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Synchronization of a unified chaotic system and the application in secure communication ☆

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Abstract

This Letter further investigates the synchronization of a unified chaotic system via different methods. Several sufficient theorems for the synchronization of the unified chaotic system are deduced. A scheme of secure communication based on the synchronization of the unified chaotic system is presented. Numerical simulation shows its feasibility.

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Keywords: Synchronization; Unified chaotic system; Secure communication

1. Introduction

In 1990, Pecora and Carroll [1,2] produced the idea for chaos synchronization and verified by the electric current later. Over the last decade, chaos synchronization has received increasing attention due to its potential application in many areas such as secure communication, information processing, biological systems, and chemical reactions [3–6].

Recently, a large variety of approaches have been proposed for chaos synchronization such as the feedback method [7], backstepping design technique [8], adaptive control approach [9], and invariant manifold means [4], among many others [4,5].

This Letter further studies the chaos synchronization of a unified chaotic system [10] via different methods. Several sufficient conditions for chaos synchronization of the unified chaotic system are gained from theory. A scheme of secure communication based on chaos synchronization of the unified chaotic system is produced. Numerical simulations show that it has wide application in secure communication.

2. Synchronization between two identical unified chaotic systems

In 1963, Lorenz found the first canonical chaotic attractor [11]. In 1999, Chen found another chaotic

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attractor [12], which is the dual to the Lorenz system. Here, the duality is in the sense defined by Vaněček and Čelikovský [13]: for the linear part of the system, $A = [a_{ij}]_{3\times 3}$, the Lorenz system satisfies the condition $a_{12}a_{21} > 0$ while Chen system satisfies $a_{12}a_{21} < 0$. In 2001, Lü found a new chaotic system [14], which satisfies the condition $a_{12}a_{21} = 0$. Very recently, Lü et al. produced a unified chaotic system [10,15] which contains the Lorenz and Chen systems as two extremes and the Lü system as a special case. The unified chaotic system is described by

$$\begin{cases} \dot{x} = (25\alpha + 10)(y - x), \\ \dot{y} = (28 - 35\alpha)x - xz + (29\alpha - 1)y, \\ \dot{z} = xy - \frac{\alpha+8}{3}z, \end{cases} \tag{1}$$

where $\alpha \in [0, 1]$. Obviously, the system (1) is the original Lorenz system [11] for $\alpha = 0$ while system (1) belongs to the original Chen system [12] for $\alpha = 1$. When $\alpha = 4/5$, the system (1) is the critical system [14]. In fact, system (1) bridges the gap between the Lorenz system and Chen system. Especially, system (1) is always chaotic for the whole interval $\alpha \in [0, 1]$. Furthermore, system (1) plays a key role in the study of the generalized Lorenz family [10].

Let the system (1) be the drive system and x be the drive variable, then the response system can be written as follows:

$$\begin{cases} \dot{u} = (25\alpha + 10)(v - u), \\ \dot{v} = (28 - 35\alpha)x - xw + (29\alpha - 1)v, \\ \dot{w} = xv - \frac{\alpha+8}{3}w. \end{cases} \tag{2}$$

Theorem 1. *When* $0 \leqslant \alpha < 1/29$, *then the response system* (2) *can synchronize the drive system* (1).

Proof. Let the state tracking error be $e = (e_1, e_2, e_3)^{\mathrm{T}} = (u - x, v - y, w - z)^{\mathrm{T}}$, then the error dynamics is

$$\begin{cases} \dot{e}_1 = (25\alpha + 10)(e_2 - e_1), \\ \dot{e}_2 = -xe_3 + (29\alpha - 1)e_2, \\ \dot{e}_3 = xe_2 - \frac{8+\alpha}{3}e_3. \end{cases} \tag{3}$$

Now the synchronization of two identical unified chaotic systems is equivalent to the asymptotic stability of the zero solution for the error system (3).

Construct a Lyapunov function of the form

$$E(e_1, e_2, e_3) = \frac{1}{2}\left(\frac{1}{c}e_1^2 + e_2^2 + e_3^2\right), \tag{4}$$

where c is a nonzero constant. Then its derivative along the solution of (3) is

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{c}e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 \\ &= \frac{1}{c}e_1(25\alpha + 10)(e_2 - e_1) \\ &\quad + e_2\big[-xe_3 + (29\alpha - 1)e_2\big] \\ &\quad + e_3\left(xe_2 - \frac{8+\alpha}{3}e_3\right). \end{aligned} \tag{5}$$

Letting $c = \frac{25\alpha+10}{4(1-29\alpha)}$, if $\alpha < 1/29$, then we have

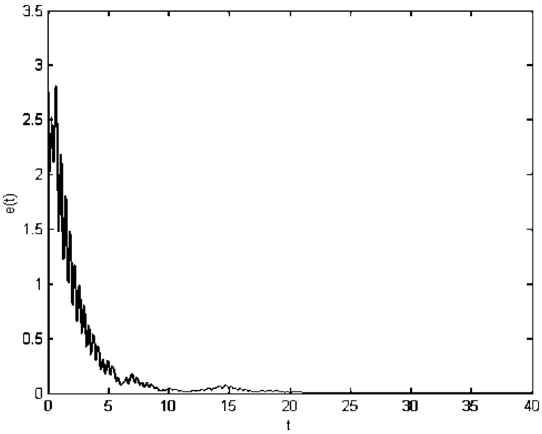
$$\frac{dE}{dt} = -(1 - 29\alpha)(2e_1 - e_2)^2 - \frac{8+\alpha}{3}e_3^2 \leqslant 0. \tag{6}$$

If $e_1 = \frac{1}{2}e_2$ and $e_3 = 0$, then $e_1 = e_2 = e_3 = 0$ from Eq. (3). That is, $\frac{dE}{dt} = 0$ implies $e_1 = e_2 = e_3 = 0$. Furthermore, error dynamical system (3) is global stability according to Lyapunov second theorem.

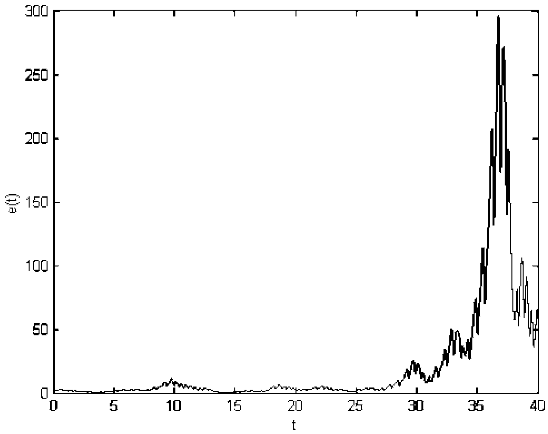
Therefore, $e \to 0$ as $t \to \infty$. Thus the systems (2) and (1) can realize synchronization when $0 \leqslant \alpha < 1/29$. □

It is noticed that the parameter interval for the synchronization is wide and contains the Lorenz system as a special case. However, the numerical simulation shows that the response system (2) can synchronize the drive system (1) for $0 \leqslant \alpha \leqslant 1/11$. That is, the condition of synchronization for Theorem 1 is sufficient, but not necessary. In fact, the largest conditional Lyapunov exponent (CLE) of the error system (3) for $\alpha = 1/11$ is -0.15. Fig. 1 shows the tracking errors for $\alpha = 1/11$ and $\alpha = 1/10$, respectively, where $e(t) = \sqrt{e_1^2 + e_2^2 + e_3^2}$. Obviously, the system (2) can synchronize system (1) for $\alpha = 1/11$, but not for $\alpha = 1/10$.

If the parameter α in systems (1) and (2) are not equivalent, then the two systems cannot realize synchronization. That is, the response system cannot synchronize the drive system when the system parameter of the receiver (response) does not match that of the transmitter (drive). Letting $\alpha = 0$ in system (1) and $\alpha = 0.01$ in system (2), Fig. 2 shows the errors graph. Obviously, the two systems are not synchronization.



(a)



(b)

Fig. 1. The tracking error $e(t)$. (a) $\alpha = 1/11$, (b) $\alpha = 1/10$.

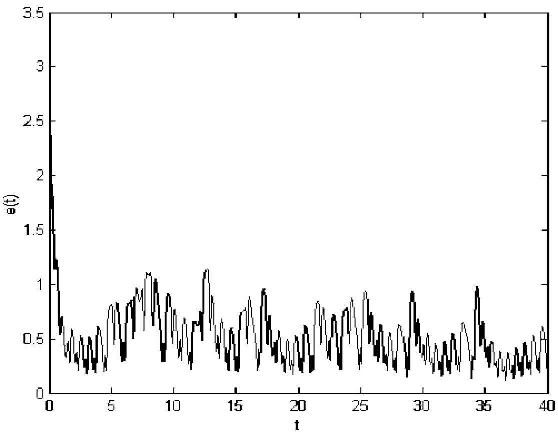


Fig. 2. The tracking error $e(t)$ $(\alpha_1 = 0, \alpha_2 = 0.01)$.

3. Using Rössler system to drive the unified chaotic system

Assume that the drive system is $\dot{x} = f(x)$, and the response $\dot{y} = g(x, y)$, where $x \in R^n$, $y \in R^m$, $f : R^n \to R^n$, $g : R^n \times R^m \to R^m$. If the output $y(t)$ of the response system is determined only by the drive signal $x(t)$, that is $y(t) = \varphi(x)$, then the two systems are called generalized synchronization. An effective method for judging the generalize synchronization is to construct an assistant system $\dot{z} = g(x, z)$ which is identical to the response system, and driving it with the same signal $x(t)$. Then the sufficient and necessary condition for generalized synchronization between $y(t)$ and $x(t)$ is $z(t) = y(t)$.

Consider the following Rössler system

$$\begin{cases} \dot{x}_1 = -x_2 - x_3, \\ \dot{x}_2 = x_1 + 0.15x_2, \\ \dot{x}_3 = 0.2 + x_3(x_1 - 10), \end{cases} \tag{7}$$

and use its output $x_1(t)$ as the drive signal to drive two identical unified chaotic systems

$$\begin{cases} \dot{y}_1 = (25\alpha + 10)(y_2 - y_1), \\ \dot{y}_2 = (28 - 35\alpha)x_1 - x_1 y_3 + (29\alpha - 1)y_2, \\ \dot{y}_3 = x_1 y_2 - \frac{\alpha+8}{3} y_3, \end{cases} \tag{8}$$

and

$$\begin{cases} \dot{z}_1 = (25\alpha + 10)(z_2 - z_1), \\ \dot{z}_2 = (28 - 35\alpha)x_1 - x_1 z_3 + (29\alpha - 1)z_2, \\ \dot{z}_3 = x_1 z_2 - \frac{\alpha+8}{3} z_3. \end{cases} \tag{9}$$

Theorem 2. *When $0 \leqslant \alpha < 1/29$, then we have $y_1(t) - z_1(t) \to 0$, $y_2(t) - z_2(t) \to 0$, $y_3(t) - z_3(t) \to 0$, as $t \to \infty$.*

Proof. Let the tracking error be $e = (e_1, e_2, e_3) = (y_1 - z_1, y_2 - z_2, y_3 - z_3)$, then the error dynamics is

$$\begin{cases} \dot{e}_1 = (25\alpha + 10)(e_2 - e_1), \\ \dot{e}_2 = -x_1 e_3 + (29\alpha - 1)e_2, \\ \dot{e}_3 = x_1 e_2 - \frac{\alpha+8}{3} e_3. \end{cases} \tag{10}$$

Construct a Lyapunov function of the form

$$E(e_1, e_2, e_3) = \frac{1}{2}\left(\frac{2(1 - 29\alpha)}{25\alpha + 10} e_1^2 + e_2^2 + e_3^2\right), \tag{11}$$

then its derivative along the solution of (10) is

$$\frac{dE}{dt} = -2(1-29\alpha)\left(e_1 - \frac{1}{2}e_2\right)^2 - \frac{1-29\alpha}{2}e_2^2 - \frac{8+\alpha}{3}e_3^2. \tag{12}$$

Obviously, $\frac{dE}{dt} = 0$ implies $e_1 = e_2 = e_3 = 0$. Moreover, error dynamical system (10) is global stability according to Lyapunov second theorem.

When $0 \leqslant \alpha < 1/29$, $\frac{dE}{dt} \leqslant 0$. Therefore, $e \to 0$ as $t \to \infty$. That is, the systems (9) and (8) can realize synchronization when $0 \leqslant \alpha < 1/29$. □

It is noticed that the systems (7), (8) and (9) are drive, response and assistant systems, respectively. Consider the systems (7) and (8) as a whole system, then it is a 6D system. When the systems (7) and (8) are generalized synchronization, then there is a function φ satisfying $y(t) = \varphi(x)$. This means that the original 6D system can shrink to a 3D submanifold.

The condition for generalized synchronization in Theorem 2 is sufficient, and not necessary. In fact, numerical simulations show that the conclusion in Theorem 2 is still right for $\alpha \in [0, 1/11]$. Furthermore, the largest CLE of the error system (10) for $\alpha = 1/11$ is -0.28. Fig. 3 shows the evolution of tracking error for $\alpha = 1/11$ using Rössler system and Chen system to drive the unified chaotic system, respectively, where $e(t) = \sqrt{e_1^2 + e_2^2 + e_3^2}$.

4. Synchronization of the unified chaotic system via occasional driving

It is not easy to realize chaos synchronization via continuous control in practical application, since its control cost is very high. Recently, the occasional control technique [16,17] has been investigated. In the following, we further study the chaos synchronization via occasional driving.

Also, we use the output $x(t)$ of the system (1) as the drive variable, and the response system is

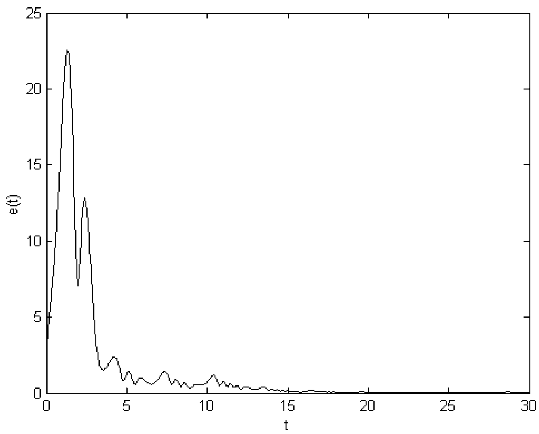
$$\begin{cases} \dot{u}_1 = (25\alpha + 10)(v_1 - u_1), \\ \dot{v}_1 = (28 - 35\alpha)r(t) - r(t)w_1 + (29\alpha - 1)v_1, \\ \dot{w}_1 = r(t)v_1 - \frac{\alpha+8}{3}w_1, \end{cases} \tag{13}$$

where

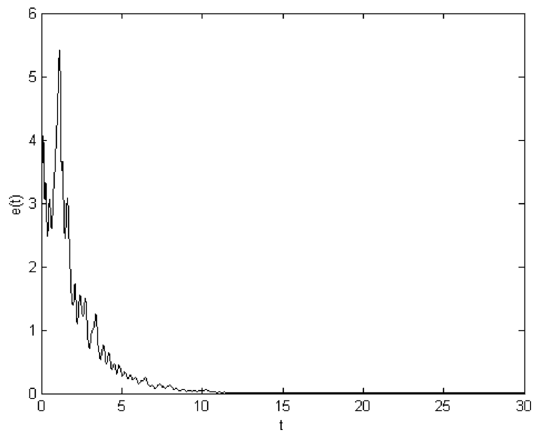
$$r(t) = \begin{cases} x(t), & t \in [nT - \varepsilon, nT + \varepsilon], \\ u_1(t), & \text{otherwise}. \end{cases}$$

Here, T is the interim period and 2ε the sampling interval $(2\varepsilon < T)$.

The theoretical basis of the discontinuous control (occasional drive) for chaos synchronization is that the response system is only driven during the interval $[nT - \varepsilon, nT + \varepsilon]$ for every period T, while runs independently in the rest time. Therefore, the drive need not act on the response continuously, which greatly reduces the control cost. Fig. 4 shows the tracking error, where $T = 0.2$, $\varepsilon = 0.05$, $\alpha = 1/11$, $e(t) = \sqrt{e_1^2 + e_2^2 + e_3^3}$.



(a)



(b)

Fig. 3. The tracking error $e(t)$. (a) Drive system is Rössler system, (b) drive system is the Chen system.

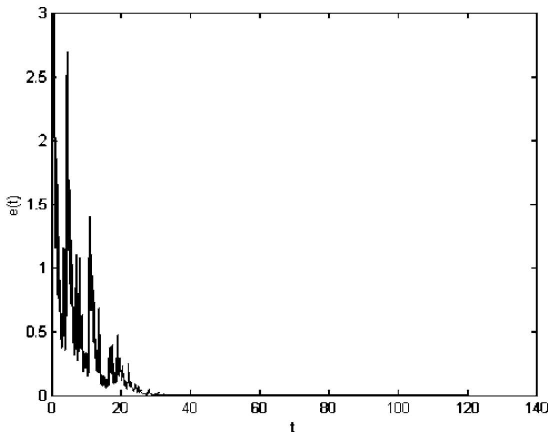


Fig. 4. The tracking error $e(t)$ via occasional drive ($\alpha = 1/11$, $T = 0.2$, $\varepsilon = 0.05$).

Compare the results via continuous control, we find that it takes much time to realize chaos synchronization via occasional drive. That is, the decreasing control cost need sacrifice the synchronization time. However, it is usually worthwhile and necessary in some physical, chemical and biological applications.

5. The application in secure communication

Assume that $m(t)$ is the message signal, adding it to the right of the first equation for the transmitter (drive system), then we have

$$\begin{cases} \dot{x} = (25\alpha + 10)(y - x) + m(t), \\ \dot{y} = (28 - 35\alpha)x - xz + (29\alpha - 1)y, \\ \dot{z} = xy - \frac{\alpha+8}{3}z. \end{cases} \tag{14}$$

Select the output $x(t)$ of the system (14) as the transmitted signal, then construct the receiver as follows:

$$\begin{cases} \dot{u} = (25\alpha + 10)(v - u) + p(t), \\ \dot{v} = (28 - 35\alpha)x - xw + (29\alpha - 1)v, \\ \dot{w} = xv - \frac{\alpha+8}{3}w, \\ \dot{p} = k(x - u), \end{cases} \tag{15}$$

where k is a parameter.

Let the tracking error be $e = (e_1, e_2, e_3, e_4) = (x - u, y - v, z - w, m - p)$, then the error dynamics is

$$\begin{cases} \dot{e}_1 = (25\alpha + 10)(e_2 - e_1) + e_4, \\ \dot{e}_2 = -xe_3 + (29\alpha - 1)e_2, \\ \dot{e}_3 = xe_2 - \frac{\alpha+8}{3}e_3, \\ \dot{e}_4 = \frac{dm}{dt} - ke_1, \end{cases} \tag{16}$$

where $k > 0$. Since the eigenfrequency of the message signal $m(t)$ is much less than the oscillating frequency of the chaotic system in practice, then $\frac{dm(t)}{dt} - ke_1(t) \approx -ke_1(t)$. Construct a Lyapunov function as follows:

$$E = \frac{1}{2}\left(ke_1^2 + e_2^2 + e_3^2 + e_4^2\right). \tag{17}$$

If $0 < \alpha \leqslant 3/116$, then its derivative along the solution of (16) is

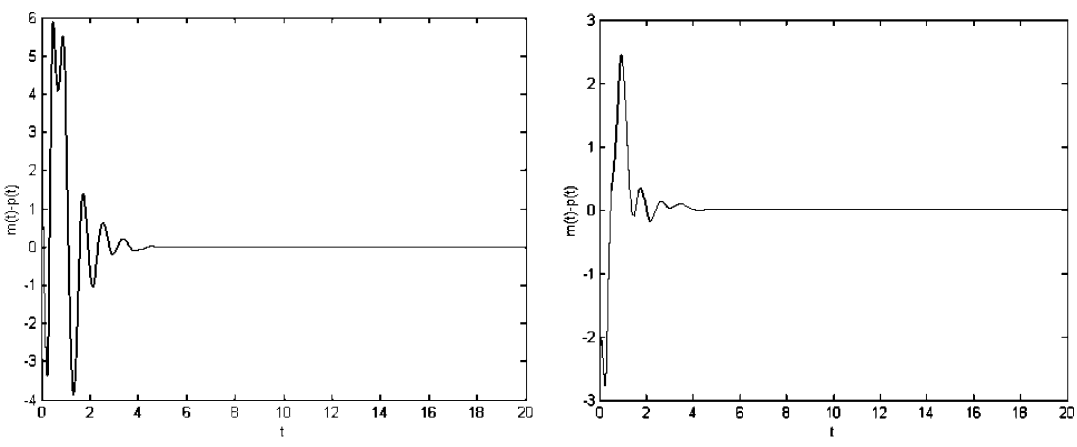


Fig. 5. The tracking error $m(t) - p(t)$ ($\alpha = 3/116$).

$$\frac{dE}{dt} = ke_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4$$
$$= -\left(\sqrt{1-29\alpha}\,e_2 - \frac{k(25\alpha+10)}{2\sqrt{1-29\alpha}}e_1\right)^2$$
$$- \frac{k(25\alpha+10)(2\sqrt{1-29\alpha}-1)}{2\sqrt{1-29\alpha}}e_1^2 \leqslant 0. \quad (18)$$

Obviously, $\frac{dE}{dt} = 0$ implies $e_1 = e_2 = e_3 = e_4 = 0$. Therefore, $E \to 0$ as $t \to \infty$. That is, $e_4 = m(t) - p(t) \to 0$ as $t \to \infty$. Thus the variable $p(t)$ can recover the message signal $m(t)$.

Taking $m(t) = 0.1\cos(0.1\pi t)$, $\alpha = 3/116$, $k = 50$, Fig. 5(a) shows the error $m(t) - p(t)$. Taking $m(t) = 0.1\,\mathrm{round}(t)$, $\alpha = 3/116$, $k = 15$, Fig. 5(b) shows the error $m(t) - p(t)$.

It is noticed that the chaos synchronization sensitively depends on the parameter interval $\alpha \in [0, 3/116]$ in above scheme for secure communication. Furthermore, parameter α is continuously adjustable in interval $[0, 3/116]$, thus it can be used as the cipher key space.

6. Conclusions

The chaos synchronization for the unified chaotic system is further investigated via different techniques in this Letter. Several sufficient conditions for chaos synchronization are gained. Furthermore, a scheme for secure communication is presented in theory, and its feasibility is verified by numerical simulations.

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