_0^2}\left[\frac{\lambda_1}{2ka}+(c-a)\frac{c(c-a)\lambda_1-akcx_0-cd^2+4ax_0^2}{2kacx_0^2}\right] \\
&\quad + \frac{\lambda_1 d(c-a)^2+2d^3(c-a)+x_0^2(d\lambda_1+2ad)}{(\lambda_1^2+4d^2)2kax_0^2} \\
&\quad \times \frac{cd^2[(c-a)^2+x_0^2-akx_0+(c-a)\lambda_1]+4a(a-c)x_0^2\lambda_1+2ka^2x_0^3}{2kacdx_0^2} \\
&\quad + i\left(\frac{cd^2[(c-a)^2+x_0^2+akx_0-(c-a)\lambda_1]-4a(a-c)x_0^2\lambda_1-2ka^2x_0^3}{2k^2a^2cdx_0^4\lambda_1}[d^2(c-a)+ax_0^2]\right. \\
&\quad - \frac{\lambda_1(c-a)d^2-2d^2(c-a)^2+x_0^2(a\lambda_1-2d^2)}{(\lambda_1^2+4d^2)2kax_0^2} \\
&\quad \times \frac{cd^2[(c-a)^2+x_0^2-akx_0+(c-a)\lambda_1]+4a(a-c)x_0^2\lambda_1+2ka^2x_0^3}{2kacdx_0^2} \\
&\quad - \frac{\lambda_1 d(c-a)^2+2d^3(c-a)+x_0^2(d\lambda_1+2ad)}{(\lambda_1^2+4d^2)2kax_0^2} \\
&\quad \left.\times\left[\frac{\lambda_1}{2ka}+(c-a)\frac{c(c-a)\lambda_1-akcx_0-cd^2+4ax_0^2}{2kacx_0^2}\right]\right). \qquad (32)
\end{aligned}
$$

Also, let

$$
C_1(0) = \frac{1}{2d}\left[g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right] + \frac{g_{21}}{2}. \qquad (33)
$$

Then we have

$$
\begin{aligned}
\mu_2 &= -\frac{\operatorname{Re} C_1(0)}{\alpha'(0)}, \\
\tau_2 &= -\frac{\operatorname{Im} C_1(0) + \mu_2\omega'(0)}{d}, \qquad (34) \\
\beta_2 &= 2\operatorname{Re} C_1(0).
\end{aligned}
$$

By some tedious manipulations, we obtain

$$
A_1 = -\frac{ac(4a-5c)(6a^4-22a^3c+16a^2c^2+7ac^3+c^4)}{4(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}, \qquad (35)
$$

$$
A_2 = -\frac{c^2(-c^2-ac+a^2)(3a^3-4a^2c-3ac^2-c^3)}{2(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}, \qquad (36)
$$

$$
A_3 = \frac{c^3(-c^2-3ac+2a^2)(-c^2-ac+a^2)}{2(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2} \qquad (37)
$$

$$A_4 = \frac{3c^3(a-c)(-c^2-ac+a^2)(-c^2-3ac+2a^2)}{2(a-2c)(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{38}$$

$$A_5 = \frac{(-c^2-ac+a^2)(-a^3c-5a^2c^2+5ac^3+3c^4+a^4)}{4c^4(a-2c)(-c^2-3ac+2a^2)(a^4-4a^3c+2a^2c^2+c^4+3ac^3)(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{39}$$

$$A_6 = \frac{c(-c^6-14c^4a^2-c^3a^3+28c^2a^4-20a^5c+4a^6)(-13a^2c^3+24a^3c^2-12a^4c-6ac^4+c^5+2a^5)}{4(a-2c)(a^4-4a^3c+2a^2c^2+c^4+3ac^3)(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{40}$$

$$\mu_2 = \frac{c^2d^2+4a^2(2c-a)^2}{c^2d^2}\sum_{i=1}^{5} A_i = \frac{14a^2c^2-16a^3c+4a^4+3ac^3+c^4}{c(c^2+3ac-2a^2)}\sum_{i=1}^{5} A_i\,, \tag{41}$$

$$\beta_2 = \sum_{i=1}^{5} A_i\,, \tag{42}$$

$$B_1 = -\frac{c(6a^4-22a^3c+16a^2c^2+7ac^3+c^4)^2}{4(a-2c)\sqrt{c^2+3ac-2a^2}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{43}$$

$$B_2 = \frac{c^3(-c^2-3ac+2a^2)(-c^2-ac+a^2)^2}{4(a-2c)\sqrt{c^2+3ac-2a^2}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2} \tag{44}$$

$$B_3 = -\frac{c(3a^4-12a^3c+11a^2c^2+2ac^3-c^4)^2}{6(a-2c)\sqrt{c^2+3ac-2a^2}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{45}$$

$$B_4 = \frac{a^2c^2(4a-5c)^2(-c^3+2a^3-4a^2c+ac^2)^2(c^2+3ac-2a^2)^{\frac{7}{2}}(6a^4-22a^3c+16a^2c^2+7ac^3+c^4)^2}{384(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^4}\,, \tag{46}$$

$$B_5 = -\frac{c(3a^3-4a^2c-3ac^2-c^3)(6a^4-22a^3c+16a^2c^2+7ac^3+c^4)}{4\sqrt{c^2+3ac-2a^2}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{47}$$

$$B_6 = -\frac{ac^2(4a-5c)(-c^2-3ac+2a^2)(-c^2-ac+a^2)}{8\sqrt{c^2+3ac-2a^2}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{48}$$

$$B_7 = -\frac{ac(4a-5c)(-c^3+2a^3-4a^2c+ac^2)(6a^4-22a^3c+16a^2c^2+7ac^3+c^4)}{8(c^2+3ac-2a^2)^{\frac{3}{2}}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{49}$$

$$B_8 = -\frac{c^2(-c^2-ac+a^2)(-c^3+2a^3-4a^2c+ac^2)(3a^3-4a^2c-3ac^2-c^3)}{4(c^2+3ac-2a^2)^{\frac{3}{2}}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{50}$$

$$B_9 = -\frac{c(-c^2-ac+a^2)(-c^2-3ac+2a^2)(4a^5-12a^4c+4a^3c^2+9a^2c^3-2ac^4-c^5)}{4a(a-2c)\sqrt{c^2+3ac-2a^2}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{51}$$

$$B_{10} = \frac{c(-c^2-3ac+2a^2)(-a^3c^3-14a^2c^4-c^6+28a^4c^2-20ca^5+4a^6)(-a^3c-5a^2c^2+5ac^3+3c^4+a^4)}{8(a^4-4a^3c+2a^2c^2+c^4+3ac^3)(a-2c)\sqrt{c^2+3ac-2a^2}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{52}$$

$$B_{11} = -\frac{c^4\sqrt{c^2+3ac-2a^2}(a-c)^2(-c^2-ac+a^2)(c^3-4a^2c+2a^3)}{8(a^4-4a^3c+2a^2c^2+c^4+3ac^3)(a-2c)(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{53}$$

$$B_{12} = -\frac{c^3(-c^2-ac+a^2)(-c^3+ac^2-4a^2c+2a^3)}{4\sqrt{c^2+3ac-2a^2}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{54}$$

$$B_{13} = -\frac{3(a-c)c^3(-c^2-ac+a^2)(-c^3+ac^2-4a^2c+2a^3)}{4(a-2c)\sqrt{c^2+3ac-2a^2}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{55}$$

$$B_{14} = -\frac{c^4(-c^2-ac+a^2)(a^4-a^3c-5a^2c^2+3c^4+5ac^3)(-c^3+ac^2-4a^2c+2a^3)}{8(a-2c)(a^4-4a^3c+2a^2c^2+c^4+3ac^3)\sqrt{c^2+3ac-2a^2}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2}\,, \tag{56}$$

$$\begin{aligned} B_{15} = &-\frac{c(-c^3+ac^2-4a^2c+2a^3)(-12a^4c+24a^3c^2-13a^2c^3-6ac^4+c^5+2a^5)}{8(a-2c)(-c^2+2a^2-3ac)\sqrt{c^2+3ac-2a^2}(14a^2c^2-16a^3c+4a^4+3ac^3+c^4)^2} \\ &\times \frac{-c^3a^3-14c^4a^2-c^6+28c^2a^4-20a^5c+4a^6}{a^4-4a^3c+2a^2c^2+c^4+3ac^3}\,, \end{aligned} \tag{57}$$

$$\tau_2 = \frac{1}{\sqrt{c^2+3ac-2a^2}}\sum_{i=1}^{15} B_i\,. \tag{58}$$

Theorem 1. *If $c > (\sqrt{17}-3)a/2$, then there is a Hopf bifurcation at $b = b_0$. Moreover,*

(1) *if $\mu_2(a, c) > 0$, then the direction of bifurcation is $b > b_0$ and the bifurcation is subcritical;*
(2) *if $\mu_2(a, c) < 0$, then the direction of bifurcation is $b < b_0$ and bifurcation is supercritical.*

Proof. According to Lemma 3, the stability of S_+ is changed from one side to other of b_0. Hence, there is a Hopf bifurcation at $b = b_0$. When $\mu_2 > 0$, since $\alpha'(0) < 0$, the direction of bifurcation is $b > b_0$. At the same time, the signs of μ_2 and β_2 are the same according to (41), that is, $\beta_2 > 0$. And the bifurcating periodic solutions are unstable. Figure 7(a) shows the subcritical bifurcation. On the contrary, when $\mu_2 < 0$, a bifurcation takes place when $b < b_0$. And the bifurcating periodic solutions are asymptotically orbitally stable. Figure 7(b) displays the supercritical bifurcation. The proof is thus completed. ■

An approximation to the periodic solutions has the following expression:

$$X = \begin{pmatrix} \sqrt{\dfrac{(2c-a)(c^2+3ac-2a^2)}{c}} \\ \sqrt{\dfrac{(2c-a)(c^2+3ac-2a^2)}{c}} \\ 2c-a \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ \dfrac{\sqrt{c^2+3ac-2a^2}}{a} & 1 & \dfrac{2a-3c}{c} \\ \dfrac{c(c-a)}{a\sqrt{c(2c-a)}} & \dfrac{c\sqrt{c^2+3ac-2a^2}}{a\sqrt{c(2c-a)}} & -\dfrac{2}{c}\sqrt{\dfrac{(2c-a)(c^2+3ac-2a^2)}{c}} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{59}$$

where

$$u_1 = \mathrm{Re}\, z\,, \quad u_2 = \mathrm{Im}\, z\,,$$
$$u_3 = w_{11}|z|^2 + \mathrm{Re}(w_{20}z^2) + O(|z|^3)\,,$$

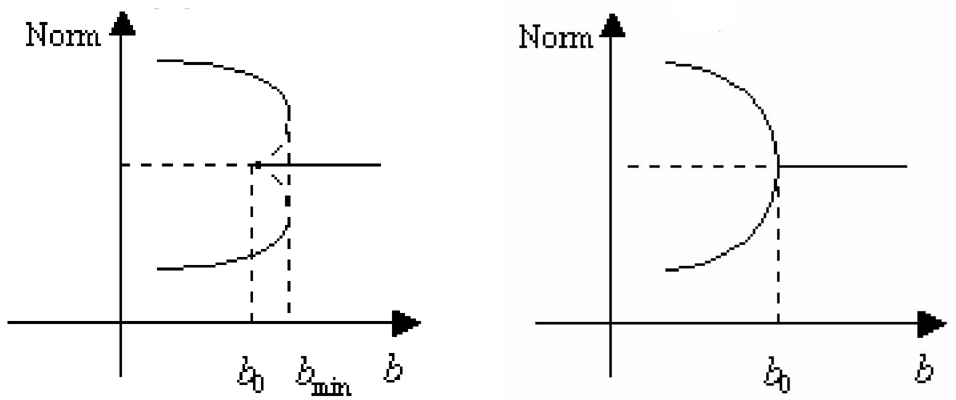
$$\begin{aligned} z = \varepsilon e^{2it\pi/T} + \frac{i\varepsilon^2}{6d}[g_{02}e^{-4it\pi/T} \\ - 3g_{20}e^{4it\pi/T} + 6g_{11}] + O(\varepsilon^3)\,, \end{aligned}$$

in which

$$\varepsilon^2 = \frac{b-b_0}{\mu_2} + O((b-b_0)^2)\,.$$

Its period and characteristic exponent are

$$\begin{aligned} T &= \frac{2\pi}{d}(1+\tau_2\varepsilon^2 + O(\varepsilon^4))\,, \\ \beta &= \beta_2\varepsilon^2 + O(\varepsilon^4)\,. \end{aligned} \tag{60}$$



(a) (b)

Fig. 7. Bifurcation graphs. (a) Subcritical bifurcation; (b) supercritical bifurcation.

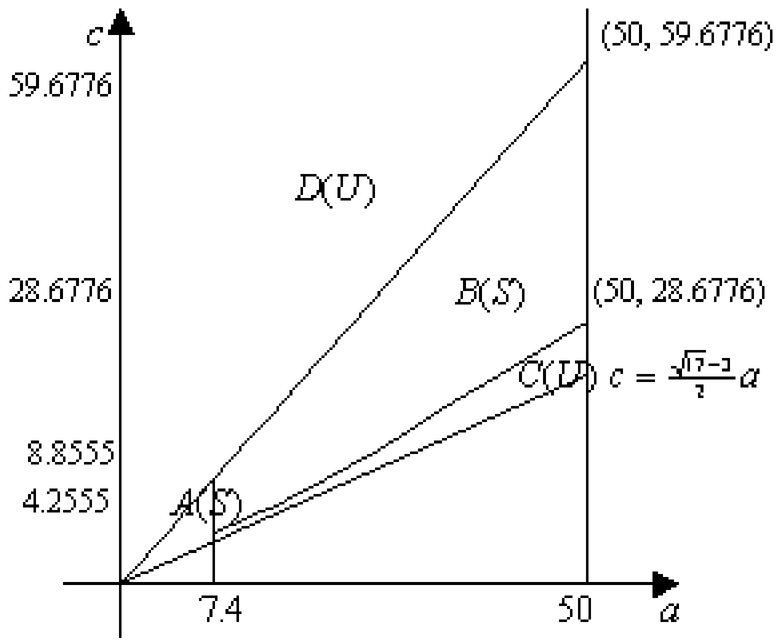


Fig. 8. Parameter domains of subcritical and supercritical bifurcations.

5. Numerical Simulations

We have carried out a large number of numerical experiments to verify the theoretical analysis and analytic conditions. The results are quite accurate and consistent.

For simplicity, we only consider the parameter domain $\{(a, c)|0 < a \leq 50,\ c > 0\}$ here. According to (41), we can draw a graph of the parameter domain of supercritical and subcritical bifurcations. In Fig. 8, for any point in domains A and B, $\mu_2 < 0$, that is, there exists supercritical bifurcation in both domains A and B. In this figure, for any point in domains C and D, $\mu_2 > 0$, so there exists subcritical bifurcation in both domains C and D. In the figure, the stability of periodic solutions are denoted by S (stable) and U (unstable), respectively. Main parts of detailed experimental results are summarized in Table 1.

It can be seen that when $c/a \to \infty$, $\mu_2 \to 0$.

For $a = 35$, $c = 28$, the periodic solutions are

$$\beta_2 = -0.004,\ \mu_2 = -0.0128,\ \tau_2 = 7.3822 \times 10^{-4}$$

and

$$g_{11} = -0.7221 + 2.4139 \times 10^{-6} i\,,$$
$$g_{02} = 0.5782 - 0.1969 i\,,$$
$$g_{20} = 0.866\,, \quad w_{11} = -0.0019\,,$$
$$w_{20} = -3.7453 \times 10^{5} - 1.2813 \times 10^{-6} i$$

Table 1. The numerical experimental results.

μ_2	Domain	c	a
-0.0222	B	5	5
0.0177	D	10	5
4.9993×10^{-6}	D	10^5	5
0.0258	C	20	35
-0.0128	A	28	35
7.6612×10^{-5}	D	42	35
4.9952×10^{-6}	D	10^5	35

therefore,

$$T = 0.176 + 0.0102(45.5 - b) + O((45.5 - b)^2)$$

and it shows that the period will increase with $b \to 45.5$. Finally, the periodic solutions are obtained as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 30.9112 \\ 30.9112 \\ 21 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1.0198 & 1 & -0.5 \\ -0.2309 & 1.1776 & -2.2079 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

where

$$u_1 = \operatorname{Re} z, \quad u_2 = \operatorname{Im} z\,,$$

$$u_3 = -0.0019|z|^2 + \operatorname{Re}((-3.7453 \times 10^5 - 1.2813 \times 10^{-6} i)z^2)$$

$$z = 8.8388\sqrt{45.5 - b}\, e^{2i(t+\phi)\pi/T} + 0.3672(45.5 - b) \times [(0.5782 - 0.1969i)e^{-4i(t+\phi)\pi/T} - 2.598e^{4i(t+\phi)\pi/T} - 4.3326 + 0.0145i]\,,$$

in which ϕ is a phase angle.

6. Conclusions and Discussion

This paper has thoroughly investigated the local bifurcations of the Chen system. There are abundant and complex dynamical behaviors still unknown about this new and interesting system, which will contribute to a better understanding of a whole family of similar and closely related chaotic systems. The intrinsic dynamics of the Chen system deserves further investigation in the near future.

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