Parameters identification and synchronization of chaotic systems based upon adaptive control

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Abstract

A novel control method is proposed for a class of chaotic systems dependent linearly on unknown parameters based upon adaptive control. With this method parameters identification and synchronization of chaotic systems can be achieved simultaneously. Simulation results with Lorentz system and Chen’s system are presented to show the effectiveness of the approach. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Since the idea of synchronizing two identical chaotic systems from different initials was introduced by Pecora and Carroll in 1990 [1], synchronization of chaotic systems has attracted much attention due to its potential application in secure communication, chemical and biological systems, information science, biotic science and so on [2]. In the broadest sense, synchronization terms the tendency to undergo resembling evolution in time. This seems impossible for chaotic system if we think of one of the main features of chaotic behavior, namely, the sensitive dependence on initial conditions which makes any two adjacent orbits separate at a exponent rate eventually. Nevertheless, many results have been produced to show how synchronization can be achieved for chaotic dynamical systems under certain conditions ever since Pecora and Carroll’s original research work and there exists a large variety of approaches for chaos synchronization, including P–C (short for Pecora and Carroll) synchronization method [3], linear and nonlinear feedback synchronization method [4,5], impulsive synchronization method [6], among many others ([7,8] and references therein).

However, the aforementioned methods and many other synchronization methods are valid for chaotic system only when the system’s parameters are known. Achieving synchronization between two chaotic system with unknown parameters is far from being straightforward, there is little work about this challenging problem because it consists of both identifi-
cation of the unknown parameters and the design of controller. Guan et al. applied an observer to identification of the unknown parameter of Lorentz system [9]. Lü et al. studied the same problem for Chen’s chaotic system with the same method [10]. But because of the design of observer is a task with technique and sensitively depends on the considered dynamical system the method can be used to the system with only one unknown parameter, for the multi-unknown-parameter system this method does not work. In this Letter, a novel parameters identification and synchronization method is proposed for a class of chaotic dynamical system with several unknown parameters. By this method, one can achieve chaos synchronization and identify the unknown parameters simultaneously.

The rest of the Letter is organized as follows: the problem formulation, the identification and synchronization method for a class of chaotic system with unknown parameters are presented in Section 2. In Section 3 and Section 4 the numerical simulation with Lorentz system and Chen’s chaotic system with all the system parameters unknown is used to show the effectiveness of the proposed approach. Section 5 contains the conclusion and discuss.

2. Design of controller for parameters identification and synchronization

Most of the synchronization methods belong to drive–response type. By one system driving another we mean that two systems are coupled so that the behavior of the second is dependent on the behavior of the first, but the first is not influenced by the behavior of the second. The first system will be called the drive and the second the response.

The drive chaotic system considered in this Letter takes the form

$$\dot{x} = f(x) + F(x)\theta,$$  \hspace{1cm} (1)

where \(x \in \mathbb{R}^n\) is the state vector of the system, \(\theta \in \mathbb{R}^m\) is the vector of the system parameters, \(f \in C^1(\mathbb{R}^n, \mathbb{R}^n)\) and \(F \in C^1(\mathbb{R}^n, \mathbb{R}^{n \times m})\). It should be stressed that the systems considered here depend linearly on parameters which is surely to be the case in many chaotic systems. The response system is given by

$$\dot{u} = f(u) + F(u)\alpha,$$  \hspace{1cm} (2)

which has the same structure as the drive system but the parameter vector \(\alpha \in \mathbb{R}^m\) is unknown. In practice, the state signal of the drive system could be transmitted to the response system, but the system parameter vector is unknown for the response system. Therefore, the goal of control is to find out a appropriate controller \(U\) such that the controlled response system

$$\dot{u} = f(u) + F(u)\alpha + U,$$

is synchronous with the drive system and the unknown parameters are identified simultaneously. For convenience, the following assumption is stated firstly.

**Assumption.** There exists a controller \(U = U(u, x, t, \theta)\) and a scalar function \(V(e)\) such that for the drive system (1) and the controlled response system

$$\dot{u} = f(u) + F(u)\theta + U,$$  \hspace{1cm} (3)

the following two conditions are satisfied:

1. \(c_1\|e\|^2 \leq V(e) \leq c_2\|e\|^2\),
2. the derivative of \(V(e)\) along the solution of (3) satisfies

$$\frac{dV}{dt} = (\nabla V(e), f(u) - f(x) + (F(u) - F(x))\theta + U) \leq -W(e),$$

where \(e = u - x\), \(W(e)\) is a positive definite function, \(c_1, c_2\) are two positive constants, \(U(u, x, t, \theta)\) is a smooth map and satisfies \(U(u, u, t, \theta) = 0\).

This Assumption seems complicated, but this is always the case if the parameter vector of the drive system is known. In fact, one can check easily that the controller

$$U(u, x, t, \theta) = -e + f(x) - f(u) + (F(x) - F(u))\theta,$$

satisfies Assumption with \(V(e) = \frac{1}{2}e^T e, W(e) = e^T e\). However, this selection of controller is so complicated that it cannot be used in practice. Fortunately for many well-known chaotic systems such as Lorentz
system, Chen’s system, Van der Pol oscillator, Rössler system and several types of Chua’s circuits (to name just a few), there exist some simple selections of the controller which could be seen from Sections 3 and 4 below.

Obviously, if the above Assumption is valid, then the controlled response system (3) is synchronous with the drive system (1). However, it should be noted that this synchronization can be achieved for chaotic systems only when the drive system’s parameters are available. If the drive system’s parameter vector is unknown, we can get the following theorem.

**Theorem 1.** If Assumption above is valid, then with the controller $U = U(u, x, t, \alpha)$ and the parameters estimation update law $\dot{\alpha}$ as below the controlled response system

$$\dot{u} = f(u) + F(u)\alpha + U(u, x, t, \alpha),$$  \hspace{1cm} (4)

is synchronous with the drive system (1) and satisfies

$$\lim_{t \to +\infty} \|\alpha(t) - \theta\| = 0,$$

where

$$\dot{\alpha} = -F^T(x)(\text{grad } V(e))T.$$  \hspace{1cm} (5)

**Proof.** Choose Lyapunov function

$$V_1(e, \alpha) = V(e) + \frac{1}{2}(\alpha - \theta)^T(\alpha - \theta),$$

where $V(e)$ is as in Assumption. From the conditions of the above Assumption and the parameters estimation update law $\dot{\alpha}$ as in (5), the derivative of $V_1(e, \alpha)$ along the solution of the controlled response system (4) satisfies

$$\frac{dV_1}{dt} = (\text{grad } V(e), f(u) - f(x) + F(u)\alpha - F(x)\theta + U(u, x, t, \alpha),$$

are uniformly stable, therefore, $e(t)$ and $\alpha(t)$ are bounded on the interval $(0, +\infty)$. Suppose $\lim_{t \to +\infty} e(t) = \bar{e}$, $\lim_{t \to +\infty} \alpha(t) = \bar{\alpha}$. If $e \neq 0$, then there exist two positive constants $\delta$, $\varepsilon$, such that $\|e(t) - \bar{e}\| < \delta$ implies $W(e) > \varepsilon$. By definition of superior limit there exists a sequence $[t_n] \subset R^+$ such that $(e(t_n), \alpha(t_n)) \to (\bar{e}, \bar{\alpha})$ as $n \to \infty$. Assume $n^*$ to be the integer such that $\|e(t_n) - \bar{e}\| < \frac{\varepsilon}{2}$ for any $n > n^*$, then by continuity of $V_1(e, \alpha)$, for $n$ sufficiently large,

$$V_1(e(t_n), \alpha(t_n)) - V_1(\bar{e}, \bar{\alpha}) < \frac{\varepsilon \delta}{4r},$$

where $\frac{\varepsilon \delta}{4r}$ is a conveniently chosen number such that on $(t_n, t_n + \frac{\delta}{r})$ we have $\|e(t) - \bar{e}\| < \delta$ and so $W(e) > \varepsilon$. Therefore,

$$V_1(e(t_n), \alpha(t_n)) - V_1(\bar{e}, \bar{\alpha}) \geq \frac{\varepsilon \delta}{3r}.$$  \hspace{1cm} (6)

Hence, we get a contradiction which implies $\lim_{t \to +\infty} e(t) = 0$. In the same way we have $\lim_{t \to +\infty} \alpha(t) = 0$. That is $\lim_{t \to +\infty} \dot{e}(t) = 0$.

Moreover, from the error system (6) we can easily see that $\dot{e}(t)$ is bounded and thus the proof of the result $\lim_{t \to +\infty} \dot{e}(t) = 0$ is trivial which together with the above results implies the conclusion of the theorem.

**3. Simulation with Lorentz chaotic system**

Lorentz chaotic system is an approximation of a partial differential equation for fluid convection, where a flat fluid layer is heated from below and cooled from above. It was the first canonical chaotic attractor in a simple three-dimensional autonomous system which has just been mathematically confirmed to exist recently [11]. Lorentz chaotic system is described by

$$\begin{cases}
\dot{x} = \theta_1(y - x), \\
\dot{y} = \theta_2 - z)x - y, \\
\dot{z} = xy - \theta_3z,
\end{cases}$$  \hspace{1cm} (7)

where $x$, $y$ and $z$ are the state variables, $\theta_1$, $\theta_2$, $\theta_3$ are positive constant parameters. We rewrite it in the
following form
\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix} = \begin{pmatrix}
0 & -y-xz \\
-y & 0 & 0 \\
x & 0 & -z
\end{pmatrix} + \begin{pmatrix}
y-x \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix}.
\]

The system (7) is viewed as the drive system (or master system). The controlled response system is as follows
\[
\begin{pmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{pmatrix} = \begin{pmatrix}
0 & -v-wu \\
-u & 0 & 0 \\
0 & u & -w
\end{pmatrix} + \begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix},
\]
where \( U = (u_1, u_2, u_3) \) is the controller to be designed. If the drive system’s parameters are available, we can select the controller \( U \) as
\[
U = (u_1, u_2, u_3) = (\theta_1 e_1 - \theta_2 e_2, -(\theta_2 - w)e_1, -ve_1 + (\theta_3 - 1)e_3),
\]
where \( e_1 = u - x, e_2 = v - y, e_3 = w - z \). One can check straightforwardly that this controller satisfies the assumption with \( V(e) = \frac{1}{2}e^T e, \ W(e) = e^T e \). Therefore, when the drive system’s parameters are not available we can achieve parameters identification and synchronization simultaneously with the controlled response system and the parameters estimation update law designed as follows from the Theorem 1.

\[
\begin{align*}
\dot{u} &= \alpha_1(t)(v-u) + (\alpha_1(t) - 1)e_1 - \alpha_1(t)e_2, \\
\dot{v} &= (\alpha_2(t) - u)(u-v - (\alpha_2(t) - w)e_1, \\
\dot{w} &= uw - \alpha_3(t)w - ve_1 + (\alpha_3(t) - 1)e_3, \\
\dot{\alpha}_1 &= (x-y)e_1, \\
\dot{\alpha}_2 &= -xe_2, \\
\dot{\alpha}_3 &= ze_3.
\end{align*}
\]

In this simulation, let the parameters of the drive system be \( \theta_1 = 10, \theta_2 = 30, \theta_3 = 3 \), the initial states of the drive system and the controlled system are \((8.0, 9.0, 10.0)^T\) and \((3.0, 4.0, 5.0)^T\), respectively, the unknown parameters have zero initial condition. The results of parameters identification and synchronization are shown in Figs. 1 and 2.

**4. Simulation with Chen’s chaotic system**

Chen’s chaotic system is given by
\[
\begin{align*}
\dot{x} &= \theta_1(y-x), \\
\dot{y} &= (\theta_2 - \theta_1)x - xz + \theta_2 y, \\
\dot{z} &= xy - \theta_3 z,
\end{align*}
\]
which has a chaotic attractor when \( \theta_1 = 35, \theta_2 = 28, \theta_3 = 3 \). This chaotic system is founded by G. Chen recently in anti-controlling chaos and is relatively difficult to control as compared to Lorenz system and Chua’s system due to its complex topological features, especially its rapid change in \( z \)-direction [12]. In this
section we view Chen’s system as the drive system. When the drive system’s parameters are available we design the controller $U$ as

$$U = (-ye_1 + ze_2 + (\theta_1 - 1)e_1, -(\theta_2 + 1)e_2 - \theta_2 e_1, (\theta_3 - 1)e_3),$$

where $e_1 = u - x$, $e_2 = v - y$, $e_3 = w - z$. Then the controlled response system as follows:

$$\dot{u} = \theta_1(v - u) - y(w - z) + z(v - y) + (\theta_1 - 1)(u - x),$$
$$\dot{v} = (\theta_2 - \theta_1)u - uw + \theta_2 v - (\theta_2 + 1)(v - y) - \theta_2 - 1(u - x),$$
$$\dot{w} = uw - \theta_3 w + (\theta_3 - 1)(w - z).$$

Subtracting system (9) from system (10) yields the error system

$$\begin{align*}
\dot{e}_1 &= \theta_1 e_2 - e_1 - ye_3 + ze_2, \\
\dot{e}_2 &= -\theta_1 e_1 - e_2 - uw + xz, \\
\dot{e}_3 &= uv - xy - e_3.
\end{align*}$$

The Lyapunov function independent on the system parameters is

$$V(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2),$$

and its derivative along the solution of (10) satisfies

$$\frac{dV}{dt} = -e_1^2 - e_2^2 - e_3^2.$$

So the simple controller for Chen’s chaotic system which satisfies the assumption is selected. Therefore, when the drive system’s parameters are not available we can achieve parameters identification and synchronization with the controlled response system and the parameters estimation update law designed as follows

$$\begin{align*}
\dot{\alpha}_1 &= (x - y)e_1(t) + xe_2(t) \\
\dot{\alpha}_2 &= -(x + y)e_2(t), \\
\dot{\alpha}_3 &= ze_3(t).
\end{align*}$$

Assume the parameters of drive system are $\theta_1 = 35$, $\theta_2 = 28$, $\theta_3 = 3$, the initial conditions as in Section 3.

Numerical simulation shows that parameters identification and synchronization are achieved successfully. Figs. 3 and 4 display the results.

5. Conclusion

We have provided a novel control method for a class of chaotic systems dependent linearly on unknown parameters based upon adaptive control. With this new and effective method parameters identifica-
tion and synchronization of chaotic system can be achieved simultaneously.

Furthermore, numerical simulation with Lorentz system and Chen’s chaotic system is given to show the effectiveness and feasibility of the developed method.

It should be noted that the parameters identification method proposed in this Letter is online estimate unlike many other existing identification method and has a wide practical application in many other complex dynamical systems dependent linearly on unknown parameters.

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