THE SMALLEST POSSIBLE INTERACTION RADIUS FOR FLOCK SYNCHRONIZATION

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Abstract. This paper investigates the synchronization behavior of a class of flocks modeled by the nearest neighbor rules. While connectivity of the associated dynamical neighbor graphs is crucial for synchronization, it is well known that the verification of such dynamical connectivity is the core of theoretical analysis. Ideally, conditions used for synchronization should be imposed on the model parameters and the initial states of the agents. One crucial model parameter is the interaction radius, and we are interested in the following natural but complicated question: What is the smallest interaction radius for synchronization of flocks? In this paper, we reveal that, in a certain sense, the smallest possible interaction radius approximately equals $\sqrt{\log n/(\pi n)}$, with $n$ being the population size, which coincides with the critical radius for connectivity of random geometric graphs given by Gupta and Kumar [Critical power for asymptotic connectivity in wireless networks, in Stochastic Analysis, Control, Optimization and Applications, Birkhäuser Boston, Boston, MA, 1999, pp. 547–566].

Key words. multiagent system, Vicsek’s model, synchronization, random geometric graph, spectral gap, percolation theory

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1. Introduction. Recent motivations in the study of complex systems have led to great interest in the collective behavior of flocks or multiagent systems; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] among many others. A central issue in the investigation of multiagent systems is to understand how local interactions among the agents lead to global behavior of the whole group.

In this paper, we focus our attention on a group of flocks, or mobile agents, modeled by the nearest neighbor rules and based on the well-known multiagent model proposed by Vicsek et al. in [3]. The model consists of $n$ autonomous agents moving in the plane with the same speed $v_n (v_n > 0)$ but with different headings. Each agent’s heading is updated according to a local rule based on the average direction of its neighbors. Two agents are called neighbors if and only if the distance between them is less than a predefined radius $r_n (r_n > 0)$. Let us assume that the $n$ agents are labeled by $1, 2, \ldots, n$. Two agents $i$ and $j$ are neighbors at time $t$ if and only if $\|X_i(t) - X_j(t)\|_2 \leq r_n$, where $\| \cdot \|_2$ denotes the Euclidean norm. For any agent $i (1 \leq i \leq n)$, the set of its neighbors at time $t (t = 0, 1, \ldots)$ is denoted by $\mathcal{N}_i(t)$. By the definition of neighbors, we see that each agent is a neighbor of itself, i.e., $i \in \mathcal{N}_i(t)$, for all $t \geq 0$ and $1 \leq i \leq n$. The position and heading of the agent $i$ at time $t$ are denoted by $X_i(t) (\in \mathbb{R}^2)$ and $\theta_i(t) (\in (-\pi, \pi])$, respectively, which are updated by

\[
X_i(t + 1) = X_i(t) + v_n (\cos \theta_i(t + 1), \sin \theta_i(t + 1)),
\]

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\begin{equation}
\theta_i(t + 1) = \arctan \frac{\sum_{j \in \mathcal{N}_i(t)} \sin \theta_j(t)}{\sum_{j \in \mathcal{N}_i(t)} \cos \theta_j(t)} + \delta_i(t),
\end{equation}
where \(\delta_i(t)\) denotes a random noise \([3]\).

As pointed out by Vicsek et al. \([3]\), such a model can be used to study the gathering and phase transition of nonequilibrium systems and may be applied to investigate the clustering and migration in some biological systems. By computer simulations, the authors of \([3]\) revealed that if the population density is large and the noise is small, all agents tend to eventually move in the same direction. Due to its simplicity and fundamental importance in the investigation of multiagent systems, this model has attracted much attention in biology, physics, computer science, control theory, and mathematics. However, the theoretical analysis of system \((1.1)–(1.2)\) is difficult because of the nonlinearity and randomness of \((1.2)\). An important step forward in analyzing the above model is given by Jadbabaie, Lin, and Morse in \([6]\), where they omitted the noise effect and linearized the heading updating rule \((1.2)\) as follows:

\begin{equation}
\theta_i(t + 1) = \frac{1}{|\mathcal{N}_i(t)|} \sum_{j \in \mathcal{N}_i(t)} \theta_j(t),
\end{equation}
where \(| \cdot |\) denotes the cardinality of the corresponding set. They proved that if the associated dynamical neighbor graphs are contiguously jointly connected, the above model will reach \textit{synchronization} in the sense that there exists a common \(\bar{\theta}\) such that for all \(i (1 \leq i \leq n)\),

\begin{equation}
\lim_{t \to \infty} \theta_i(t) = \bar{\theta}.
\end{equation}

After that, Savkin in \([7]\) investigated the model with discrete headings and showed that if the limit of the neighbor graphs is connected, then synchronization can also be achieved. In \([9]\), Ren and Beard studied the case where the neighbor graphs are directed and showed that synchronization can be achieved if the union of the interaction graphs has a spanning tree frequently enough.

In fact, most existing studies resort to certain connectivity conditions on the dynamical neighbor graphs, and these conditions are hard to verify. Therefore the corresponding analysis is not theoretically complete. One notable exception in the study of flocks is the interesting paper by Cucker and Smale \([4]\), where global interactions are considered with weights of interactions decaying with the distances among agents. However, an unresolved central issue is how to guarantee the connectivity of the dynamical neighbor graphs resulting from local interactions using conditions imposed on only the initial states, the moving speed \(v_n\), and the interaction radius \(r_n\).

To give a complete analysis for the synchronization behavior of the system, Tang and Guo \([8]\) introduced a random framework, assuming that the initial positions and headings of all agents are uniformly and independently distributed, as those in \([3]\). They show that for any given positive model parameters, the flocking model based on \((1.1)\) and \((1.3)\) will synchronize with large probability, giving the first complete theoretical result in this direction. Furthermore, in \([12]\) they proved that if \(\sqrt{\log n}/n = \omega(r_n)\) and \(v_n = O \left( \frac{r_n^5}{\log n} \right)\), then the model will synchronize.\(^1\)

\(^1\)For two positive sequences \(\{a_n, n \geq 1\}\) and \(\{b_n, n \geq 1\}\), \(a_n = o(b_n)\) means that \(\lim_{n \to \infty} (a_n/b_n) = 0\).

\(^2\)For two positive sequences \(\{a_n, n \geq 1\}\) and \(\{b_n, n \geq 1\}\), \(a_n = O(b_n)\) means that there exists a positive constant \(c\) independent of \(n\) such that \(a_n \leq cb_n\) for large enough \(n\).
on their results, Liu and Guo [10] investigated the system (1.1)–(1.2) without noise and provided a similar condition for synchronization. However, the theoretical analysis of the (linearized) Vicsek’s model with the radius \( r_n = O(\sqrt{\log n/n}) \) is still lacking, and the question concerning the smallest possible radius for synchronization is never investigated in this context.

We will carry out our analysis under the assumption that all agents are independently and uniformly distributed in \([0, 1]^2\) with arbitrary headings in \((−\pi, \pi]\) at the initial time. As pointed out by Jadbabaie, Lin, and Morse in [6], the connectivity of the neighbor graphs is important for synchronization. Gupta and Kumar in [13] proved that the initial neighbor graph with radius \( r_n \) is important for synchronization. However, the theoretical analysis of their results, Liu and Guo [10] investigated the system (1.1)–(1.2) without noise and provided a similar condition for synchronization. Nevertheless, the theoretical analysis of the auxiliary results will be given in section 4. A simulation example is put in section 5. Section 6 concludes the paper with remarks.

2. Main results. The objective of this paper is to study the synchronization behavior of the dynamical system (1.1) and (1.3). From the description of the model, we know that the initial states of all agents and the model parameters will determine the trajectories of all agents. Throughout this paper, we assume that the initial positions of all agents are independently and uniformly distributed in \([0, 1]^2\) with arbitrary initial headings in \((−\pi, \pi]\). All analysis proceeds under the above assumption without further explanations.

Similar to [10], we will use a graph sequence \( \{G(t), t = 0, 1, \ldots\} \) to describe the relationship among neighbors. For \( t \geq 0 \), define

\[
G(t) = G(\{X_1(t), \ldots, X_n(t)\}, E(t))
\]

to be the position graph of the model at time \( t \), where \( E(t) = \{(i, j) : \|X_i(t) - X_j(t)\| \leq r_n\} \). Obviously, the graphs formed in this way are undirected, and for all \( 1 \leq i \leq n \) and \( t \geq 0 \), \((i, i) \in E(t)\). Denote by \( P(t) \) the average matrix of the graph \( G(t) \), i.e.,

\[
\forall i, j = 1, 2, \ldots, n, \quad (P(t))_{ij} = \begin{cases} \frac{1}{|N_i(t)|} & \text{if } (i, j) \in E, \\ 0 & \text{otherwise.} \end{cases}
\]

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3We say that a sequence of events \( E_n \) occurs w.h.p. if \( \lim_{n \to \infty} P[E_n] = 1 \).
Let $\theta(t) := (\theta_1(t), \theta_2(t), \ldots, \theta_n(t))^T$; then the iteration rule of the headings and positions of the model based on (1.1) and (1.3) can be rewritten as

\begin{equation}
(2.1) \begin{cases}
\theta(t+1) = P(t)\theta(t) \\
X_i(t+1) = X_i(t) + v_n(\cos \theta_i(t+1), \sin \theta_i(t+1)) \quad \forall t \geq 0, 1 \leq i \leq n.
\end{cases}
\end{equation}

Note that under the assumption on the initial positions, the graph $G(0)$ is a random geometric graph, which has been studied in detail in, e.g., [15]. One of the classical result concerning the connectivity of the random geometric graph can be stated as follows.

**Lemma 2.1 (see [13]).** The initial random geometric graph $G(0)$ is connected w.h.p. if and only if $r_n$ satisfies

\begin{equation}
\lim_{n \to \infty} \left( \pi n r_n^2 - \log n \right) = \infty.
\end{equation}

Based on this lemma, Gupta and Kumar in [14] called $\sqrt{\log n / (\pi n)}$ the critical radius for connectivity of $G(0)$. In this paper, we will show that in a probability sense, this critical radius can be regarded as the smallest possible radius for synchronization of the flocks. The main results of this paper are formulated as the following theorem.

**Theorem 2.2.** Suppose that the $n$ agents are independently and uniformly distributed in $[0, 1]^2$ at the initial time $t = 0$. If $r_n$ satisfies (2.2) and $v_n$ satisfies

\begin{equation}
(2.3) \quad v_n = o \left( r_n (\log n)^{-1} n^{-2} \right),
\end{equation}

then the system (2.1) will synchronize w.h.p. for arbitrary initial headings. Moreover, if $r_n$ satisfies

\begin{equation}
(2.4) \quad \lim_{n \to \infty} \left( \pi n r_n^2 + 3 \log \log n - \log n \right) = -\infty,
\end{equation}

then w.h.p. there exist some initial headings such that the system (2.1) cannot reach synchronize for any speed $v_n \geq 0$.

The proof of this theorem is in section 3.

Before closing this section, we propose a conjecture (which is intuitively correct) on the system (2.1), in terms of the values of the speed and the radius for synchronization.

**Conjecture 2.3.** Suppose $n$ agents are distributed in a plane and the initial positions are given. If the system (2.1) can synchronize with speed $v$ and radius $r$, then it will also uniformly synchronize with speed $v_1 \in (0, v)$ and radius $r$, or with speed $v$ and radius $r_1 > r$.

**3. Proof of Theorem 2.2.** To prove Theorem 2.2, we need to estimate the maximum degree, the minimum degree, and the eigenvalues of the average matrix of the random geometric graph $G(0)$. For this purpose, we need to introduce some notation.

Define the large deviations rate function $H : [0, \infty) \to \mathbb{R}$ by $H(0) = 1$ and

\[ H(a) = 1 - a + a \log a, \quad a > 0. \]

Note that $H(1) = 0$ and that the unique turning point of $H$ is the minimum at 1. Also, $H(a)/a$ is increasing on $(1, \infty)$. Let $H^{-1} : [0, 1] \to [0, 1]$ be the unique inverse of the restriction of $H$ to $[0, 1]$, and let $H^{-1}_+ : [0, \infty) \to [1, \infty)$ be the inverse of the
restriction of $H$ to $[1, \infty)$; see [15] for the properties of $H$. Denote by $d_i$ the degree of the vertex $i$ in $G(0)$, i.e., the number of neighbors of the agent $i$ at the initial time instant. Set

$$d_{\max} := \max_{1 \leq i \leq n} d_i \quad \text{and} \quad d_{\min} := \min_{1 \leq i \leq n} d_i.$$ 

The estimation for the maximum and minimum degrees of the initial random geometric graph $G(0)$ have been given by Penrose [15], as will be described by the following lemma.

**Lemma 3.1.** Suppose that $\pi n r_n^2 / \log n \to w \in (1, \infty)$ and $r_n \to 0$ as $n \to \infty$. Then with probability 1,

$$\lim_{n \to \infty} \left( \frac{d_{\max}}{n \pi r_n^2} \right) = H_+^{-1} \left( \frac{1}{w} \right)$$

and

$$\lim_{n \to \infty} \left( \frac{d_{\min}}{n \pi r_n^2} \right) = \min \left( H_+^{-1} \left( \frac{1}{w} \right), \frac{1}{4} \right).$$

**Proof.** The assertions (3.1) and (3.2) are indicated by Theorems 6.14 and 7.14 of [15].

**Corollary 3.2.** If $r_n$ satisfies (2.2), then $d_{\max} < 3 d_{\min} \log n$ w.h.p.

**Proof.** For the case where $\pi n r_n^2 \geq 3 \log n/e$, by Lemma 3.1 we see that $d_{\max} < d_{\min} \log n$ holds almost surely for large $n$. Next, we will discuss the case where $\pi n r_n^2 < 3 \log n/e$. Note that $d_{\max}$ increases with $r_n$; by Lemma 3.1, the following inequality holds almost surely for large $n$:

$$d_{\max} \leq \frac{3 \log n}{e} H_+^{-1} \left( \frac{e}{3} \right) \left( 1 + o(1) \right) = \frac{3 \log n}{e} H_+^{-1} (1) = 3 \log n.$$

Also, by Lemma 2.1, $d_{\min} \geq 1$ w.h.p., and thus our result yields.

Next, we will estimate the eigenvalues of $G(0)$. Let $D = (d_{ij})_{n \times n}$ denote the degree matrix of $G(0)$, which is a diagonal matrix with diagonal entries $d_{ii} = d_i$. Obviously, the matrix $D^{1/2} P(0) D^{-1/2}$ is symmetric, so all eigenvalues of $P(0)$ are real numbers. On the other hand, all entries of $P(0)$ are nonnegative, and $\sum_{i=1}^n (P(0))_{ij} = 1$, $i = 1, 2, \ldots, n$, so the average matrix $P(0)$ is a stochastic matrix. The eigenvalues of $P(0)$, denoted by $\lambda_i$, $1 \leq i \leq n$, with $\lambda_i$ being the $i$-largest eigenvalues of $P(0)$, satisfy the inequalities

$$|\lambda_i| \leq 1, \quad 1 \leq i \leq n,$$

which means that

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1.$$

Define the **essential spectral radius** $\bar{\lambda}$ of $G(0)$ as

$$\bar{\lambda} = \bar{\lambda}(P(0)) := \max \{|\lambda_2|, |\lambda_n|\}.$$ 

We remark that for the case where \(\lim_{n \to \infty} (nr_n^2 / \log n) = \infty\), Tang and Guo [8] proved that the essential spectral radius of $G(0)$ satisfies the following inequality w.h.p. for large $n$:

$$\bar{\lambda} \leq 1 - \frac{\pi r_n^2}{512(r_n + \sqrt{6})^4} (1 + o(1)).$$
However, the methods used in [8] cannot be applied to estimate the spectral gap of $G(0)$ for the case of $r_n = O(\sqrt{\log n/n})$ since the interaction radius is too small to satisfy the condition of Lemma 4 in [8], which plays a key role in the estimation of $\bar{\lambda}$.

In this paper, we will use some methods from percolation theory to study the essential spectral radius of $G(0)$ for the case where $r_n$ satisfies (2.2).

**Theorem 3.3.** Assume that $r_n \leq 1$. Then there exists a constant $c > 0$ such that the inequality $\bar{\lambda} \leq 1 - c r_n^2$ holds w.h.p. if and only if $r_n$ satisfies (2.2).

The proof of Theorem 3.3 is given in section 4.

For $\eta > 0$, we write $Po(\eta)$ for any Poisson random variable with parameter $\eta$. Define a Poisson point process $\mathcal{P}_\eta$ by $\mathcal{P}_\eta := \{Y_1, Y_2, \ldots, Y_{Po(\eta)}\}$, where $\{Y_1, Y_2, \ldots\}$ is the set of vertices independently and uniformly distributed in $[0, 1]^2$ and $Po(\eta)$ is independent of $\{Y_1, Y_2, \ldots\}$; see section 1.7 in [15]. For a Borel set $A \subseteq [0, 1]^2$, $|\mathcal{P}_\eta \cap A|$, the number of the vertices lying in $A$ is a Poisson random variable with parameter $\eta|\mathcal{L}(A)$, where $\mathcal{L}(\cdot)$ denotes the Lebesgue measure in this paper. For any Borel set $A_1, A_2 \subseteq [0, 1]^2$, if $\mathcal{L}(A_1 \cap A_2) = 0$, then the random variables $|\mathcal{P}_\eta \cap A_1|$ and $|\mathcal{P}_\eta \cap A_2|$ are mutually independent. This property is called spatial independence of a Poisson point process.

**Proof of Theorem 2.2.** We will first prove the sufficient part of Theorem 2.2.

For $r_n > 1$, under the condition (2.3), we can directly deduce that the system (2.1) can reach synchronization by Theorem 1 of [8]. Thus, we just need to consider the case where $r_n \leq 1$. By Theorem 3.3 and (2.2), we see that there exists a constant $c > 0$ such that

\[(3.4) \quad \lim_{n \to \infty} P (\bar{\lambda} \leq 1 - c r_n^2) = 1.\]

Let $E_n$ denote the event $\bar{\lambda} \leq 1 - c r_n^2$, and let $\bar{E}_n$ denote the event $d_{\text{max}} < 3d_{\text{min}} \log n$. Define $F_n$ to be the event

\[\bigcap_{1 \leq i, j \leq n} \left\{ \|X_i(0) - X_j(0)\|_2 \notin \left( r_n - o\left(\frac{1}{n^2 r_n}\right), r_n + o\left(\frac{1}{n^2 r_n}\right) \right) \right\}.\]

Using Boole’s inequality, we have

\[P(F_n^c) \leq \sum_{i \neq j} P \left( \|X_i(0) - X_j(0)\|_2 \in \left( r_n - o\left(\frac{1}{n^2 r_n}\right), r_n + o\left(\frac{1}{n^2 r_n}\right) \right) \right) \]

\[< n^2 \int_{x \in [0, 1]^2} P \left( \|x - X_j(0)\|_2 \in \left( r_n - o\left(\frac{1}{n^2 r_n}\right), r_n + o\left(\frac{1}{n^2 r_n}\right) \right) \right) dx \]

\[< n^4 \pi r_n \cdot o\left(\frac{1}{n^2 r_n}\right) \to 0 \quad \text{as} \quad n \to \infty,\]

where the property that the initial positions are independently and uniformly distributed in $[0, 1]^2$ is used in the last inequality. Combining (3.5) with (3.4) and Corollary 3.2, we can deduce that

\[(3.5) \quad P(E_n \cap \bar{E}_n \cap F_n) \to 1 \quad \text{as} \quad n \to \infty.\]

We assert that if the speed $v_n$ satisfies (2.3), then for all $t \geq 0$, the topology of $G(t)$ remains unchanged given $E_n \cap \bar{E}_n \cap F_n$. We will prove this assertion by induction.

For $t = 0$, the assertion is obviously true. Assume that the assertion holds for all $s \leq t$, that is, $P(s) = P(0)$ for all $s \leq t$. Thus, by (2.1), we have

\[\theta(s + 1) = P^n(0)\theta(0) \quad \forall \quad 0 \leq s \leq t.\]
Combining this with Proposition 3 in [16], for all integers \(s \in [0, t]\) and \(i, j \in [1, n]\) we have

\[
|\theta_i(s + 1) - \theta_j(s + 1)| = \left| \sum_{k=1}^{n} \left( (P^s(0))_{ik} - (P^s(0))_{jk} \right) \theta_k(0) \right|
\]

\[
\leq \pi \sum_{k=1}^{n} \left| (P^s(0))_{ik} - (P^s(0))_{jk} \right| \leq \pi \sqrt{n} \left( \sqrt{d_{\max}^i} + \sqrt{d_{\max}^j} \right) \tilde{\lambda}^s
\]

\[
\leq 2\pi \sqrt{3n \log n} \cdot \tilde{\lambda}^s,
\]

where the assertion conditions \(E_n\) and \(\tilde{E}_n\) are used in the last inequality. Set

\[
d_{ij}(t + 1) := \|X_i(t + 1) - X_j(t + 1) - X_i(0) + X_j(0)\|_2.
\]

Subsequently using (2.1), the triangle inequality, and standard goniometric formulae, we have

\[
d_{ij}(t + 1) = \left\| v_n \sum_{s=1}^{t+1} (\cos \theta_i(s), \sin \theta_i(s)) - v_n \sum_{s=1}^{t+1} (\cos \theta_j(s), \sin \theta_j(s)) \right\|_2
\]

\[
\leq v_n \sum_{s=1}^{t+1} \| (\cos \theta_i(s) - \cos \theta_j(s), \sin \theta_i(s) - \sin \theta_j(s)) \|_2
\]

\[
= v_n \sum_{s=1}^{t+1} \sqrt{2 - 2\cos[\theta_i(s) - \theta_j(s)]} \leq v_n \sum_{s=1}^{t+1} |\theta_i(s) - \theta_j(s)|,
\]

where the inequality \(\cos x \geq 1 - x^2/2\) is also used. Set \(t_0 := \min\{t : 2\pi \sqrt{3n \log n} \cdot \tilde{\lambda}^t \leq 2\pi\}.\) Then

\[
t_0 = \left\lfloor \frac{\log 1 \log \lambda}{\log \lambda} \right\rfloor - \frac{\log(3n \log n)}{2 \log \lambda} + 1,
\]

where \([x]\) denotes the smallest integer no less than \(x\). Hence, by (3.6) and the inequality \(1 - x < -\log x\) for \(x \in (0, 1)\), we have

\[
\max_{i, j} \sum_{s=1}^{t+1} |\theta_i(s) - \theta_j(s)| \leq 2\pi t_0 + \sum_{s=t_0+1}^{t+1} 2\pi \sqrt{3n \log n} \cdot \tilde{\lambda}^s
\]

\[
< 2\pi \left( \frac{-\log(3n \log n)}{2 \log \lambda} + 1 \right) + \left( \frac{2\pi \sqrt{3n \log n}}{1 - \lambda} \right) \tilde{\lambda} \frac{-\log(3n \log n)}{2 \log \lambda}
\]

\[
= O \left( \frac{\log n}{1 - \lambda} \right) = O \left( r_n^{-2} \log n \right).
\]

Substituting this inequality and (2.3) into (3.7), we can obtain that

\[
\max_{i, j} d_{ij}(t + 1) \leq v_n \max_{i, j} \sum_{s=1}^{t+1} |\theta_i(s) - \theta_j(s)|
\]

\[
= o \left( \frac{r_n}{n^2 \log n} \cdot \frac{\log n}{r_n^2} \right) = o \left( \frac{1}{n^2 r_n} \right),
\]

\[
(3.8)
\]
SMALLEST POSSIBLE INTERACTION RADIUS

FIG. 1. If $A_k$ happens, then the system will not synchronize by setting the initial headings of the agents lying in $B(x_k, \varepsilon_n)$ to be $-\pi/2$ and the others to be $\pi/2$.

which means that the position between any two agents changed at time $t$ is bounded by $o(1/n)$ in comparison with that at the initial time. Combining (3.8) with the condition $F_n$, we know that, compared with $G(0)$, the topology of the graph $G(t+1)$ is unchanged w.h.p.

By induction, our assertion holds for all $t \geq 0$, which means that the inequality (3.6) holds for all $t \geq 0$. Thus, the system (2.1) can reach synchronization.

It remains to prove the necessary part of the theorem. Set

$$M_n := \left\lfloor \sqrt{\pi n/(4 \log n)} \right\rfloor - 1,$$

where $\lfloor x \rfloor$ denotes the largest integer no bigger than $x$. Define the point

$$x_k := \left( (2k + 1) \sqrt{\log n/\pi n}, 0 \right) \in [0, 1]^2, \quad k = 0, \ldots, K_n.$$

Let $b_n := \log n - 3 \log \log n - \pi n r_n^2$; then by (2.4)

$$b_n < \log n - 3 \log \log n \quad \text{and} \quad \lim_{n \to \infty} b_n = \infty.$$ 

Take $\varepsilon_n = \sqrt{1/(\pi n \log n)}$. Let

$$X_n := \{X_1(0), X_2(0), \ldots, X_n(0)\}$$

denote the $n$ vertices independently and uniformly distributed in $[0, 1]^2$. For any integer $k \in [0, M_n]$, define the event

$$A_k := \{X_n \cap B(x_k, \varepsilon_n) \neq \emptyset, X_n \cap [B(x_k, r_n + \varepsilon_n) \setminus B(x_k, \varepsilon_n)] = \emptyset\},$$

where $B(x, r) := \{y \in \mathbb{R}^2 : \|x - y\|_2 \leq r\}$ denotes the ball centered at $x$ with radius $r$. If the event $A_k (k \in [0, M_n])$ happens, then the agents lying in $B(x_k, \varepsilon_n)$ do not have any neighbor at the initial time. For such a case, the system (2.1) will not synchronize by setting the initial headings of the agents lying in $B(x_k, \varepsilon_n)$ to be $-\pi/2$ and the others to be $\pi/2$; see Figure 1. Thus, to prove the necessary part we just need to verify the following equation:

$$\lim_{n \to \infty} P \left( \bigcup_{0 \leq k \leq M_n} A_k \right) = 1.$$
Set \(\eta(n) := n + n^{3/4}\) and \(\lambda(n) := n - n^{3/4}\). Let \(\mathcal{P}_{\eta(n)}\) and \(\mathcal{P}_{\lambda(n)}\) denote a Poisson point process in \([0, 1]^2\) with parameters \(\eta(n)\) and \(\lambda(n)\), respectively. Using Lemma 1.4 in [15], for large \(n\) we can get

\[
\Pr(\mathcal{X}_n \subseteq \mathcal{P}_{\eta(n)}) = \Pr(\{\text{Po}(\eta(n)) \geq n\}) > 1 - e^{-n^{3/4}}
\]

(3.11) and

\[
\Pr(\mathcal{P}_{\lambda(n)} \subseteq \mathcal{X}_n) = \Pr(\{\text{Po}(\lambda(n)) \leq n\}) > 1 - e^{-n^{3/4}}.
\]

(3.12)

Define the event

\[
\tilde{A}_k := \{\mathcal{P}_{\lambda(n)} \cap B(x_k, \epsilon_n) \neq \emptyset, \mathcal{P}_{\eta(n)} \cap [B(x_k, r_n + \epsilon_n) \setminus B(x_k, \epsilon_n)] = \emptyset\};
\]

then by (3.11) and (3.12),

\[
P\left(\bigcup_{0 \leq k \leq M_n} \tilde{A}_k\right) \geq P\left(\bigcup_{0 \leq k \leq M_n} \tilde{A}_k, \mathcal{P}_{\lambda(n)} \subseteq \mathcal{X}_n, \mathcal{X}_n \subseteq \mathcal{P}_{\eta(n)}\right)
\]

(3.13)

\[
\geq P\left(\bigcup_{0 \leq k \leq M_n} \tilde{A}_k\right) + P(\mathcal{P}_{\lambda(n)} \subseteq \mathcal{X}_n) + P(\mathcal{X}_n \subseteq \mathcal{P}_{\eta(n)}) - 2
\]

\[
> P\left(\bigcup_{0 \leq k \leq M_n} \tilde{A}_k\right) - 2e^{-n^{3/4}}.
\]

Also, using the spatial independence of the Poisson point process and Taylor’s expansion,

\[
P\left(\bigcup_{0 \leq k \leq M_n} \tilde{A}_k\right) = 1 - P\left(\bigcap_{0 \leq k \leq M_n} \tilde{A}_k^c\right) = 1 - \prod_{0 \leq k \leq M_n} \left[1 - P(\tilde{A}_k)\right]
\]

\[
= 1 - \left[1 - \left(1 - e^{-\lambda(n)\pi\epsilon_n^2/2}\right) e^{-\eta(n)\pi(r_n^2 + 2r_n\epsilon_n)/2}\right]^{M_n+1}
\]

\[
= 1 - \left[1 - \frac{1}{2\log n} n^{-\frac{1}{2}} (\log n)^{\frac{1}{2}} e^{\frac{bn}{\log n}} - \sqrt{1 - \frac{bn}{\log n}} (1 + o(1))\right]^{M_n+1}
\]

\[
= 1 - \exp\left(-\frac{1}{2} (M_n + 1) n^{-\frac{1}{2}} (\log n)^{\frac{1}{2}} e^{\frac{bn}{\log n}} - \sqrt{1 - \frac{bn}{\log n}} (1 + o(1))\right)
\]

\[
\to 1 \quad \text{as} \quad n \to \infty.
\]

Combining this with (3.13) yields (3.10).

Remark 3.4. From the proof of Theorem 2.2, we see that the speed \(v_n\) is so small that the topology of the neighbor graph remains unchanged during the evolution of the system. However, the relaxation of the restriction on the speed is very hard, since the estimation of the essential spectral radius of \(P(t)P(t-1) \cdots P(0)\) is still open in the inhomogeneous Markov chain theory even if only one edge is changed in the neighbor graph; see Problem 1.1 in [17]. The restriction on the speed may be relaxed if the above open problem is resolved.

4. Proof of Theorem 3.3. First, we will provide the proof of the sufficient part of Theorem 3.3. For the case where \(\pi vr_n^2 \geq (\log n)^2\), the inequality \(\lambda \leq 1 - cr_n^2\) holds w.h.p. by Theorem 3 in [8]. Therefore, we just need to consider the case where

\[
\pi vr_n^2 \leq (\log n)^2 \quad \text{and} \quad \lim_{n \to \infty} (\pi vr_n^2 - \log n) = \infty.
\]

(4.1)
In this section we use \(G(x_n; r_n)\) to denote the initial random geometric graph \(G(0)\). Divide the unit square \([0, 1]^2\) into \(K_n^2\) small squares with the length of each side equal to \(1/K_n\), where \(K_n := \lceil \sqrt{5}/r_n \rceil\). Denote these small squares by \(S_1, S_2, \ldots, S_{K_n^2}\). Set

\[
\alpha_n := E[|X_n \cap S_1|] = \frac{n}{\lceil \sqrt{5}/r_n \rceil^2},
\]

where \(X_n\) is defined by (3.9). Define

\[
\Delta_n := \max_{1 \leq i \leq K_n^2} |X_n \cap S_i|.
\]

We will consider the upper bound of \(\Delta_n\) first.

**Lemma 4.1.** Assume that \(r_n\) satisfies (4.1). Then, with probability 1, \(\Delta_n < 21 \alpha_n\) for large enough \(n\).

**Proof.** Since the initial positions \(X_j(0), j = 1, 2, \ldots, n\), are independently and uniformly distributed in \([0, 1]^2\), \(P(X_j(0) \in S_i) = 1/K_n^2, 1 \leq j \leq n, i \in [1, K_n^2]\), and \(|X_n \cap S_i|\) is a binomial random variable. By (1.7) in [15], for large enough \(n\),

\[
P(|X_n \cap S_i| \geq 21 \alpha_n) \leq \exp\left(\frac{-21 \alpha_n \log \left(\frac{21 \alpha_n}{E[|X_n \cap S_i|]}\right)}{2}\right) \\
\leq \exp\left(\frac{-21n}{2(\frac{\sqrt{5}}{r_n})^2 + 1}\right) \\
< \exp(-2.03 \cdot \log n) = n^{-2.03}.
\]

Thus, by the definition of \(\Delta_n\), for large enough \(n\) we have

\[
P(\Delta_n \geq 21 \alpha_n) = P\left(\bigcup_{i=1}^{K_n^2} \{|X_n \cap S_i| \geq 21 \alpha_n\}\right) \\
\leq \sum_{i=1}^{K_n^2} P(|X_n \cap S_i| \geq 21 \alpha_n) \\
< n \cdot n^{-2.03} = n^{-1.03}.
\]

Hence, using the Borel–Cantelli lemma yields our result. \(\square\)

**Remark 4.2.** Using a method similar to that of Theorem 6.14 in [15], we can get that, with probability 1, the inequality

\[
\alpha_n H^{-1}\left(\frac{\log n}{\alpha_n}\right) (1 - o(1)) \leq \Delta_n \leq \alpha_n H^{-1}\left(\frac{\log n}{\alpha_n}\right) (1 + o(1))
\]

holds for large \(n\). However, the proof of this result is complicated, so we do not include it in this paper.

In the following we need to introduce some definitions. Let \(\| \cdot \|_1\) denote the \(l_1\)-norm, and let \(\| \cdot \|_\infty\) denote the infinity norm. For any \(x, y \in \mathbb{Z}^2\), if \(\|x - y\|_1 = 1\), then we say that \(x\) and \(y\) are adjacent, and we write \(x \sim y\). Also, given \(A \subseteq \mathbb{Z}^2\), if for any \(x, y \in A\), there exists a vertex sequence \(x_1, x_2, \ldots, x_n\) in \(A\) such that \(x \sim x_1, x_1 \sim x_2, x_2 \sim x_3, \ldots, x_n \sim y\), then we say \(A\) is connected. Similarly, if
\[ \| x - y \|_\infty \leq k, \ k \geq 1, \] we say that \( x \) and \( y \) are \( k \)-adjacent, and we write \( x \sim_k y \).

Given \( A \subseteq \mathbb{Z}^2 \), if for any \( x, y \in A \), there exists a vertex sequence \( x_1, x_2, \ldots, x_n \) in \( A \) such that \( x \sim_k x_1, x_1 \sim_k x_2, x_2 \sim_k x_3, \ldots, x_n \sim_k y \), then we say \( A \) is \( k \)-connected.

It can be seen that if \( A \) is connected, then \( A \) must be \( k \)-connected for all \( k \geq 1 \). In particular, a single vertex set \( \{ x \} \subseteq \mathbb{Z}^2 \) is both connected and \( k \)-connected.

We define the lattice box \( B_{\mathbb{Z}}(K_n) \) by \( B_{\mathbb{Z}}(K_n) := \prod_{i=1}^{2}([1, K_n] \cap \mathbb{Z}) \). If \( A \subset B_{\mathbb{Z}}(K_n) \), set \( A^c := B_{\mathbb{Z}}(K_n) \setminus A \), and let \( \partial A \) denote the internal vertex-boundary of \( A \), that is, the set of vertex \( z \in A \) such that \( \{ y \in A^c : \| z - y \|_1 = 1 \} \) is nonempty. To prove Theorem 3.3, several lemmas are needed.

**Lemma 4.3.** Let \( \beta \in (0, 1) \). If \( A \) is a subset of \( B_{\mathbb{Z}}(K_n) \) (not necessarily connected), with \( |A| \leq \beta K_n^2 \), then

\[ |\partial A| \geq \frac{1}{4}(1 - \sqrt{\beta})|A|. \]

**Proof.** Replacing 2/3 with \( \beta \) into the proof of Lemma 9.9 of [15], the result can be deduced. \( \square \)

**Lemma 4.4.** (Lemma 9.6 in [15]). Suppose \( A \subset B_{\mathbb{Z}}(K_n) \) is such that both \( A \) and \( A^c \) are connected. Then \( \partial A \) is 1-connected.

**Remark 4.5.** If both \( A \) and \( A^c \) are connected, by Lemma 4.4 both \( \partial A \) and \( \partial (A^c) \) are 1-connected since \( (A^c)^c = A \).

**Lemma 4.6.** Suppose \( A \subset B_{\mathbb{Z}}(K_n) \). If \( A \) is 3-connected and \( A^c \) is connected, then \( \partial A \) is 3-connected, and \( \partial (A^c) \) is 2-connected.

**Proof.** Let \( A_1, A_2, \ldots, A_m \) denote the connected components of \( A \), which indicates that \( A_1, \ldots, A_m \) are connected, but \( A_i \cup A_j, 1 \leq i \neq j \leq m, \) is not connected. The fact that \( B_{\mathbb{Z}}(K_n) \) is connected, \( A_i, i \in [1, m] \), are all connected with \( A^c \). Note that \( A^c \) is connected, so for any \( i \in [1, m], A_i^c = A^c \cup A_1 \cup \cdots \cup A_{i-1} \cup A_{i+1} \cup A_m \) is also connected. By Lemma 4.4, we know that both \( \partial A_i \) and \( \partial (A_i^c) \) are 1-connected.

Moreover, if \( A_i \cup A_j (i \neq j) \) is 3-connected, then there exists a pair \( (z_i, z_j) \in (\partial A_i, \partial A_j) \) such that \( z_i \) and \( z_j \) are 3-connected, and there exists another pair \( (\bar{z}_i, \bar{z}_j) \in (\partial (A_i^c), \partial (A_j^c)) \) such that \( \bar{z}_i \) and \( \bar{z}_j \) are 2-connected; see Figure 2. Thus, \( \partial A_i \cup \partial A_j \) is 3-connected, and \( \partial (A_i^c) \cup \partial (A_j^c) \) is 2-connected since \( \partial A_i, \partial A_j, \partial (A_i^c), \) and \( \partial (A_j^c) \) are 1-connected. Combining this with the fact that \( A = \bigcup_{i=1}^{m} A_i \) is 3-connected, we have that \( \partial A = \bigcup_{i=1}^{m} \partial A_i \) is 3-connected, and \( \partial (A^c) = \bigcup_{i=1}^{m} \partial (A_i^c) \) is 2-connected. \( \square \)
LEMMA 4.7 (Corollary 9.4 in [15]). Given integer $k \geq 1$, the number of $k$-connected subsets of the lattice box $B_2(K_n)$ of cardinality $m$ is at most $K_n^{2(4k+1)m}$.

For each small square $S_i$, $1 \leq i \leq K_n^2$, let $z_i$ denote its center point. Set $z_i := K_n x_i + \frac{1}{2} \in \mathbb{Z}^2$; see Figure 3. By the definition of $B_2(K_n)$, we can get that the set $\{z_i : 1 \leq i \leq K_n^2\}$ is equal to $B_2(K_n)$.

Recall that $\lambda(n) = n - n^{\beta}$, and $P_{\lambda(n)}$ denotes a Poisson point process in $[0, 1]^2$ with parameter $\lambda(n)$. Define the function

$$f_1(A) := \sum_{z_i \in A, z_j \in A', z_i \sim z_j} |P_{\lambda(n)} \cap S_i| \cdot |P_{\lambda(n)} \cap S_j|;$$

we can get the following lemmas.

LEMMA 4.8. Assume that $r_n$ satisfies (4.1). Suppose $A \subset B_2(K_n)$ and integer $k \geq 1$. Then for any constant $\beta \in (0, 1)$, there exists a constant $\eta = \eta(k, \beta) > 0$ such that for large enough $n$,

$$P \left[ \inf_{\substack{\beta n^{-2} K_n \leq |A| \leq (1-\beta)K_n^2 \\partial A \text{ is } k\text{-connected}}} \frac{f_1(A)}{|A|} \leq \frac{\eta \alpha_n^2}{K_n} \right] < e^{-n^{1/5}}.$$

Proof. This proof partly uses the ideas appearing in [18]. Let

$$c_1 := \frac{1 - \sqrt{1 - \beta}}{4 \sqrt{1 - \beta}} \quad \text{and} \quad c_2 := \frac{1}{4} (1 - \sqrt{1 - \beta}) \sqrt{\beta}.$$

If $\beta n^{-2} K_n \leq |A| \leq (1-\beta)K_n^2$, then by Lemma 4.3,

$$|\partial A| \geq \frac{1}{4} (1 - \sqrt{1 - \beta}) \sqrt{|A|} \geq c_1 |A| K_n,$$

and also

$$|\partial A| \geq \frac{1}{4} (1 - \sqrt{1 - \beta}) \sqrt{|A|} \geq \frac{c_2}{\alpha_n} \sqrt{K_n}.$$ 

For any $\varepsilon > 0$, by the definition of $f_1$ we can get

$$f_1(A) \geq (\varepsilon \alpha_n)^2 \sum_{z_i \in \partial A, z_j \in \partial (A'), z_i \sim z_j} I(|P_{\lambda(n)} \cap S_i| \geq \varepsilon \alpha_n, |P_{\lambda(n)} \cap S_j| \geq \varepsilon \alpha_n).$$

Fig. 3. The relationships of $S_i, x_i$, and $z_i$ are shown. If $z_i \sim z_j$, then any two vertices $x, y$ in $S_i \cup S_j$ will satisfy $\|x - y\| \leq r_n$.
1962

GE CHEN, ZHIXIN LIU, AND LEI GUO

For any set \( \Lambda, \Gamma \subset B_2(K_n) \), let

\[
\xi(\Lambda, \Gamma) := \sum_{z_i \in \Lambda, z_j \in \Gamma, z_i \sim z_j} I_{\{ |\partial \Lambda \cap S_i| \geq \varepsilon \alpha_n, |\partial \Gamma \cap S_j| \geq \varepsilon \alpha_n \}}.
\]

Therefore, by (4.2) and (4.4) we have

\[
\frac{f_1(A)}{|A|} \geq \frac{c_1(\varepsilon \alpha_n)^2 \xi(\partial A, \partial (A^c))}{K_n |\partial A|}.
\]

Combining the above inequality with (4.3) yields

\[
\inf_{\substack{\beta = 2, K_n \leq |A| \leq (1-\beta)K_n^2 \\
\partial A \text{ is } k\text{-connected}}} \frac{f_1(A)}{|A|} \geq \frac{c_1(\varepsilon \alpha_n)^2}{K_n} \inf_{|\partial A| \geq \varepsilon \alpha_n^2 \sqrt{K_n}} \frac{\xi(\partial A, \partial (A^c))}{|\partial A|}. \tag{4.5}
\]

Note that for any \( z \in \partial (A^c) \), there exists at least one vertex \( \tilde{z} \in \partial A \) such that \( z \sim \tilde{z} \), so if \( \partial A \) is \( k \)-connected, then \( \partial A \cup \partial (A^c) \) is also \( k \)-connected. Let \( (\Lambda_{1M,n}, \Gamma_{1M,n}), (\Lambda_{2M,n}, \Gamma_{2M,n}), \ldots, (\Lambda_{M,n}, \Gamma_{M,n}) \) denote all possible pairs of \( \partial A, \partial (A^c) \) satisfying (i) \( \partial A \) is \( k \)-connected, (ii) \( |\partial A \cup \partial (A^c)| = M \), and (iii) \( |\partial A| = m \).

Then by Lemma 4.7,

\[
\sum_{m=1}^{M} i_{M,n} \leq K_n^2 2^{k(k+1)M} \sum_{m=1}^{M} \left( \begin{array}{c} M \\ m \end{array} \right)
\]

\[
= K_n^2 2^{k(k+1)M} \cdot 2^M = K_n^{2(2k+1)^2} M. \tag{4.6}
\]

Thus, for any constant \( c_3 > 0 \), using Boole’s inequality we can get

\[
P \left( \inf_{\substack{\partial A \text{ is } k\text{-connected}}} \frac{\xi(\partial A, \partial (A^c))}{|\partial A|} \leq c_3 \right)
\]

\[
= P \left( \bigcup_{m \geq \varepsilon \alpha_n^2 \sqrt{K_n}} \bigcup_{M \geq m} \bigcup_{l=1}^{i_{M,n}} \left\{ \frac{\xi(\Lambda_{M,n}^l, \Gamma_{M,n}^l)}{M} \leq c_3 \right\} \right)
\]

\[
\leq P \left( \bigcup_{m \geq \varepsilon \alpha_n^2 \sqrt{K_n}} \bigcup_{M \geq m} \bigcup_{l=1}^{i_{M,n}} \left\{ \frac{\xi(\Lambda_{M,n}^l, \Gamma_{M,n}^l)}{M} \leq c_3 \right\} \right)
\]

\[
\leq \sum_{m \geq \varepsilon \alpha_n^2 \sqrt{K_n}} \sum_{M \geq m} \sum_{l=1}^{i_{M,n}} P \left( \xi(\Lambda_{M,n}^l, \Gamma_{M,n}^l) \leq c_3 M \right).
\]

For any \( z \in \Lambda_{1M,n} \) (or \( \Gamma_{1M,n} \)), \( 1 \leq l \leq i_{M,n} \), there exist at least one and at most four vertices in \( \Gamma_{1M,n} \) (or \( \Lambda_{1M,n} \)) which are connected with \( z \), and thus we can choose the vertex pairs \( (z_{i_1}, z_{i_2}), (z_{i_3}, z_{i_4}), \ldots, (z_{i_j}, z_{i_{j+1}}) \) \( \in (\Lambda_{1M,n}^l, \Gamma_{1M,n}^l) \), \( j(l) \geq M/8 \), such that \( z_{i_1} \sim z_{i_2} \sim z_{i_3} \sim z_{i_4} \sim \ldots \sim z_{i_{j(l)}} \sim z_{i_{j(l)}(k)} \) and \( z_{i_1}, z_{i_2}, z_{i_3}, \ldots, z_{i_{j(l)}}, z_{i_{j(l)(k)}} \) are mutually different. Thus, by the spatial independence of the Poisson point process, for

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any $1 \leq k_1 \neq k_2 \leq j(l)$, the corresponding events $I(\{P_{\lambda(n)} \cap S_{i_{k_1}} \geq \varepsilon \alpha_n, P_{\lambda(n)} \cap S_{i_{k_2}} \geq \varepsilon \alpha_n\}$ and $I(\{P_{\lambda(n)} \cap S_{i_{k_1}} \geq \varepsilon \alpha_n, P_{\lambda(n)} \cap S_{i_{k_2}} \geq \varepsilon \alpha_n\}$ are mutually independent. Let $E_k = I(\{P_{\lambda(n)} \cap S_{i_k} \geq \varepsilon \alpha_n, P_{\lambda(n)} \cap S_{i_k} \geq \varepsilon \alpha_n\}$, then

\begin{equation}
(4.8) \quad P\left(\xi(\Lambda_i^{M,m}, \Gamma_i^{M,m}) \leq c_3 M\right) \leq P\left(\sum_{k=1}^{j(l)} E_k \leq c_3 M\right),
\end{equation}

where $j(l) \geq M/8$ and the events $E_k$, $1 \leq k \leq j(l)$, are mutually independent.

Choose $\varepsilon = 1/2$; then for all large $n$ and $1 \leq k \leq j(l)$,

\begin{equation}
P(\xi(\Lambda_i^{M,m}, \Gamma_i^{M,m}) \leq c_3 M) = P\left(\left|\Lambda_i^{M,m} \cap S_{i_k}\right| \geq \frac{\alpha_n}{2}\right) = P\left(\left|\Lambda_i^{M,m} \cap S_{i_k}\right| \geq \frac{\alpha_n}{2}\right).
\end{equation}

where the last inequality follows from Lemma 1.2 in [15]. Therefore, for any $\rho > 0$ and large enough $n$, by Markov’s inequality we have

\begin{equation}
P\left(\sum_{k=1}^{j(l)} E_k \leq c_3 M\right) = P\left(\sum_{k=1}^{j(l)} E_k \geq -\rho \sum_{k=1}^{j(l)} E_k\right) \leq e^{-\rho c_3 M}\prod_{k=1}^{j(l)} E\left[e^{-\rho E_k}\right]
\end{equation}

Choose $c_3 > 0$ small enough; then there exist constants $\rho > 0$ and $c_4 > 0$ such that for large enough $n$,

\begin{equation}
(2k+1)^2 \log 2 + \rho c_3 \leq e^{-\rho c_3 M} \left(1 - n^{-H(\psi)/5}\right)^2 e^{-\rho} + 1 - \left(1 - n^{-H(\psi)/5}\right)^2 \leq -c_4.
\end{equation}

Combining (4.6)–(4.9) with (4.10), for large enough $n$ we have

\begin{equation}
P\left(\inf_{\delta A \text{ is } k\text{-connected}} \frac{\xi(\partial A, \partial(A^c))}{|\partial A|} \leq c_3\right) \leq \sum_{M \geq c_3 a_n} \sum_{m=1}^{M} \sum_{l=1}^{i_{M,m}} P\left(\sum_{k=1}^{j(l)} E_k \leq c_3 M\right).
\end{equation}
\[ \begin{align*}
&\leq \sum_{M \geq c_2 n^{1/4} \sqrt{n}} K_n^{2e} e^{2k+1} M e^{\rho c_3 M} \\
&\quad \cdot \left( \left( 1 - n^{-\left(\frac{\beta}{3}\right)/5}\right)^{2} e^{-\rho} + 1 - \left( 1 - n^{-\left(\frac{\beta}{3}\right)/5}\right)^{2}\right)^{M/8} \\
&\leq \sum_{M \geq c_2 n^{1/4} \sqrt{n}} K_n^{2e} e^{-c_4 M} \leq \exp \left( \frac{-c_4 c_2^{-1} \sqrt{n}}{2} \right) < e^{-n^{1/5}}.
\end{align*} \]

The above inequality and (4.5) yield our result.

For any \( z_i \in B_{\mathbb{Z}}(K_n) \), we call \( z_i \) open if \( S_i \cap \mathcal{P} \neq \emptyset \) and call \( z_i \) closed otherwise. Let \( \mathcal{O}_n \) denote the set of open vertices in \( B_{\mathbb{Z}}(K_n) \), and let \( \mathcal{C}_n \) denote the largest open clusters of \( \mathcal{O}_n \).

**Lemma 4.9.** Assume that \( n \) satisfies (4.1). Then with probability 1, \( |\mathcal{C}_n| = (1 - o(1))K_n^2 \) for all large enough \( n \).

**Proof.** For any \( z \in B_{\mathbb{Z}}(K_n) \),

\[ P(\{ z \text{ is closed} \}) = \exp \left( -\frac{\lambda(n)}{K_n^2} \right) \to 0 \quad \text{as } n \to \infty. \]

By Theorem 8.65 in [19] and Theorem 1 in [20] our result can be deduced.

**Lemma 4.10.** Assume that \( n \) satisfies (4.1). Suppose \( A \subset B_{\mathbb{Z}}(K_n) \). Then for any constant \( \beta \in (0, 1) \), there exists a constant \( \eta = \eta(\beta) > 0 \) such that for large enough \( n \),

\[ \inf_{\beta \leq \frac{\alpha_n^2 K_n}{|A|} \leq (1 - \beta)\frac{K_n^2}{M_A}, \ A \text{ is 3-connected}} \frac{f_1(A)}{|A|} \geq \frac{\eta \alpha_n^2}{K_n} \quad \text{a.s.} \]

**Proof.** For any \( A \subset B_{\mathbb{Z}}(K_n) \) with \( \beta \alpha_n^{-2} K_n \leq |A| \leq (1 - \beta)\frac{K_n^2}{M_A} \), let \( \Lambda_1, \ldots, \Lambda_m \) denote the connected components of \( A \), taken in decreasing order. In other words, \( \Lambda_1, \ldots, \Lambda_m \) are connected, but \( \Lambda_i \cup \Lambda_j, \ 1 \leq i \neq j \leq m, \) is not connected, and \( |\Lambda_1| \geq |\Lambda_2| \geq \cdots \geq |\Lambda_m| \). Since \( \Lambda_1, \ldots, \Lambda_m \) are all connected with \( A \), and \( A \) is 3-connected, \( \Lambda_i, \ 1 \leq i \leq m \) are all 3-connected. By Lemma 4.6, for \( 1 \leq i \leq m, \partial(\Lambda_i) \) is 3-connected and \( \partial(\Lambda_i) \) is 2-connected. By the definition of \( f_1 \) we can get

\[ f_1(A) = \sum_{i=1}^{m} f_1(\Lambda_i) = \sum_{i=1}^{m} f_1(\Lambda_i^c). \]

If \( |\Lambda_1| > K_n^2/2 \), then \( |\Lambda_1^c| \leq K_n^2/2 \). Note that \( A \supseteq \Lambda_1^c \), and by (4.11) and Lemma 4.8 we have

\[ \inf_{|A| \geq \beta \alpha_n^2 K_n, |A| \geq \frac{1}{2} \frac{K_n^2}{M_A}, \ A \text{ is 3-connected}} \frac{f_1(A)}{|A|} \geq \inf_{\frac{1}{2} \frac{K_n^2}{M_A} \leq |\Lambda_i^c|} \frac{f_1(\Lambda_i^c)}{|\Lambda_i^c|} \geq \frac{\eta \alpha_n^2}{K_n} \quad \text{a.s.} \]

Next we consider the case of \( |\Lambda_1| \leq K_n^2/2 \). Without loss of generality, we assume that \( |\Lambda_1| \geq \frac{1}{2} \alpha_n^{-2} K_n \) for \( 1 \leq i \leq i_A \), and \( |\Lambda_i| < \frac{1}{2} \alpha_n^{-2} K_n \) for \( i_A + 1 \leq i \leq m_A \), where
\( i_A \in [1, m_A]. \) Since \( \partial \Lambda_i \) is 2-connected, by Lemma 4.8 and the Borel–Cantelli lemma, with probability 1,

\[
\frac{f_1(\Lambda_i)}{|\Lambda_i|} > \frac{\eta \alpha_n^2}{K_n} \quad \forall 1 \leq i \leq m_A
\]

for all large \( n. \) Thus,

\[
\inf_{A \text{ is 3-connected}, |\Lambda_i| \leq \frac{1}{2} K_n^2} \frac{\sum_{i=1}^{m_A} f_1(\Lambda_i)}{\sum_{i=1}^{m_A} |\Lambda_i|} \geq \min_{1 \leq i \leq m_A} \left\{ \inf_{\forall i_A, \Lambda_i \text{ is 2-connected}} \frac{f_1(\Lambda_i)}{|\Lambda_i|} \right\} > \frac{\eta \alpha_n^2}{K_n} \quad \text{a.s.}
\]

(4.13)

holds for large enough \( n. \)

For \( \Lambda_i, i_A + 1 \leq i \leq m_A, \) if \( \Lambda_i \cap C_n \neq \emptyset, \) then \( f_1(\Lambda_i) \geq 1, \) which indicates that

\[
f_1(\Lambda_i) > \frac{1}{2\alpha_n^2 K_n} = \frac{2\alpha_n^2}{K_n}
\]

(4.14)

Let \( \eta' = \min\{\eta, 2\}. \) By (4.13) and (4.14) we can get, with probability 1,

\[
\inf_{A \text{ is 3-connected}, |\Lambda_i| \leq \frac{1}{2} K_n^2} \frac{\sum_{i=1}^{m_A} f_1(\Lambda_i) + \sum_{i_A+1 \leq i \leq m_A, \Lambda_i \cap C_n \neq \emptyset} f_1(\Lambda_i)}{\left( \sum_{i=1}^{m_A} + \sum_{i_A+1 \leq i \leq m_A, \Lambda_i \cap C_n \neq \emptyset} \right) |\Lambda_i|} \geq \frac{\eta' \alpha_n^2}{K_n}
\]

(4.15)

For the case of \( \Lambda_i \cap C_n = \emptyset, \) by Lemma 4.9, for large enough \( n, \)

\[
\sum_{i_A+1 \leq i \leq m_A, \Lambda_i \cap C_n = \emptyset} |\Lambda_i| \leq K_n^2 - |C_n| = o(K_n^2) \quad \text{a.s.}
\]

Moreover, note that \( \sum_{i=1}^{m_A} |\Lambda_i| = |A^c| \geq \beta K_n^2, \) so we have

\[
\left( \sum_{i=1}^{m_A} + \sum_{i_A+1 \leq i \leq m_A, \Lambda_i \cap C_n = \emptyset} \right) |\Lambda_i| = |A^c| - \sum_{i_A+1 \leq i \leq m_A, \Lambda_i \cap C_n = \emptyset} |\Lambda_i| \geq \frac{\beta}{2} K_n^2 \quad \text{a.s.}
\]

Combining the above inequality with (4.11) and (4.15), for large enough \( n, \) we have

\[
\inf_{|A| \leq (1-\beta) K_n^2, A \text{ is 3-connected}} f_1(A) \geq \frac{\eta' \alpha_n^2}{K_n} \cdot \frac{\beta K_n^2}{2} = \frac{\eta' \beta \alpha_n^2 K_n}{2} \quad \text{a.s.}
\]

By the above inequality, we can deduce that, with probability 1,

\[
\inf_{|A| \leq (1-\beta) K_n^2, A \text{ is 3-connected}} \frac{f_1(A)}{|A|} \geq \frac{\eta' \beta \alpha_n^2 K_n}{2} \cdot \frac{1}{(1-\beta) K_n^2} = \frac{\eta' \beta \alpha_n^2}{2(1-\beta) K_n}
\]
for large $n$. Combining this with (4.12) yields our result. \hfill \qed

Recall that $d_i$ denotes the degree of vertex $i$ in $G(0)$, and set $d^* := \sum_{i=1}^n d_i$. We can get the following lemma.

**Lemma 4.11.** Assume that $r_n$ satisfies (4.1). Then for any constant $s > 1/\pi$, with probability 1, $d^* > n^2 r_n^2 / s$ for large enough $n$.

**Proof.** Given a constant $s' \in (1/\pi, s)$, let $Z_n(s')$ denote the number of vertices of $G(0)$ of degree at least $n r_n^2 / s'$. By Theorem 4.2 in [15], $n^{-1} Z_n(s')$ converges completely to 1 as $n \to \infty$. Since $d^* \geq Z_n(s') n^2 r_n^2 / s'$, this yields our result. \hfill \qed

**Proof of Theorem 3.3.** If $G(0)$ is not connected, then $P(0)$ is reducible and therefore $\lambda_0 = 1$, so our necessary condition can be deduced directly by Lemma 2.1. Also, for the case of $\pi n r_n^2 \geq (\log n)^2$, our sufficient condition has been indicated by (3.3). Thus, we just need to consider the sufficient condition for the case that $r_n$ satisfies (4.1).

Given $\lambda \in \mathbb{R}$, if $\lambda < \frac{1}{2 \pi r_n^2} - 1$, then $P(0) - \lambda I_n$ is a strictly diagonally dominant matrix and $\det(P(0) - \lambda I_n) \neq 0$. Therefore $\lambda$ is not an eigenvalue of $P(0)$. By Lemma 3.1 we can get that for all large enough $n$, with probability 1,

$$
\lambda_n \geq \frac{1}{e \pi n r_n^2 (1 + o(1))} - 1.
$$

Note that $1 \geq \lambda_2 \geq \lambda_n \geq -1$, so we just need to estimate $\lambda_2$ to get our result.

Let $F \subseteq \{1, 2, \ldots, n\}$ denote a subset of agents and define $\tilde{F} := \{X_i(0) : i \in F\} \subseteq X_n$ to be the initial positions of agents in $F$. Let $F^c = \{1, 2, \ldots, n\} \setminus F$. For any area $D_1, D_2 \subseteq [0, 1]^2$, set

$$
f_{D_1, D_2}(F) := \sum_{x \in D_1, y \in D_2} I_{\{|x-y| \leq r_n\}}
$$

and take $f(F) = f_{[0,1]^2,[0,1]^2}(F)$. Define Cheeger’s constant $\Phi$ of $P(0)$ by

$$
\Phi = \inf_{\sum_{i \in F} d_i \leq \frac{1}{2} d^*} \frac{\sum_{i \in F} d_i}{\sum_{i \in F} f_i}.
$$

We assert that there exists a constant $\eta > 0$ such that w.h.p., $\Phi \geq \eta r_n$ for large enough $n$. Next we will prove this assertion.

For any $F \subseteq \{1, 2, \ldots, n\}$, set

$$
A_F := \left\{ z_i : |S_i \cap \tilde{F}| > \frac{1}{2} |S_i \cap X_n| \right\} \subseteq B_2(K_n),
$$

and define

$$
\tilde{A}_F := \bigcup_{z \in A_F} S_i \cap X_n.
$$

By the condition of $\sum_{i \in F} d_i \leq \frac{1}{2} d^*$, we have

$$
|F^c| d_{\max} \geq \sum_{i \in F^c} d_i \geq \frac{1}{2} d^*;
$$

then by Corollary 3.2 and Lemma 4.11, for all large enough $n$,

$$
|F^c| \geq \frac{d^*}{2 d_{\max}} \geq \frac{2 e \pi n r_n^2}{6} \cdot \frac{1}{2 e \pi n r_n^2} = \frac{n}{6} \quad \text{a.s.}
$$

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Set $\beta := \frac{1}{252}$. If $|A_F| > (1 - \beta)K_n^2$, then $|A_F| \leq |A_F| \Delta_n \leq \frac{K_n^2}{252}$. By Lemma 4.1,

$$\sum_{z_i \in A_F} |S_i| \leq |A_F| \Delta_n \leq \frac{K_n^2}{252} 21n = \frac{n}{12}$$

holds almost surely for large enough $n$. Note that $|\overline{F^c}| = |F^c| > \frac{n}{2}$; then there exist at least $\frac{n}{2}$ vertices of $\overline{F^c}$ contained in $A_F$. For $x \in \overline{F^c} \cap A_F$, without loss of generality we assume that $x \in S_i$ with $z_i \in A_F$; then by the definition of $A_F$ we can get $|F \cap S_i| \geq |\overline{F^c} \cap S_i| \geq 1$, which indicates that there exists at least one vertex $y \in F \cap S_i$ such that $\|x - y\| \leq r_n$. Thus, by Lemma 3.1,

$$\sum_{i \in F} d_i \leq \frac{1}{d^*} |A_F| > (1 - \beta)K_n^2 \sum_{i \in F} d_i \geq \frac{n}{12} \eta \max_{i} d_i$$

(4.16)

holds almost surely for large enough $n$.

Now, we will consider the case of $|A_F| \leq (1 - \beta)K_n^2$. Let $A_1, A_2, \ldots, A_{m_F}$ be the 3-connected components of $A_F$, taken in decreasing order of size. In other words, $A_1, \ldots, A_{m_F}$ are all 3-connected, but $A_i \cup A_j$, $1 \leq i \neq j \leq m_F$, is not 3-connected, and $|A_1| \geq |A_2| \geq \cdots \geq |A_{m_F}|$. Without loss of generality, we assume that $|A_i| \geq \beta \alpha_n^{-2}K_n$ for $1 \leq i \leq i_F$, and $|A_i| < \beta \alpha_n^{-2}K_n$ for $i_F + 1 \leq i \leq m_F$, where $i_F \in [1, m_F]$. Then by Lemma 4.10, there exists a constant $\eta' > 0$ such that

$$\inf_{|A_F| \leq (1 - \beta)K_n^2, 1 \leq i \leq i_F} \frac{f_1(A_i)}{|A_i|} \geq \frac{\eta' \alpha_n^2}{K_n} \ a.s.$$ 

(4.17)

For $i \in [1, i_F]$, it is easy to see that if $z_k \in A_i$ and $z_j \in A_F$ with $z_k \sim z_j$, then $z_j \in A_F$, and all pairs of vertices in $S_k \cup S_j$ are neighbors. So by the definition of $A_F$, we have

$$f_{S_k, S_j}(F) = \sum_{x \in S_k \cap F, y \in S_j \cap F^c} I_{\{\|x - y\| \leq r_n\}} \geq \frac{1}{4} |\mathcal{X}_n \cap S_k| \cdot |\mathcal{X}_n \cap S_j|.$$ 

Therefore, if $\mathcal{P}_{\lambda(n)} \subseteq \mathcal{X}_n$, then

$$\sum_{z_k \in A_i, z_j \in A_F, z_k \sim z_j} f_{S_k, S_j}(F) = \sum_{z_k \in A_i, z_j \in A_F, z_k \sim z_j} f_{S_k, S_j}(F) \geq \frac{1}{4} f_1(A_i).$$ 

(4.18)

Moreover, by (3.12) and the Borel–Cantelli lemma, we know that $\mathcal{P}_{\lambda(n)} \subseteq \mathcal{X}_n$ holds almost surely for large enough $n$. Set

$$S^1_F := \bigcup_{i=1}^{i_F} \bigcup_{z_k \in A_i} S_k.$$ 

By (4.18), for large enough $n$, we have

$$f_{S^1_F, [0,1]^2 \setminus S^1_F}(F) \geq \sum_{i=1}^{i_F} \sum_{z_k \in A_i, z_j \in A_F, z_k \sim z_j} f_{S_k, S_j}(F) \geq \frac{1}{4} f_1(A_i) \quad \text{a.s.}$$ 

(4.19)
For $i \in [i_{F} + 1, m_{F}]$, if $\bigcup_{j \in A_{i}} S_{j} \cap \mathring{F} \neq \emptyset$, let $D_{i} = \bigcup_{j \in A_{i}} S_{j}$. Then we have $f_{D_{i}, D_{i}}(F) \geq 1$; otherwise, by Lemma 2.1, with high probability there exists at least one vertex $x^{*} \in (\bigcup_{j \in A_{i}} S_{j})^{c} \cap X_{n}$ such that the set

$$\left\{ y : y \in \bigcup_{j \in A_{i}} S_{j} \cap \mathring{F}, \|x^{*} - y\| \leq r_{n} \right\}$$

is not empty. Assume that $x^{*} \in S_{k}(1 \leq k \leq K_{n}^{2})$ and $z_{k}$ is the corresponding integer point of $S_{k}$. Then $z_{k}$ must be 3-connected with $A_{i}$, and $z_{k} \in A_{F}$. Set $D_{i} = \bigcup_{j \in A_{i}} S_{j} \cup S_{k}$. If $x^{*} \in \mathring{F}^{c}$, then $f_{D_{i}, D_{i}}(F) \geq 1$; otherwise, by the definition of $A_{F}$ we have $S_{k} \cap \mathring{F} \neq \emptyset$, so

$$f_{D_{i}, D_{i}}(F) \geq f_{S_{k}, S_{k}}(F) \geq 1.$$ 

Let $S_{F}^{2} = \bigcup_{i = i_{F} + 1}^{m_{F}} D_{i}$. For $z \in \mathbb{Z}^{2}$, it is easy to see that the number of 3-connected components that $z$ is 3-connected with is not more than 8. By the above argument we have w.h.p.

$$f_{S_{F}^{3}, S_{F}^{3}}(F) \geq \frac{1}{8}(m_{F} - i_{F}).$$ 

Let $S_{F}^{3} = [0, 1]^{2} \setminus (S_{F}^{1} \cup S_{F}^{2})$. For $x \in S_{F}^{3} \cap \mathring{F}$, assume that $x \in S_{k}(1 \leq k \leq K_{n}^{2})$ and $z_{k} \in B_{2}(K_{n})$ is the corresponding integer point of $S_{k}$. Obviously $z_{k} \in A_{F}$, so the set $S_{k} \cap \mathring{F}$ is not empty. Thus,

$$f_{S_{F}^{3}, S_{F}^{3}}(F) \geq \sum_{x \in S_{F}^{3} \cap \mathring{F}} 1 \geq |S_{F}^{3} \cap \mathring{F}|.$$ 

Recall that $\mathcal{L}(\cdot)$ denotes the Lebesgue measure. By the definitions of $S_{F}^{1}$ and $S_{F}^{2}$, we have $\mathcal{L}(S_{F}^{1} \cap S_{F}^{2}) = 0$. So by (4.19), (4.20), and (4.21), we have

$$f(F) \geq f_{S_{F}^{1}, [0, 1]^{2} \setminus S_{F}^{1}}(F) + f_{S_{F}^{2}, S_{F}^{2}}(F) + f_{S_{F}^{3}, S_{F}^{3}}(F) \geq \sum_{i = 1}^{i_{F}} \frac{1}{4} f_{1}(A_{i}) + \frac{1}{8}(m_{F} - i_{F}) + |S_{F}^{3} \cap \mathring{F}| \text{ w.h.p.}$$

Thus, w.h.p.,

$$\inf_{|A_{F}| \leq (1 - \beta)K_{n}^{2}} \frac{f(F)}{\sum_{i \in F} d_{i}} \geq \frac{\sum_{i = 1}^{i_{F}} \frac{1}{4} f_{1}(A_{i}) + \frac{1}{8}(m_{F} - i_{F}) + |S_{F}^{3} \cap \mathring{F}|}{d_{\max}(|S_{F}^{1} \cap \mathring{F}| + |S_{F}^{2} \cap \mathring{F}| + |S_{F}^{3} \cap \mathring{F}|)} \geq \frac{\sum_{i = 1}^{i_{F}} \frac{1}{4} f_{1}(A_{i}) + \frac{1}{8}(m_{F} - i_{F}) + |S_{F}^{3} \cap \mathring{F}|}{d_{\max}\Delta_{n} \sum_{i = 1}^{i_{F}} |A_{i}| + (m_{F} - i_{F})\Delta_{n}\beta\alpha_{n}^{2}K_{n} + |S_{F}^{3} \cap \mathring{F}|)} \geq \min \left\{ \frac{\frac{1}{4} \sum_{i = 1}^{i_{F}} f_{1}(A_{i})}{d_{\max}\Delta_{n} \sum_{i = 1}^{i_{F}} |A_{i}|}, \frac{\frac{1}{8}(m_{F} - i_{F})}{d_{\max}(m_{F} - i_{F})\Delta_{n}\beta\alpha_{n}^{2}K_{n} + d_{\max}|S_{F}^{3} \cap \mathring{F}|}, \frac{1}{4d_{\max}\Delta_{n}K_{n}}, \frac{8d_{\max}\Delta_{n}\beta K_{n}^{2}}{\max|S_{F}^{3} \cap \mathring{F}|} \right\} \geq \min \left\{ \frac{1}{4d_{\max}\Delta_{n}K_{n}}, \frac{1}{8d_{\max}\Delta_{n}\beta K_{n}^{2}}, \frac{1}{\max|S_{F}^{3} \cap \mathring{F}|} \right\}.$$
where the last inequality can be deduced by (4.17).

Combining (4.1), (4.16), (4.22) with Lemmas 3.1 and 4.1, our assertion holds.

By Cheeger’s inequality (Proposition 6 in [16]), we have \( \lambda_2 \leq 1 - \Phi^2 \). Hence with probability 1, \( \lambda_2 \leq 1 - \eta^2 r^2 n \) holds for large enough \( n \). This completes the proof of our result.

5. Simulation example. In this section, we will provide a simulation example. Here, the number of agents is taken as \( n = 1000 \), and the interaction radius is \( r_n = \sqrt{1.1 \log n / (\pi n) \} \). The initial positions and headings of the \( n \) agents are independent, with positions uniformly and independently distributed in \( [0, 1]^2 \), and with headings uniformly and independently distributed in \( (-\pi, \pi] \). Figure 4 shows how the probability of synchronization changes with moving speed. From this simulation, we see that if the speed is small, the system can synchronize w.h.p., and the probability of synchronization will tend to zero as the speed increases.

![Simulation results for the system with \( n = 1000 \), \( r_n = \sqrt{1.1 \log n / (\pi n) \} \), and the random initial states.](image)

6. Concluding remarks. For the multiagent systems or flocks studied in this paper, it is intuitively obvious that the smaller the interaction radius is, the harder it is for the synchronization to happen. Thus, an important and interesting problem is how small the interaction radius can be in order to guarantee synchronization. This paper shows that in a certain sense, the smallest possible interaction radius for synchronization can be considered as the same as the critical radius for connectivity of the initial random geometric graph. We remark that an important step of this paper is to provide an estimation of the spectral gap of the average matrix of the random geometric graph. This result may be applied to other interesting problems, such as the mixing times and the hitting times of random walk on random geometric graphs.

REFERENCES