Hierarchical Mean Field Games for Multiagent Systems With Tracking-Type Costs: Distributed ε-Stackelberg Equilibria

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***Abstract*—In this technical note, hierarchical games are investigated for multi-agent systems involving a leader and a large number of followers with infinite horizon tracking-type costs. By jointly analyzing dynamic equations and index functions of all agents, a set of centralized Stackelberg equilibrium strategies is given. Then, by using the mean field approach and the brute force method, a set of distributed strategies is designed. Under mild conditions, it is shown that the closed-loop system is uniformly stable and the set of distributed strategies is an ε-Stackelberg equilibrium.**

***Index Terms*—Distributed strategy, mean field approach, multi-agent system, Stackelberg equilibrium, tracking control.**

I. Introduction

In recent years, much attention has been drawn to the game-theoretic framework for control and optimization of multi-agent systems (MASs). Under this framework, each rational agent needs to consider all possible interactions to make its decisions, which results in a great challenge on computational complexity of designing distributed strategies, particularly for large population systems. To reduce the complexity, mean field (MF) approaches were applied to the kind of problems and some asymptotic equilibrium solutions were obtained and studied [1]–[10].

For MF games of MASs, most previous works considered the case where agents are with equal roles. Recently, there have been some results on the case that agents are with different influences. Huang [11] investigated continuous-time stochastic dynamic games for large-population systems with a major player, and provided ε-Nash equilibria for the systems under some consistency conditions. This work was extended to mixed games with continuum-parameterized minor players [12] and nonlinear major-minor MF systems [13], respectively. Wang and Zhang [14] considered the discrete-time case with a major agent and random parameters, and gave a set of ε-Nash strategies in an explicit form. Benkard *et al.* [15] studied oblivious equilibria with dominant firms in industry models.

All the above-mentioned papers assumed that no agents can enforce their strategies on others or have priority to move first. However, this does not fit into some practical situations. For example, in a market with a monopoly investor and many private investors, due to complexity and volatility of markets private investors may not dare to act rashly and often make decisions at the heels of the monopoly investor. Another example is in a game for the central government and local governments the higher authority holds the dominant position and first announces a policy, and then the localities attempt to seek their rational countermeasures to the top-down policy. A well-known mathematical description for the above models is the hierarchical (Stackelberg) game [19]. In this game, the leaders are in a dominant place and able to impose their strategies on followers in subordinate places. Generally speaking, the leaders first announce the decisions, and then the followers give their response strategies. For hierarchical games with a leader and many followers, readers are referred to [16]–[20]. Kydland [17], [18] discussed three types of equilibrium solutions, and gave the computation details of feedback solutions for finite horizon dominant-player games. However, the above works mainly considered the case of centralized strategies, and there are few results on distributed strategies. [21] investigated the hierarchical control of Markov chains, and provided a near-optimal algorithm for open-loop distributed strategies.

In this technical note, hierarchical games are investigated for MASs involving a major agent and many minor agents with infinite horizon tracking-type costs. In each stage of the game, the major agent has priority to first announce its decision, and then all the minor agents give strategies simultaneously. Compared with the previous works, the model in this note is characterized by the following features: (a) In accordance with the role difference, agents are divided into a leader and many followers; there is a two-level hierarchy. The leader dominates in the game and can enforce its strategy on followers. (b) Different from the previous works [14], [22], the initial values of agents' states are arbitrary square-integrable random variables, without assuming that their expectations are equal. Instead of tracking with one-step lag, the objective of each agent is to track the state average of all minor agents in real time, which is more difficult to tackle. In this note, we first provide a set of centralized Stackelberg equilibrium strategies by jointly analyzing the dynamic equations and index functions of all agents. Then, based on the MF approach and the BF method, we obtain a fixed-point equation satisfied by the MF aggregate quantity, from which a set of distributed strategies is given. Under mild conditions, we show that the closed-loop system is uniformly stable and the set of distributed strategies is an ε-Stackelberg equilibrium.

The technical note is organized as follows. In Section II, we describe the model and basic assumptions. In Section III, we provide a set of centralized Stackelberg equilibrium strategies. In Section IV, we first design a set of distributed strategies by the MF theory and the BF method, and then analyze the stability of the closed-loop system and the optimality of the distributed strategies. In Section V, through a numerical example, we verify asymptotical optimality of the distributed strategies. In Section VI, we conclude the note.

Notation

I_n denotes an n-dimensional identity matrix; 1_n denotes the n-dimensional column vector whose elements are 1; e_j denotes the column vector whose elements are 0 except that the jth element is 1. $\|\cdot\|$ denotes the Euclidean vector norm (or induced matrix norm); $\rho(\cdot)$ denotes the spectral radius. $\otimes$ denotes the Kronecker product. For a given set collection $\mathcal{C}$, $\sigma(\mathcal{C})$ denotes the σ algebra generated by $\mathcal{C}$. For a family of $\mathbb{R}^n$-values random variables $\{\xi_\lambda, \lambda \in \Lambda\}$, $\sigma(\xi_\lambda, \lambda \in \Lambda)$ denotes the σ algebra $\sigma\{[\xi_\lambda \in B], B \in \mathcal{B}^n, \lambda \in \Lambda\}$, where $\mathcal{B}^n$ is an n dimensional Borel σ algebra.

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II. Problem Description

Consider the MAS described by the following dynamics:

$$x_0(k+1) = f_0(k, x_0(k)) + u_0(k) + F_0 x^{(N)}(k) + D_0 w_0(k+1) \quad (1)$$

$$x_i(k+1) = f_i(k, x_i(k)) + u_i(k) + F x^{(N)}(k) + G x_0(k) + D w_i(k+1),\ 1 \le i \le N \quad (2)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^n$ and $w_i \in \mathbb{R}^d$, $0 \le i \le N$ are the state, input and stochastic disturbance of the agent i, respectively. x_0 denotes the state of the major agent, and x_i, $1 \le i \le N$ denotes the state of the ith minor agent. $x^{(N)}(k) = \frac{1}{N}\sum_{i=1}^{N} x_i(k)$ is state average of all minor agents. $f_i : \mathbb{Z}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$, $0 \le i \le N$ are Borel measurable functions.

The cost functions of $N+1$ agents are described by

$$J_0^N(u_0, u_{-0}) = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=1}^{T} E\left\| x_0(k) - \Phi(x^{(N)}) \right\|^2 \quad (3)$$

$$J_i^N(u_i, u_{-i}) = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=1}^{T} E\left\| x_i(k) - \Psi(x_0, x^{(N)}) \right\|^2 \quad (4)$$

where $i = 1, \cdots, N$, $\Phi(x^{(N)}) = \hat{H}_k x^{(N)}(k) + \alpha_0$, and $\Psi(x_0, x^{(N)}) = \bar{H}_k x_0(k) + H_k x^{(N)}(k) + \alpha$. Here, $u_{-i} = \{u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N\}$, and $\alpha_0, \alpha \in \mathbb{R}^n$.

Remark 1: Model (1)–(4) is not only closely related with the adaptive tracking control [23], [24], but also has wide application backgrounds. For example, consider a stock market with a monopoly investor and many private investors. x_0 denotes the earnings of the monopoly investor, and x_i denotes the earnings of the ith private investor. For this model, the previous works [14], [22] mainly considered the case where the earnings of each investor attain some function of the average earnings of the market at the preceding instant, i.e., the tracking model with one-step lag. Instead, here each investor anticipates its earnings get to some function of the average earnings of the market in real time, which fits into the practical background better. However, to achieve real-time tracking, each investor not only needs to estimate the average earnings of the market, but also should consider all possible actions of all the investors in the next step. Thus, it is more difficult to analyze (1)–(4).

For convenience of reference, we list the main assumptions:

A1): $\{\{w_i(k)\}, 0 \le i \le N\}$ is a family of independent d-dimensional white noise sequences on a probability space $(\Omega, \mathcal{F}, P)$, $E[w_i(k)] = 0$, $E[w_i(k)w_i^T(j)] = \delta_{kj} I_d$, $0 \le i \le N$, where if $k = j$, $\delta_{kj} = 1$; otherwise, $\delta_{kj} = 0$.

A2): Initial states $\{x_{i0}, 0 \le i \le N\}$ are independent random variables. $\sup_{0\le i\le N} E\|x_{i0}\|^2 < \infty$.

A3): $\sup_{k\ge1}\|\hat{H}_k\| + \sup_{k\ge1}\|H_k\| + \sup_{k\ge1}\|\bar{H}_k\| < \infty$. F is stable, i.e., all its eigenvalues lie inside the unit circle.

Throughout the technical note, assume that the state and parameters of each agent are known to itself. Parameters in costs of minor agents are available to the major agent. The real-time state and strategy of the major agent can be observed by all the minor agents. In each stage of the game, the major agent first announces decisions, and then N minor agents give their strategies simultaneously and noncooperatively. Thus, this game is a sequential game, and every information set of followers contains only one element.

Remark 2: The above assumptions have wide practical backgrounds. For instance, consider a market consisting of a monopoly investor and many private investors. Due to complexity and volatility of markets, private investors may not dare to act rashly, and they often make decisions at the heels of the monopoly investor. Meanwhile, with a high degree of market transparency, actions of the monopoly investor could be available to private investors.

First, we provide two groups of strategy sets:

$$\mathcal{U}_{c,i} = \left\{ u_i | u_i(k) \in \sigma\Big\{ \bigcup_{0\le i\le N} \sigma(x_i(j), u_0(j), j \le k) \Big\} \right\}$$

$$\mathcal{U}_{d,i} = \left\{ u_i | u_i(k) \in \sigma(x_i(j), x_0(j), u_0(j), j \le k) \right\}, 0 \le i \le N.$$

$\mathcal{U}_{c,i}$ is called the centralized strategy set, which corresponds to the information structure $\{x_l(j), 0 \le l \le N, 0 \le j \le k; u_0(j), 0 \le j \le k\}$. $\mathcal{U}_{d,i}$ is called the distributed strategy set, which corresponds to the information structure $\{x_i(j), x_0(j), u_0(j), 0 \le j \le k\}$. The main objective of this technical note is to seek centralized and distributed Stackelberg strategies for the game above.

Remark 3: Distributed games of MAS with a major agent were investigated in [11]–[14], while no leaders or followers are involved in their games and all agents give strategies simultaneously. However, in the game (1)–(4) the major agent as a leader is dominant and has priority to announce a decision in advance, and the minor agent cannot take any actions before getting the leader's instruction. Thus, an essential difference arises from the proactiveness of the leader between the game (1)–(4) and the models in [11]–[14].

III. Centralized Strategies

In this section, we give centralized Stackelberg equilibrium strategies (i.e., each agent can obtain the states and parameters of all the agents, which is an ideal case). This result contributes to design and analyze distributed strategies.

A. The Case of Equal Roles

Consider the following game problem, named (P1), for convenience of reference. The dynamic equations of agents are

$$x_i(k+1) = f_i(k, x_i(k)) + u_i(k) + F x^{(N)}(k) + D w_i(k+1),$$

and the index functions of Agent i, $1 \le i \le N$ is

$$J_i^N(u_i, u_{-i}) = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=1}^{T} E\left\| x_i(k) - (H_k x^{(N)}(k) + \alpha) \right\|^2.$$

We first provide the following result of optimality.

Theorem 3.1: For Problem (P1), if **A1)-A2)** hold, and for any $k > 0$, $I_n - H_k$ is invertible then under any set of strategies $\{u_i \in \mathcal{U}_i, 1 \le i \le N\}$, it follows that for any $1 \le i \le N$

$$J_i^N(u_i, u_{-i}) \ge tr(D^T D) + \frac{1}{N} \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=1}^{T} [tr(D^T H_k^T H_k D) - 2tr(D^T H_k D)]$$

and the equality holds if and only if

$$\hat{u}_i(k) = e_i^T \otimes I_n (I_{nN} - \frac{1}{N} 1_N 1_N^T \otimes H_{k+1})^{-1} (1_N \otimes \alpha) - f_i(k, x_i(k)) - F x^{(N)}(k). \quad (5)$$

Proof: Let

$$x(k) = (x_1^T(k), \ldots, x_N^T(k))^T,\ u(k) = (u_1^T(k), \ldots, u_N^T(k))^T,$$
$$f(k, x(k)) = (f_1^T(k, x_1(k)), \ldots, f_N^T(k, x_N(k)))^T,$$
$$Dw(k) = (w_1^T(k) D^T, \ldots, w_N^T(k) D^T)^T.$$

From **A1)**, **A2)** and independence of $\{w_i(k)\}, i = 1, \ldots, N$

$$J_i^N(u_i, u_{-i}) = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=0}^{T-1} E\left\| (e_i^T \otimes I_n - \frac{1}{N} 1_N^T \otimes H_{k+1}) x(k+1) - \alpha \right\|^2$$

$$= \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=0}^{T-1} \Big\{ E\Big\| (e_i^T \otimes I_n - \frac{1}{N} 1_N^T \otimes H_{k+1}) \times [u(k) + f(k, x(k)) + (1_N \otimes I_n) F x^{(N)}(k)] - \alpha \Big\|^2 + E\Big\| (e_i^T \otimes I_n - \frac{1}{N} 1_N^T \otimes H_{k+1}) D w(k+1) \Big\|^2 \Big\} \geq tr(D^T D) + \frac{1}{N} \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=1}^{T} [tr(D^T H_k^T H_k D) - 2tr(D^T H_k D)]. \quad (6)$$

This implies that the equality holds for $\hat{u}_i$ given by (5). □

Remark 4: By Theorem 3.1, we can obtain that (5) is a unique (strong) Nash equilibrium [25], [26]. Thus, (5) is the unique optimal (rational) response of (P1).

B. Stackelberg Equilibrium Solutions

We now seek a centralized Stackelberg equilibrium for the system (1)–(4) by the brute force (BF) method [19]. Kydland [18] provided a derivation for finite-horizon Stackelberg games. In contrast, here we need to tackle the infinite-horizon game with time-varying tracking-type costs.

Suppose the major agent first provides the strategy $u_0(k) = \bar{u}_0(k)$. Then the optimization problem faced by each minor agent is to minimize $\bar{J}_i(u_i, u_{-i})$ over $\mathcal{U}_{g,i}, 1 \leq i \leq N$, where

$$\bar{J}_i(u_i, u_{-i}) = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=0}^{T-1} \Big[E\Big\| x_i(k+1) - H_{k+1} x^{(N)}(k+1) - \beta(k) \Big\|^2 + E\Big\| \bar{H}_{k+1} D_0 w_0(k+1) \Big\|^2 \Big]. \quad (7)$$

Here, $\beta(k) \triangleq \bar{H}_{k+1}[f_0(k, x_0(k)) + \bar{u}_0(k) + F_0 x^{(N)}(k)] + \alpha$. By Theorem 3.1, the optimal response of Agent i is

$$\bar{u}_i(k) = (e_i^T \otimes I_n)[I_{nN} - \frac{1}{N} 1_N 1_N^T \otimes H_{k+1}]^{-1} (1_N \otimes \beta(k)) - f_i(k, x(k)) - [F x^{(N)}(k) + G x_0(k)]. \quad (8)$$

Applying (8) into (2) yields the closed-loop equation

$$\bar{x}_i(k+1) = (e_i^T \otimes I_n)[I_{nN} - \frac{1}{N} 1_N 1_N^T \otimes H_{k+1}]^{-1} \times (1_N \otimes \beta(k)) + D w_i(k+1). \quad (9)$$

From this and (4), the optimization for Agent 0 is to minimize $J_0(\bar{u}_0, \bar{u}_{-0})$ over $\mathcal{U}_{g,0}$, where

$$J_0^N(\bar{u}_0, \bar{u}_{-0}) = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=0}^{T-1} E\Big\| x_0(k+1) - \frac{1}{N} (1_N^T \otimes \hat{H}_{k+1}) \bar{x}(k+1) - \alpha_0 \Big\|^2$$

and $\bar{x}(k) = (\bar{x}_1^T(k), \ldots, \bar{x}_n^T(k))^T$. By (1), (9) and **A1)**,

$$J_0^N(\bar{u}_0, \bar{u}_{-0}) = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=0}^{T-1} E\Big\| \bar{u}_0(k) + f_0(k, x_0(k)) + F_0(k) x^{(N)}(k) - \alpha_0 - \frac{1}{N} 1_N^T \otimes \hat{H}_{k+1} [I_{nN} - \frac{1}{N} 1_N 1_N^T \otimes H_{k+1}]^{-1} \times (1_N \otimes \beta(k)) \Big\|^2 + E\Big\| D_0 w_0(k+1) \Big\|^2 + E\Big\| \frac{1}{N} (1_N^T \otimes \hat{H}_{k+1}) D w(k+1) \Big\|^2 \geq tr(D_0^T D_0) + \frac{1}{N} \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=0}^{T-1} tr(D^T \hat{H}_{k+1}^T \hat{H}_{k+1} D)$$

and the equality holds when

$$\bar{u}_0(k) + f_0(k, x_0(k)) + F_0 x^{(N)}(k) - \frac{1}{N} 1_N^T \otimes \hat{H}_{k+1} [I_{nN} - \frac{1}{N} 1_N 1_N^T \otimes H_{k+1}]^{-1} (1_N \otimes \beta(k)) - \alpha_0 = 0. \quad (10)$$

This implies

$$\bar{u}_0(k) = -f_0(k, x_0(k)) + [I_n - Q_{k+1}(1_N \otimes \bar{H}_{k+1})]^{-1} \times \Big\{ Q_{k+1}[1_N \otimes (\bar{H}_{k+1} F_0 x^{(N)}(k) + \alpha)] - F_0 x^{(N)}(k) - \alpha_0 \Big\} \quad (11)$$

where $Q_k \triangleq \frac{1}{N} 1_N^T \otimes \hat{H}_k [I_{nN} - \frac{1}{N} 1_N 1_N^T \otimes H_k]^{-1}$.

From the above analysis, we get the following theorem.

Theorem 3.2: If **A1)-A3)** hold, and for any $k > 0$, $I_n - H_k$ and $I_n - \hat{H}_k (I_n - H_k)^{-1} \bar{H}_k$ are invertible, then there exists a centralized Stackelberg equilibrium solution for (1)–(4), which is given by (8) and (11). The corresponding index values are

$$J_0^N(\bar{u}_0, \bar{u}_{-0}) = tr(D_0^T D_0) + \frac{1}{N} \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=1}^{T} tr(D^T \hat{H}_k^T \hat{H}_k D) \quad (12)$$

$$\bar{J}_i(\bar{u}_i, \bar{u}_{-i}) = tr(D^T D) + \frac{1}{N} \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=1}^{T} [tr(D^T H_k^T H_k D) - 2tr(D^T H_k D)] + \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{k=1}^{T} tr(D_0^T \bar{H}_k^T \bar{H}_k D_0), \quad 1 \leq i \leq N. \quad (13)$$

Proof: We only need to prove that $I_{nN} - \frac{1}{N} 1_N 1_N^T \otimes H_k$ and $I_{nN} - Q_k(1_N \otimes \bar{H}_k)$ are invertible. Notice $\det(I_{nN} - \frac{1}{N} 1_N 1_N^T \otimes H_k) = \det(I_n - H_k)$. From the invertibility of $I_n - H_k$, $I_{nN} - \frac{1}{N} 1_N 1_N^T \otimes H_k$ are invertible. Furthermore, $(I_{nN} - \frac{1}{N} 1_N 1_N^T \otimes H_k)^{-1} = I_{nN} + \frac{1}{N} 11^T \otimes [(I - H_k) H_k]$. By direct computation, $I_{nN} - Q_k(1_N \otimes \bar{H}_k) = I_n - \hat{H}_k (I_n - H_k)^{-1} \bar{H}_k$, which implies that $I_{nN} - Q_k(1_N \otimes \bar{H}_k)$ is invertible. □

IV. Distributed Strategies

A. Design of Strategies

We now design distributed strategies for the system (1)–(4) by using the MF approach and the BF method. The key idea of MF approaches is to replace the overall effect of all agents to a single agent by the aggregate effect [2], [9].

First, we construct the auxiliary system described by

$$\check{x}_0(k+1) = f_0(k, \check{x}_0(k)) + u_0(k) + F_0 g(k) + D_0 w_0(k+1) \quad (14)$$

$$\check{x}_i(k+1) = f_i(k, \check{x}_i(k)) + u_i(k) + F g(k) + G \check{x}_0(k) + D w_i(k+1), \ 1 \leq i \leq N \quad (15)$$

with the index functions

$$\check{J}_0(u_0,u_{-0}) = \overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=1}^{T}E\left\|\check{x}_0(k)-\hat{H}_k g(k)-\alpha_0\right\|^2 \tag{16}$$

$$\check{J}_i(u_i,u_{-i}) = \overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=1}^{T}E\Big\|\check{x}_i(k)-(\bar{H}_k\check{x}_0(k) + H_k g(k)+\alpha)\Big\|^2, 1\le i\le N. \tag{17}$$

In (14)–(17), $g(k), k\ge 0$ is called the aggregate effect function of all the minor agents, which can be obtained by each agent by solving a fixed-point equation.

Suppose Agent 0 first announces $u_0(k) = \check{u}_0(k)$. Then the optimization problem faced by all the minor agents is to minimize $\check{J}_i(\check{u}_0,u_1,\ldots,u_N)$ over $\check{\mathcal{U}}_{l,i}$, where $\check{\mathcal{U}}_{l,i} \triangleq \Big\{u_i|u_i(k)\in\sigma(\check{x}_i(j),\check{x}_0(j),\check{u}_0(j),j\le k)\Big\}$, and

$$\begin{aligned}&\check{J}_i(\check{u}_0,u_1,\ldots,u_N)\\ &= \overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=0}^{T-1}E\Big\|\check{x}_i(k+1)-[\bar{H}_{k+1}(f_0(k,\check{x}_0(k))\\ &\quad+\check{u}_0(k)+F_0 g(k))+H_{k+1}g(k+1)+\alpha]\\ &\quad-\hat{H}_{k+1}D_0w_0(k+1)\Big\|^2,\qquad 1\le i\le N.\end{aligned} \tag{18}$$

By Assumption **A1)** and (15)

$$\begin{aligned}&\check{J}_i(\check{u}_0,u_1,\ldots,u_N)\\ &= \overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=0}^{T-1}\Big\{E\Big\|u_i(k)+f_i(k,\check{x}_i(k))+Fg(k)+G\check{x}_0(k)\\ &-[\bar{H}_{k+1}(f_0(k,\check{x}_0(k))+\check{u}_0(k)+F_0g(k))+H_{k+1}g(k+1)\\ &+\alpha]\Big\|^2+tr(DD^T)+tr(D_0^T\hat{H}_{k+1}^T\hat{H}_{k+1}D_0)\Big\}.\end{aligned}$$

This gives the optimal response strategies of minor agents

$$\begin{aligned}\check{u}_i(k) &= \bar{H}_{k+1}(f_0(k,\check{x}_0(k))+\check{u}_0(k)+F_0g(k))+\alpha\\ &\quad+H_{k+1}g(k+1)-f_i(k,\check{x}_i(k))-Fg(k)-G\check{x}_0(k).\end{aligned} \tag{19}$$

Applying (19) into (15), we get the closed-loop equation

$$\begin{aligned}\check{x}_i(k+1) &= \bar{H}_{k+1}(f_0(k,\check{x}_0(k))+\check{u}_0(k)+F_0g(k))+\alpha\\ &\quad+H_{k+1}g(k+1)+Dw_i(k+1),\ 1\le i\le N\end{aligned} \tag{20}$$

which implies

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\check{x}_i(k+1) &= \bar{H}_{k+1}(f_0(k,\check{x}_0(k))+\check{u}_0(k)+F_0g(k))\\ &\quad+H_{k+1}g(k+1)+\alpha+\frac{1}{N}\sum_{i=1}^{N}Dw_i(k+1).\end{aligned}$$

Note that from the law of large numbers, it follows that:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}Dw_i(k+1)=0,\ a.s.$$

and $x^{(N)}$ is replaced by g when constructing the auxiliary system (14)–(17). By the MF approach the aggregate function should satisfy

$$\begin{aligned}g(k+1) &= \bar{H}_{k+1}(f_0(k,\check{x}_0(k))+\check{u}_0(k)+F_0g(k))\\ &\quad+H_{k+1}g(k+1)+\alpha.\end{aligned} \tag{21}$$

On the other hand, by (14) and (16), we have

$$\check{u}_0(k)=\hat{H}_{k+1}g(k+1)+\alpha_0-f_0(k,\check{x}_0(k))-F_0g(k).$$

This together with (21) leads to the fixed-point equation

$$\begin{aligned}g^*(k+1) &= (\bar{H}_{k+1}\hat{H}_{k+1}+H_{k+1})g^*(k+1)\\ &\quad+\bar{H}_{k+1}\alpha_0+\alpha,\quad k\ge 0.\end{aligned} \tag{22}$$

Thus, we obtain the set of distributed strategies

$$u_0^*(k)=\hat{H}_{k+1}g^*(k+1)+\alpha_0-F_0g^*(k)-f_0(k,x_0(k)) \tag{23}$$

$$u_i^*(k)=g^*(k+1)-Fg^*(k)-f_i(k,x_i(k))-Gx_0(k) \tag{24}$$

where $1\le i\le N$; $g^*(k), k\ge 1$ is given by (22) and $g^*(0)$ can be taken as an arbitrary value.

B. Analysis of the Closed-Loop System

Applying (23) and (24) into the dynamic (1)–(2), we get the closed-loop system

$$\begin{aligned}x_0(k+1) &= \hat{H}_{k+1}g^*(k+1)+\alpha_0+F_0(x^{(N)}(k)-g^*(k))\\ &\quad+D_0w_0(k+1)\end{aligned} \tag{25}$$

$$\begin{aligned}x_i(k+1) &= g^*(k+1)+F(x^{(N)}(k)-g^*(k))\\ &\quad+Dw_i(k+1),\ 1\le i\le N\end{aligned} \tag{26}$$

where $g^*(k)$ is given by (22).

Noting the MF aggregate function g^* is only determined by (22), g^* may not be existent or unique. For the former case, the MF approach does not work for our problem; for the later case, further investigation needs to be done. In the remainder of this paper, we only consider the case where the MF aggregate function is existent and unique. Specifically, we assume:

A4): $\overline{\lim}_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}\Big\|[I_n-M_k]^{-1}\Big\|^2<\infty$, where $M_k\triangleq\bar{H}_k\hat{H}_k+H_k$.

Generally speaking, **A4)** is hard to verify directly. We now provide an easier-to-be-verified condition that ensures **A4)**.

Proposition 4.1: If $\rho(M_k)<1$ for any $k\ge 1$, and there exists a sufficiently large l_0, such that for all $l>l_0$,

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=1}^{T}\|M_k\|^l\le\frac{1}{l^r},\ r>2,$$

then Assumption **A4)** is guaranteed. Particularly, if $\sup_{k\ge 1}\rho(M_k)<1$, then **A4)** holds.

Proof: If $\rho(M_k)<1$, then from [27], $(I_n-M_k)^{-1}$ is existent, and $(I_n-M_k)^{-1}=\sum_{l=0}^{\infty}M_k^l$. By a straightforward calculation with properties of matrix norms, we have

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=1}^{T}\|(I_n-M_k)^{-1}\|^2$$

$$\leq \overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=1}^{T}\Big(\sum_{l=0}^{\infty}\|M_k\|^l\Big)^2$$
$$= \overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=1}^{T}\sum_{l=0}^{\infty}(l+1)\|M_k\|^l$$
$$= \sum_{l=0}^{\infty}(l+1)\overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=1}^{T}\|M_k\|^l.$$

This with $\overline{\lim}_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}\|M_k\|^l \leq \frac{1}{l^r}, r>2, l>l_0$ gives

$$\overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=1}^{T}\|(I_n-M_k)^{-1}\|^2 \leq \sum_{l=l_0}^{\infty}\frac{l+1}{l^r}+C_1<\infty.$$

□

We first provide a result of the approximation error.

Theorem 4.1: For the system (1)–(4), if **A1)-A4)** hold, then under (23)–(24), there exists a constant C_2 such that

$$\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}E\|x^{(N)}(k)-g^*(k)\|^2 \leq \frac{C_2}{N}. \quad (27)$$

Proof: Let $\eta_N(k)=x^{(N)}(k)-g^*(k)$. By (26), we have

$$\eta_N(k+1)=F\eta_N(k)+\xi_N(k+1) \quad (28)$$

where $\xi_N(k)=\frac{1}{N}\sum_{i=1}^{N}Dw_i(k)$. From **A1)**, it follows that

$$E\left[\xi_N(k)\xi_N^T(k)\right]=\frac{1}{N}(D^TD),$$
$$E\left[\xi_N(k)\xi_N^T(l)\right]=0,\ \text{for } k\neq l.$$

From (28)

$$\eta_N(k)=F^k\eta_N(0)+\sum_{j=0}^{k}F^j\xi_N(k-j).$$

Thus

$$\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}E\|\eta_N(k)\|^2$$
$$=\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}tr\Big[(F^k)^TF^kE(\eta_N(0)\eta_N^T(0)) + \frac{1}{N}\sum_{j=0}^{k}(F^j)^TF^jDD^T\Big]$$
$$=\frac{1}{N}\sum_{k=1}^{\infty}tr(D^T(F^k)^TF^kD)\leq\frac{C_2}{N}.$$

□

We now give uniform stability of the closed-loop system.

Theorem 4.2: For (1)–(4), if **A1)-A4)** hold, then under (23)–(24), the closed-loop system has the following property:

$$\sup_{N\geq1}\max_{0\leq i\leq N}\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}E\|x_i(k)\|^2<\infty. \quad (29)$$

Proof: By (22) and Assumptions **A3)-A4)**, we have

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=1}^{T}\|g^*(k)\|^2$$
$$=\overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=1}^{T}\Big\|[I_n-M_k]^{-1}(\bar H_k\alpha_0+\alpha)\Big\|^2$$
$$\leq C_1\Big(2\|\alpha_0\|^2\sup_{k\geq1}\|\bar H_k\|^2+2\|\alpha\|^2\Big)\triangleq C_4 \quad (30)$$

where $M_k=\bar H_k\hat H_k+H_k$. This together with (25), **A1)**, **A3)** and Theorem 4.1 implies that

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=0}^{T-1}E\|x_0(k+1)\|^2$$
$$=\overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=0}^{T-1}\Big[E\|\hat H_{k+1}g^*(k+1)+\alpha_0 + F_0(x^{(N)}(k)-g^*(k))\|^2+E\|D_0w_0(k+1)\|^2\Big]$$
$$\leq\overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=0}^{T-1}\Big[3E\|\hat H_{k+1}g^*(k+1)\|^2+3\|\alpha_0\|^2 + 3E\|F_0(x^{(N)}(k)-g^*(k))\|^2+tr(D_0D_0^T)\Big]$$
$$\leq 3[C_4\sup_{k\geq1}\|\hat H_k\|^2+\|\alpha_0\|^2+\|F_0\|^2\frac{C_2}{N}]+tr(D_0D_0^T). \quad (31)$$

From (26), (30), **A1)**, **A3)**, and Theorem 4.1, it follows that:

$$\max_{1\leq i\leq N}\overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=0}^{T-1}E\|x_i(k+1)\|^2$$
$$\leq\overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=0}^{T-1}\Big[2E\|F(x^{(N)}(k)-g^*(k))\|^2 + 2E\|g^*(k+1)\|^2+tr(DD^T)\Big]$$
$$\leq 2\|F\|^2\frac{C_2}{N}+2C_4+tr(DD^T) \quad (32)$$

which together with (31) gives the theorem. □

Below we extend the Stackelberg strategy concept in [19] from the optimal response to the ε-optimal response for followers and give the definition of the ε-Stackelberg equilibrium.

Definition 4.1: Let $\varepsilon\geq0$. For the system (1)–(4), a set of strategies $\{u_i\in\mathcal{U}_{g,i},1\leq i\leq N\}$ is an ε-optimal response of minor agents to the strategy $\{u_0\}$, if for any $1\leq i\leq N$

$$J_i^N(u_0,u_1,\ldots,u_N)\leq J_i^N(u_0,u_1',\ldots,u_N')+\varepsilon$$

where $\{u_1',\ldots,u_N'\}$ is an optimal response of the minor agents to $\{u_0\}$. A set of strategies $\{u_i\in\mathcal{U}_{g,i},0\leq i\leq N\}$ is an ε-Stackelberg equilibrium, if $\{u_1,\ldots,u_N\}$ is an ε-optimal response of the minor agents to $\{u_0\}$, and

$$J_0^N(u_0,u_{-0})\leq J_0^N(\bar u_0,\bar u_{-0})+\varepsilon,$$

where $\{\bar u_0,\ldots,\bar u_N\}$ is a Stackelberg equilibrium.

We are now in a position to show the asymptotic optimality of the distributed strategies.

Theorem 4.3: For the system (1)–(4), if **A1)-A4)** hold, then under (23)–(24), the corresponding index values satisfy

$$J_0^N(u_0^*,u_{-0}^*)\leq tr(D_0D_0^T)+O(\frac{1}{N}) \quad (33)$$
$$J_i^N(u_i^*,u_{-i}^*)\leq tr(DD^T) + \overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{k=1}^{T}tr(D_0^T\bar H_k^T\bar H_kD_0)+O(\frac{1}{N}). \quad (34)$$

Furthermore, $\{u_0^*,\ldots,u_N^*\}$ constitutes an ε-Stackelberg equilibrium, where $\varepsilon=O(1/N)$.

Proof: By Assumption **A1)**, (3), (25) and (27), we have

$$
\begin{aligned}
&J_0^N(u_0^*, u_{-0}^*)\\
&= \overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=0}^{T-1} E\Big\| \hat{H}_{k+1}(g^*(k+1) - x^{(N)}(k+1))\\
&\quad + F_0[x^{(N)}(k) - g^*(k)] + D_0 w_0(k+1)\Big\|^2\\
&\le \overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=0}^{T-1} \Big\{ 2E\Big\| \hat{H}_{k+1}(g^*(k+1) - x^{(N)}(k+1))\Big\|^2\\
&\quad + 2E\Big\| F_0[x^{(N)}(k) - g^*(k)]\Big\|^2 + E\Big\| D_0 w_0(k+1)\Big\|^2\Big\}\\
&\le \Big[2\sup_{k\ge 1}\|\hat{H}_k\|^2 + 2\|F_0\|^2\Big]\frac{C_2}{N} + tr(D_0 D_0^T)\\
&= tr(D_0 D_0^T) + O(\frac{1}{N}).
\end{aligned} \tag{35}
$$

From **A1)**, (4), (22) and (25)–(27), it follows that

$$
\begin{aligned}
&J_i^N(u_i^*, u_{-i}^*)\\
&= \overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=0}^{T-1} E\Big\{ \Big\| H_{k+1}[g^*(k+1) - x^{(N)}(k+1)]\\
&\quad + (F_0 - F)[x^{(N)}(k) - g^*(k)] + D w_i(k+1)\Big\|^2\\
&\quad + \Big\| \bar{H}_{k+1} D_0 w_0(k+1)\Big\|^2\Big\}\\
&\le \Big[2\sup_{k\ge 1}\|H_k\|^2 + 2\|F_0 - F\|^2\Big]\frac{C_2}{N} + tr(DD^T)\\
&\quad + \overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=1}^{T} tr(D_0^T \bar{H}_k^T \bar{H}_k D_0)\\
&= tr(DD^T) + \overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=1}^{T} tr(D_0^T \bar{H}_k^T \bar{H}_k D_0) + O(\frac{1}{N}).
\end{aligned}
$$

This together with (7), (13), and Theorem 3.1 gives

$$
\begin{aligned}
&J_i^N(u_i^*, u_{-i}^*)\\
&\le tr(DD^T) + \overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=1}^{T} tr(D_0^T \bar{H}_k^T \bar{H}_k D_0)\\
&+ \frac{1}{N}\overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=1}^{T}[tr(D^T H_k^T H_k D) - 2tr(H_k D)] + O(\frac{1}{N})\\
&= J_i^N(u_0^*, \tilde{u}_1, \ldots, \tilde{u}_N) + O(\frac{1}{N})
\end{aligned} \tag{36}
$$

where $\{\tilde{u}_1, \ldots, \tilde{u}_N\}$ is the optimal response of the minor agents to $\{u_0^*\}$. From (12), (35) and **A3)**, one can get

$$
\begin{aligned}
&J_0^N(u_0^*, u_{-0}^*)\\
&\le tr(D_0 D_0^T) + \frac{1}{N}\overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{k=1}^{T} tr(D^T H_k^T H_k D) + O(\frac{1}{N})\\
&= J_0^N(\bar{u}_0, \bar{u}_{-0}) + O(\frac{1}{N})
\end{aligned}
$$

which together with (36) implies $\{u_0^*, \ldots, u_N^*\}$ is an ε-Stackelberg equilibrium with $\varepsilon = O(1/N)$. □

V. Numerical Example

We now use a numerical example to illustrate the asymptotical optimality of distributed strategies.

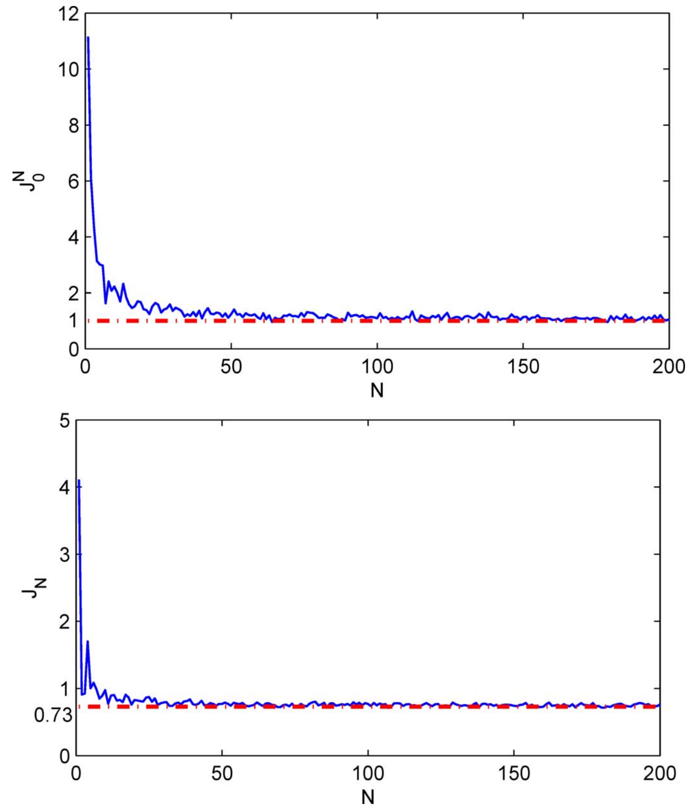

Fig. 1. Trajectories of J_0^N and J^N with respect to N.

The dynamic equations of $N+1$ agents are given by

$$
\begin{aligned}
x_0(k+1) &= 0.9x_0(k) + u_0(k) - 0.45x^{(N)}(k) + 0.6w_0(k+1),\\
x_i(k+1) &= 0.8x_i(k) + u_i(k) - 0.93x^{(N)}(k) + 0.2x_0(k)\\
&\quad + 0.8w_i(k+1),\ 1 \le i \le N
\end{aligned}
$$

where $\{w_i(k), 0 \le i \le N\}$ is a white noise sequence with the normal distribution $N(0,1)$. Let $x_{00} = 5$, $\{x_{i0}, i = 1, \ldots, N\}$ be independent and identically distributed random variables with $N(1, 0.2)$. Parameters in index functions are taken as $\hat{H}_k = \frac{k+4}{k+3}$, $\bar{H}_k = 0.3$, $H_k = 0.4 + \frac{1}{k+5}$, $k = 1, 2, \ldots$. It can be verified that **A1)-A4)** hold. From (23) and (24)

$$
u_0^*(k) = \frac{k+4}{k+3}g^*(k+1) - 0.45g^*(k) - 0.9x_0(k) + 5 \tag{37}
$$
$$
u_i^*(k) = g^*(k+1) - 0.93g^*(k) - 0.8x_i(k) - 0.2x_0(k) \tag{38}
$$

where $g^*(k) = 35/\Big(3 - \frac{3}{k+3} - \frac{10}{k+5}\Big)$.

We now check the costs of all the agents under (37) and (38). Let $J^N = \max_{1\le i\le N} J_i^N$. Then, by Theorems 3.2 and 4.3, we get that for the case of large population J_0^N and J^N are approximately $D_0^2 = 1$ and $D^2 + \overline{\lim}_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T} D_0^2\bar{H}_k^2 = 0.73$, respectively, which are the index values under the centralized Stackelberg equilibrium. When the number of agents grows from 1 to 200, the trajectories of J_0^N and J^N are shown in Fig. 1, from which one can see that the costs tend to the upper bounds 1 and 0.73.

VI. Concluding Remarks

This note studies Stackelberg games for MASs involving a leader and many followers with infinite horizon tracking-type costs. We first provide a set of centralized strategies for this case, and then give a set of distributed strategies by the MF approach and the BF method.

It is shown that the set of distributed strategies is an ε-Stackelberg equilibrium.

For Stackelberg games based on the MF approach, there are a lot of interesting problems worthy of investigating. The MF approach may be used to tackle the case where the costs are general functions of x_i, $x^{(N)}$ and u_i. Furthermore, we would ask how robust the results on MF games are, or whether MF approaches can tackle the model with non-white noises (e.g., general bounded noises).

In this note, we assume minor agents can get the information of the major agent freely. However, in some case the cost of communication cannot be neglected, which needs to be considered further. This technical note considers MASs with time-varying dynamics and costs. If the model specializes to a time-invariant one, some connections with ergodic control and stationary distribution may be an interesting issue.

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