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Decentralized tracking-type games for multi-agent systems with coupled ARX models: Asymptotic Nash equilibria[☆]

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Abstract

A class of decentralized tracking-type games is considered for large population multi-agent systems (MAS). The agents are described by stochastic discrete-time auto-regressive models with exogenous inputs (ARX models), and coupled together through their individual dynamics and performance indexes by terms of the unknown population state average (PSA). The performance index of each agent to minimize is a stochastic long term averaged group-tracking-type functional, in which there is a nonlinear term of the unknown PSA. The control law is decentralized and implemented via the Nash certainty equivalence principle. By probability limit theory, under mild conditions it is shown that: (a) the estimate of the PSA is strongly consistent; (b) the closed-loop system is stable almost surely, and the stability is independent of the number N of agents; (c) the decentralized control law is an asymptotic Nash equilibrium almost surely or in probability according to the property of the nonlinear coupling function in the performance indexes.

Keywords: Multi-agent systems; Decentralized control; Optimal tracking; ARX model; Discrete-time systems; Nash certainty equivalence principle; Stochastic dynamic game

1. Introduction

In this paper, a class of individual–population interacting stochastic multi-agent systems (MAS) is considered, which has a wide practical background in engineering (Huang, Caines, & Malhamé, 2004a), biological (Erdmann, Ebeling, & Mikhailov, 2005; Mach & Schweitzer, 2003), social and economic systems (Erickson, 1995; Huang, Caines, & Malhamé, 2007). This class of MAS has the following prominent characteristics: (a) Each agent has an integrated ability of sensing, decision-making and communicating, also a control objective of its own interest, so it can be viewed as a rational decision-maker. (b) Interactions between the individual states and population state average (PSA) exist in both agents' dynamics and control objectives. (c) Dynamics of agents are often influenced by stochastic disturbances. (d) The number N of agents is usually very large. Though individual agent behaves as a stochastic process, the whole system often takes on certain deterministic pattern at the macroscopic level, therefore the asymptotic property of the system is an interesting and important issue to be investigated when N increases to infinity. This class of systems belongs to large population MAS (Altman, Basar, & Srikant, 2002; Green, 1984; Morale, Capasso, & Oelschläger, 2005) and can be viewed as large scale coupled stochastic systems. The optimization of such systems can be viewed as a stochastic distributed game.

Under the game-theoretic framework, the optimization of coupled systems can be divided into two categories: centralized control (Engwerda, 2000; Lim & Gajic, 1999; Mukaidani, 2006; Mukaidani & Xu, 2004; Weeren, Schumacher, & Engwerda, 1999) and decentralized control (Bauso, Giarré, & Pesenti, 2006; Huang, Caines, & Malhamé, 2003, 2004b, 2007; Li & Zhang, 2006, 2007a; Ma, Li, & Zhang, 2007)

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Centralized control refers to the case where the states (or outputs) of all agents are available for each agent to design its control law, while decentralized control refers to the case where only local states (under certain circumstances, including those in its sensing or communicating neighborhood) are available. In MAS, there are often no central control stations, and agents have only limited sensing ability. Therefore, the control laws for MAS are usually required to be decentralized. Bauso et al. (2006) considered consensus protocol design of cost-coupled deterministic MAS described by non-coupled first order scalar differential equations. A receding horizon method was used to decouple the indexes, and construct the approximate optimal strategies. Huang et al. (2003, 2004b, 2007) and Li and Zhang (2007a) considered the decentralized control of cost-coupled stochastic MAS described by non-coupled stochastic differential equations. Inspired by the work of Huang et al. (2003, 2004b, 2007) on deterministic indexes, Li and Zhang (2007a) investigated the stochastic index case where each agent possesses a stochastic linearly coupled quadratic performance index. Based on the introduced notions of *asymptotic Nash equilibria in the probabilistic sense*, a decentralized control law was given and shown to be asymptotically optimal almost surely in the sense of Nash equilibrium as the number N of agents increases to infinity. But, how to get the convergence rate of the sub-optimal index values to the optimal one is still open, irrespective of the performance indexes being deterministic (Huang et al., 2003, 2004b, 2007) or stochastic (Li & Zhang, 2007a). This is worth studying since the convergence rate is a measure of evaluating the performance of the decentralized control law.

In this paper, the decentralized game is investigated for individual–population interacting stochastic MAS. Compared with the previous works, this paper is characterized by the following features: (a) The dynamic equation of each agent is described by a discrete-time ARX model, and coupled with each other by the PSA. Different from the non-coupled dynamics considered in Li and Zhang (2007a), here the closed-loop stability is closely dependent on the convergence of the estimate of the PSA. (b) The performance indexes of the agents are coupled together nonlinearly, which may result in more complex population behaviors in the closed-loop system. Under mild conditions, we not only prove that the decentralized control law is asymptotically optimal almost surely, but also obtain the rate of the associated performance indexes converging to the corresponding optimal values. It is shown that when the nonlinear coupling function is Hölder continuous with exponent μ, the convergence rate is $O(1/N^{\mu})$ as N increases to infinity. Some preliminary results of this work have been reported in Li and Zhang (2007b).

The remainder of this paper is organized as follows. In Section 2, the individual performance based optimization problem is formulated. In Section 3, a detailed design procedure of the decentralized control law is presented, which is based on the estimate of the PSA and the so-called Nash certainty equivalence (NCE) principle. In Section 4, by using probability limit theory, the stability and optimality of the closed-loop system are analyzed. In Section 5, two numerical examples are given to illustrate our results. In Section 6, some concluding remarks and further research topics are discussed.

The following notations will be used throughout this paper. $\mathbb{R}^m$ denotes the set of all m-dimensional real column vectors; $\mathbb{R}^{m\times d}$ denotes the set of all $m \times d$ dimensional real matrices; I_m denotes the m-dimensional unit matrix. For a given vector or matrix X, X^{T} denotes its transpose; $\|X\|$ denotes the Euclidean-norm of X; when X is square, $\rho(X)$ denotes its spectral radius; $\mathrm{tr}(X)$ denotes its trace; $\lambda_{\max}(X)$ denotes the maximum eigenvalue of X. For a given random variable (r.v.) ξ on a probability space $(\Omega, \mathscr{F}, P)$, $E(\xi)$ denotes the mathematical expectation of ξ. For a given class $\mathfrak{A}$ of sets, $\sigma(\mathfrak{A})$ denotes the σ-algebra generated by $\mathfrak{A}$. For a family $\{\xi_\lambda, \lambda \in \Lambda\}$ of $\mathbb{R}^m$-valued r.v.s, $\sigma(\xi_\lambda, \lambda \in \Lambda)$ denotes the σ-algebra $\sigma(\{\xi_\lambda \in B\}, B \in \mathscr{B}^m, \lambda \in \Lambda)$, where $\mathscr{B}^m$ denotes the m-dimensional Borel sets. For a sequence $\{\mathscr{F}_t, t \geqslant 0\}$ of non-decreasing σ-algebras and a sequence $\{\xi(t), t \geqslant 0\}$ of r.v.s, we say $\xi(t)$ is adapted to $\mathscr{F}_t$ or $\{\xi(t), \mathscr{F}_t\}$ is an adapted sequence, if for any $t \geqslant 0$, $\xi(t)$ is $\mathscr{F}_t$ measurable.

2. Preliminaries and problem formulation

We denote a system of N agents by $\mathbf{S}^N$, and the dynamic equation for the ith agent is given by

$$x_i^N(t+1) = g_i(x_i^N(t), t) + u_i^N(t) + \Lambda \bar{x}_N(t) + \omega_i(t+1), \quad t \geqslant 0, \tag{1}$$

where $x_i^N \in \mathbb{R}^m$, $u_i^N \in \mathbb{R}^m$ are state and control input, respectively; $\bar{x}_N(t) \triangleq (1/N)\sum_{j=1}^N x_j^N(t)$ is the PSA; $\omega_i(t) \in \mathbb{R}^m$ is the random noise; $g_i(\cdot,\cdot) : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is a Borel measurable function; $\Lambda \in \mathbb{R}^{m\times m}$ is called the coupling parameter matrix satisfying $\rho(\Lambda) < 1$.

For model (1), we have the following assumptions:

(A1) $\{\{\omega_i(t), \mathscr{F}_t^i\}, i \geqslant 1\}$ is a family of independent martingale difference sequences defined on a probability space $(\Omega, \mathscr{F}, P)$ with the following properties:

$$\sup_{t\geqslant 0} E[\|\omega_i(t)\|^2 | \mathscr{F}_{t-1}^i] < \infty \quad \text{a.s.},$$
$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n} \omega_i(t)\omega_i^{\mathrm{T}}(t) = R_\sigma \quad \text{a.s.}, \tag{2}$$

where $\mathscr{F}_t^i \triangleq \sigma(\omega_i(s), 0 \leqslant s \leqslant t)$, $R_\sigma \in \mathbb{R}^{m\times m}$ is an m-dimensional nonnegative definite matrix.

(A2) $\{x_i^N(0), 1 \leqslant i \leqslant N, N \geqslant 1\}$ is independent of $\{\{\omega_i(t), \mathscr{F}_t^i\}, i \geqslant 1\}$, with a common mathematical expectation $x_0 \triangleq E x_1^1(0) < \infty$.

The performance index of agent i is described by

$$J_i^N(u_i^N, u_{-i}^N) = \limsup_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n} \|x_i^N(t+1) - \Phi(\bar{x}_N(t))\|^2, \tag{3}$$

where $u_{-i}^N = (u_1^N, \ldots, u_{i-1}^N, u_{i+1}^N, \ldots, u_N^N)$, $\Phi(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is a Borel measurable function.

Remark 1. The model (1) with index (3) has a wide background in biological and engineering systems. The Brownian agent swarm systems (Erdmann et al., 2005; Mach & Schweitzer, 2003) are such examples, where the acceleration of agent i depends on not only its own state variables (e.g. position r_i, velocity v_i, energy e_i), control u_i, Gaussian white noise ω_i, but also the population position average $(1/N)\sum_{j=1}^{N} r_j$. The dynamic equations are coupled together via the population position average $(1/N)\sum_{j=1}^{N} r_j$. Other index-coupled examples can be found in wireless communication networks (Huang et al., 2004a) and production output adjustment problems (Huang et al., 2007). In the former example, the changing rate of the received power p_i for user i depends on p_i, control u_i, random noise ω_i. Each user makes its own strategy u_i to ensure the signal-to-interference-ratio to be around a target level γ. This can be formulated by the following coupled index group:

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n}\left[p_i(t)-\gamma\left(\frac{1}{N}\sum_{j=1}^{N}p_j(t)+\eta\right)\right]^2,\quad i=1,2,\ldots,N.$$

Here η is the constant background noise intensity. In the latter example, the changing rate of production output level x_i of firm i depends on control u_i and random noise ω_i. Each firm makes its strategy u_i to ensure its production level to be approximately proportional to the price provided by the current market. This can be formulated by the following coupled index group:

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n}[x_i(t)-\beta p]^2,\quad i=1,2,\ldots,N,$$

where $p=\overline{\eta}-(\overline{\gamma}/N)\sum_{j=1}^{N}x_j(t)$ is the price depending on the production level average $(1/N)\sum_{j=1}^{N}x_j(t)$, and $\beta,\overline{\gamma},\overline{\eta}$ are positive constants.

Remark 2. The model (1) is a discrete-time first order nonlinear ARX model, where x_i is the output, u_i is the control input, ω_i is the noise input. Nonlinear ARX models can be found in the analysis and design of many discrete-time systems (Chen & Tsay, 1993; De Nicolao, Magni, & Scattolini, 1997; Fan & Yao, 2005).

For convenience of citation, for agent i, we denote the *global-measurement-based admissible control set* by

$$\mathscr{U}_{g,i}^{N}\triangleq\left\{u\mid u(t)\text{ is adapted to }\sigma\left(\bigcup_{j=1}^{N}\sigma(x_j^N(s),0\leqslant s\leqslant t)\right)\right\},$$

local-measurement-based admissible control set by

$$\mathscr{U}_{l,i}^{N}\triangleq\{u\mid u(t)\text{ is adapted to }\sigma(x_i^N(s),0\leqslant s\leqslant t)\},$$

and admissible control set by $\mathscr{U}_i^N$. The so-called decentralized game means that agent i synthesizes u_i^N only based on the local measurement (i.e. $\mathscr{U}_i^N=\mathscr{U}_{l,i}^N$) to minimize its index $J_i^N(u_i^N,u_{-i}^N)$.

For investigating the asymptotic property of the whole system when $N\to\infty$, we have to analyze a sequence of systems $\{\mathbf{S}^N, N\geqslant 1\}$. To do so, we denote a control group of $\mathbf{S}^N$ by $\mathbf{U}^N=\{u_i^N, 1\leqslant i\leqslant N\}$, and its corresponding index group by $\mathbf{J}^N=\{J_i^N(u_i^N,u_{-i}^N), 1\leqslant i\leqslant N\}$.

Before designing the decentralized control law for (1), we present the optimal index value for each agent under the centralized control law. To this end, we need the following lemma.

Lemma 2.1 (Chen and Guo, 1991). *Let $\{W(t),\mathscr{F}_t\}$ be a matrix martingale difference sequence, $\{M(t),\mathscr{F}_t\}$ an adapted sequence of random matrices, $\|M(t)\|<\infty$, $\forall t\geqslant 0$. If*

$$\sup_{t\geqslant 0} E[\|W(t)\|^2|\mathscr{F}_{t-1}]<\infty\quad\text{a.s.},$$

then for any given $\varepsilon>0$,

$$\sum_{t=0}^{n}M(t)W(t+1)=O\left(\left(\sum_{t=0}^{n}\|M(t)\|^2\right)^{1/2+\varepsilon}\right)\quad\text{a.s.}$$

Theorem 2.1. *For system (1) with index (3), if Assumptions (A1) and (A2) hold, then under any control group $\mathbf{U}^N=\{u_i^N\in U_{g,i}^N, 1\leqslant i\leqslant N\}$, the corresponding index group $\mathbf{J}^N=\{J_i^N(u_i^N,u_{-i}^N), 1\leqslant i\leqslant N\}$ has the following property:*

$$J_i^N(u_i^N,u_{-i}^N)\geqslant \mathrm{tr}(R_\sigma)\quad\text{a.s.},\quad i=1,2,\ldots,N.$$

Moreover, under the control group

$$u_i^N(t)=\Phi(\overline{x}_N(t))-g_i(x_i^N(t),t)-\Lambda\overline{x}_N(t),\quad t\geqslant 0,\quad i=1,2,\ldots,N,\tag{4}$$

the corresponding index group satisfies:

$$J_i^N(u_i^N,u_{-i}^N)=\mathrm{tr}(R_\sigma)\quad\text{a.s.},\quad i=1,2,\ldots,N.$$

Proof. Denote $\mathscr{F}_0^N=\sigma(x_i^N(0), i=1,2,\ldots,N)$. Then, from Assumptions (A1) and (A2), it is known that $\{\omega_i(t),\sigma(\mathscr{F}_0^N\cup(\bigcup_{j=1}^N\mathscr{F}_t^j))\}$ is a martingale difference sequence, and

$$\sup_{t\geqslant 0} E\left[\|\omega_i(t)\|^2|\sigma\left(\mathscr{F}_0^N\cup\left(\bigcup_{j=1}^{N}\mathscr{F}_{t-1}^i\right)\right)\right]<\infty\quad\text{a.s.},$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\omega_i(t)\omega_i^{\mathrm{T}}(t)=R_\sigma\quad\text{a.s.}$$

Noticing that $u_i^N\in\mathscr{U}_{g,i}^N$, from model (1) we have $x_i^N(t)$ is adapted to $\sigma(\mathscr{F}_0^N\cup(\bigcup_{j=1}^N\mathscr{F}_t^j))$, $u_i^N(t)$ is adapted to $\sigma(\mathscr{F}_0^N\cup(\bigcup_{j=1}^N\mathscr{F}_t^j))$, $\overline{x}_N(t)$ is adapted to $\sigma(\mathscr{F}_0^N\cup(\bigcup_{j=1}^N\mathscr{F}_t^j))$, $\forall t\geqslant 0$. Thus, similar to Chen and Guo (1991, Theorem 3.6), by Lemma 2.1 we can get Theorem 2.1. □

From Theorem 2.1, it can be seen that the optimal index value of each agent can attain the lower bound $\mathrm{tr}(R_\sigma)$, provided that the states of all agents are available for the control design. However, due to information restriction, the decentralized control law may only make the indexes sub-optimal. Below we will present some notions of asymptotic Nash equilibria in the probabilistic sense to quantitatively describe the sub-optimality of the control groups with respect to the stochastic indexes. First we present the notion of ε-Nash equilibrium with respect to deterministic indexes for comparison.

Definition 2.1 (*Baser and Olsder, 1982*). A control group $\{u_i^N \in \mathscr{U}_i^N, 1 \leqslant i \leqslant N\}$ is called an ε-Nash equilibrium with respect to the index group $\{J_i, 1 \leqslant i \leqslant N\}$ if there exists $\varepsilon > 0$ such that for any $1 \leqslant i \leqslant N$,

$$J_i(u_i^N, u_{-i}^N) \leqslant \inf_{v_i \in \mathscr{U}_{g,i}^N} J_i(v_i, u_{-i}^N) + \varepsilon.$$

Below we give the definitions of asymptotic Nash equilibria in the probabilistic sense.

Definition 2.2. For system (1), a sequence of control groups $\{\mathbf{U}^N = \{u_i^N \in \mathscr{U}_i^N, 1 \leqslant i \leqslant N\}, N \geqslant 1\}$ is called an asymptotic Nash equilibrium in probability with respect to the corresponding sequence of index groups $\{\mathbf{J}^N = \{J_i^N, 1 \leqslant i \leqslant N\}, N \geqslant 1\}$, if for any $\varepsilon > 0$, $\delta > 0$, there exists $M > 0$ such that for any $N > M$,

$$P\left\{\max_{1 \leqslant i \leqslant N}\left\{J_i^N(u_i^N, u_{-i}^N) - \inf_{v_i \in \mathscr{U}_{g,i}^N} J_i^N(v_i, u_{-i}^N)\right\} \geqslant \varepsilon\right\} \leqslant \delta. \tag{5}$$

Definition 2.3. For system (1), a sequence of control groups $\{\mathbf{U}^N = \{u_i^N, 1 \leqslant i \leqslant N\}, N \geqslant 1\}$ is called an almost sure asymptotic Nash equilibrium with respect to the corresponding sequence of index groups $\{\mathbf{J}^N = \{J_i^N, 1 \leqslant i \leqslant N\}, N \geqslant 1\}$, if there exists a sequence of non-negative r.v.s $\{\varepsilon_N(\omega), N \geqslant 1\}$ on the probability space $(\Omega, \mathscr{F}, P)$, such that $\varepsilon_N \to 0$ a.s., as $N \to \infty$, and for sufficiently large N,

$$J_i^N(u_i^N, u_{-i}^N) \leqslant \inf_{v_i \in \mathscr{U}_{g,i}^N} J_i^N(v_i, u_{-i}^N) + \varepsilon_N \quad \text{a.s.}, \qquad i = 1, 2, \ldots, N. \tag{6}$$

Remark 3. From Theorem 2.1, it can be seen that

$$\inf_{v_i \in \mathscr{U}_{g,i}^N} J_i^N(v_i, u_{-i}^N) = \mathrm{tr}(R_\sigma) \quad \text{a.s.}$$

Therefore, the $\inf_{v_i \in \mathscr{U}_{g,i}^N} J_i^N(v_i, u_{-i}^N)$ in (5) and (6) can be replaced with the optimal index value based on global measurement $\mathrm{tr}(R_\sigma)$. An asymptotic Nash equilibrium in probability is to say that for any given $\varepsilon > 0$, when the number N of agents is sufficiently large, there is a large probability to ensure that the control group $\mathbf{U}^N = \{u_i^N, 1 \leqslant i \leqslant N\}$ is an ε-Nash equilibrium with respect to the index group $\mathbf{J}^N = \{J_i^N, 1 \leqslant i \leqslant N\}$. It can be interpreted intuitively as, if agent i changes its strategy u_i^N unilaterally even based on global measurement information, the probability of its gaining a cost reduction by ε is still very small.

An almost sure asymptotic Nash equilibrium is to say that when N is sufficiently large, the control group $\mathbf{U}^N = \{u_i^N, 1 \leqslant i \leqslant N\}$ is an ε_N-Nash equilibrium with respect to $\mathbf{J}^N = \{J_i^N(u_i^N, u_{-i}^N), 1 \leqslant i \leqslant N\}$ with probability 1. The index $J_i^N(u_i^N, u_{-i}^N)$ of agent i deviates from the optimal index value based on global measurement $\mathrm{tr}(R_\sigma)$ by only a small quantity ε_N, which is convergent to zero almost surely as $N \to \infty$.

For the above two notions, we have the following two theorems on their relationship, whose proofs are straightforward, and so, omitted here.

Theorem 2.2. *If a sequence of control groups $\{\mathbf{U}^N = \{u_i^N, 1 \leqslant i \leqslant N\}, N \geqslant 1\}$ is an almost sure asymptotic Nash equilibrium with respect to the corresponding sequence of index groups $\{\mathbf{J}^N = \{J_i^N, 1 \leqslant i \leqslant N\}, N \geqslant 1\}$, then $\{\mathbf{U}^N, N \geqslant 1\}$ is also an asymptotic Nash equilibrium in probability with respect to $\{\mathbf{J}^N, N \geqslant 1\}$.*

Theorem 2.3. *If a sequence of control groups $\{\mathbf{U}^N = \{u_i^N, 1 \leqslant i \leqslant N\}, N \geqslant 1\}$ is an asymptotic Nash equilibrium in probability with respect to the corresponding sequence of index groups $\{\mathbf{J}^N = \{J_i^N, 1 \leqslant i \leqslant N\}, N \geqslant 1\}$, then for any sub-sequence of $\{\mathbf{U}^N = \{u_i^N, 1 \leqslant i \leqslant N\}, N \geqslant 1\}$, there exists a sub-sequence $\{\mathbf{U}^{N_k} = \{u_i^{N_k}, 1 \leqslant i \leqslant N_k\}, k \geqslant 1\}$, which is an almost sure asymptotic Nash equilibrium with respect to $\{\mathbf{J}^{N_k} = \{J_i^{N_k}, 1 \leqslant i \leqslant N_k\}, k \geqslant 1\}$.*

If the sequence of control groups $\{\mathbf{U}^N = \{u_i^N \in \mathscr{U}_i^N = \mathscr{U}_{l,i}^N, 1 \leqslant i \leqslant N\}, N \geqslant 1\}$ of the system sequence $\{\mathbf{S}^N, N \geqslant 1\}$ is an almost sure (in probability) asymptotic Nash equilibrium with respect to $\{\mathbf{J}^N = \{J_i^N, 1 \leqslant i \leqslant N\}, N \geqslant 1\}$, then we call it almost surely (in probability) asymptotically optimal decentralized control in the sense of Nash equilibrium.

Remark 4. Unlike single-agent systems (one-player games), in multi-player games the optimality may have different meanings for different problems. For instance, Nash equilibrium is a specific form of "optimality" often considered in a non-cooperative game, which says that one player cannot reduce its cost by altering his strategy unilaterally (Baser & Olsder, 1982); while in a cooperative game, what is often adopted is Pareto optimality, which says that no other joint strategy can reduce the cost of at least one player, without increasing the cost of the others. In this paper, we will focus on the non-cooperative case, and design a decentralized control law to achieve a Nash equilibrium (asymptotically). For (stochastic) cooperative games, the readers are referred to Yeung and Petrosyan (2006).

3. Decentralized control design

In the centralized control law (4), the control of agent i depends on the PSA $\bar{x}_N$. Since in general, u_i^N does not belong to the admissible control set $\mathscr{U}_{l,i}^N$, to ensure the control law designed to be decentralized, we will adopt the methodology of the NCE principle (Huang, Malhamé, & Caines, 2006; Li & Zhang, 2007a). Firstly we construct an estimate $f(t)$ of the PSA with the following property: if every agent takes $f(t)$ as the estimate of the PSA, and according to $f(t)$, makes the optimal decision, then the expectation of the closed-loop PSA is just $f(t)$ or convergent to it when N increases to infinity. Secondly if the $f(t)$ with the above property indeed exists, then we can construct the decentralized control law by using $f(t)$ instead of $\bar{x}_N(t)$.

Based on the NCE principle, we now design the decentralized control law.

The auxiliary equation of agent i is given by

$$\begin{aligned}\widehat{x}_i^N(t+1) &= g_i(\widehat{x}_i^N(t),t) + \widehat{u}_i^N(t) + \Lambda f(t) \\ &\quad + \omega_i(t+1), \quad t \geqslant 0, \quad i = 1,2,\ldots,N,\end{aligned} \tag{7}$$

with a tracking-type performance index:

$$J_i^N(\widehat{u}_i^N) = \limsup_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n} \|\widehat{x}_i^N(t+1) - \Phi(f(t))\|^2.$$

In this case, the optimal control is obviously

$$\widehat{u}_i^N(t) = \Phi(f(t)) - g_i(\widehat{x}_i^N(t),t) - \Lambda f(t). \tag{8}$$

Substituting control (8) into model (7), we have

$$E\widehat{x}_i^N(t+1) = \Phi(f(t)), \quad E\widehat{x}_i^N(0) = x_0. \tag{9}$$

As mentioned above, the mathematical expectation of the closed-loop PSA ought to be $f(t)$, that is

$$\frac{1}{N}\sum_{j=1}^{N} E\widehat{x}_j^N(t) = f(t), \quad t \geqslant 0. \tag{10}$$

Therefore, the unique solution of the auxiliary system (9) and (10) can be used as the estimate of the PSA. We denote it by $f^*(t)$, which is iteratively given by

$$f^*(t+1) = \Phi(f^*(t)), \quad t \geqslant 0, \quad f^*(0) = x_0. \tag{11}$$

By (8) and the NCE principle, the control law for agent i can be taken as

$$u_i^0(t) = \Phi(f^*(t)) - g_i(x_i^N(t),t) - \Lambda f^*(t). \tag{12}$$

Here and hereafter, the superscript N of $u_i^{0^N}(t)$ is omitted for conciseness of expression. Comparing (4) with (12), it can be seen that $\bar{x}_N$ in (12) is replaced by f^* for control design. Since f^* given by (11) is only related to the nonlinear coupling function $\Phi(\cdot)$ and the expectation of initial states, independent of the states of agents in real time, $u_i^0(t)$ indeed belongs to $\mathscr{U}_{l,i}^N$, that is, the control law (12) is decentralized.

Substituting (12) into (1) leads to the closed loop equation of agent i,

$$\begin{aligned}x_i^N(t+1) &= \Phi(f^*(t)) - \Lambda f^*(t) + \Lambda\bar{x}_N(t) + \omega_i(t+1) \\ &= \Phi(f^*(t)) + \Lambda\xi_N(t) + \omega_i(t+1),\end{aligned} \tag{13}$$

where $\xi_N(t) = \bar{x}_N(t) - f^*(t)$ is the estimation error of the PSA.

From above, it can be seen that the control design procedure has two steps: (a) construct the estimate $f^*(t)$ for the PSA; (b) construct the decentralized controllers (12) by using $f^*(t)$ instead of the PSA $\bar{x}_N(t)$.

For the decentralized control law designed and the resulting closed-loop system, the following three questions are naturally put forward:

(i) Whether the estimate of the PSA is strongly consistent, or whether $\xi_N(t)$ converges to zero with respect to some metric almost surely, as $N \to \infty$.
(ii) Whether the closed-loop system is stable, and whether the stability depends on N.
(iii) Whether the decentralized control law designed is asymptotically optimal almost surely or in probability. If the answer is affirmative, what is the convergence rate of the sub-optimal index of each agent to the optimal value, as $N \to \infty$.

In Section 4, the properties of the closed-loop system will be analyzed so as to answer these questions.

4. Closed-loop system analysis

For the closed-loop analysis, we need the following lemma.

Lemma 4.1. *Let $A \in \mathbb{R}^{m\times m}$ and $D \in \mathbb{R}^{m\times d}$, $\{W(t), \mathscr{F}_t\}$ be a d-dimensional martingale difference sequence, satisfying*

$$\sup_{t\geqslant 0} E[\|W(t)\|^2|\mathscr{F}_{t-1}] < \infty \quad \text{a.s.},$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n} W(t)W^{\mathrm{T}}(t) = R_W \quad \text{a.s.},$$

where $R_W \in \mathbb{R}^{d\times d}$ is a d-dimensional non-negative definite matrix. If $\rho(A) < 1$, then the solution of the following stochastic difference equation:

$$X(t+1) = AX(t) + DW(t+1) \tag{14}$$

satisfies

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n} X(t)X^{\mathrm{T}}(t) = \sum_{k=0}^{\infty} A^k D R_W D^{\mathrm{T}}(A^k)^{\mathrm{T}} \quad \text{a.s.} \tag{15}$$

Proof. See Appendix A.

Theorem 4.1. *For system* (1), *if Assumptions* (A1)–(A2) *hold, then under the control law* (12), *the closed-loop system has the following properties*:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\xi_N(t)\|^2 \leqslant \alpha_{1N} \quad \text{a.s.}, \tag{16}$$

$$\lim_{N\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\xi_N(t)\|^2 = 0 \quad \text{a.s.}, \tag{17}$$

where $\xi_N(t)$, R_σ, Λ are given in (13), (2), (1), *respectively, and*

$$\alpha_{1N} = \frac{m\|R_\sigma\|\|(I_m - \Lambda\Lambda^{\mathrm{T}})^{-1}\|}{N}.$$

Proof. By (13) and (11) one can get

$$\xi_N(t+1) = \Lambda\xi_N(t) + \frac{1}{N}\sum_{j=1}^{N}\omega_j(t+1). \tag{18}$$

We first prove that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\omega_i(t)\omega_j^{\mathrm{T}}(t) = 0 \quad \text{a.s.}, \quad i \neq j. \tag{19}$$

Denote $\widetilde{\omega}_j(t) = \omega_j(t-1)$, $\mathscr{F}_t^{ij} = \sigma(\mathscr{F}_t^i \cup \mathscr{F}_{t-1}^j)$. Then, from Assumption (A1), both $\{\omega_i(t), \mathscr{F}_t^{ij}\}$ and $\{\widetilde{\omega}_j(t), \mathscr{F}_t^{ij}\}$ are martingale difference sequences, satisfying

$$\sup_{t\geqslant 0} E[\|\widetilde{\omega}_j(t)\|^2|\mathscr{F}_{t-1}^{ij}] < \infty \quad \text{a.s.}$$

This together with Lemma 2.1 implies that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\omega_i(t)\omega_j^{\mathrm{T}}(t) = \lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\omega_i(t)\widetilde{\omega}_j^{\mathrm{T}}(t+1)$$

$$= \lim_{n\to\infty}\frac{1}{n}O(n^{1/2+\varepsilon}) = 0 \quad \text{a.s.}$$

Hence, (19) holds. Furthermore, by (2), we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\left(\frac{1}{N}\sum_{j=1}^{N}\omega_j(t)\right)\left(\frac{1}{N}\sum_{j=1}^{N}\omega_j(t)\right)^{\mathrm{T}}$$

$$= \frac{1}{N}R_\sigma \quad \text{a.s.},$$

which together with (18) and Lemma 4.1 leads to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\xi_N(t)\|^2 = \frac{1}{N}\mathrm{tr}\left(\sum_{k=0}^{\infty}\Lambda^k R_\sigma(\Lambda^k)^{\mathrm{T}}\right)$$

$$\leqslant \frac{m\lambda_{\max}(R_\sigma)}{N}\left\|\sum_{k=0}^{\infty}\Lambda^k(\Lambda^k)^{\mathrm{T}}\right\|$$

$$= \frac{m\|R_\sigma\|\|(I_m - \Lambda\Lambda^{\mathrm{T}})^{-1}\|}{N} \quad \text{a.s.}$$

Thus, (16) and (17) are true. □

Remark 5. Theorem 4.1 characterizes the estimation accuracy of f^* as an estimate of the PSA. In Li and Zhang (2007a), the error between the closed-loop PSA $(1/N)\sum_{j=1}^{N} y_j^0$ and its estimate y^* is measured by $(\limsup_{T\to\infty}(1/T)\int_0^{\mathrm{T}}\|y^*(t) - (1/N)\sum_{j=1}^{N} y_j^0(t)\|^2\,\mathrm{d}t)^{1/2}$. Here, since all the signals considered are discrete-time sequences, we can construct a linear space

$$l_{\mathrm{Pb}} = \left\{x \,\middle|\, \limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|x(t)\|^2 < \infty\right\},$$

which is the family of all the sequences with finite power average. Define an equivalence relationship on l_{Pb} denoted by $\sim$: for any $x, y \in l_{\mathrm{Pb}}$, $x \sim y$, if and only if,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|x(t) - y(t)\|^2 = 0.$$

Denote the equivalent class of x by $[x]$, and define a norm on the quotient space $l_{\mathrm{Pb}}/\sim$:

$$\|[x]\|_{\mathrm{Pb}} \triangleq \left(\limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|x(t)\|^2\right)^{1/2}, \quad \forall [x] \in l_{\mathrm{Pb}}.$$

Then, it can be shown that $(l_{\mathrm{Pb}}/\sim, \|\cdot\|_{\mathrm{Pb}})$ is a normed space. Theorem 4.1 tells us that $\lim_{N\to\infty}\|\xi_N(t)\|_{\mathrm{Pb}} = 0$ with probability 1. Namely, along with the increasing of N, the estimation error of the PSA converges to zero in the sense of $\|\cdot\|_{\mathrm{Pb}}$-norm almost surely. Thus, we say $f^*(t)$ is a strongly consistent estimate of the PSA in the sense of long term average or $\|\cdot\|_{\mathrm{Pb}}$-norm, with the convergence rate $O(1/N)$.

Remark 6. From the point of view of decentralized control law design, f^* can be regarded as the estimate of the PSA. On the other hand, it characterizes the macroscopic behavior. Since the individual states and PSA are coupled nonlinearly in the performance indexes, complex macroscopic behaviors may emerge. A concrete example for this phenomenon will be given in Section 5.

Theorem 4.2. *For system* (1), *if Assumptions* (A1)–(A2) *hold, and the solution of the nonlinear iteration* $x(t+1) = \Phi(x(t))$ *with* $x(0) = x_0$ *is bounded, then under the control law* (12), *the closed-loop system satisfies*

$$\sup_{N\geqslant 1}\max_{1\leqslant i\leqslant N}\limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|x_i^N(t)\|^2 < \infty \quad \text{a.s.} \tag{20}$$

Proof. By (13), Theorem 4.1, Assumption (A1) and Lemma 2.1, we have

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|x_i^N(t+1)\|^2$$

$$= \limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\Phi(f^*(t)) + \Lambda\xi_N(t)\|^2$$

$$+ \lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\omega_i(t)\|^2$$

$$\leqslant 2\overline{\kappa} + 2\|\Lambda\|^2\alpha_{1N} + \mathrm{tr}(R_\sigma) \quad \text{a.s.},$$

where

$$\overline{\kappa} = \limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|f^*(t)\|^2.$$

Noticing that $\alpha_{1N} = O(N^{-1})$ and Λ, R_σ and $\overline{\kappa}$ are independent of N, i, we have (20). □

Remark 7. Theorem 4.2 tells us that, if some condition on the nonlinear function $\Phi(\cdot)$ is satisfied, then under the decentralized control law designed, the closed-loop system is stable almost surely, and the stability is uniform with respect to N, that is, there exists $A \in \mathscr{F}$, $P(A)=1$, and a non-negative finite random variable $C(\omega)$ independent of N, i, such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n} \|x_i^N(t)\|^2 \leqslant C(\omega), \quad \forall \omega \in A.$$

Hence, the closed-loop stability is retained with probability 1 even if N becomes arbitrarily large. The upper bound $C(\omega)$ of $\limsup_{n\to\infty}(1/n)\sum_{t=0}^{n}\|x_i^N(t)\|^2$ is dependent on the nonlinear function in their group-tracking-type indexes, but independent of N.

Theorem 4.3. *For system* (1) *with index* (3), *if Assumptions* (A1)–(A2) *hold and $\Phi(\cdot)$ is Hölder continuous with exponent $\mu \in (0,1]$ in the sense that there exists a constant $\gamma>0$ such that $\|\Phi(x)-\Phi(y)\| \leqslant \gamma\|x-y\|^\mu$, $\forall x, y \in \mathbb{R}^m$, then under the control law* (12), *the corresponding index group satisfies*

$$J_i^N(u_i^0, u_{-i}^0) \leqslant \mathrm{tr}(R_\sigma) + \varepsilon_N \quad \text{a.s.}, \quad i=1,2,\ldots,N, \tag{21}$$

where

$$\varepsilon_N = 2\gamma^2(\alpha_{1N})^\mu + 2\|\Lambda\|^2\alpha_{1N}.$$

Proof. By Assumption (A1), (3) and (13) we have

$$\begin{aligned} J_i^N(u_i^0, u_{-i}^0) &= \limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\Phi(f^*(t)) - \Phi(\overline{x}_N(t)) \\ &\quad + \Lambda\xi_N(t) + \omega_i(t+1)\|^2 \\ &\leqslant \mathrm{tr}(R_\sigma) + I_1^N + I_2^N, \end{aligned} \tag{22}$$

where

$$I_1^N = \limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\Phi(f^*(t)) - \Phi(\overline{x}_N(t)) + \Lambda\xi_N(t)\|^2, \tag{23}$$

$$\begin{aligned} I_2^N &= 2\limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}[\Phi(f^*(t)) - \Phi(\overline{x}_N(t)) \\ &\quad + \Lambda\xi_N(t)]^{\mathrm{T}}\omega_i(t+1). \end{aligned} \tag{24}$$

From the condition of the theorem, Jensen inequality (Chow & Teicher, 1997) and Theorem 4.1, it follows that

$$\begin{aligned} I_1^N &\leqslant 2\limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\Phi(f^*(t)) - \Phi(\overline{x}_N(t))\|^2 + 2\|\Lambda\|^2\alpha_{1N} \\ &\leqslant 2\gamma^2\limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\xi_N(t)\|^{2\mu} + 2\|\Lambda\|^2\alpha_{1N} \\ &\leqslant 2\gamma^2\left(\limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\xi_N(t)\|^2\right)^{\mu} + 2\|\Lambda\|^2\alpha_{1N} \\ &\leqslant 2\gamma^2(\alpha_{1N})^\mu + 2\|\Lambda\|^2\alpha_{1N}. \end{aligned} \tag{25}$$

This together with (24) and Lemma 2.1 leads to $I_2^N = 0$ a.s. Therefore, by (22) and (25) one can get (21). □

Remark 8. From Theorem 4.3, the decentralized control law (12) is sub-optimal, and the maximum error between the sub-optimal index values and the optimal one

$$\Delta_N J^* \triangleq \max_{1\leqslant i\leqslant N} J_i^N(u_i^0, u_{-i}^0) - \mathrm{tr}(R_\sigma) \tag{26}$$

satisfies

$$\Delta_N J^* \leqslant \varepsilon_N. \tag{27}$$

Thus, the performance index of each agent is almost surely asymptotically optimal with the convergence rate $O(1/N^\mu)$, as $N\to\infty$.

This theorem implies that the sequence of control groups $\{\mathbf{U}^N = \{u_i^0, 1\leqslant i\leqslant N\}, N\geqslant 1\}$ is an almost sure asymptotic Nash equilibrium with respect to the corresponding sequence of index groups.

In Theorem 4.3, under the condition of μ-Hölder continuity on $\Phi(\cdot)$, the control law designed is shown to be almost surely asymptotically optimal. When the nonlinear function $\Phi(\cdot)$ is only locally Lipschitz continuous, to get asymptotic optimality of the decentralized control law, we need the following assumption:

(**A3**) The martingale difference sequences $\{\{\omega_i(t), \mathscr{F}_t^i\}, i\geqslant 1\}$ in model (1) satisfy

$$\sup_{i\geqslant 1} E\left(\limsup_{t\to\infty}\|\omega_i(t)\|^2\right) < \infty.$$

Theorem 4.4. *For system* (1) *with index* (3), *if Assumptions* (A1)–(A3) *hold, the solution of the nonlinear iteration $x(t+1) = \Phi(x(t))$ with $x(0)=x_0$ is bounded, and for any given $R>0$, there exists $\gamma_R>0$ such that $\|\Phi(x)-\Phi(y)\| \leqslant \gamma_R\|x-y\|$, $\forall \|x\|\leqslant R$, $\|y\|\leqslant R$, then under the control law* (12), *the corresponding index group satisfies*:

$$\lim_{N\to\infty} P\{\Delta_N J^* > \widetilde{\varepsilon}_N\} = 0, \tag{28}$$

where $\Delta_N J^*$ is defined by (26),

$$\widetilde{\varepsilon}_N = 2\gamma_{R_0}^2(\alpha_{1N})^\mu + 2\|\Lambda\|^2\alpha_{1N},$$

and R_0 is a constant independent of N.

Proof. See Appendix B.

Remark 9. Under the conditions of Theorem 4.4, by the definition of $\widetilde{\varepsilon}_N$, it is known that for any given $\varepsilon>0$, $\delta>0$, there exists $N_1>0$ such that $\widetilde{\varepsilon}_N < \varepsilon$, $\forall N > N_1$. Furthermore, by (28), there exists $N_2>0$, such that

$$P\{\Delta_N J^* > \widetilde{\varepsilon}_N\} \leqslant \delta, \quad \forall N \geqslant N_2.$$

Thus, we have

$$P\{\Delta_N J^* \geqslant \varepsilon\} \leqslant \delta, \quad \forall N \geqslant \max\{N_1, N_2\}.$$

From the Definition 2.2, we know that the sequence of control groups $\{\mathbf{U}^N = \{u_i^0, 1\leqslant i\leqslant N\}, N\geqslant 1\}$ is an asymptotic Nash equilibrium in probability with respect to the corresponding sequence of index groups.

Remark 10. If $\{\{\omega_i(t), \mathscr{F}_t^i\}, i \geqslant 1\}$ is uniformly bounded, namely, there exists a constant $M > 0$, such that $\sup_{i\geqslant 1} \sup_{t\geqslant 0} \|\omega_i(t)\| \leqslant M$, then Assumption (A3) holds.

In some applications, the agents may start the estimate of the PSA with false assumptions on x_0. For instance, due to the limitation of numerical precision, there may be small errors between x_0 and the initial value of the estimate of each agent. This naturally leads to some interesting questions: given a bound on initial value errors, what can we conclude on the general behavior of the MAS? Under what conditions is the decentralized control law based on NCE principle still asymptotically optimal (as N increases to infinity)? Below we will discuss these questions. In this case, instead of (12), the control law for the ith agent should be given by

$$u_i^0(t) = \Phi(f_i(t)) - g_i(x_i^N(t), t) - \Lambda f_i(t), \tag{29}$$

where $f_i(t)$ is generated by

$$f_i(t+1) = \Phi(f_i(t)), \quad t \geqslant 0, \qquad f_i(0) = f_{i0}. \tag{30}$$

In the following analysis, $\delta_N \triangleq \max_{1\leqslant i\leqslant N} \|f_{i0} - x_0\|$ denotes the maximum error of the initial values. We assume that

(**A4**) There exist constants $C_\phi \geqslant 0$, $r_\phi > 0$, such that for any initial value $x(0)$ satisfying $\|x(0) - x_0\| \leqslant r_\phi$, the solution $x(t)$ of the nonlinear iteration $x(t+1) = \Phi(x(t))$ starting from $x(0)$ has the following property:

$$\limsup_{t\to\infty} \|x(t) - f^*(t)\| \leqslant C_\phi,$$

where $f^*(t)$ is the solution of (11) with initial value $f^*(0)=x_0$.

Remark 11. If $\Phi(x)$ is a bounded function, then Assumption (A4) holds. In addition, if $f^*(t)$ is an attractive solution of nonlinear iteration $x(t+1) = \Phi(x(t))$, then Assumption (A4) holds. In this case, $C_\phi = 0$, r_ϕ represents the radius of the attractive domain.

We have the following theorems, whose proofs are similar to those of Theorems 4.2–4.4, and so, omitted here.

Theorem 4.5. *For system* (1), *suppose Assumption* (A4) *and the conditions of Theorem* 4.2 *hold. If the maximum error* δ_N *of the initial values satisfies* $\delta_N \leqslant r_\phi$, *then under the control law* (29)–(30), *the closed-loop system has the following property*:

$$\max_{1\leqslant i\leqslant N} \limsup_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n} \|x_i^N(t)\|^2 \leqslant M_1 \quad a.s., \qquad N = 1, 2, \ldots,$$

where M_1 *is a positive constant independent of* N.

Remark 12. This theorem is concerned with the closed-loop stability, and tells us that, if the maximum error between x_0 and the initial value of the estimate of each agent is not too large, then under some conditions and the control law (29)–(30), the closed-loop system is stable almost surely, and the stability is uniform with respect to N.

Theorem 4.6. *For system* (1), *suppose Assumption* (A4) *and the conditions of Theorem* 4.3 *hold. If the maximum error* δ_N *of the initial values satisfies* $\delta_N \leqslant r_\phi$, *then under the control law* (29)–(30), *the corresponding index group has the following property*:

$$J_i^N(u_i^0, u_{-i}^0) \leqslant \mathrm{tr}(R_\sigma) + \beta_{1N} \quad a.s., \quad i = 1, 2, \ldots, N,$$

where

$$\begin{aligned}\beta_{1N} &= 2\gamma^2(4C_\phi^2 + \alpha_{2N} + 2C_\phi\|\Lambda\|\sqrt{\alpha_{1N}})^\mu \\ &\quad + 2\|\Lambda\|^2(4C_\phi^2 + \alpha_{2N} + 2C_\phi\|\Lambda\|\sqrt{\alpha_{1N}}) \\ &= O(1), \quad N \to \infty,\end{aligned}$$

$$\begin{aligned}\alpha_{2N} &= \frac{m\|R_\sigma\|(\|\Lambda\|^2\|(I_m - \Lambda\Lambda^{\mathrm{T}})^{-1}\| + 1)}{N} \\ &= O(N^{-1}), \quad N \to \infty.\end{aligned}$$

Theorem 4.7. *For system* (1), *suppose Assumption* (A4) *and the conditions of Theorem* 4.4 *hold. If the maximum error* δ_N *of the initial values satisfies* $\delta_N \leqslant r_\phi$, *then under the control law* (29)–(30), *the corresponding index group has the following property*:

$$\lim_{N\to\infty} P\{\Delta_N J^* > \beta_{2N}\} = 0,$$

where $\Delta_N J^*$ *is defined by* (26),

$$\begin{aligned}\beta_{2N} &= 2\gamma_{R_1}^2(4C_\phi^2 + \alpha_{2N} + 2C_\phi\|\Lambda\|\sqrt{\alpha_{1N}})^\mu \\ &\quad + 2\|\Lambda\|^2(4C_\phi^2 + \alpha_{2N} + 2C_\phi\|\Lambda\|\sqrt{\alpha_{1N}}) \\ &= O(1), \quad N \to \infty,\end{aligned}$$

and R_1 *is a constant independent of* N.

Remark 13. Theorems 4.6–4.7 are concerned with the asymptotic optimality of the control law (29)–(30). They says that, if the maximum error between x_0 and the initial value of the estimate of each agent is not too large, then the control law is sub-optimal. Under the conditions of Theorem 4.6 (Theorem 4.7), the maximum error between the sub-optimal index values and optimal one is bounded almost surely (in probability) as N increases to infinity. Especially, if $C_\phi = 0$, then β_{1N} (β_{2N}) $= O(N^{-\mu})$ and the control law (29)–(30) is asymptotically optimal almost surely (in probability).

5. Numerical examples

In this section, two numerical examples are given to verify the asymptotic optimality of the decentralized control law designed when the nonlinear coupling function Φ is globally Lipschitz continuous or locally Lipschitz continuous, respectively. In addition, the second example is also used to show the emergent complex population behavior and the consistency of the estimate for the PSA.

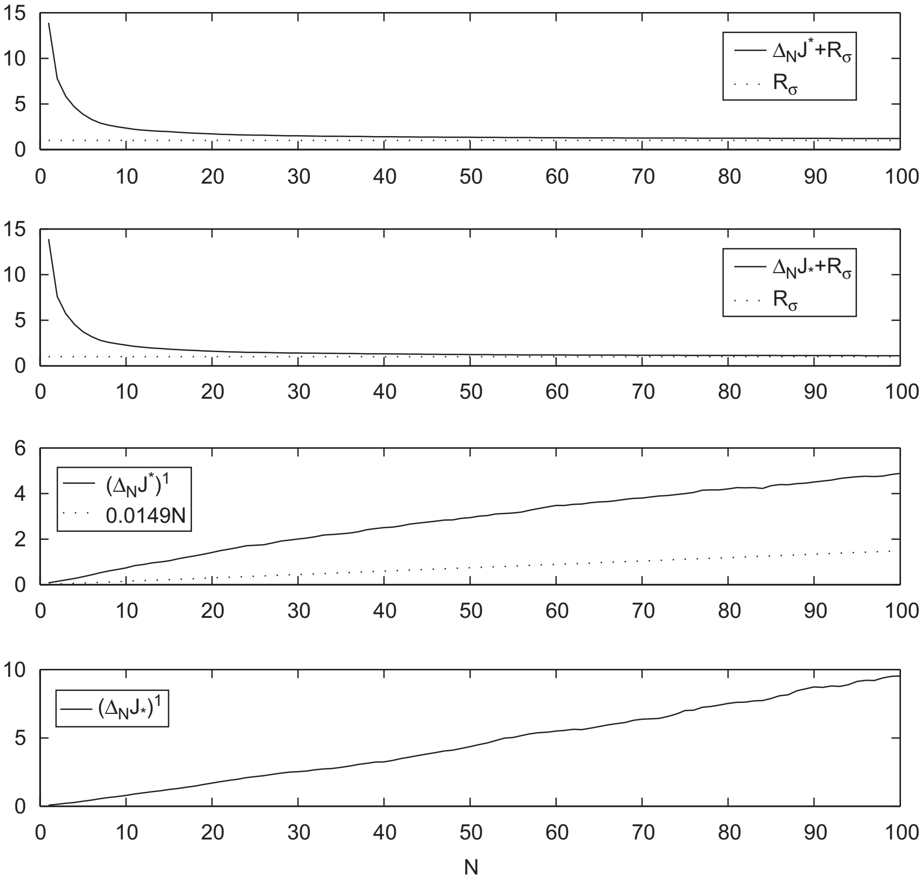

Fig. 1. Curves of $\Delta_N J^*$, $\Delta_N J_*$ with respect to N.

Example 1. The dynamic equation for the ith agent is given by

$$x_i^N(t+1) = 0.8x_i^N(t) + u_i^N(t) + 0.5\bar{x}_N(t) + \omega_i(t+1),$$

where the initial value $x_i^N(0)$ has the normal distribution $N(5, 0.5)$, $\{\omega_i(t), t \geqslant 0\}$ is a sequence of Gaussian white noise with distribution $N(0, 1)$, that is, $\Lambda = 0.5$, $R_\sigma = 1$. The nonlinear coupling function in the indexes is $\Phi(x) = 5\sin(x) + 10$.

By (12), the decentralized controllers can be taken as

$$u_i^0(t) = 5\sin(f^*(t)) + 10 - 0.8x_i^N(t) - 0.5f^*(t), \tag{31}$$

where $f^*(t)$ is iteratively given by

$$f^*(t+1) = 5\sin(f^*(t)) + 10, \quad t \geqslant 0, \quad f^*(0) = 5.$$

Since $|\mathrm{d}\Phi(x)/\mathrm{d}x| \leqslant 5$, $\Phi(\cdot)$ is global Lipschitz continuous (Hölder continuous with exponent 1), and the Lipschitz constant γ can be taken as 5.

Letting the number of agents N increase from 1 to 100, we have the variation of the index differences $\Delta_N J^*$, $\Delta_N J_*$ shown in Fig. 1. Here, $\Delta_N J^*$ is defined by (26), and

$$\Delta_N J_* \triangleq \min_{1 \leqslant i \leqslant N} J_i^N(u_i^0, u_{-i}^0) - \mathrm{tr}(R_\sigma).$$

From Fig. 1, the following properties of the sequence of index groups can be seen:

(i) When $N \to \infty$, both $\max_{1 \leqslant i \leqslant N} J_i^N(u_i^0, u_{-i}^0)$ and $\min_{1 \leqslant i \leqslant N} J_i^N(u_i^0, u_{-i}^0)$ converge to $1\,(=\mathrm{tr}(R_\sigma))$. This indicates that the decentralized control (31) is asymptotically optimal.

(ii) Curve $(\Delta_N J^*)^{-1}$ has a larger slope than that of curve $0.0149N$, that is,

$$\Delta_N J^* \leqslant \frac{2(\gamma^2 + \Lambda^2)R_\sigma}{N(1-\Lambda^2)} = \frac{67.1141}{N}.$$

This indicates that both (21) and (27) hold.

(iii) Both $\Delta_N J^*$ and $\Delta_N J_*$ have the order $O(1/N)$, or the performance index of each agent is convergent to the optimal value with the rate $O(1/N)$.

Example 2. The parameters of all agents are taken the same values as in Example 1, except that $x_i^N(0)$ has the normal distribution $N(0.5, 0.1)$ and $\{\omega_i(t), t \geqslant 0\}$ is a sequence of white noise with the uniform distribution on $[-1, 1]$. The nonlinear coupling function in the performance indexes is $\Phi(x) = x(1 + 2.99(1-x))$.

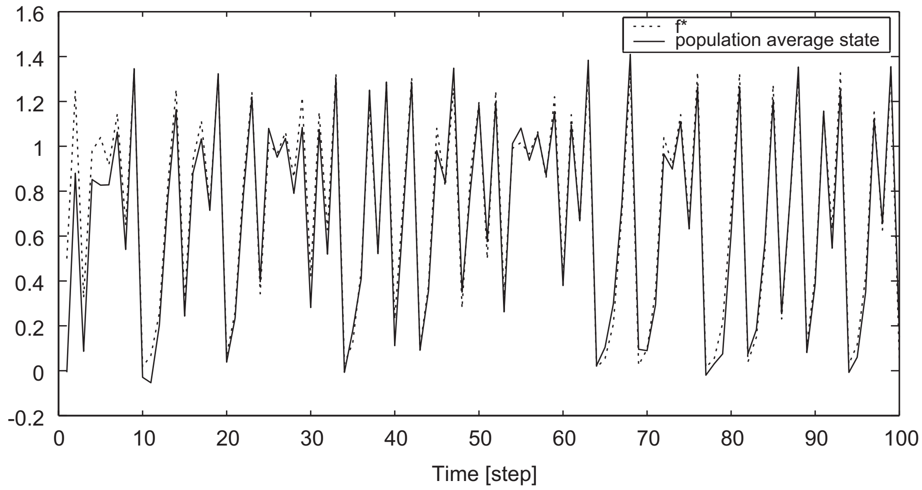

Fig. 2. Curves of $f^*, \overline{x}_N$ with respect to t when $N = 100$.

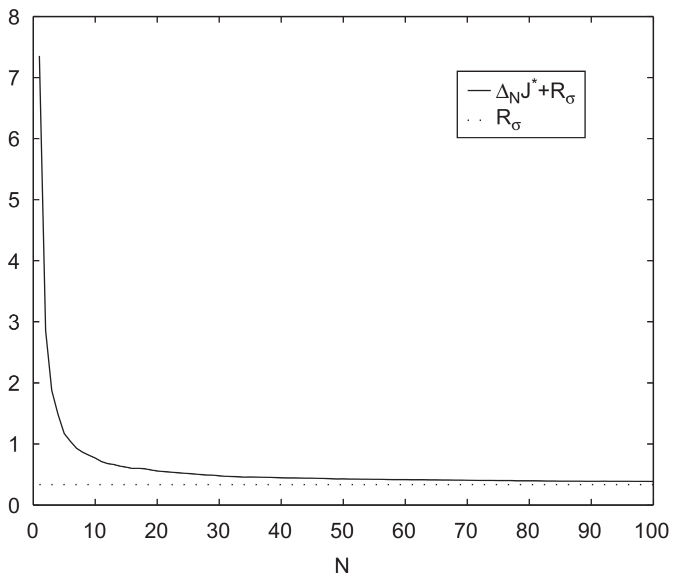

Fig. 3. Curves of $\Delta_N J^*$ with respect to N.

Letting the number of agents $N = 100$, the curves of f^*, $\overline{x}_N$ with respect to t are shown in Fig. 2. The error between f^* and $\overline{x}_N$ can be quantitatively characterized by Theorem 4.1: the estimation error in the long term averaged sense is equal to $0.0667\ (=\sqrt{\mathrm{tr}(R_\sigma)/N(1-\Lambda^2)})$. So, when the number of agents is sufficiently large, the trajectory of the PSA is almost identical to f^*. In addition, the variation of the performance index differences $\Delta_N J^*$ with respect to N is shown in Fig. 3, from which, it can be seen that the index values of agents converge to the optimal value $\mathrm{tr}(R_\sigma)$ as N increases to infinity.

6. Concluding remarks

A decentralized tracking-type game has been considered in this paper for a class of MAS, in which the individual agent and the overall population interact via dynamics and performance indexes. The information available for each agent's control design is local, and so, the control is decentralized. The control is designed based on the NCE principle: the estimate of the PSA is first given, and then, used as if it is the real PSA. By probability limit theory, the following three properties have been obtained: strong consistency of the estimate of the PSA, almost surely uniform stability of the closed-loop system, and asymptotic optimality in the sense of Nash equilibrium of the decentralized controls. It is shown that: (a) when the nonlinear coupling function in the indexes is Hölder continuous with exponent μ, the decentralized control law is asymptotically optimal almost surely, and the performance index of each agent converges to the optimal value almost surely with the rate $O(1/N^{\mu})$; (b) when the function is locally Lipschitz continuous, the decentralized control law is asymptotically optimal in probability, and the probability for the individual performance to exceed its corresponding optimal value by any given small positive quantity converges to zero, as $N \to \infty$.

For MAS control, there are a lot of problems worth investigating, including optimal control of the case where the control energies of the agents are incorporated in the performance indexes, and adaptive control of the case where unknown parameters or unmodeled dynamics appear. How to exploit the learning ability of agents (Panait & Luke, 2005) to adaptively improve the models and system performances in a distributed way is also an important issue to be studied for MAS.

Acknowledgments

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Appendix A. Proof of Lemma 4.1

Proof. From $\rho(A)<1$ we know that the algebraic Lyapunov equation $P=A^{\mathrm T}PA+I_m$ has a unique positive definite solution $P=\sum_{k=0}^{\infty}(A^k)^{\mathrm T}A^k\geqslant I_m$. From (14) we have

$$\begin{aligned}
&X^{\mathrm T}(t+1)PX(t+1)\\
&\quad=X^{\mathrm T}(t)A^{\mathrm T}PAX(t)+W^{\mathrm T}(t+1)D^{\mathrm T}PDW(t+1)\\
&\qquad+2X^{\mathrm T}(t)A^{\mathrm T}PDW(t+1)\\
&\quad=X^{\mathrm T}(t)(P-I_m)X(t)+W^{\mathrm T}(t+1)D^{\mathrm T}PDW(t+1)\\
&\qquad+2X^{\mathrm T}(t)A^{\mathrm T}PDW(t+1).
\end{aligned}$$

Summing both sides of the above equation from $t=0$ to n gives

$$\begin{aligned}
&X^{\mathrm T}(n+1)PX(n+1)\\
&\quad=X^{\mathrm T}(0)PX(0)+\sum_{t=0}^{n}W^{\mathrm T}(t+1)D^{\mathrm T}PDW(t+1)\\
&\qquad-\sum_{t=0}^{n}\|X(t)\|^2+2\sum_{t=0}^{n}X^{\mathrm T}(t)A^{\mathrm T}PDW(t+1).
\end{aligned}$$

This together with (14) and Lemma 2.1 leads to

$$\begin{aligned}
\sum_{t=0}^{n}\|X(t)\|^2&\leqslant\sum_{t=0}^{n}W^{\mathrm T}(t+1)D^{\mathrm T}PDW(t+1)+X^{\mathrm T}(0)PX(0)\\
&\quad+2\sum_{t=0}^{n}X^{\mathrm T}(t)A^{\mathrm T}PDW(t+1)\\
&=X^{\mathrm T}(0)PX(0)+O\left(\sum_{t=0}^{n}\|X(t)\|^2\right)^{1/2+\varepsilon}\\
&\quad+O(n)\quad\text{a.s. }\forall\varepsilon>0.
\end{aligned}$$

Therefore,

$$\sum_{t=0}^{n}\|X(t)\|^2=O(n)\quad\text{a.s.}\tag{A.1}$$

By (14) we have

$$\begin{aligned}
&X(t+1)X^{\mathrm T}(t+1)\\
&\quad=AX(t)X^{\mathrm T}(t)A^{\mathrm T}+DW(t+1)X^{\mathrm T}(t)A^{\mathrm T}\\
&\qquad+AX(t)W^{\mathrm T}(t+1)D^{\mathrm T}+DW(t+1)W^{\mathrm T}(t+1)D^{\mathrm T}.
\end{aligned}$$

This together with (A.1), (14) and Lemma 2.1 gives

$$\begin{aligned}
&\limsup_{n\to\infty}\frac1n\sum_{t=0}^{n}X(t)X^{\mathrm T}(t)\\
&\quad=A\left(\limsup_{n\to\infty}\frac1n\sum_{t=0}^{n}X(t)X^{\mathrm T}(t)\right)A^{\mathrm T}+DR_WD^{\mathrm T}\quad\text{a.s.},
\end{aligned}$$

$$\begin{aligned}
&\liminf_{n\to\infty}\frac1n\sum_{t=0}^{n}X(t)X^{\mathrm T}(t)\\
&\quad=A\left(\liminf_{n\to\infty}\frac1n\sum_{t=0}^{n}X(t)X^{\mathrm T}(t)\right)A^{\mathrm T}+DR_WD^{\mathrm T}\quad\text{a.s.}
\end{aligned}$$

So, by $\rho(A)<1$, (15) holds. □

Appendix B. Proof of Theorem 4.4

Proof. From the conditions of the theorem it follows that

$$\limsup_{N\to\infty}\frac1N\sum_{j=1}^{N}E\left(\limsup_{t\to\infty}\|\omega_j(t)\|\right)<\infty,$$

and $\sup_{t\geqslant0}\|f^*(t)\|<\infty$. By $\rho(\Lambda)<1$, there exists two constants $c_1>0$, $c_2>0$ and norm $\|\cdot\|_\Lambda$, such that $\|\Lambda\|_\Lambda<1$, and $c_1\|\cdot\|\leqslant\|\cdot\|_\Lambda\leqslant c_2\|\cdot\|$.

From (18), we have

$$\limsup_{t\to\infty}\|\xi_N(t)\|_\Lambda\leqslant\frac{1}{1-\|\Lambda\|_\Lambda}\limsup_{t\to\infty}\left\|\frac1N\sum_{j=1}^{N}\omega_j(t)\right\|_\Lambda$$

and

$$\limsup_{t\to\infty}\|\xi_N(t)\|\leqslant\frac{c_2}{c_1(1-\|\Lambda\|_\Lambda)}\limsup_{t\to\infty}\left\|\frac1N\sum_{j=1}^{N}\omega_j(t)\right\|.\tag{B.1}$$

Take a constant R_0 such that

$$\begin{aligned}
R_0>&\sup_{t\geqslant0}\|f^*(t)\|\\
&+\left[c_2\limsup_{N\to\infty}\frac1N\sum_{j=1}^{N}E\left(\limsup_{t\to\infty}\|\omega_j(t)\|\right)\right]\\
&\times[c_1(1-\|\Lambda\|_\Lambda)]^{-1}.
\end{aligned}\tag{B.2}$$

Denote $\mu_0=R_0-\sup_{t\geqslant0}\|f^*(t)\|$ and $\mathscr{B}_N=\{\limsup_{t\to\infty}\|\xi_N(t)\|<\mu_0\}$. Then, by (B.1) we have

$$\begin{aligned}
&P\{\Omega-\mathscr{B}_N\}\\
&\quad\leqslant P\left\{\frac1N\sum_{j=1}^{N}\limsup_{t\to\infty}\|w_j(t)\|\geqslant\frac{c_1(1-\|\Lambda\|_\Lambda)\mu_0}{c_2}\right\}\\
&\quad\leqslant P\left\{\left|\frac1N\sum_{j=1}^{N}\limsup_{t\to\infty}\|w_j(t)\|-\frac1N\sum_{j=1}^{N}E\limsup_{t\to\infty}\|w_j(t)\|\right|\right.\\
&\qquad\left.+\frac1N\sum_{j=1}^{N}E\limsup_{t\to\infty}\|w_j(t)\|\geqslant\frac{c_1(1-\|\Lambda\|_\Lambda)\mu_0}{c_2}\right\}.
\end{aligned}\tag{B.3}$$

By (B.2), there exist $\delta_0>0$, $N_{\delta_0}>0$, such that

$$\frac{c_1(1-\|\Lambda\|_\Lambda)\mu_0}{c_2}-\frac1N\sum_{j=1}^{N}E\limsup_{t\to\infty}\|w_j(t)\|>\delta_0,\quad\forall N\geqslant N_{\delta_0}.\tag{B.4}$$

Then, from (B.3) and (B.4) it follows that

$$\begin{aligned}
&P\{\Omega-\mathscr{B}_N\}\\
&\quad\leqslant P\left\{\left|\frac1N\sum_{j=1}^{N}\left(\limsup_{t\to\infty}\|w_j(t)\|\right.\right.\right.\\
&\qquad\left.\left.\left.-E\limsup_{t\to\infty}\|w_j(t)\|\right)\right|\geqslant\delta_0\right\},\quad N\geqslant N_{\delta_0}.
\end{aligned}\tag{B.5}$$

By Assumption (A3) and Lemma 2.1, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\left(\limsup_{t\to\infty}\|w_j(t)\|-E\limsup_{t\to\infty}\|w_j(t)\|\right)=0 \quad \text{a.s.},$$

which together with (B.5) leads to

$$\lim_{N\to\infty} P(\mathscr{B}_N)=1. \tag{B.6}$$

Denote $\Omega_0=\{\lim_{n\to\infty}(1/n)\sum_{t=0}^{n}\|\xi_N(t)\|^2\leqslant \alpha_{1N}\}$. Then, by Theorem 4.1 we have $P(\Omega_0)=1$. By the definition of $\mathscr{B}_N$, we know that for any $\omega\in\Omega_0\cap\mathscr{B}_N$, there exists $n_0(\omega)>0$, such that $\|\xi_N(t)\|<\mu_0$, $\forall t\geqslant n_0(\omega)$. Thus,

$$\begin{aligned}\|\overline{x}_N(t)\|&\leqslant \sup_{t\geqslant 0}\|f^*(t)\|+\mu_0\\ &=R_0, \quad \forall t\geqslant n_0(\omega),\ \ \forall\omega\in\Omega_0\cap\mathscr{B}_N.\end{aligned}$$

This together with the conditions of the theorem gives

$$\begin{aligned}&\|\Phi(\overline{x}_N(t))-\Phi(f^*(t))\|\\ &\quad\leqslant \gamma_{R_0}\|\xi_N(t)\|, \quad \forall t\geqslant n_0(\omega),\ \ \forall\omega\in\Omega_0\cap\mathscr{B}_N.\end{aligned}$$

Therefore, by the definition of Ω_0 and Jensen inequality, we have

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\Phi(f^*(t))-\Phi(\overline{x}_N(t))\|^2\\ &\quad=\limsup_{n\to\infty}\frac{1}{n}\left\{\sum_{t=0}^{n_0(\omega)-1}\|\Phi(f^*(t))-\Phi(\overline{x}_N(t))\|^2\right.\\ &\qquad\left.+\sum_{t=n_0(\omega)}^{n}\|\Phi(f^*(t))-\Phi(\overline{x}_N(t))\|^2\right\}\\ &\quad=\limsup_{n\to\infty}\frac{1}{n}\sum_{t=n_0(\omega)}^{n}\|\Phi(f^*(t))-\Phi(\overline{x}_N(t))\|^2\\ &\quad\leqslant\gamma_{R_0}^2\limsup_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\|\xi_N(t)\|^{2\mu}\\ &\quad\leqslant\gamma_{R_0}^2\left(\frac{m\|R_\sigma\|\|(I_m-\Lambda\Lambda^{\mathrm{T}})^{-1}\|}{N}\right)^{\mu}, \quad \forall\omega\in\Omega_0\cap\mathscr{B}_N.\end{aligned}$$

Noticing that $P(\Omega_0)=1$, by the above inequality, (23) and Theorem 4.1 one can get

$$I_1^N(\omega)\leqslant\widetilde{\varepsilon}_N, \quad \text{a.a.} \quad \omega\in\mathscr{B}_N. \tag{B.7}$$

Furthermore, from (24) and Lemma 2.1 we have

$$I_2^N(\omega)=\limsup_{n\to\infty}\frac{1}{n}O(n^{1/2+\varepsilon})=0, \quad \text{a.a.} \quad \omega\in\mathscr{B}_N.$$

This together with (22) and (B.7) leads to

$$\Delta_N J^*\leqslant\widetilde{\varepsilon}_N, \quad \text{a.a.} \quad \omega\in\mathscr{B}_N$$

and

$$P\{\Delta_N J^*\leqslant\widetilde{\varepsilon}_N\}\geqslant P(\mathscr{B}_N).$$

Combing this with (B.6) gives (28). □

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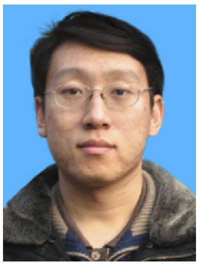

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