Distributed Parameter Estimation Over Unreliable Networks With Markovian Switching Topologies

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***Abstract*—Due to the existence of various uncertainties, the design of distributed estimation algorithms with robustness and high accuracy is an urgent demand for sensor network applications. This paper is aimed at investigating the design of distributed parameter estimation algorithms and the analysis of their convergence properties in uncertain sensing and communication environments. Consensus-based distributed parameter estimation algorithms for both discrete-time and continuous-time cases are established, which are suitable for unreliable communication networks with stochastic communication noises, random link gains and Markovian signal losses. Under mild conditions on stochastic noises, gain function and topology-switching Markov chain, we establish both the mean square and almost sure convergence of the designed algorithms by use of probability limit theory, algebraic graph theory, stochastic differential equation theory and Markov chain theory. The effect of sensor-dependent gain functions on the convergence of the algorithm is also analyzed.**

***Index Terms*—Consensus, distributed estimation, multi-agent systems, sensor network, stochastic approximation.**

I. Introduction

A. Background and Motivation

COMPRISED of large numbers of spatially distributed electronic nodes with certain capability of sensing, computation, and communication, sensor network has obtained wide span of applications due to its flexibility, high accuracy, low cost, fault tolerance, and ease of deployment characteristics, for example, environment monitoring, disaster surveillance, military reconnaissance, etc. ([2] and [3]). Among the theoretic researches for sensor network, one fundamental problem is the distributed parameter estimation in uncertain environments, which is proposed because of the following practical demands. On the one hand, due to the limited sensing capability, each sensor can only sense or observe partial information of the unknown parameter. If we solely use these local measurements to conduct parameter estimation, the observability for the whole information of the unknown parameter often cannot be assured. On the other hand, due to the limited communication ability, each sensor can only exchange information within its neighbors. Thus, a natural idea is whether we can design a scalable distributed estimation algorithm based on local measurements of each sensor and local communications among neighbor sensors, and finish the estimation task for the unknown parameter cooperatively. Coordination among sensors can enhance the global observability of the sensor network. Compared with centralized ones where local measurements need to be transmitted to a fusion sensor for processing, the distributed estimation scheme has the advantages of robustness for single node failure and reductions of communication and computation costs, thus can extend the lifetime of the sensor network. As a result, the distributed parameter estimation algorithm design and its convergence analysis are of significance, and have gained more and more studies recently. Consensus-based distributed estimation algorithm is one of such important algorithms.

As the name suggests, consensus-based distributed estimation algorithm contains a consensus scheme, which reflects the coordination among sensors. By introducing the consensus scheme in algorithm, it can be expected to achieve the following goals: making the algorithm more robust to various uncertainties such as communication noises, measurement noises, communication link failures, etc., and improving the overall estimation accuracy, by use of the coordination nature of the consensus scheme; increasing the convergence rate and noise immunity of the algorithm by use of the "convexification" procedure of the consensus scheme for the information of each sensor's neighbors ([4]). Thus, the study of consensus-based distributed estimation algorithm is closely related to that of consensus scheme. Actually, due to its wide applications in distributed control and distributed estimation, distributed consensus problem of multi-agent systems has become a relatively independent research field, obtaining deep and comprehensive results. The interested readers are referred to the literatures [5]–[14] and references therein.

Meanwhile, the results on consensus-based distributed estimation algorithm and its convergence analysis are gradually coming up. [15] proposed a distributed estimation algorithm based on diffusion strategies. This scheme can help to percolate new data across sensor network, but increases communication cost and degrades estimation performance when the data is corrupted by uncertainties such as communication noises. To deal with these issues, [16], [17] introduced the "bridge" sensor subset, transformed the global convex optimization problem into an equivalent constrained form suitable for distributed implementations, and by a distributed optimization method, developed a distributed least squares method ([17]) and a distributed least mean-squares method ([16]). They proved that in the noise-free case, the estimate was exponentially convergent to the unknown parameter; and in the case of measurement

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noises and communication noises, the estimation error was weakly stochastic bounded. [18] designed distributed estimation algorithms for both linear and nonlinear models, dealing with the uncertainties such as quantization errors and random link failures. It is shown that the algorithms are almost surely convergent, asymptotically unbiased and normal. To ensure the quantization error has a good statistical property, a random dither is added to the signal to be quantized and transmitted. Besides, it is assumed that the communication link failures are temporally independent, which can be seen from the condition that requires the Laplacian matrices of the communication topology graphs are independent and identically distributed (i.i.d.). [4] proposed a distributed estimation algorithm by combining local stochastic approximation and consensus scheme to deal with the uncertainties such as independent communication noises and independent switches of communication topologies. The estimate is shown to be mean square convergent. The literatures about distributed Kalman filter algorithms are referred to [19], [20], where both the state model and measurement model of the sensors are required to be known.

B. Main Contribution

This paper is about to consider the design and convergence analysis of distributed parameter estimation algorithm over unreliable communication networks. The uncertainties include measurement and communication noises, random link gains, and temporally correlated signal losses. Random link gains often result from uncoded analog signal transmissions during large sensor network applications ([8]). Signal losses are often caused by temporary faults of the sender and receiver, or active topology switches to adapt to high level commands and environment changes. [4], [18] have analyzed the discrete-time distributed estimation problem by using i.i.d. random graph sequence to capture the loss of connectivity. However, in practice it is often the case for randomly switching topologies to be temporally correlated. Thus, in this paper we introduce a discrete-time Markov chain to capture the temporal correlation of the random topology switches. These additional uncertainties encourage us to design effective algorithms and analysis methods. In this paper, a stochastic approximation-type distributed parameter estimation algorithm is proposed. Besides the mean square convergence as discussed in [4], the almost sure convergence is also obtained by use of probability limit theory, algebraic graph theory and Markov chain theory.

In addition, we notice in the existing literatures, the design and convergence analysis of continuous-time distributed estimation algorithms have received relatively little attention. Thus, a stochastic approximation-type continuous-time estimation algorithm is presented and analyzed rigorously in this paper. It should be noticed that the discrete and continuous-time algorithms are designed in a unified way, but are analyzed by using very different methods. For the continuous-time case, after assuming the random measurement matrices, random measurement and communication noises as white noise sequences, the analysis on the convergence of the algorithm is transformed to that on the stability of the Markov switching stochastic differential equation describing the dynamics of the parameter estimation error. By using probability limit theory, algebraic graph theory, Markov chain theory and stability analysis methods of the continuous-time Markovian jump parameter systems ([21]), the proposed continuous-time algorithm is shown to be convergent both in mean square and almost surely. The effects of different gain functions on the convergence of the designed algorithms are also analyzed in this paper.

C. Organization of the Paper and Notations

The rest of this paper is organized as follows. In Section II, we present some related graph notations, and formulate the distributed parameter estimation problem to be investigated. In Section III, we propose a discrete-time parameter estimation algorithm, and prove it converges both in mean square and almost surely. In Section IV, we consider the continuous-time counterpart. In Section V, we use two numerical examples to illustrate the theoretic results obtained. In Section VI, we give some concluding remarks and further research topics.

The following notations will be used throughout this paper. I_n denotes the n dimensional identity matrix. 1_n is an n dimensional vector whose elements are all ones.[1] For a given vector or matrix X, X^T denotes its transpose, $\|X\|$ denotes its Euclid norm, $Null(X)$ denotes its null-space. The Hadamard product of two $m\times n$ dimensional matrices A and B is denoted by $A\odot B$ with each entry $[A\odot B]_{ij}=[A]_{ij}[B]_{ij}, 1\leq i\leq m, 1\leq j\leq n$. The Kronecker sum of two square matrices $C\in\mathbb{R}^{m\times m}$ and $D\in\mathbb{R}^{n\times n}$ is defined by $C\oplus D=I_n\otimes C+D\otimes I_m$, where $\otimes$ denotes the Kronecker product.

II. Problem Statement

In this section, we first present some preliminary notations on graph theory which will be used throughout this paper. The distributed parameter estimation problem is then formulated, including the modeling for local noisy measurements and unreliable communication network.

A. Graph Theory Notations

Consider a sensor network with N sensors. The communications between sensors are modeled by a weighted digraph $\mathcal{G}=\{\mathcal{V},\mathcal{E}_{\mathcal{G}},\mathcal{A}_{\mathcal{G}}\}$, where $\mathcal{V}=\{1,\cdots,N\}$ is the node set, and node $i\in\mathcal{V}$ represents sensor i; $\mathcal{E}_{\mathcal{G}}\subseteq\mathcal{V}\times\mathcal{V}$ is the edge set, and a direct edge $(i,j)\in\mathcal{E}_{\mathcal{G}}$ if and only if there is a communication link from i to j, where i is defined as the parent node, and j is defined as the child node. $\mathcal{A}_{\mathcal{G}}=[a_{ij}]\in\mathbb{R}^{N\times N}$ is the weighted adjacency matrix of $\mathcal{G}$ with $a_{ij}\geq 0$, and $a_{ij}>0$ if and only if $(j,i)\in\mathcal{E}_{\mathcal{G}}$. $\mathcal{N}_i=\{j\in\mathcal{V}:(j,i)\in\mathcal{E}_{\mathcal{G}}\}$ denotes the neighborhood of node i. Here, we assume the self-edge (i,i) is not allowed.

$\mathcal{G}$ is called a balanced digraph, if $\sum_{j=1}^N a_{ij}=\sum_{j=1}^N a_{ji}$ for all i. $\mathcal{G}$ is called an undirected graph, if $\mathcal{A}_{\mathcal{G}}$ is symmetric. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L}_{\mathcal{G}}=\mathcal{D}_{\mathcal{G}}-\mathcal{A}_{\mathcal{G}}$, where $\mathcal{D}_{\mathcal{G}}=diag\{\sum_{j=1}^N a_{1j},\cdots,\sum_{j=1}^N a_{Nj}\}$. For a given matrix $\Lambda=[\lambda_{ij}]\in\mathbb{R}^{N\times N}$, $\mathcal{G}_\Lambda=\{\mathcal{V},\mathcal{E}_{\mathcal{G}_\Lambda},\Lambda_{\mathcal{G}_\Lambda}\}$ is called the digraph generated by Λ, where $\Lambda_{\mathcal{G}_\Lambda}=[\lambda'_{ij}]$, $\lambda'_{ii}=0$, $\lambda'_{ij}=\lambda_{ij}, i\neq j$. The mirror graph of the digraph $\mathcal{G}$ is an undirected graph, denoted by $\hat{\mathcal{G}}=\{\mathcal{V},\mathcal{E}_{\hat{\mathcal{G}}},\mathcal{A}_{\hat{\mathcal{G}}}\}$ with $\mathcal{A}_{\hat{\mathcal{G}}}=[\hat{a}_{ij}]$, $\hat{a}_{ij}=\hat{a}_{ji}=(1/2)(a_{ij}+a_{ji})$ ([6]). For a given positive integer k, the union of k digraphs $\{\mathcal{G}(i),i=1,\cdots,k\}$ is denoted by $\sum_{i=1}^k\mathcal{G}(i)=\{\mathcal{V},\cup_{i=1}^k\mathcal{E}_{\mathcal{G}(i)},\sum_{i=1}^k\mathcal{A}_{\mathcal{G}(i)}\}$. A sequence of edges $(i_1,i_2),(i_2,i_3),\cdots,(i_{k-1},i_k)$ is called a path from i_1 to i_k. $\mathcal{G}$ is called strongly connected if for any $i,j\in\mathcal{V}$, there is a

[1]When the dimension of the vector is clear, the subscript n will be omitted.

path from i to j. A directed tree is a digraph, where each node except the root has exactly one parent node. A spanning tree of $\mathcal{G}$ is a directed tree whose node set is $\mathcal{V}$ and whose edge set is a subset of $\mathcal{E}_{\mathcal{G}}$.

B. Sensing and Communication Models

Consider N sensors with a common task to estimate an n dimensional unknown parameter θ^* via coordination among sensors. Assume each sensor receives local noisy measurements of θ^* independently. Due to the limited sensing ability of each sensor, it is often the case that the local measurements obtained only reflect partial information of θ^*, i.e., $n_i \ll n$. In such a situation, if only local measurements are used to conduct the estimation task, then at most a part of the parameter θ^* is estimated. Thus, we need to design an efficient distributed estimation algorithm for each sensor, and make their estimation states converge to θ^* via local communications among sensors and iterations of the algorithm.

For sensor i $(i = 1, \cdots, N)$, its local measurement at each time instant for the unknown parameter vector θ^* is assumed to be a stochastic linear function with θ^* and corrupted by random noises. In other words, we model the local measurement of sensor i at time t as follows:

$$z_i(t) = C_i(t)\theta^* + v_i(t), \quad t = 0, 1, \cdots \tag{1}$$

where $z_i(t)$ and $v_i(t) \in \mathbb{R}^{n_i}$ denote the measurement vector and random measurement noise of sensor i at time t, respectively; $C_i(t) \in \mathbb{R}^{n_i \times n}$ denotes the time-varying random measurement matrix.

Denote the ideal communication network of sensors by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}\}$. In practice, due to various uncertainties, some links are failure-prone with positive probability. Assuming $(l, k) \in \mathcal{E}_{\mathcal{G}}$ is failure-prone, we will use a Markov chain m_t^{ji} with the state space {0,1} to describe its temporal evolution, where state "1" denotes the link is normal or created, while "0" denotes the channel is lost. Let us make a fixed ordering for all links in $\mathcal{E}_{\mathcal{G}}$, according to which the Markov chains of all failure-prone links can be listed into a vector m_t. Obviously, m_t has a finite state space $\mathcal{S}$, each state of which is an n_f-dimensional vector with 0 or 1 as its element, where n_f is the number of all failure-prone links. Label the states of $\mathcal{S}$ in sequence as a set $I = \{1, \cdots, s\}$, $s = 2^{n_f}$, corresponding to the communication topology graph set $\mathcal{C} = \{\mathcal{G}^{(1)}, \cdots, \mathcal{G}^{(s)}\}$, where $\mathcal{G}^{(k)} = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}^{(k)}}, \mathcal{A}_{\mathcal{G}^{(k)}}\}$ is a digraph. Without loss of generality, we assume the first state is $[1, \cdots, 1]^T$, which amounts to saying $\mathcal{G}^{(1)} = \mathcal{G}$. Denote the Markovian random communication graph at time t by $\mathcal{G}_t$. We say $m_t = k$[2] if and only if $\mathcal{G}_t = \mathcal{G}^{(k)}$. Thus, it can be seen that the random process m_t completely describes dynamic changes of the communication topology. In addition, the Markov chain model can also capture the uncertain aspects of quantization error and packet loss in digital communications ([8]). Notice that, no constraint is made on the spatial evolution of unreliable links, i.e., the link failures may be spatially correlated, which is significant for the application of erasure communication networks.

Each pair of adjacent sensors exchanges information in the following way: at the sender side of the communication channel $(j, i) \in \mathcal{E}_{\mathcal{G}}$, sensor j sends its parameter estimate $x_j(t)$ to sensor i. Due to the uncertainties of communication channels, at the receiver side of the channel (j, i), sensor i receives an estimate of $x_j(t)$, denoted by

$$y_{ji}(t) = \begin{cases} L_{ji}(t)x_j(t) + w_{ji}(t), & \text{if } (j, i) \in \mathcal{E}_{\mathcal{G}_t}, \\ 0, & \text{if } (j, i) \in \mathcal{E}_{\mathcal{G}} \setminus \mathcal{E}_{\mathcal{G}_t} \end{cases} \tag{2}$$

where $L_{ji}(t) \in \mathbb{R}^{n \times n}$ denotes the random link gain of (j, i), which often results from uncoded analog communication channels in large sensor network applications due to the simplicity and low delay properties ([8], [22]); $w_{ji}(t)$ is the stochastic additive communication noise, which can be used to model the thermal noise, channel fading, etc. ([9] and [10]). The combination of the Markovian link failure model and model (2) is reflected by the random edge set sequence $\mathcal{E}_{\mathcal{G}_t}$. In this paper, we assume sensor i knows the statistical information $\bar{L}_{ji} = E[L_{ji}(t)]$ of the random link gain $L_{ji}(t)$. Notice that the random link gain $L_{ji}(t)$ and communication noise $w_{ji}(t)$ are defined at all times for all links $(j, i) \in \mathcal{E}_{\mathcal{G}}$. When the link (j, i) is not active, $L_{ji}(t)$ and $w_{ji}(t)$ can be understood as dummy random variables ([8]). According to the fixed ordering of all links in $\mathcal{E}_{\mathcal{G}}$, $\{w_{ji}(t), (j, i) \in \mathcal{E}_{\mathcal{G}}\}$ and $\{L_{ji}(t), (j, i) \in \mathcal{E}_{\mathcal{G}}\}$ can be listed into the random vectors w_t and L_t, respectively.

III. Discrete-Time Distributed Estimation Algorithm and Convergence Analysis

In this section, we assume each sensor can obtain local noisy measurements at discrete time instants $t = 0, 1, \cdots$. Based on the unreliable sensing and communication models in Section II, by using the local noisy measurements and local communications within neighbor sensors, a discrete-time distributed estimation algorithm is proposed for each sensor in Section III-A. The convergence properties of the designed algorithm are then analyzed in Section III-B.

A. Distributed Estimation Algorithm

For sensor i $(i = 1, \cdots, N)$, we propose the following distributed algorithm to estimate the unknown parameter θ^*:

$$x_i(t+1) = x_i(t) + b(t)\left\{ \sum_{j \in \mathcal{N}_{it}} a_{ij}(t) \left(\bar{L}_{ji}^{-1} y_{ji}(t) - x_i(t) \right) + \bar{C}_i^T \left(z_i(t) - \bar{C}_i x_i(t) \right) \right\} \tag{3}$$

where $x_i(t) \in \mathbb{R}^n$ is the parameter estimate of sensor i at time t; $\mathcal{N}_{it}$ denotes the neighborhood of sensor i at time t; $\bar{C}_i$ is the mathematical expectation of $C_i(t)$; $y_{ji}(t)$ is the estimate of $x_j(t)$ received by sensor i, which is defined by (2); $b(t)$ is called gain function; $a_{ij}(t)$ is decided by the weighted adjacency matrix $\mathcal{A}_{\mathcal{G}_t}$ of the communication topology graph $\mathcal{G}_t$, i.e., $a_{ij}(t) = a_{ij}^{(m_t)}$. When $\mathcal{N}_{it}$ is empty, $\sum_{j \in \mathcal{N}_{it}} (\cdot)$ is defined as 0.

To facilitate convergence analysis of (3), we rewrite it in the following compact form:

$$\begin{aligned} X(t+1) = {} & X(t) - b(t) \left(\mathcal{L}_{\mathcal{G}}^{(m_t)} \otimes I_n \right) X(t) \\ & + b(t)\bar{C}^T \bar{C} \left(1_N \otimes \theta^* - X(t) \right) + b(t) w_t^{(m_t)} \\ & + b(t) \triangle L_{\mathcal{G}}^{(m_t)} X(t) + b(t)\bar{C}^T \tilde{C}(t)(1_N \otimes \theta^*) \\ & + b(t)\bar{C}^T v(t), \quad t \geq 0 \end{aligned} \tag{4}$$

[2]For convenience of expression, in this paper, "random process m_t takes the kth state at time t" is simply written as $m_t = k$, regardless of the inconsistence of their dimension.

where $X(t) = [x_1^T(t), \cdots, x_N^T(t)]^T$ with the initial state satisfying $E[\|X(0)\|^2] < \infty$, $\mathcal{L}_{\mathcal{G}}^{(m_t)}$ denotes the Laplacian matrix of the communication topology graph $\mathcal{G}_t$, $\bar{C} = diag\{\bar{C}_1, \cdots, \bar{C}_N\}$, $\tilde{C}(t) = diag\{C_1(t) - \bar{C}_1, \cdots, C_N(t) - \bar{C}_N\}$, $v(t) = [v_1^T(t), \cdots, v_N^T(t)]^T$, $\triangle L_{\mathcal{G}}^{(m_t)} = [\delta l_{ij}(t)]_{1\le i,j\le N}$, $w_t^{(m_t)} = [w_1^T(t), \cdots, w_N^T(t)]^T$

$$\begin{aligned}\delta l_{ij}(t) &= a_{ij}(t)\left(\bar{L}_{ji}^{-1}L_{ji}(t) - I\right)\\ &= \sum_{k=1}^{s} 1_{[m_t=k]} a_{ij}^{(k)}\left(\bar{L}_{ji}^{-1}L_{ji}(t) - I\right)\\ w_i(t) &= \sum_{j\in\mathcal{N}_{it}} a_{ij}(t)\bar{L}_{ji}^{-1}w_{ji}(t)\\ &= \sum_{k=1}^{s} 1_{[m_t=k]} \sum_{j\in\mathcal{N}_i} a_{ij}^{(k)}\bar{L}_{ji}^{-1}w_{ji}(t). \end{aligned} \quad (5)$$

Remark 1: Algorithm (3) can be seen as a combination of a "filtering" part to obtain the local estimate by stochastic approximation algorithm based on each sensor's own measurements, and a "prediction" part to obtain the final estimate by convexifying the local estimates of neighbor sensors based on stochastic approximation-type consensus scheme. Specifically, algorithm (3) can be represented in the following two steps:

$$\hat{x}_i(t) = x_i(t) + b(t)\bar{C}_i^T\left(z_i(t) - \bar{C}_i x_i(t)\right) \quad (6)$$

$$x_i(t+1) = \hat{x}_i(t) + b(t)\sum_{j\in\mathcal{N}_{it}} a_{ij}(t)\left(\bar{L}_{ji}^{-1}y_{ji}(t) - x_i(t)\right). \quad (7)$$

Here, we do not make "convexification" to the local estimation increment $b(t)\bar{C}_i^T(z_i(t) - \bar{C}_i x_i(t))$ in (6). This is because, if we do so, there will be an additional term with $b^2(t)$ as its coefficient in the "prediction" part (7), whose influence on the convergence of the algorithm will be dominated by those terms with coefficient $b(t)$ (see the proof below). The stochastic approximation-type consensus scheme ([8]–[10]) is introduced in the algorithm to make the parameter estimates of all the sensors reach a common value in the case of noisy communication channels. When the local estimate of some sensor does not update, this scheme can use coordination among sensors to improve the estimate, and thus, improve the convergence rate and estimation accuracy of the algorithm.

Remark 2: Similar algorithms as (3) can be found in [18]. The key difference between them is the introduction of more uncertainties including Markovian random switches of communication topologies and random link gains. To be specific, in the algorithm (3), $\{a_{ij}(t)\}$ represents the weight sequence of unreliable link (j, i) driven by a Markov chain; $\{\bar{L}_{ji}^{-1}y_{ji}(t)\}$ denotes the estimate of sensor j's estimation state received by sensor i, taking into account the effect of random link gain.

Below, we will prove the mean square and almost sure convergence of the algorithm (4) under some mild conditions on stochastic noises, gain functions and topology-switching Markov chain.

B. Convergence Analysis

Define the σ-algebra

$$\mathcal{F}_t = \sigma\{X(0), w_k, L_k, v(k), C(k), m_l, 0 \le k \le t, 0 \le l \le t+1\}.$$

To analyze the convergence of algorithm (4), we make the following assumptions:

A1) The digraph $\mathcal{G}^{(i)}, 1 \le i \le s$, is balanced, and the union of the communication topology set $\mathcal{C} = \{\mathcal{G}^{(i)}; 1 \le i \le s\}$, denoted by $\mathcal{G}_{\mathcal{C}}$, contains a spanning tree.

A2) $\{w_t, \mathcal{F}_t, t \ge 0\}$ and $\{v(t), \mathcal{F}_t, t \ge 0\}$ are two martingale difference sequences (m.d.s), with bounded second moments $\sigma_w^2 \triangleq \sup_{t\ge 0} E[\|w_t\|^2] < \infty$ and $\sigma_v^2 \triangleq \sup_{t\ge 0} E[\|v(t)\|^2] < \infty$.

A3) The random measurement matrix $C(t)$ is independent of $\mathcal{F}_{t-1}$, and has the mathematical expectation $E[C_i(t)] = \bar{C}_i$, and bounded second moment $\sigma_C^2 \triangleq \sup_{t\ge 0} E[\|C(t)\|^2] < \infty$. The matrix $Q_o = \sum_{i=1}^{N} \bar{C}_i^T \bar{C}_i$ is of full rank.

A4) The random link gain matrix L_t is independent of $\mathcal{F}_{t-1}$. $L_{ji}(k)$ are mutually independent (w.r.t. i, j, k), with full-rank mathematical expectation $E[L_{ji}(t)] = \bar{L}_{ji}$ and bounded second moment $\sup_{j,i,t} E[\|L_{ji}(t)\|^2] < \infty$.

A5) $\{m_t, t \ge 0\}$ is a homogeneous ergodic Markov chain with the transition probability matrix $[p_{ij}]_{1\le i,j\le s}$, where $= p_{ij} = P\{m_{t+1} = j | m_t = i, m_0, \cdots, m_{t-1}, w_k, v(k), L_k, \tilde{C}(k), 0 \le k \le t\}$.

A6) The gain function $b(t)$ satisfies: $b(t) > 0$; $\sum_{t=0}^{\infty} b(t) = \infty$; $\sum_{t=0}^{\infty} b^2(t) < \infty$.

A6′) The gain function $b(t)$ satisfies: $b(t) > 0$; $\sum_{t=0}^{\infty} b(t) = \infty$; $\lim_{t\to\infty} b(t) = 0$.

Remark 3: A1) is a joint connectivity condition on the communication topology. Intuitively, it means that if the communication connectivity relation among the sensors visits all digraphs of $\mathcal{C}$ in certain time interval, then for any pair of sensors i and j, sensor i can influence sensor j in this time interval only by local interactions among sensors. A2) is a condition on temporal correlation of communication and measurement noise sequence, and does not make any constraints on spatial correlation at any time instant. The m.d.s assumption for the noise sequences is weaker than the independent assumption in [4], [18]. In A3), Q_o being of full rank is a global observability condition ([18]), which together with the statistical assumption on individual measurement matrices will play a crucial role in convergence analysis. The coordination nature of distributed estimation requires conditions in both temporal and spatial scale. Thus, to take advantage of coordination among sensors in spatial scale, we will not use the typical persistent excitation condition in temporal scale on measurement matrices as the classical regression models, which can ensure each individual sensor to obtain accurate estimates based solely on its own measurements. A5) is on the ergodicity of the Markovian random switches of communication topologies. From Markov chain theory ([23]), a discrete-time Markov chain with finite states is ergodic if and only if it is irreducible and aperiodic. From the discussions below, for a more general case where m_t has finite number of irreducible, recurrent closed sets, and the union graph corresponding to each closed set contains a spanning tree, the convergence of the designed algorithm can also be proved.

Before conducting the convergence analysis of the algorithm (4), we need the following lemma.

Lemma 3.1: Assume $\mathcal{G}$ is a balanced digraph containing a spanning tree. For any positive constant d, the matrix $\mathcal{L}_{\mathcal{G}}^T \otimes I_n +$

$\mathcal{L}_{\mathcal{G}}\otimes I_n+2d\bar{C}^T\bar{C}$ is positive definite if and only if $\sum_{i=1}^N \bar{C}_i^T\bar{C}_i$ is of full rank, i.e., the global observability condition holds.

Proof: By [6, Th. 7], we know that $(\mathcal{L}_{\mathcal{G}}^T+\mathcal{L}_{\mathcal{G}})/2$ is the Laplacian matrix of the mirror graph $\hat{\mathcal{G}}$, i.e., $\mathcal{L}_{\hat{\mathcal{G}}}=(\mathcal{L}_{\mathcal{G}}^T+\mathcal{L}_{\mathcal{G}})/2$. Thus, we need only to prove that $\mathcal{L}_{\hat{\mathcal{G}}}\otimes I_n+d\bar{C}^T\bar{C}$ is positive definite if and only if $\sum_{i=1}^N \bar{C}_i^T\bar{C}_i$ is of full rank. Noticing the property of $\mathcal{G}$, we know that $\hat{\mathcal{G}}$ is an undirected connected graph, and if a nonzero vector x satisfies $x^T(\mathcal{L}_{\hat{\mathcal{G}}}\otimes I_n)x=0$, then it must have the form $x=1_N\otimes u, \forall u\neq 0\in\mathbb{R}^n$. Thus, from the positive semi-definiteness of $\mathcal{L}_{\hat{\mathcal{G}}}$ and $\bar{C}^T\bar{C}$ it follows that $\mathcal{L}_{\hat{\mathcal{G}}}\otimes I_n+d\bar{C}^T\bar{C}$ is positive definite if and only if $(1_N^T\otimes u^T)\bar{C}^T\bar{C}(1_N\otimes u)>0, \forall u\neq 0\in\mathbb{R}^n$. Obviously, the latter is equivalent to the global observability condition. □

For convenience of further analyzing the convergence property of the algorithm (4) under the joint connectivity condition A1), similar to [8], we first consider a little simpler topology connectivity condition. In other words, we make the following assumption:

A1′) The digraph $\mathcal{G}^{(i)}$, $1\leq i\leq s$, is balanced, $\mathcal{G}=\mathcal{G}^{(1)}$ contains a spanning tree; and $\min_{1\leq i\leq s} p_{i1}>0$.

Assumption A1′) says at each time instance the network topology will be connected with a positive probability. Since we will use the Lyapunov method to conduct the algorithm analysis, this condition guarantees the energy function $E\|X(t)-1_N\otimes\theta^*\|^2$ can decay with the rate $1-Cb(t)$ after one step, where C is some positive constant. The algorithm analysis under this condition will give an insight to that under the joint connectivity condition. This is because, essentially speaking, the joint connectivity condition together with the ergodicity of m_t guarantees the energy function decay with the rate $1-Cb(t)$ over a time interval, which leads to the more complex convergence analysis of algorithm (4). Under the above assumptions, we can prove that the distributed algorithm (4) is both in mean square and almost surely convergent.

Theorem 3.1: Under Assumptions A1′), A2)–A5), A6′), the distributed estimation algorithm (4) converges in mean square, i.e.,

$$\lim_{t\to\infty} E\left[\|\delta(t)\|^2\right]=0 \tag{8}$$

where $\delta(t)\triangleq X(t)-1_N\otimes\theta^*$ is the parameter estimation error.

Proof: See Appendix A. □

From the mean convergence criterion and dominated convergence theorem ([24]), a gap exists between the almost sure and mean square convergence of a random variable sequence unless the uniform integrability and certain moment conditions are satisfied. Thus, based on Theorem 3.1 and the slightly stronger condition A6) on the gain function, we now come to the almost sure convergence of the algorithm (4).

Theorem 3.2: Under Assumptions A1′), A2)–A6), the distributed estimation algorithm (4) converges almost surely, i.e.,

$$\lim_{t\to\infty} x_i(t)=\theta^*\ a.s.,\quad i=1,\cdots,N. \tag{9}$$

In addition, if $b(t)\downarrow 0$, $t\to\infty$, then

$$\frac{1}{t}\sum_{k=0}^{t}\|\delta(k)\|=o\left((b(t)t)^{-1/2}\right)\ a.s.,\quad \text{as } t\to\infty. \tag{10}$$

Proof: See Appendix A. □

Remark 4: Compared with [9, Th. 3.6], the noise condition is weakened in this paper. Theorem 3.2 gives a rough estimate for the convergence rate of the time-average estimation error. As we know, if $\|\delta(t)\|\sqrt{b(t)t}\underset{t\to\infty}{\longrightarrow}0$, then (10) naturally holds. In addition, (10) together with some conditions on the nonnegative sequence $\{\|\delta(t)\|\sqrt{b(t)t}, t\geq 0\}$ can also lead $\|\delta(t)\|\sqrt{b(t)t}\underset{t\to\infty}{\longrightarrow}0$. Thus, the convergence rate of the time-average estimation error reflects in a certain sense the convergence rate of the estimation error itself. If the gain function $b(t)$ is chosen to satisfy $(c_1/t^\gamma[\ln(t)]^\beta)\leq b(t)\leq(c_2/t^\gamma[\ln(t)]^\beta)$, where $c_1,c_2>0$, $\beta\leq 1$, $0.5<\gamma\leq 1$, then $b(t)$ monotonically decreases and A6) holds. From (10), the convergence rate of algorithm (4) is faster than $c_0 t^{-((1-\gamma)/2)}[\ln(t)]^{\beta/2}$, where c_0 is a positive constant. Thus, we can improve the convergence rate by properly choosing the coefficient γ and β. The issue for more accurate estimate of the convergence rate about the estimation error rather than the time-average of estimation error, and more rigorous theoretic optimization methods on the gain function is of significance, and needs further investigations.

Theorem 3.1 and Theorem 3.2 provide the convergence analysis for the case where the distributed estimation algorithm of each sensor has the same gain function $b(t)$. However, in practice, it is often the case that the gain functions are different. Thus, it is significant to analyze whether the designed estimation algorithm is still convergent when the gain functions are different. In this case, we can express the distributed parameter estimation algorithm in the following compact form:

$$\begin{aligned} X(t+1)=&X(t)-(B(t)\otimes I_n)\Big(\mathcal{L}_{\mathcal{G}}^{(m_t)}\otimes I_n\Big)X(t)\\ &+(B(t)\otimes I_n)\Big[\bar{C}^T\bar{C}(1_N\otimes\theta^*-X(t))+\triangle L_{\mathcal{G}}^{(m_t)}X(t)\\ &+w_t^{(m_t)}+\bar{C}^T\tilde{C}(t)(1_N\otimes\theta^*)\\ &+\bar{C}^T v(t)\Big],\quad t\geq 0 \end{aligned} \tag{11}$$

where $B(t)=diag\{b_1(t),\cdots,b_N(t)\}$, $X(t)$, $\triangle L_{\mathcal{G}}^{(m_t)}$, and $w_t^{(m_t)}$ have the same definitions as in (4).

For the algorithm (11), we have the following convergence results.

Theorem 3.3: Under Assumptions A1′), A2)–A5), if for each sensor i $(i=1,\cdots,N)$, its algorithm gain function $b_i(t)$ satisfies Assumption A6′) and

$$\max_{1\leq i,j\leq N}|b_i(t)-b_j(t)|=o\left(\sum_{i=1}^N b_i(t)\right),\quad \text{as } t\to\infty \tag{12}$$

then the distributed algorithm (11) converges in mean square.

Furthermore, if each algorithm gain function $b_i(t)$ satisfies Assumption A6) and (12), then the distributed estimation algorithm (11) converges almost surely. In addition, if $b_i(t)\downarrow 0$, $t\to\infty$, then we have the following estimate for the convergence rate:

$$\frac{1}{t}\sum_{k=0}^{t}\|\delta(k)\|=o\left(\left(\bar{b}(t)t\right)^{-1/2}\right)\ a.s.,\quad \text{as } t\to\infty \tag{13}$$

where $\bar{b}(t)=(1/N)\sum_{i=1}^N b_i(t)$.

Proof: Due to the similarity with Theorem 3.1–3.2, the proof is outlined in Appendix B. □

We now prove the corresponding convergence results as in Theorem 3.1–Theorem 3.3 under the joint connectivity condition A1). In this case, the communication topology graph at any given time instant may be not connected. Similar to Assumption A4) in [9] and Assumption iii) of [8, Th. 7], to deal with the time-varying coefficient matrices caused by topology switches, we assume the gain function $b(t)$ does not change too fast. To be specific, we make the following assumption:

A6″) In addition to A6), there are constants $\alpha_1, \alpha_2 > 0$, such that $\alpha_1 b(t) \le b(t+1) \le \alpha_2 b(t), \forall t > 0$.

Theorem 3.4: Under Assumptions A1)–A5), A6″), the distributed parameter estimation algorithm (4) is convergent both in mean square and almost surely. In addition, if $b(t) \downarrow 0$, $t \to \infty$, then $(1/t)\sum_{k=0}^{t}\|\delta(k)\| = o((b(t)t)^{-1/2})$ a.s., as $t \to \infty$.

Proof: See Appendix C. □

Theorem 3.5: Under Assumptions A1)–A5), if for each sensor i $(i = 1, \cdots, N)$, its algorithm gain function $b_i(t)$ satisfies A6) and (12), and there exist $\alpha_1, \alpha_2 > 0$, such that $\alpha_1 b_i(t) \le b_i(t+1) \le \alpha_2 b_i(t), \forall t > 0$, then the distributed algorithm (11) converges in mean square. Furthermore, if (12) is replaced by

$$\sum_{t=0}^{\infty} \max_{1\le i\le N} |b_i(t) - b_j(t)| < \infty \tag{14}$$

then the distributed estimation algorithm (11) converges almost surely. In addition, if $b_i(t) \downarrow 0$, $t \to \infty$, then we have the following estimate for the convergence rate: $(1/t)\sum_{k=0}^{t}\|\delta(k)\| = o((\bar{b}(t)t)^{-1/2})$ a.s., as $t \to \infty$, where $\bar{b}(t) = (1/N)\sum_{i=1}^{N} b_i(t)$.

Proof: The proof is similar to Theorem 3.4, and is outlined in Appendix C. □

Remark 5: At first glance, the algorithm (4) may have a faster convergence rate under A1′) than that under the joint connectivity condition A1), since the energy function $E\|\delta(t)\|^2$ decays after a certain time interval under A1), but decays after only one step under A1′). However, from Theorem 3.2–Theorem 3.5, actually we can only obtain the same convergence rate estimate under the two different connectivity conditions. This is because, by assuming the gain function $b(t)$ slowly time-varying (Assumption A6″)), the energy function for both the two cases will decay with the rate $1 - Cb(t)$. The only difference lies in the constant C for the two cases, which does not affect the estimate of convergent rate about the gain function. Thus, the faster convergence rate under A1′) may be reflected in the less multiples of $(b(t)t)^{-1/2}$ than that under A1).

By Remark 3, we further relax the ergodic condition on m_t in Assumption A5). Similar to the multi-agent consensus control under Markovian switching communication topologies ([11]), we will discuss the convergence of the designed estimation algorithm by classifying the states of the Markov chain m_t. From the decomposition theorem of the states of a Markov chain ([23]), we know that the state space $\mathcal{S}$ of m_t can be partitioned uniquely as

$$\mathcal{S} = \mathcal{T} \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_p$$

where $\mathcal{T}$ is the set of transient states, $\mathcal{C}_i$ $(i = 1, \cdots, p)$ is the irreducible closed set of (positive) recurrent states. We make the following assumptions:

A1″) The digraph $\mathcal{G}^{(i)}$, $1 \le i \le s$, is balanced, and $\mathcal{G}_{\mathcal{C}_j}$, $1 \le j \le p$, contains a spanning tree, where $\mathcal{G}_{\mathcal{C}_j}$ is the union of the communication topology graph set corresponding to the state set $\mathcal{C}_j$.

A5′) $\{m_t, t \ge 0\}$ is a homogeneous Markov chain with the transition probability matrix $[p_{ij}]_{1\le i,j\le s}$, $P\{m_{t+1} = j | m_t = i, m_0, \cdots, m_{t-1}, w_k, v(k), L_k, \tilde{C}(k), 0 \le k \le t\} = p_{ij}$.

Obviously, when $p = 1$, $\mathcal{T}$ is empty, and m_t is aperiodic, Assumptions A1′) and A5′) are reduced to Assumptions A1) and A5), respectively. According to the initial distribution, the initial state of the Markov chain m_t belongs to either the set of transient states $\mathcal{T}$, or a certain irreducible closed set of recurrent states $\mathcal{C}_i$, $1 \le i \le p$. For the former case, m_t will finally reach an irreducible closed set of recurrent states $\mathcal{C}_i$ after finite time, and will not jump out of it from then on. For the latter case, m_t will not jump out of the irreducible closed set $\mathcal{C}_i$ from the initial instant. Thus, for both cases, m_t will take values in some irreducible closed set of recurrent states after finite time. Based on the above assumptions and Theorem 3.4, we can obtain the following corollary addressing the convergence properties of the distributed estimation algorithm (4).

Corollary 3.1: Under Assumptions A1″), A2)–A4), A5′), A6), the distributed estimation algorithm (4) is convergent both in mean square and almost surely.

IV. Continuous-Time Distributed Estimation Algorithm and Convergence Analysis

In this section, we assume each sensor obtains local noisy measurements continuously. The continuous-time distributed parameter estimation problem is considered in uncertain environments with measurement noises, communication noises and Markovian random signal losses. To be specific, in Section IV-A, after modifying the sensing and communication models (1), (2) to make them suitable for discussions in continuous-time setting, we propose an estimation algorithm described by a Markovian switching stochastic differential equation. In Section IV-B, we prove the mean square and almost sure convergence of the designed continuous-time algorithm. The problem in this section has been previously discussed in our recent work [1].

A. Distributed Estimation Algorithm

Assume each sensor in the network observes the unknown parameter vector $\theta^* \in \mathbb{R}^n$ independently. The local continuous-time noisy measurement obtained by sensor i $(i = 1, \cdots, N)$ at time t is expressed as

$$z_i(t) = C_i(t)\theta^* + \sigma_{vi} n_{vi}(t), \quad t \in \mathbb{R}^+ \tag{15}$$

where $z_i(t) \in \mathbb{R}^{n_i}$ is the measurement vector of sensor i at time t; $n_{vi}(t) \in \mathbb{R}^{n_i}$ is the measurement noise, which is modeled as a standard white noise, $\sigma_{vi} = diag\{\sigma_{vi}^1, \cdots, \sigma_{vi}^{n_i}\}$, $\sigma_{vi}^k > 0$, $k = 1, \cdots, n_i$ denotes its noise intensity matrix; $C_i(t) = \bar{C}_i + \sigma_{Ci} \odot \tilde{C}_i(t)$ is the random time-varying measurement matrix, $\bar{C}_i = E[C_i(t)]$, each element of the matrix $\tilde{C}_i(t)$ is modeled as a standard white noise, $\sigma_{Ci} \in \mathbb{R}^{n_i \times n}$ denotes its noise intensity matrix.

Corresponding to the discrete-time case with Markovian topology switches in Section II-B, here the random switches of communication topologies are driven by a continuous-time Markov chain m_t, with the state space $\mathcal{S} = \{1, \cdots, s\}$. When $m_t = i$ $(i \in \mathcal{S})$, the communication topology of the sensor network at time t is $\mathcal{G}_t = \mathcal{G}^{(i)}$ accordingly. At the sender side of the channel $(j,i) \in \mathcal{E}_{\mathcal{G}}$, sensor j sends its estimate $x_j(t) \in \mathbb{R}^n$ to sensor i. At the receiver side of this channel, sensor i receives an estimate of $x_j(t)$, denoted by

$$y_{ji}(t) = \begin{cases} x_j(t) + \sigma_w^{ji} n_w^{ji}(t), & \text{if } (j,i) \in \mathcal{E}_{\mathcal{G}_t}, \\ 0, & \text{if } (j,i) \in \mathcal{E}_{\mathcal{G}} \setminus \mathcal{E}_{\mathcal{G}_t} \end{cases} \quad (16)$$

where $n_w^{ji}(t)$ is the communication noise, which is modeled as standard white noise, $\sigma_w^{ji} = diag\{\sigma_{w1}^{ji}, \cdots, \sigma_{wn}^{ji}\}$, $\sigma_{wk}^{ji} > 0$, $k = 1, \cdots, n$ denotes its noise intensity matrix.

Under the Markovian random switching communication topologies, and the communication scheme defined by (16), for each sensor i $(i = 1, \cdots, N)$, we propose the following continuous-time distributed real-time estimation algorithm to estimate the unknown parameter θ^*:

$$\dot{x}_i(t) = b(t)\left\{ \sum_{j \in \mathcal{N}_{it}} a_{ij}(t)\left(y_{ji}(t) - x_i(t)\right) + \bar{C}_i^T \left(z_i(t) - \bar{C}_i x_i(t)\right) \right\} \quad (17)$$

where $x_i(t)$, $\mathcal{N}_{it}$, $a_{ij}(t)$ and $b(t)$ have the same meanings as in (3). $z_i(t)$ and $y_{ji}(t)$ are defined by (15) and (16), respectively. When $\mathcal{N}_{it}$ is empty, $\sum_{j \in \mathcal{N}_{it}}(\cdot)$ is understood as zero.

Substituting (15), (16) into (17), we obtain the following compact form of the distributed estimation algorithm:

$$\begin{aligned} \dot{X}(t) = & -b(t)\left(\mathcal{L}_{\mathcal{G}}^{(m_t)} \otimes I_n\right) X(t) \\ & + b(t)\bar{C}^T \bar{C}\left(1_N \otimes \theta^* - X(t)\right) + b(t)\Sigma_w^{(m_t)} n_w(t) \\ & + b(t)\Sigma_{\tilde{C}} n_{\tilde{C}}(t) + b(t)\Sigma_v n_v(t), \quad t \geq 0 \end{aligned} \quad (18)$$

where $X(t) = [x_1^T(t), \cdots, x_N^T(t)]^T$, $\mathcal{L}_{\mathcal{G}}^{(m_t)}$ is the Laplacian matrix of $\mathcal{G}_t$, $\bar{C} = diag\{\bar{C}_1, \cdots, \bar{C}_N\}$, $n_w(t) = [n_w^{11^T}(t), \cdots, n_w^{N1^T}(t), \cdots, n_w^{1N^T}(t), \cdots, n_w^{NN^T}(t)]^T$; $\Sigma_w^{(m_t)} = diag\{\Sigma_w^1, \cdots, \Sigma_w^N\}$, $\Sigma_w^i = (\alpha_i^T(t) \otimes I_n) diag\{\sigma_w^{1i}, \cdots, \sigma_w^{Ni}\}$, $\alpha_i^T(t)(i = 1, \cdots, N)$ denotes the ith row of the matrix $\mathcal{A}_{\mathcal{G}_t}$; $\Sigma_{\tilde{C}} = \bar{C}^T diag\{\Sigma_{\tilde{C}}^1, \cdots, \Sigma_{\tilde{C}}^N\}$, $\Sigma_{\tilde{C}}^i = \sigma_{C_i} diag\{\theta^{*T}, \cdots, \theta^{*T}\}$; $n_{\tilde{C}}(t) = [\tilde{C}_{11}^T, \cdots, \tilde{C}_{1n_1}^T, \cdots, \tilde{C}_{N1}^T, \cdots, \tilde{C}_{Nn_N}^T]^T$, $\tilde{C}_{ij}^T$ denotes the jth row of matrix $\tilde{C}_i(t)$; $\Sigma_v = \bar{C}^T diag\{\sigma_{v1}, \cdots, \sigma_{vN}\}$, $n_v(t) = [n_{v1}^T(t), \cdots, n_{vN}^T(t)]^T$.

Assume the σ-algebras $\mathcal{F}(n_w(t), t \geq 0)$, $\mathcal{F}(n_{\tilde{C}}(t), t \geq 0)$ and $\mathcal{F}(n_v(t), t \geq 0)$, generated, respectively, by the noise sequences $\{n_w(t)\}$, $\{n_{\tilde{C}}(t)\}$ and $\{n_v(t)\}$, are independent. Then the algorithm (18) is a stochastic system driven by an $nN^2 + (n+1)\sum_{i=1}^N n_i$ dimensional standard white noise, which can be written in the following Itô stochastic differential equation:

$$\begin{aligned} dX(t) = & \Big[-b(t)\left(\mathcal{L}_{\mathcal{G}}^{(m_t)} \otimes I_n\right) X(t) \\ & + b(t)\bar{C}^T \bar{C}\left(1_N \otimes \theta^* - X(t)\right)\Big] dt + b(t)\Sigma^{(m_t)} dW(t) \end{aligned} \quad (19)$$

where $\Sigma^{(m_t)} = [\Sigma_w^{(m_t)}, \Sigma_{\tilde{C}}, \Sigma_v]$, $W(t)$ is an $nN^2 + (n+1)\sum_{i=1}^N n_i$ dimensional standard Brownian motion.

B. Convergence Analysis

The discussions in this subsection are based on the complete probability space $(\Omega, \mathcal{F}, P)$, equipped with a filtration $\{\mathcal{F}_t, t \in \mathbb{R}^+\}$. To proceed the convergence analysis of the distributed estimation algorithm (19), we need the following preparations.

For given matrix $L = (L_1, \cdots, L_s)$ with $L_i = [l_{i1} \cdots l_{im}]$, $l_{ij} \in \mathbb{R}^n$, $i = 1, \cdots, s$, $j = 1, \cdots, m$, we define the following operators φ and $\hat{\varphi}$: $\varphi(L_i) = [l_{i1}^T, \cdots, l_{im}^T]^T \in \mathbb{R}^{nm}$, $\hat{\varphi}(L) = [\varphi^T(L_1), \cdots, \varphi^T(L_s)] \in \mathbb{R}^{mns}$.

It can be seen that the above operators actually make a long vector out of a matrix by stacking up its columns from left to right. The operator φ has the following property ([25]):

$$\varphi(LKH) = (H^T \otimes L)\varphi(K) \quad (20)$$

where L, H and K are matrices with proper dimensions.

In order to prove the convergence of the algorithm (19), we make the following assumptions:

B1) $\{(m_t, \mathcal{F}_t), t \in \mathbb{R}^+\}$ is a homogeneous ergodic Markov chain with right continuous trajectories, taking values on the set $\mathcal{S} = \{1, \cdots, s\}$. $Q = [q_{ij}]_{1 \leq i,j \leq s}$ is the transition rate matrix of m_t, satisfying $\sum_{j:j \neq i} q_{ij} = \sum_{j:j \neq i} q_{ji}$. $\{(m_t, \mathcal{F}_t), t \in \mathbb{R}^+\}$ has the initial distribution $\{q_i, i \in \mathcal{S}\}$.

B2) The initial estimation state $X(0)$ satisfies $E[\|X(0)\|^2] < \infty$. $\{W(t), \mathcal{F}_t\}$, $\{m_t, \mathcal{F}_t\}$, and $X(0)$ are independent.

B3) The matrix $Q_o = \sum_{i=1}^N \bar{C}_i^T \bar{C}_i$ is of full rank.

B4) The gain function $b(t)$ satisfies: $b(t) \geq 0$; $\int_0^\infty b(t)dt = \infty$; $\int_0^\infty b^2(t)dt < \infty$.

Remark 6: From the Markov chain theory ([23]), if m_t takes values in a finite state space, and its transition probability matrix $P(t) = [p_{ij}(t)]$ is standard, i.e., $\lim_{t \to 0+} P(t) = I$, $p_{ij}(t) = P(m_{t+s} = j | m_s = i)$, $\forall s \geq 0$, then the transition rate matrix Q is conservative, i.e., $q_{ii} = -\sum_{j:j \neq i} q_{ij}$, $0 \leq q_{ij} < \infty$. This together with the condition $\sum_{j:j \neq i} q_{ij} = \sum_{j:j \neq i} q_{ji}$ in B1) ensures Q is doubly stochastic, which amounts to saying the digraph generated by Q is balanced. Thus, $(1/2)(Q + Q^T)$ is a symmetric transition rate matrix. In addition to the ergodicity of m_t (i.e., m_t is irreducible, positive recurrent), we know that $(1/2)(Q + Q^T)$ is irreducible and the digraph generated is strongly connected. Denote $p_i(t) = P(m_t = i)$, $i \in \mathcal{S}$. By the ergodicity of m_t, there exist limiting probabilities $\{\pi_i, i \in \mathcal{S}\}$ which do not depend on the initial distribution, and satisfy $\sum_{i=1}^s \pi_i = 1$ and $\lim_{t \to \infty} \sup_{j \in \mathcal{S}} \{|p_j(t) - \pi_j|\} = 0$.

Now we come to analyze the convergence of the distributed estimation algorithm (19). It can be seen that algorithm (19) is actually a time-varying stochastic system with Markov switches. One prominent feature of this kind of systems is that the state $\{X(t), \mathcal{F}_t, t \geq 0\}$ is not Markov, while the augmented state $\{(X(t), m_t), \mathcal{F}_t, t \geq 0\}$ is Markov. Therefore, we can use tools such as infinitesimal generator of the Markov process $(X(t), m_t)$ and generalized Itô formula ([26]) to study the convergence properties of the designed algorithm (19).

Theorem 4.1: Under Assumptions A1), B1)–B4), the continuous-time distributed estimation algorithm (19) converges in mean square, i.e.,

$$\lim_{t \to \infty} E\left[\|\delta(t)\|^2\right] = 0 \quad (21)$$

where $\delta(t) = X(t) - 1_N \otimes \theta^*$ is the parameter estimation error.

Proof: See Appendix D. □

Based on Theorem 4.1, we can prove the almost sure convergence of the algorithm (19).

Theorem 4.2: Under Assumptions A1), B1)–B4), the continuous-time distributed estimation algorithm (19) converges almost surely.

Proof: See Appendix E. □

Remark 7: From Theorems 4.1 and 4.2, we can see that different from the discrete-time case, the gain function $b(t)$ is not required to satisfy a condition like A6$''$). In addition, from the proof of Theorem 3.4, we can see that the Markovian nature of the time-varying network topology is not strictly necessary for the discrete-time case as long as the union graph of deterministically switching network topology without any probabilistic assumptions is assumed to be connected ([9]), or there is a positive probability of visiting all the states of the chain over an interval of length h. In contrast, the Markovian property plays an important role in the convergence analysis of the continuous-time algorithm (19). The essential difference between the analysis methods of the discrete-time stochastic systems and that of the continuous-time stochastic systems may arise from the transient limitation analysis requirement (e.g., the infinitesimal generator) caused by the continuum nature of the continuous-time stochastic systems.

Similar to Theorem 3.3, we move on to analyze the convergence of the distributed estimation algorithm with sensor-dependent gain functions $b_i(t)$, $i = 1, \cdots, N$. In this case, the continuous-time distributed estimation algorithm (19) becomes

$$dX(t) = (B(t) \otimes I_n)\Big[-\left(\mathcal{L}_{\mathcal{G}}^{(m_t)} \otimes I_n\right)X(t) + \bar{C}^T\bar{C}\left(1_N \otimes \theta^* - X(t)\right)\Big]dt + (B(t)\otimes I_n)\Sigma^{(m_t)}dW(t) \tag{22}$$

where $B(t) = diag\{b_1(t), \cdots, b_N(t)\}$, $\Sigma^{(m_t)}$, $W(t)$ are defined as in (19).

On the convergence of the algorithm (22), we have the following theorem.

Theorem 4.3: Suppose Assumptions A1), B1)–B3) hold, and for each i $(i = 1, \cdots, N)$

$$\int_0^\infty b_i(t)dt = \infty, \tag{23}$$

$$\int_0^\infty b_i^2(t)dt < \infty \tag{24}$$

$$\max_{1\le i,j\le N} |b_i(t) - b_j(t)| = o\left(\sum_{i=1}^N b_i(t)\right) \text{ as } t \to \infty. \tag{25}$$

Then the continuous-time distributed estimation algorithm (22) converges in mean square. In addition, if (25) is replaced by

$$\int_{t=0}^\infty \max_{1\le i,j\le N} |b_i(t) - b_j(t)|\, dt < \infty \tag{26}$$

then the algorithm (22) converges almost surely.

Proof: See Appendix F. □

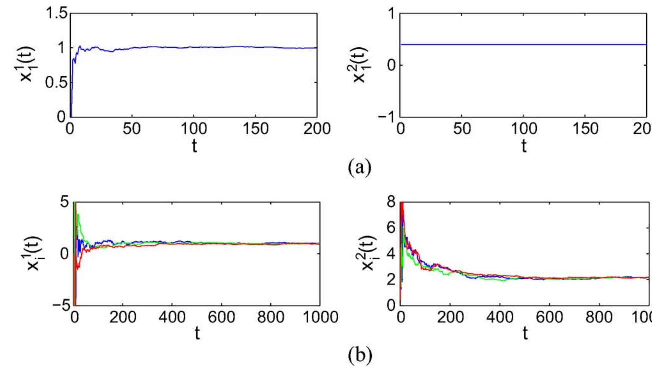

Fig. 1. Trajectories of parameter estimates in sensor network (a) SN_1 and (b) SN_2.

V. Numerical Examples

In this section, we will give two numerical examples to illustrate the convergence results of Sections III and IV in both discrete-time and continuous-time settings.

Example 1: Consider two sensor networks SN_1 and SN_2, whose sensors are denoted by (s_1) and (s_1, s_2, s_3), respectively. Each sensor network carries out a common task to estimate the unknown parameter vector $\theta^* = [1, 2]^T$.

The local noisy measurement $z_i(t)$ of sensor s_i $(i = 1, 2, 3)$ is taken as $z_i(t) = (C_i + n_i(t))\theta^* + v_i(t)$, where $C_1 = [1, 0]$, $C_2 = [0, 1]$, $C_3 = [1, 1]$, $n_i(t) = [n_{i1}(t), n_{i2}(t)]$, $n_{i1}(t)$, $n_{i2}(t)$, $v_i(t)$ are independent white noises with uniform distribution on $[-0.2, 0.2]$. The signal at the receiver side of a communication channel is described by (2), where $E[L_{ji}(t)] = 0.2I$, and the elements of $L_{ji}(t) - 0.2I$ are white noises with uniform distribution on $[-0.05, 0.05]$. The elements of all communication noises $w_{ji}(t)$ are Gaussian white noises with distribution $N(0, 0.1)$. The random switches of communication topologies for sensor network SN_2 are driven by the homogeneous ergodic Markov chain m_t. The state space of m_t is $\mathcal{S} = \{1, 2, 3\}$, and the transition probability matrix is taken as $P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$, where state i $(i = 1, 2, 3)$ stands for the communication topology graph $\mathcal{G}_i$ with its weighted adjacency matrix $\mathcal{A}_1 = \begin{pmatrix} 0 & 5 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix}$, $\mathcal{A}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 \end{pmatrix}$. We use the algorithm (4) to estimate the unknown parameter θ^* with the initial states $x_i(0) = [0, 0.4]^T$, $i = 1, 2, 3$. The gain function is $b(t) = 1/t$.

In Fig. 1(a), it is evident that the single sensor s_1 in SN_1 fails to estimate the second element of θ^*. This shows the necessity to conduct parameter estimate via coordination among sensors. Fig. 1(b) shows that the trajectories of the estimates under noisy local measurements converge to the true value of the unknown parameter, which verify the convergence results of Theorem 3.4. Thus, we conclude that the designed algorithm (4) possesses robustness to random measurement noises, random communication noises, random link gains and Markovian switching communication topologies.

Example 2: The convergence properties of the distributed continuous-time parameter estimation algorithm (17) will be

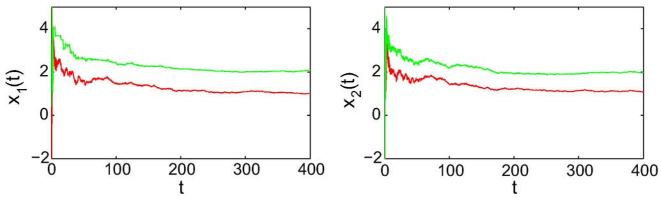

Fig. 2. Trajectories of parameter estimates of continuous-time algorithms by sensor s_1 and s_2.

illustrated. The unknown parameter vector to be estimated is $\theta^* = [1,2]^T$. We consider a sensor network with two sensors s_1 and s_2. The local noisy measurements of sensor s_i $(i = 1,2)$ are given by (15), where $C_1 = [1,0]$, $C_2 = [1,1]$, $\sigma_{C_i} = [1,1]$, $\sigma_{vi} = 1$. All communication noise intensity matrices are taken as $\sigma_w^{ji} = I$. The state space of the continuous-time homogeneous ergodic Markov chain m_t is denoted by $\mathcal{S} = \{1,2\}$, where state i $(i = 1,2)$ corresponds to the communication topology graph $\mathcal{G}_i$ with the Laplacian matrix $\mathcal{L}_{\mathcal{G}_1} = \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix}$, $\mathcal{L}_{\mathcal{G}_2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The transition rate matrix of m_t is given by $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, and the initial probability distribution of m_t is taken as $q_1 = q_2 = 1/2$. The gain function is in the form $b(t) = 550/t$, and the initial values of parameter estimates are $x_1(0) = x_2(0) = [0, 0.4]$. By choosing algorithm coefficients as above, all conditions of Theorem 4.1 and Theorem 4.2 hold. The trajectories of parameter estimates by sensor s_1 and s_2 are shown in Fig. 2. It can be seen that as time goes on, the parameter estimate of each sensor's algorithm converges to the true value of θ^*.

VI. Concluding Remarks

The purpose of distributed estimation over sensor networks is to study how to design an estimator for each sensor so that a nice estimate of the unknown parameter can be given cooperatively via local communications and interactions among sensors. In this paper, we have considered the design and convergence of distributed estimation algorithms under various uncertainties during sensing measurements and communicating signals among sensors. The sensing and communication models address uncertainties of measurement and communication noises, random signal losses, and random link gains. The temporal correlations of random topology switches are described by a Markov chain. Distributed algorithms in both discrete-time and continuous-time case are proposed. By using probability limit theory, stochastic differential equation theory, algebraic graph theory and Markov chain theory, under mild conditions on measurement and communication noise, gain function, and topology-switching Markov chain, we prove the designed algorithms converges both in mean square and almost surely.

Interesting extensions of this paper might concern the convergence rate estimation of the continuous-time algorithm, the effects of other parameters on convergence rate such as the transition probabilities of the Markov chain and the link gains, the relaxation of independence assumption on measurement matrices to certain correlation assumptions, and the use of recursive least squares (RLS)-type local filtering algorithms instead of stochastic gradient algorithms. In addition, when the parameter to be estimated is time-varying, how to combine the existing local estimation algorithms and dynamic consensus algorithms ([27]) to produce efficient distributed dynamic estimation algorithms, and how to conduct their convergence analysis are important issues. We believe dynamic consensus schemes will play an active role in estimating unknown dynamic parameters cooperatively, and enhancing the global observability of the sensor network. These topics require further investigation.

Appendix A
Proofs of Theorem 3.1 and Theorem 3.2

Lemma A.1: ([28]) Let $\{u(k), k = 0,1,\cdots\}$, $\{\alpha(k), k = 0,1,\cdots\}$ and $\{q(k), k = 0,1,\cdots\}$ be real sequences, satisfying $0 < q(k) \le 1$, $\alpha(k) \ge 0$, $k = 0,1,\cdots$, $\sum_{k=0}^{\infty} q(k) = \infty$, $(\alpha(k)/q(k)) \underset{k\to\infty}{\longrightarrow} 0$, and $u(k+1) \le (1-q(k))u(k) + \alpha(k)$. Then $\limsup_{k\to\infty} u(k) \le 0$. Particularly, if $u(k) \ge 0$, $k = 0,1,\cdots$, then $\lim_{k\to\infty} u(k) = 0$.

Proof of Theorem 3.1: By Assumption A1$'$) and properties of the Kronecker product ([25]), one can obtain $(\mathcal{L}_{\mathcal{G}}^{(m_t)} \otimes I_n)(1_N \otimes \theta^*) = 0$. Thus, (4) can be written in the following difference equation associated with the parameter estimation error $\delta(t)$:

$$\begin{aligned}\delta(t+1) = &\Big[I - b(t)\Big(\mathcal{L}_{\mathcal{G}}^{(m_t)} \otimes I_n + \bar{C}^T\bar{C}\Big)\Big]\delta(t) \\ &+ b(t)\Big[\triangle L_{\mathcal{G}}^{(m_t)}(\delta(t) + 1_N \otimes \theta^*) \\ &\quad + \bar{C}^T\tilde{C}(t)(1_N \otimes \theta^*) + w_t^{(m_t)} + \bar{C}^T v(t)\Big]. \quad \text{(A.1)}\end{aligned}$$

Let $V(t) = \|\delta(t)\|^2$. Then, by (A.1) we have

$$\begin{aligned}V(t+1) = &\,\delta^T(t)\left[I - b(t)\left(\mathcal{L}_{\mathcal{G}}^{(m_t)T} \otimes I_n + \bar{C}^T\bar{C}\right)\right] \\ &\times \left[I - b(t)\left(\mathcal{L}_{\mathcal{G}}^{(m_t)} \otimes I_n + \bar{C}^T\bar{C}\right)\right]\delta(t) \\ &+ 2b(t)I_1(t) + b^2(t)I_2(t), \quad \text{(A.2)}\end{aligned}$$

where $I_i(t)$, $i = 1,2$ are defined by $I_1(t) = \delta^T(t)[I - b(t)(\mathcal{L}_{\mathcal{G}}^{(m_t)T} \otimes I_n + \bar{C}^T\bar{C})][\triangle L_{\mathcal{G}}^{(m_t)}\delta(t) + \triangle L_{\mathcal{G}}^{(m_t)}(1_N \otimes \theta^*) + \bar{C}^T\tilde{C}(t)(1_N \otimes \theta^*) + w_t^{(m_t)} + \bar{C}^T v(t)]$, $I_2(t) = \|\triangle L_{\mathcal{G}}^{(m_t)}\delta(t) + \triangle L_{\mathcal{G}}^{(m_t)}(1_N \otimes \theta^*) + \bar{C}^T\tilde{C}(t)(1_N \otimes \theta^*) + w_t^{(m_t)} + \bar{C}^T v(t)\|^2$.

Define $\mathcal{F}_t' = \sigma\{x(0), w_k, m_k, v(k), L_k, \tilde{C}(k), 0 \le k \le t-1\}$. Then, $\delta(t)$ is adapted to $\mathcal{F}_t'$. Below, we want to estimate $E[V(t+1)|\mathcal{F}_t']$.

From Assumptions A1$'$), A5) and Lemma 3.1 we know that, no matter what the value m_{t-1} takes, the probability of the event $\{m_t = 1\}$ is always greater than zero. Thus, there is a constant $\lambda_0 > 0$, such that $E[(1/2)(\mathcal{L}_{\mathcal{G}}^{(m_t)T} + \mathcal{L}_{\mathcal{G}}^{(m_t)}) \otimes I_n + \bar{C}^T\bar{C}|\mathcal{F}_t'] \ge \lambda_0 I$ and

$$\begin{aligned}E\Big\{&\delta^T(t)\left[I - b(t)\left(\mathcal{L}_{\mathcal{G}}^{(m_t)T} \otimes I_n + \bar{C}^T\bar{C}\right)\right] \\ &\times \left[I - b(t)\left(\mathcal{L}_{\mathcal{G}}^{(m_t)} \otimes I_n + \bar{C}^T\bar{C}\right)\right]\delta(t)|\mathcal{F}_t'\Big\} \\ &\le \left[1 - 2\lambda_0 b(t) + C_{01}b^2(t)\right]V(t) \quad \text{(A.3)}\end{aligned}$$

where $C_{01} = 2\max_{1\le k\le s}\left\|\mathcal{L}_{\mathcal{G}}^{(k)} \otimes I_n\right\|^2 + 2\|\bar{C}^T\bar{C}\|^2$.

By Assumptions A2)–A4) and (5) we have $E[\triangle L_{\mathcal{G}}^{(m_t)}|\mathcal{F}_{t-1}] = E[w_t^{(m_t)}|\mathcal{F}_{t-1}] =$

$E[\tilde{C}(t)|\mathcal{F}_{t-1}] = E[v(t)|\mathcal{F}_{t-1}] = 0$, which together with the definition of $I_1(t)$ and $\mathcal{F}'_t \subset \mathcal{F}_{t-1}$ leads to

$$\begin{aligned}&E\left[I_1(t)|\mathcal{F}'_t\right]\\ &\quad= E\left[E\left[I_1(t)|\mathcal{F}_{t-1}\right]|\mathcal{F}'_t\right]\\ &\quad= E\Big\{\delta^T(t)\Big(I-b(t)\Big(\mathcal{L}_{\mathcal{G}}^{(m_t)T}\otimes I_n+\bar{C}^T\bar{C}\Big)\Big)\\ &\qquad\times\Big(E\Big[\triangle L_{\mathcal{G}}^{(m_t)}|\mathcal{F}_{t-1}\Big](\delta(t)+1_N\otimes\theta^*)\\ &\qquad+\bar{C}^TE\Big[\tilde{C}(t)|\mathcal{F}_{t-1}\Big](1_N\otimes\theta^*)+E\Big[w_t^{(m_t)}|\mathcal{F}_{t-1}\Big]\\ &\qquad+\bar{C}^TE\left[v(t)|\mathcal{F}_{t-1}\right]\Big)|\mathcal{F}'_t\Big\}=0. \end{aligned}\tag{A.4}$$

From the definition of $\triangle L_{\mathcal{G}}^{(m_t)}$ and Assumption A4) it follows that

$$\begin{aligned}&E\left[\triangle L_{\mathcal{G}}^{(m_t)T}\triangle L_{\mathcal{G}}^{(m_t)}|\mathcal{F}_{t-1}\right](i,j)\\ &\quad=\begin{cases}0, & \text{if } i\neq j\\ \sum_{k=1}^{s}1_{[m_t=k]}a_{li}^{(k)^2}\sum_{i=1}^{N} & \\ \times E\left(L_{il}^T(t)\bar{L}_{il}^{-1}-I\right)\times\left(\bar{L}_{il}^{-1}L_{il}(t)-I\right), & \text{otherwise.}\end{cases}\end{aligned}\tag{A.5}$$

Thus, by Assumption A4) and $\mathcal{F}'_t \subset \mathcal{F}_{t-1}$, there is $C_{02} > 0$ such that

$$\begin{aligned}&E\left[\delta^T(t)\triangle L_{\mathcal{G}}^{(m_t)T}\triangle L_{\mathcal{G}}^{(m_t)}\delta(t)|\mathcal{F}'_t\right]\\ &\quad=E\left\{E\left[\delta^T(t)\triangle L_{\mathcal{G}}^{(m_t)T}\triangle L_{\mathcal{G}}^{(m_t)}\delta(t)|\mathcal{F}_{t-1}\right]|\mathcal{F}'_t\right\}\\ &\quad\le C_{02}V(t).\end{aligned}\tag{A.6}$$

Similarly, there exists $C_{03} > 0$ such that

$$\left(1_N^T\otimes\theta^{*T}\right)E\left[\triangle L_{\mathcal{G}}^{(m_t)T}\triangle L_{\mathcal{G}}^{(m_t)}|\mathcal{F}'_t\right](1_N\otimes\theta^*)\le C_{03}.\tag{A.7}$$

By (5) we know that there exists $C_{04} > 0$ such that

$$E\left[w_t^{(m_t)T}w_t^{(m_t)}|\mathcal{F}'_t\right]\le C_{04}E\left[\|w_t\|^2|\mathcal{F}'_t\right].\tag{A.8}$$

From (A.6)–(A.8) and Cauchy inequality it follows $E[I_2(t)|\mathcal{F}'_t] \le 5(C_{02}V(t) + C_{03} + \|\bar{C}\|^2E[\|v(t)\|^2|\mathcal{F}'_t]+\|\bar{C}\|^2\|1_N\otimes\theta^*\|^2E[\|\tilde{C}(t)\|^2|\mathcal{F}'_t] + C_{04}E[\|w_t\|^2|\mathcal{F}'_t])$. This together with (A.2), (A.3) and (A.4) implies

$$\begin{aligned}&E\left[V(t+1)|\mathcal{F}'_t\right]\\ &\quad\le\left[1-2\lambda_0b(t)+b^2(t)(C_{01}+5C_{02})\right]V(t)\\ &\qquad+5b^2(t)\Big(C_{03}+C_{04}E\left[\|w_t\|^2|\mathcal{F}'_t\right]+\|\bar{C}\|^2E\Big[\|v(t)\|^2|\mathcal{F}'_t\Big]\\ &\qquad+\|\bar{C}\|^2\|1_N\otimes\theta^*\|^2E\Big[\Big\|\tilde{C}(t)\Big\|^2|\mathcal{F}'_t\Big]\Big)\end{aligned}\tag{A.9}$$

and hence, by Assumptions A2)–A3),

$$EV(t+1)\le\left[1-2\lambda_0b(t)+b^2(t)(C_{01}+5C_{02})\right]EV(t)+b^2(t)C_0\tag{A.10}$$

where $C_0 = 5C_{03} + 5\sigma_w^2C_{04} + 5\sigma_v^2\|\bar{C}\|^2 + 5\sigma_C^2\|\bar{C}\|^2\|1_N\otimes\theta^*\|^2$. Noticing that $\lambda_0 > 0$, $b(t)\underset{t\to\infty}{\longrightarrow}0$, there exists $t_0 > 0$ such that $b(t)(C_{01} + 5C_{02}) \le \lambda_0$, and $2b(t)\lambda_0 \le 1, \forall t \ge t_0$. Thus, by A6′) we have

$$0\le1-2\lambda_0b(t)+b^2(t)(C_{01}+5C_{02})<1,\quad t\ge t_0\tag{A.11}$$

$$\sum_{t=t_0}^{\infty}\left[2\lambda_0b(t)-b^2(t)(C_{01}+5C_{02})\right]\ge\lambda_0\sum_{t=t_0}^{\infty}b(t)=\infty\tag{A.12}$$

$$\frac{b^2(t)C_0}{2\lambda_0b(t)-b^2(t)(C_{01}+5C_{02})}\to0,\quad\text{as } t\to\infty.\tag{A.13}$$

which together with (A.10) and Lemma A.1 conclude the proof of the theorem. □

Proof of Theorem 3.2: From Assumptions A2), A3), A6), and the monotone convergence theorem ([24]), we have

$$\begin{aligned}&E\sum_{t=0}^{\infty}b^2(t)E\left[\|w_t\|^2|\mathcal{F}'_t\right]=\sum_{t=0}^{\infty}b^2(t)E\left[\|w_t\|^2\right]<\infty\\ &E\sum_{t=0}^{\infty}b^2(t)E\Big[\|v(t)\|^2|\mathcal{F}'_t\Big]=\sum_{t=0}^{\infty}b^2(t)E\Big[\|v(t)\|^2\Big]<\infty\\ &E\sum_{t=0}^{\infty}b^2(t)E\Big[\Big\|\tilde{C}(t)\Big\|^2|\mathcal{F}'_t\Big]=\sum_{t=0}^{\infty}b^2(t)E\Big[\Big\|\tilde{C}(t)\Big\|^2\Big]<\infty.\end{aligned}\tag{A.14}$$

This implies that $\sum_{t=0}^{\infty}b^2(t)E[\|w_t\|^2|\mathcal{F}'_t] < \infty$, *a.s.*, $\sum_{t=0}^{\infty}b^2(t)E[\|\tilde{C}(t)\|^2|\mathcal{F}'_t] < \infty$, *a.s.*, and $\sum_{t=0}^{\infty}b^2(t)E[\|v(t)\|^2|\mathcal{F}'_t] < \infty$, *a.s.* Thus, by (A.9) in the proof of Theorem 3.1 and nonnegative supermartingale convergence theorem ([29, Lem. D.5.3]) we know that $V(t)$ converges almost surely as $t\to\infty$, and

$$\sum_{t=0}^{\infty}b(t)V(t)<\infty,\quad a.s.\tag{A.15}$$

which together with Theorem 3.1 leads to (9). Furthermore, if $b(t) \downarrow 0$, by (A.15), Cauchy inequality, and Kronecker lemma ([24]) we have $(1/t)\sum_{k=0}^{t}\|\delta(k)\| \le (tb(t))^{-1/2}(b(t)\sum_{k=0}^{t}\|\delta(k)\|^2)^{1/2} = o((b(t)t)^{-1/2})$, as $t\to\infty$. □

APPENDIX B
PROOF OF THEOREM 3.3

Proof of Theorem 3.3: Let $\bar{b}(t) = (1/N)\sum_{i=1}^{N}b_i(t)$, $\Delta_i(t) = \bar{b}(t) - b_i(t)$, $\Delta(t) = diag\{\Delta_1(t),\cdots,\Delta_N(t)\}$, $\xi_1(t) = [I-\bar{b}(t)(\mathcal{L}_{\mathcal{G}}^{(m_t)}\otimes I_n+\bar{C}^T\bar{C})]\delta(t)+\bar{b}(t)[\triangle L_{\mathcal{G}}^{(m_t)}\delta(t)+\triangle L_{\mathcal{G}}^{(m_t)}(1_N\otimes\theta^*)+\bar{C}^T\tilde{C}(t)(1_N\otimes\theta^*)+w_t^{(m_t)}+\bar{C}^Tv(t)]$, and $\xi_2(t) = (\Delta(t)\otimes I_n)[(\mathcal{L}_{\mathcal{G}}^{(m_t)}\otimes I_n+\bar{C}^T\bar{C})\delta(t)-\triangle L_{\mathcal{G}}^{(m_t)}\delta(t)-\triangle L_{\mathcal{G}}^{(m_t)}(1_N\otimes\theta^*)-\bar{C}^T\tilde{C}(t)(1_N\otimes\theta^*)-w_t^{(m_t)}-\bar{C}^Tv(t)]$. Then, (11) can be rewritten into the following form:

$$\delta(t+1)=\xi_1(t)+\xi_2(t).\tag{B.1}$$

Below we will estimate $E[V(t+1)|\mathcal{F}'_t]$. Since $b_i(t)$ satisfies A6′), it can be seen that $\bar{b}(t)\underset{t\to\infty}{\longrightarrow}0$, $\sum_{t=0}^{\infty}\bar{b}(t)=\infty$. Similar to the derivation of (A.9), one can get

$$\begin{aligned}&E\Big[\|\xi_1(t)\|^2|\mathcal{F}'_t\Big]\\ &\quad\le\Big[1-2\lambda_0\bar{b}(t)+\bar{b}^2(t)(C_{01}+5C_{02})\Big]V(t)+5\bar{b}^2(t)\Upsilon(t),\end{aligned}$$

$$E\left[\|\xi_2(t)\|^2|\mathcal{F}'_t\right] \le (C_{01}+5C_{02})\|\Delta(t)\|^2 V(t)+5\|\Delta(t)\|^2\Upsilon(t),$$
$$E\left[\xi_1^T(t)\xi_2(t)|\mathcal{F}'_t\right] \le C_{11}\|\Delta(t)\| V(t)+\bar{b}(t)\max_{1\le i\le N}\{-\Delta_i(t)\}(C_{01}+5C_{02})V(t) +5\Upsilon(t), \quad \text{(B.2)}$$

where $\Upsilon(t) = C_{03} + C_{04}E[\|w_t\|^2|\mathcal{F}'_t] + \|\bar{C}\|^2E[\|v(t)\|^2|\mathcal{F}'_t]+\|\bar{C}\|^2\|1_N\otimes\theta^*\|^2E[\|\tilde{C}(t)\|^2|\mathcal{F}'_t]$, $C_{11} = \max_{1\le k\le s}\|\mathcal{L}_{\mathcal{G}}^{(k)}\otimes I_n\| + \|\bar{C}^T\bar{C}\|$. By (B.1) and (B.2) we have

$$E\left[V(t+1)|\mathcal{F}'_t\right]\le(1-q(t))V(t) +5\left(\bar{b}^2(t)+2\bar{b}(t)\max_{1\le i\le N}\{-\Delta_i(t)\}+\|\Delta(t)\|^2\right)\Upsilon(t) \quad \text{(B.3)}$$

where $q(t) = 2\lambda_0\bar{b}(t) - \bar{b}^2(t)(C_{01} + 5C_{02})-2\bar{b}(t)\max_i\{-\Delta_i(t)\}(C_{01} + 5C_{02}) - (C_{01} + 5C_{02})\|\Delta(t)\|^2-2C_{11}\|\Delta(t)\|$. Taking mathematical expectation on both sides of (B.3), by Assumptions A2)–A3) we have $E[V(t + 1)] \le (1 - q(t))E[V(t)] + (\bar{b}^2(t)+2\bar{b}(t)\max_{1\le i\le N}\{-\Delta_i(t)\} + \|\Delta(t)\|^2)C_0$, where C_0 is defined in (A.10). Since each gain function $b_i(t)$ satisfies A6′) and (12), it can be seen that $\sum_{t=0}^{\infty}\bar{b}(t) = \infty$, $\lim_{t\to\infty}\bar{b}(t) = 0$, $\|\Delta(t)\| = o(\bar{b}(t))$, $t \to \infty$. Thus, similar to (A.11)–(A.13), there is $t_1 > 0$ such that $0 < q(t) \le 1$, $\forall t \ge t_1$; $\sum_{t=0}^{\infty}q(t) = \infty$; $(\bar{b}^2(t)/q(t)) \to 0$ as $t \to \infty$. This together with Lemma A.1 leads to the mean square convergence of the algorithm (11).

By Cauchy inequality, we have $\sum_{t=0}^{\infty}\bar{b}^2(t) \le (1/N)\sum_{i=1}^{N}\sum_{t=0}^{\infty}b_i^2(t) < \infty$. This together with (A.14) and the nonnegative supermartingale convergence theorem ([29, Lem. D.5.3]) implies that $V(t)$ converges almost surely as $t \to \infty$, and $\sum_{t=0}^{\infty}\bar{b}(t)V(t) < \infty$, a.s.. Thus, similar to the proof of Theorem 3.2, we can obtain (13).

Appendix C
Proof of Theorem 3.4 and Theorem 3.5

Proof of Theorem 3.4: By (A.1) we have

$$\delta((m+1)h) = \Phi((m+1)h, mh)\delta(mh) + \overline{W}_{mh}^h \quad \text{(C.1)}$$

where h is a positive integer, $\Phi(k+1,i) = [I - b(k)(\mathcal{L}_{\mathcal{G}}^{(m_k)}\otimes I_n + \bar{C}^T\bar{C})]\Phi(k,i)$, $\Phi(i,i) = I_{nN}$, $\forall k \ge i$, $i \ge 0$; $\overline{W}_t^h = \sum_{j=t}^{t+h-1}\Phi(t+h-1,j)b(j)[\triangle L_{\mathcal{G}}^{(m_j)}\delta(j)+\triangle L_{\mathcal{G}}^{(m_j)}(1_N\otimes\theta^*)+\bar{C}^T\tilde{C}(j)(1_N\otimes\theta^*)+w_j^{(m_j)}+\bar{C}^Tv(j)]$. By Assumption A1) and [6, Th. 7] we know that $(1/2)(\mathcal{L}_{\mathcal{G}^{(i)}} + \mathcal{L}_{\mathcal{G}^{(i)}})$ is the Laplacian matrix of the mirror graph $\hat{\mathcal{G}}^{(i)}$, i.e., $\mathcal{L}_{\hat{\mathcal{G}}^{(i)}} = (1/2)(\mathcal{L}_{\mathcal{G}^{(i)}} + \mathcal{L}_{\mathcal{G}^{(i)}})$, $\forall i \in \mathcal{S}$. By Assumption A6″) we know that there is $C_{20} > 0$ such that

$$\Phi^T((m+1)h,mh)\Phi((m+1)h,mh) \le I - 2\sum_{i=mh}^{(m+1)h-1}b(i)\left[\mathcal{L}_{\hat{\mathcal{G}}^{(m_i)}}\otimes I_n + \bar{C}^T\bar{C}\right] + C_{20}b^2(mh)I$$

which together with Assumption A6″) gives

$$\begin{aligned}V((m+1)h) &= \delta^T(mh)\Phi^T((m+1)h,mh)\Phi((m+1)h,mh)\delta(mh)\\ &\quad + 2\delta^T(mh)\Phi^T((m+1)h,mh)\overline{W}_{mh}^h + \overline{W}_{mh}^{hT}\overline{W}_{mh}^h\\ &\le V(mh) - C_{21}b(mh)\delta^T(mh)\\ &\quad\times\sum_{i=mh}^{(m+1)h-1}\left[\mathcal{L}_{\hat{\mathcal{G}}^{(m_i)}}\otimes I_n + \bar{C}^T\bar{C}\right]\delta(mh)\\ &\quad + C_{20}b^2(mh)V(mh) + \overline{W}_{mh}^{hT}\overline{W}_{mh}^h\\ &\quad + 2\delta^T(mh)\Phi^T((m+1)h,mh)\overline{W}_{mh}^h \end{aligned} \quad \text{(C.2)}$$

where $C_{21} > 0$ is a constant related to h.

Below we want to show

$$E\left[2\delta^T(mh)\Phi^T((m+1)h,mh)\overline{W}_{mh}^h|\mathcal{F}'_{mh}\right] = 0. \quad \text{(C.3)}$$

To this end, define the σ-algebras $\mathcal{F}''_{(m+1)h} = \sigma\{x(0), w_k, v(k), L_k, \tilde{C}(k), m_l, k = 0,\cdots,mh-1, l = 0,\cdots,(m+1)h-1\}$ and $\bar{\mathcal{F}}'_{mh+j} = \sigma\{x(0), w_k, v(k), L_k, \tilde{C}(k), m_l, k = 0,\cdots,mh+j-1, l = 0,\cdots,(m+1)h-1\}$. Then, it follows that

$$\begin{aligned}&E\left[2\delta^T(mh)\Phi^T((m+1)h,mh)\overline{W}_{mh}^h|\mathcal{F}'_{mh}\right]\\ &=2\delta^T(mh)E\Big[\Phi^T((m+1)h,mh)\\ &\qquad\times\sum_{j=mh}^{(m+1)h-1}\Phi((m+1)h-1,j)\\ &\qquad\times b(j)E\Big\{\triangle L_{\mathcal{G}}^{(m_j)}(\delta(j)+1_N\otimes\theta^*)\\ &\qquad\quad+\bar{C}^T\tilde{C}(j)(1_N\otimes\theta^*)+w_j^{(m_j)}\\ &\qquad\quad+\bar{C}^Tv(j)|\mathcal{F}''_{(m+1)h}\Big\}|\mathcal{F}'_{mh}\Big]. \end{aligned} \quad \text{(C.4)}$$

Noticing that $E[\triangle L_{\mathcal{G}}^{(m_j)}(1_N \otimes \theta^*)|\mathcal{F}''_{(m+1)h}] = 0$, $E[\triangle L_{\mathcal{G}}^{(m_j)}\delta(j)|\mathcal{F}''_{(m+1)h}] = E[E[\triangle L_{\mathcal{G}}^{(m_j)}|\bar{\mathcal{F}}'_{mh+j}]\delta(j)|\mathcal{F}''_{(m+1)h}] = 0$, $E[\bar{C}^T\tilde{C}(j)(1_N \otimes \theta^*)|\mathcal{F}''_{(m+1)h}] = E[w_j^{(m_j)}|\mathcal{F}''_{(m+1)h}] = E[\bar{C}^Tv(j)|\mathcal{F}''_{(m+1)h}] = 0$, by (C.4) we have (C.3).

By (A.5), (A.8), the definition of $\overline{W}_{mh}^h$, Minkowski inequality and Cauchy inequality, there exist $C_{22} > 0$ and $C_{23} > 0$, such that

$$\begin{aligned}&E\left[\overline{W}_{mh}^{hT}\overline{W}_{mh}^h|\mathcal{F}'_{mh}\right]\\ &\le 5E\Bigg[\sum_{j=mh}^{(m+1)h-1}b^2(j)\|\Phi((m+1)h-1,j)\|^2\\ &\qquad\times\bigg(\left\|\triangle L_{\mathcal{G}}^{(m_j)}\right\|^2\|\delta(j)\|^2+\left\|\triangle L_{\mathcal{G}}^{(m_j)}\right\|^2\|1_N\otimes\theta^*\|^2\\ &\qquad\quad+\|\bar{C}^T\|^2\left\|\tilde{C}(j)\right\|^2\|1_N\otimes\theta^*\|^2+\left\|w_j^{(m_j)}\right\|^2\\ &\qquad\quad+\|\bar{C}\|^2\|v(j)\|^2\bigg)|\mathcal{F}'_{mh}\Bigg]\\ &\le C_{22}E\Bigg[\sum_{j=mh}^{(m+1)h-1}b^2(j)\bigg(\|\delta(j)\|^2E\left[\left\|\triangle L_{\mathcal{G}}^{(m_j)}\right\|^2|\mathcal{F}_{j-1}\right]\\ &\qquad\quad+1+\|w_j\|^2+\|v(j)\|^2\bigg)|\mathcal{F}'_{mh}\Bigg]\end{aligned}$$

$$\leq C_{23}\sum_{j=mh}^{(m+1)h-1} b^2(j)\Big(1+E\left[V(j)|\mathcal{F}'_{mh}\right]+E\left[\|w_j\|^2|\mathcal{F}'_{mh}\right] +E\left[\|v(j)\|^2\,|\mathcal{F}'_{mh}\right]\Big). \qquad \text{(C.5)}$$

By Assumption A5), there is $h > 0$ such that for any initial time $t \geq 0$ and initial state, the Markov chain m_t will visit all its states in the time interval $[t, t+h]$ with positive probability $\epsilon_0 > 0$, where ϵ_0 does not depend on t. Thus, by Lemma 3.1 and the definition of the union graph, there is $C_{24} > 0$ such that

$$\begin{aligned}\Xi_m^h &\triangleq C_{21}E\left(\sum_{i=mh}^{(m+1)h-1}\left[\mathcal{L}_{\hat{\mathcal{G}}}^{(m_i)}\otimes I_n+\bar{C}^T\bar{C}\right]|\mathcal{F}'_{mh}\right)\\ &\geq C_{21}E\Big(\mathcal{L}_{\sum_{i=mh}^{(m+1)h-1}\hat{\mathcal{G}}^{(m_i)}}\otimes I_n+h\min_{1\leq i\leq N}p_i\bar{C}^T\bar{C}|\mathcal{F}'_{mh}\Big)\\ &\geq C_{24}I\end{aligned}$$

which together with (C.2), (C.3) and (C.5) leads to

$$\begin{aligned}&E\left[V\left((m+1)h\right)|\mathcal{F}'_{mh}\right]\\ &\quad\leq\left(1-C_{24}b(mh)+C_{20}b^2(mh)\right)V(mh)\\ &\qquad+C_{23}\sum_{j=mh}^{(m+1)h-1}b^2(j)\Big(1+E\left[V(j)|\mathcal{F}'_{mh}\right] + E\left[\|w_j\|^2|\mathcal{F}'_{mh}\right] + E\left[\|v(j)\|^2\,|\mathcal{F}'_{mh}\right]\Big).\end{aligned} \qquad \text{(C.6)}$$

Similar to (A.9) in Theorem 3.1, we know that there exist $C_{25}, C_{26} > 0$, such that

$$\begin{aligned}E\left[V(t+1)\right] &\leq \left[1+C_{25}b^2(t)\right]E\left[V(t)\right]+C_{26}b^2(t)\\ &=\prod_{k=0}^{t}\left(1+C_{25}b^2(k)\right)E\left[V(0)\right]\\ &\quad+\sum_{k=0}^{t}\prod_{j=k+1}^{t}\left(1+C_{25}b^2(j)\right)C_{26}b^2(k)\\ &\leq \exp\left\{C_{25}\sum_{k=0}^{\infty}b^2(k)\right\}\left(E\left[V(0)\right]+C_{26}\sum_{k=0}^{\infty}b^2(k)\right)\\ &<\infty.\end{aligned} \qquad \text{(C.7)}$$

This together with Assumption A2) and (C.6) implies that there is $C_{27} > 0$ such that

$$\begin{aligned}&E\left[V\left((m+1)h\right)\right]\\ &\quad\leq\left(1-C_{24}b(mh)+C_{20}b^2(mh)\right)E\left[V(mh)\right]+C_{27}b^2(mh).\end{aligned} \qquad \text{(C.8)}$$

By Assumption A6″), there is $\bar{\alpha}_1 > 0$ such that $\sum_{m=0}^{\infty} b(mh) \geq \bar{\alpha}_1\sum_{m=0}^{\infty}\sum_{i=mh}^{(m+1)h-1} b(i) = \bar{\alpha}_1\sum_{t=0}^{\infty} b(t) = \infty$, $\sum_{m=0}^{\infty} b^2(mh) \leq \sum_{t=0}^{\infty} b^2(t) < \infty$. Thus, similar to Theorem 3.1, by Lemma A.1 and (C.8) we have $\lim_{m\to\infty} E[V(mh)] = 0$.

For any $t > 0$, define $\tau_t = \lfloor t/h \rfloor$. Then, we have $0 \leq t-\tau_t \leq h$. Similar to the estimation of (C.5), by (A.1) and (C.7) there is $C_{28} > 0$ such that

$$\begin{aligned}&E\left[V(t)\right]\\ &\quad\leq 2E\left[\left\|\Phi^T(t,\tau_t)\right\|^2V(\tau_t)\right]\\ &\qquad+10\sum_{k=\tau_t}^{t-1}E\left[b^2(k)\left\|\Phi(t,k+1)\right\|^2\left[\left\|\triangle L_{\mathcal{G}}^{(m_k)}\right\|^2V(k)\right.\right.\\ &\qquad\quad+\left\|\triangle L_{\mathcal{G}}^{(m_k)}\right\|^2\|1_N\otimes\theta^*\|^2\\ &\qquad\quad+\left\|\bar{C}^T\tilde{C}(k)\right\|^2\|1_N\otimes\theta^*\|^2\\ &\qquad\quad\left.\left.+\left\|w_k^{(m_k)}\right\|^2+\left\|\bar{C}^Tv(k)\right\|^2\right]\right]\\ &\quad\leq C_{28}\left(E\left[V(\tau_t)\right]+\sum_{k=\tau_t}^{t}b^2(k)\right).\end{aligned}$$

Thus, by Assumption A6″) and $E[V(mh)] \underset{m\to\infty}{\longrightarrow} 0$, we have $E[V(t)] \underset{t\to\infty}{\longrightarrow} 0$.

We now prove the almost sure convergence of the algorithm (4). For any $l = 0, 1, \cdots, h-1$, similar to (C.1), (C.2), (C.5), and (C.6), we have

$$\begin{aligned}&E\big[V\left(l+(m+1)h\right)|\mathcal{F}'_{l+mh}\big]\\ &\quad\leq\left(1-C_{24}b(l+mh)+C_{20}b^2(l+mh)\right)V(l+mh)\\ &\qquad+C_{23}\sum_{j=l+mh}^{l+(m+1)h-1}b^2(j)\Big(1+E\big[V(j)|\mathcal{F}'_{l+mh}\big] +E\big[\|w_j\|^2|\mathcal{F}'_{l+mh}\big] +E\Big[\|v(j)\|^2\,|\mathcal{F}'_{l+mh}\Big]\Big).\end{aligned} \qquad \text{(C.9)}$$

Noticing that $\sum_{m=0}^{\infty} b^2(l+mh) < \infty$, $\sum_{m=0}^{\infty}\sum_{j=l+mh}^{l+(m+1)h-1} b^2(j) < \infty$, by (C.7) and Assumptions A2)–A3), and applying the nonnegative supermartingale convergence theorem ([29, Lem. D.5.3]) to (C.9), we obtain that $V(l+(m+1)h)$ converges almost surely as $m \to \infty$, and

$$\sum_{m=0}^{\infty} b(l+mh)V(l+mh) < \infty, \quad l = 0, 1, \cdots, h-1. \qquad \text{(C.10)}$$

From $\lim_{t\to\infty} E[V(t)] = 0$ it follows that $\lim_{t\to\infty} V(l+(m+1)h) = 0$, which implies $\lim_{t\to\infty} V(t) = 0$ a.s. Furthermore, (C.10) implies $\sum_{t=0}^{\infty} b(t)V(t) = \sum_{l=0}^{h-1}\sum_{m=0}^{\infty} b(mh+l)V(mh+l) < \infty$. Thus, if $b(t) \downarrow 0$, by Cauchy inequality, and Kronecker lemma ([24]) we have $(1/t)\sum_{k=0}^{t}\|\delta(k)\| \leq (tb(t))^{-1/2}(b(t)\sum_{k=0}^{t}\|\delta(k)\|^2)^{1/2} = o((b(t)t)^{-1/2})$ a.s., as $t \to \infty$. □

Proof of Theorem 3.5: By (B.1) we have

$$\delta\left((m+1)h\right) = \bar{\Phi}\left((m+1)h, mh\right)\delta(mh) + \tilde{W}_{mh}^h \qquad \text{(C.11)}$$

where $\bar{\Phi}(k+1, i) = [I - \bar{b}(k)(\mathcal{L}_{\mathcal{G}}^{(m_k)} \otimes I_n + \bar{C}^T\bar{C})]\bar{\Phi}(k, i)$, $\bar{\Phi}(i, i) = I_{nN}, \forall k \geq i, i \geq 0$; $\tilde{W}_t^h = \sum_{j=t}^{t+h-1}\bar{\Phi}(t+h-1, j)[\bar{b}(j)(\triangle L_{\mathcal{G}}^{(m_j)}\delta(j) + \triangle L_{\mathcal{G}}^{(m_j)}(1_N \otimes \theta^*) + \bar{C}^T\tilde{C}(j)(1_N \otimes$

$\theta^*)+w_j^{(m_j)}+\bar{C}^T v(j))+\xi_2(j)], \xi_2(t)$ is defined in (B.1). Similar to (C.3) we have

$$\begin{aligned} E&\left[2\delta^T(mh)\bar{\Phi}^T\left((m+1)h, mh\right)\tilde{W}^h_{mh}|\mathcal{F}'_{mh}\right] \\ &= 2\delta^T(mh)E\left[\bar{\Phi}^T\left((m+1)h, mh\right)\right. \\ &\quad\times\sum_{j=mh}^{(m+1)h-1}\bar{\Phi}\left((m+1)h-1, j\right) \\ &\quad\times\left(\Delta(j)\otimes I_n\right)\left(\mathcal{L}_{\mathcal{G}}^{(m_j)}\otimes I_n+\bar{C}^T\bar{C}\right) \\ &\quad\times\delta(j)|\mathcal{F}'_{mh}\left.\right]. \end{aligned} \tag{C.12}$$

Similar to (C.5), by Minkowski inequality and Cauchy inequality, there exist $C_{30} > 0$, such that

$$\begin{aligned} E&\left[\tilde{W}^{hT}_{mh}\tilde{W}^h_{mh}|\mathcal{F}'_{mh}\right] \\ &\le C_{30}E\left[\sum_{j=mh}^{(m+1)h-1}\left(\bar{b}^2(j)+\|\Delta(j)\|^2\right)\right. \\ &\quad\left.\times\left(1+\|\delta(j)\|^2+\|w_j\|^2+\|v(j)\|^2\right)|\mathcal{F}'_{mh}\right]. \end{aligned} \tag{C.13}$$

Thus, similar to (C.6), by (C.11)–(C.13), there exist C_{31}, C_{32}, C_{33}, $C_{34} > 0$, such that

$$\begin{aligned} E&\left[V\left((m+1)h\right)|\mathcal{F}'_{mh}\right] \\ &\le\left(1-C_{31}\bar{b}(mh)+C_{32}\bar{b}^2(mh)\right. \\ &\quad\left.+C_{33}\max_{1\le i\le N}|\Delta_i(mh)|\right)V(mh) \\ &\quad+C_{34}\max_{1\le i\le N}|\Delta_i(mh)|\sum_{j=mh}^{(m+1)h-1}E\left[V(j)|\mathcal{F}'_{mh}\right] \\ &\quad+C_{30}\sum_{j=mh}^{(m+1)h-1}\left(\bar{b}^2(j)+\|\Delta(j)\|^2\right) \\ &\quad\times\left(1+E\left[V(j)|\mathcal{F}'_{mh}\right]+E\left[\|w_j\|^2|\mathcal{F}'_{mh}\right]\right. \\ &\quad\left.+E\left[\|v(j)\|^2|\mathcal{F}'_{mh}\right]\right). \end{aligned} \tag{C.14}$$

Similar to (C.7), we have $\sup_{t\ge 0}E[V(t)] < \infty$. Then from (C.14) and Lemma A.1, by the similar derivation of Theorem 3.4, we have $E[V(t)]\underset{t\to\infty}{\longrightarrow}0$. The almost sure convergence property and the convergence rate estimate of the algorithm can be obtained by almost the same deduction as Theorem 3.4, just noticing that (14) implies (12), and $E[V(l+(m+1)h)|\mathcal{F}'_{mh}]$, $l=0,\cdots,h-1$ has the same recursive estimate as (C.14).

APPENDIX D
PROOF OF THEOREM 4.1

Proof of Theorem 4.1: The algorithm (19) can be written in the following stochastic differential equation associated with the parameter estimation error $\delta(t)$:

$$d\delta(t)=-b(t)\left[\mathcal{L}_{\mathcal{G}}^{(m_t)}\otimes I_n+\bar{C}^T\bar{C}\right]\delta(t)dt+b(t)\Sigma^{(m_t)}dW(t) \tag{D.1}$$

where $\Sigma^{(m_t)}$ is defined in (18). In the sequel, we want to analyze[3] $V(t) = E[\delta(t)\delta^T(t)]$. To this end, define $V_i(t) = E[\delta(t)\delta^T(t)1_{[m_t=i]}]$. Then, $V(t) = \sum_{i=1}^s V_i(t)$. By (D.1), [21, Lem. 4.2] and Itô formula ([30]) we have

$$\begin{aligned} dV_i(t) &= E\left[(d\delta(t))\,\delta^T(t)1_{[m_t=i]}+\delta(t)\left(d\delta^T(t)\right)1_{[m_t=i]}\right. \\ &\quad\left.+d\delta(t)d\delta^T(t)1_{[m_t=i]}+\delta(t)\delta^T(t)d1_{[m_t=i]}\right] \\ &= -b(t)\left[\left(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n+\bar{C}^T\bar{C}\right)V_i(t)\right. \\ &\quad\left.+V_i(t)\left(\mathcal{L}_{\mathcal{G}}^{(i)T}\otimes I_n+\bar{C}^T\bar{C}\right)\right]dt \\ &\quad+b^2(t)\Sigma^{(i)}\Sigma^{(i)T}p_i(t)dt+\sum_{j=1}^s q_{ji}V_j(t)dt+o(dt). \end{aligned}$$

or equivalently,

$$\begin{aligned} \frac{dV_i(t)}{dt} &= -b(t)\left(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n+\bar{C}^T\bar{C}\right)V_i(t) \\ &\quad-b(t)V_i(t)\left(\mathcal{L}_{\mathcal{G}}^{(i)T}\otimes I_n+\bar{C}^T\bar{C}\right) \\ &\quad+b^2(t)\Sigma^{(i)}\Sigma^{(i)T}p_i(t)+\sum_{j=1}^s q_{ji}V_j(t). \end{aligned} \tag{D.2}$$

Denote $R_i(t)=\Sigma^{(i)}\Sigma^{(i)T}p_i(t)$ and $\bar{V}(t)=[V_1(t),\cdots,V_s(t)]$. From (20), (D.2) and the definition of Kronecker sum it follows that

$$\begin{aligned} \frac{d\varphi\left(V_i(t)\right)}{dt} &= -b(t)\left[\left(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n\right)\oplus\left(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n\right)\right. \\ &\quad\left.+\left(\bar{C}^T\bar{C}\oplus\bar{C}^T\bar{C}\right)\right]\varphi\left(V_i(t)\right) \\ &\quad+\left(Q_i^T\otimes I_{(nN)^2}\right)\hat{\varphi}\left(\bar{V}(t)\right)+b^2(t)\varphi\left(R_i(t)\right) \end{aligned}$$

which implies

$$\begin{aligned} \frac{d\hat{\varphi}\left(\bar{V}(t)\right)}{dt} &= \left(-b(t)\Gamma_L-b(t)\Gamma_C\right. \\ &\quad\left.+Q^T\otimes I_{(nN)^2}\right)\hat{\varphi}\left(\bar{V}(t)\right)+b^2(t)\hat{\varphi}\left(\bar{R}(t)\right) \end{aligned} \tag{D.3}$$

where $\bar{R}(t)=[R_1(t),\cdots,R_s(t)]$, $\hat{\varphi}(\bar{V}(0))=[q_1,\cdots,q_s]^T\otimes\varphi(V(0))$, and Γ_L,Γ_C are defined by

$$\begin{aligned} \Gamma_L &= diag\left\{\left(\mathcal{L}_{\mathcal{G}}^{(1)}\otimes I_n\right)\oplus\left(\mathcal{L}_{\mathcal{G}}^{(1)}\otimes I_n\right),\cdots,\right. \\ &\quad\left.\left(\mathcal{L}_{\mathcal{G}}^{(s)}\otimes I_n\right)\oplus\left(\mathcal{L}_{\mathcal{G}}^{(s)}\otimes I_n\right)\right\} \\ \Gamma_C &= I_s\otimes\left(\bar{C}^T\bar{C}\oplus\bar{C}^T\bar{C}\right). \end{aligned} \tag{D.4}$$

Noticing that $\{\mathcal{G}^{(i)}, i\in\mathcal{S}\}$ is a set of balanced graphs, by [6, Th. 7] and the definition of Kronecker sum, we know that $(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n)\oplus(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n)+(\mathcal{L}_{\mathcal{G}}^{(i)T}\otimes I_n)\oplus(\mathcal{L}_{\mathcal{G}}^{(i)T}\otimes I_n)=2(\mathcal{L}_{\hat{\mathcal{G}}}^{(i)}\otimes I_n)\oplus(\mathcal{L}_{\hat{\mathcal{G}}}^{(i)}\otimes I_n)$, $i\in\mathcal{S}$, where $\hat{\mathcal{G}}^{(i)}$ is the mirror graph of $\mathcal{G}^{(i)}$, $\mathcal{L}_{\hat{\mathcal{G}}}^{(i)}$ is the Laplacian matrix of $\hat{\mathcal{G}}^{(i)}$. Thus, by (D.3) we have

$$\begin{aligned} &\frac{d\hat{\varphi}^T\left(\bar{V}(t)\right)\varphi\left(\bar{V}(t)\right)}{dt} \\ &= 2\hat{\varphi}^T\left(\bar{V}(t)\right)\left(-b(t)\Gamma_{\hat{L}}-b(t)\Gamma_C+\hat{Q}\otimes I_{(nN)^2}\right)\hat{\varphi}\left(\bar{V}(t)\right) \\ &\quad+2b^2(t)\hat{\varphi}^T\left(\bar{V}(t)\right)\hat{\varphi}\left(\bar{R}(t)\right) \end{aligned} \tag{D.5}$$

[3]Different from the scalar random process $V(t)$ defined in Section III, here $V(t)\in\mathbb{R}^{nN}$ is a deterministic function.

where $\Gamma_{\hat{L}}$ is the Γ_L with $\mathcal{L}_{\mathcal{G}}^{(i)}$ replaced by $\mathcal{L}_{\hat{\mathcal{G}}}^{(i)}$. As the transition rate matrix Q is doubly stochastic, we know that $\hat{Q} = (1/2)(Q+Q^T)$ corresponds to a symmetric irreducible transition rate matrix.

Below, we will prove that the matrix $\Gamma_{\hat{L}} + \Gamma_C - \hat{Q}\otimes I_{(nN)^2}$ is positive definite. First, similar to [31, Cor. 3.7], we prove that $-\Gamma_{\hat{L}} + \hat{Q}\otimes I_{(nN)^2}$ has a zero eigenvalue with algebraic multiplicity n^2, with corresponding left and right eigenvectors given by $1_s\otimes[(1_N\otimes\mu)\otimes(1_N\otimes\nu)]$, $\forall\mu,\nu\in\mathbb{R}^n$. In fact, it can be seen that $-\Gamma_{\hat{L}} + \hat{Q}\otimes I_{(nN)^2}$ is a transition rate matrix, and $-\Gamma_{\hat{L}}$ is block diagonalizable, i.e., there exists a similarity transformation P such that $P^T(-\Gamma_{\hat{L}})P$ is a block diagonal matrix with irreducible blocks. If v is a eigenvector corresponding to the zero eigenvalue of $-\Gamma_{\hat{L}}+\hat{Q}\otimes I_{(nN)^2}$, then by [31, Lem. 3.2], we have $v\in Null(-\Gamma_{\hat{L}})\cap Null(\hat{Q}\otimes I_{(nN)^2})$. By the definition of $\hat{Q}\otimes I_{(nN)^2}$, we get $v = 1_s\otimes u$, $\forall u\in\mathbb{R}^{(nN)^2}$. Thus, from $-\Gamma_{\hat{L}}\nu = 0$ we have $(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n)\oplus(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n)u = 0$, $i\in\mathcal{S}$. Furthermore, noticing that $\mathcal{G}^{(i)}$ is balanced, $(\mathcal{L}_{\sum_{i=1}^N\mathcal{G}^{(i)}}\otimes I_n)\oplus(\mathcal{L}_{\sum_{i=1}^N\mathcal{G}^{(i)}}\otimes I_n)u = 0$. By Assumption A1), the balanced graph $\sum_{i=1}^N\mathcal{G}^{(i)}$ contains a spanning tree, which amounts to saying that $\mathcal{L}_{\sum_{i=1}^N\mathcal{G}^{(i)}}$ is an irreducible transition rate matrix. Thus, by [31, Lem. 3.2], the eigenvectors of the null-space of the matrix $(\mathcal{L}_{\sum_{i=1}^N\mathcal{G}^{(i)}}\otimes I_n)\oplus(\mathcal{L}_{\sum_{i=1}^N\mathcal{G}^{(i)}}\otimes I_n) = (\mathcal{L}_{\sum_{i=1}^N\mathcal{G}^{(i)}}\otimes I_n)\otimes I_{nN} + I_{nN}\otimes(\mathcal{L}_{\sum_{i=1}^N\mathcal{G}^{(i)}}\otimes I_n)$ has the following form: $u = (1_N\otimes\mu)\otimes(1_N\otimes\nu)$, $\forall\mu,\nu\in\mathbb{R}^n$.

Since $\Gamma_{\hat{L}} + \Gamma_C - \hat{Q}\otimes I_{(nN)^2}$ is a nonnegative matrix, if there is a nonzero vector $x\in\mathbb{R}^{s(nN)^2}$ such that $x^T[\Gamma_{\hat{L}}+\Gamma_C - \hat{Q}\otimes I_{(nN)^2}]x = 0$, then by the positive semi-definiteness of the matrix $\Gamma_{\hat{L}} - \hat{Q}\otimes I_{(nN)^2}$ and Γ_C we know that

$$x^T\left[\Gamma_{\hat{L}} - \hat{Q}\otimes I_{(nN)^2}\right]x = 0,\quad x^T\Gamma_C x = 0$$

which implies $x = 1_s\otimes[(1_N\otimes\mu)\otimes(1_N\otimes\nu)]$, $\forall\mu,\nu\in\mathbb{R}^n$. Thus, $[(1_N^T\otimes\mu^T)\otimes(1_N^T\otimes\nu^T)](\bar{C}\otimes I_{nN} + I_{nN}\otimes\bar{C})[(1_N\otimes\mu)\otimes(1_N\otimes\nu)] = 0$. By properties of the Kronecker product ([25]), the above equation can be simplified as: $(N\nu^T\nu)(\mu^T\sum_{i=1}^N\bar{C}_i^T\bar{C}_i\mu) + (N\mu^T\mu)(\nu^T\sum_{i=1}^N\bar{C}_i^T\bar{C}_i\nu) = 0$, $\forall\mu,\nu\in\mathbb{R}^n$. This contradicts the global observability condition B3). Thus, $\Gamma_{\hat{L}} + \Gamma_C - \hat{Q}\otimes I_{(nN)^2}$ is positive definite.

Since $b(t)\underset{t\to\infty}{\longrightarrow}0$, there is t_0 such that $b(t)\le\min\{1,\lambda\}$, $\forall t\ge t_0$, where $\lambda>0$ is the minimal eigenvalue of $\Gamma_{\hat{L}}+\Gamma_C - \hat{Q}\otimes I_{(nN)^2}$. Noticing that $\hat{Q}$ is negative semi-definite, by the inequality $2x^Ty\le\epsilon x^Tx + \epsilon^{-1}y^Ty$, $\forall x,y\in\mathcal{R}^{s(nN)^2}$, $\epsilon>0$, and (D.5), for any $t\ge t_0$ we have

$$\begin{aligned}
&\frac{d\hat{\varphi}^T\big(\bar{V}(t)\big)\hat{\varphi}\big(\bar{V}(t)\big)}{dt}\\
&\le 2\hat{\varphi}^T\big(\bar{V}(t)\big)\Big(-b(t)\Gamma_{\hat{L}} + b(t)\hat{Q}\otimes I_{(nN)^2} - b(t)\Gamma_C\Big)\hat{\varphi}\big(\bar{V}(t)\big)\\
&\quad + b^2(t)\hat{\varphi}^T\big(\bar{V}(t)\big)\hat{\varphi}\big(\bar{V}(t)\big) + b^2(t)\hat{\varphi}^T\big(\bar{R}(t)\big)\hat{\varphi}\big(\bar{R}(t)\big)\\
&\le -\lambda b(t)\hat{\varphi}^T\big(\bar{V}(t)\big)\hat{\varphi}\big(\bar{V}(t)\big) + b^2(t)\hat{\varphi}^T\big(\bar{R}(t)\big)\hat{\varphi}\big(\bar{R}(t)\big)
\end{aligned}\tag{D.6}$$

which together with the comparison theorem ([32]) gives

$$\hat{\varphi}^T\left(\bar{V}(t)\right)\hat{\varphi}\left(\bar{V}(t)\right)\le\left\|\hat{\varphi}\left(\bar{V}(t_0)\right)\right\|^2 I_1(t) + I_2(t)\tag{D.7}$$

where $I_1(t) = \exp\{-\lambda\int_{t_0}^t b(u)du\}$, $I_2(t) = \int_{t_0}^t\exp\{-\lambda\int_s^t b(u)du\}b^2(s)\hat{\varphi}^T(\bar{R}(s))\hat{\varphi}(\bar{R}(s))ds$. From (D.5) and $E[\|X(0)\|^2]<\infty$, one can obtain the boundedness of $\|\hat{\varphi}(\bar{V}(t_0))\|^2$. Thus, by $\int_{t_0}^\infty b(s)ds = \infty$ and $\lambda>0$, we have $\lim_{t\to\infty}I_1(t) = 0$. In addition, by Remark 6 we know that there is $t_1>0$ such that for any $t\ge t_1$

$$\left|\hat{\varphi}^T\left(\bar{R}(t)\right)\hat{\varphi}\left(\bar{R}(t)\right) - \hat{\varphi}^T(\bar{R})\hat{\varphi}(\bar{R})\right|<\varepsilon$$

where $\bar{R} = [R_1,\cdots,R_s]$, $R_i = \Sigma^{(i)}\Sigma^{(i)T}\pi_i$, $i\in\mathcal{S}$. Since $\int_0^\infty b^2(t)dt<\infty$, there is $t_2>0$ such that for any $t\ge t_2$, $\int_{t_2}^\infty b^2(t)dt<\varepsilon$. Let $t_3 = \max\{t_1,t_2\}$. Then, we have

$$\begin{aligned}
I_2(t) &= \int_{t_0}^{t_3}\exp\left\{-\lambda\int_s^t b(u)du\right\}b^2(s)\left\|\hat{\varphi}\left(\bar{R}(s)\right)\right\|^2 ds\\
&\quad + \int_{t_3}^{t}\exp\left\{-\lambda\int_s^t b(u)du\right\}b^2(s)\left\|\hat{\varphi}\left(\bar{R}(s)\right)\right\|^2 ds\\
&\le\exp\left\{-\lambda\int_{t_3}^t b(u)du\right\}\int_{t_0}^{t_3}b^2(s)\left\|\hat{\varphi}\left(\bar{R}(s)\right)\right\|^2 ds\\
&\quad + \left(\left\|\hat{\varphi}(\bar{R})\right\|^2+\varepsilon\right)\int_{t_3}^t b^2(s)ds\\
&\to\left(\left\|\hat{\varphi}(\bar{R})\right\|^2+\varepsilon\right)\varepsilon,\quad\text{as } t\to\infty.
\end{aligned}$$

This together with the arbitrariness of ε implies that $\lim_{t\to\infty}I_2(t) = 0$. Thus, from (D.7) it follows $\lim_{t\to\infty}\|\hat{\varphi}(\bar{V}(t))\| = 0$, $\lim_{t\to\infty}\|V(t)\| = 0$. Since $E[\|\delta(t)\|^2] = tr(V(t))$, we get (21). □

APPENDIX E
PROOF OF THEOREM 4.2

Proof of Theorem 4.2: Let $\hat{V}(t) = \delta^T(t)\delta(t)$. Noticing that $\mathcal{G}_t$ is balanced, by (D.1) and Itô formula we have $d\hat{V}(t) = -2b(t)\delta^T(t)(\mathcal{L}_{\hat{\mathcal{G}}}^{(m_t)}\otimes I_n + \bar{C}^T\bar{C})\delta(t)dt + b^2(t)tr(\Sigma^{(m_t)T}\Sigma^{(m_t)})dt + 2b(t)\delta^T(t)\Sigma^{(m_t)}dW(t)$.

Taking integral from 0 to t on both sides of the above equation gives

$$\begin{aligned}
\hat{V}(t) &= \hat{V}(0) - 2\int_0^t b(s)\delta^T(s)\left(\mathcal{L}_{\hat{\mathcal{G}}}^{(m_s)}\otimes I_n + \bar{C}^T\bar{C}\right)\delta(s)ds\\
&\quad + \int_0^t b^2(s)tr\Big(\Sigma^{(m_s)T}\Sigma^{(m_s)}\Big)ds + 2\int_0^t b(s)\delta^T(s)\Sigma^{(m_s)}dW(s).
\end{aligned}\tag{E.1}$$

By [26], there exists a unique solution $\delta(t)$ for the Markovian switching stochastic differential equation (D.1), which satisfies $E[\sup_{0\le s\le t}\|\delta(s)\|^2]<\infty$, $\forall t\ge 0$. Thus, $\{\int_0^t b(s)\delta^T(s)\Sigma^{(m_s)}dW(s),\mathcal{F}_t\}$ is a martingale, i.e., $E[\int_s^t b(u)\delta^T(u)\Sigma^{(m_u)}dW(u)|\mathcal{F}_s] = 0$, $\forall s\le t$. Denote $\tilde{V}(t) = \hat{V}(t) - \int_0^t b^2(s)tr(\Sigma^{(m_s)T}\Sigma^{(m_s)})ds$. Then, by (E.1) we have $E[\tilde{V}(t) - \tilde{V}(s)|\mathcal{F}_s] = -2E[\int_s^t b(u)\delta^T(u)(\mathcal{L}_{\hat{\mathcal{G}}}^{(m_u)}\otimes I_n + \bar{C}^T\bar{C})\delta(u)du|\mathcal{F}_s]\le 0$.

Therefore, $\{\tilde{V}(t),\mathcal{F}_t,t\ge 0\}$ is a continuous supermartingale. Here, we have used the positive semi-definiteness of $\mathcal{L}_{\hat{\mathcal{G}}}^{(m_t)}\otimes I_n + \bar{C}^T\bar{C}$, $t\ge 0$. Noticing that $\hat{V}(t)\ge 0$, by Assumption B4) we have $\sup_{t\ge 0}E[\tilde{V}(t)^-]\le$

$\int_0^\infty b^2(t)\max_{1\le i\le s} tr(\Sigma^{(i)T}\Sigma^{(i)})dt< \infty$. Then, by continuous supermartingale convergence theorem ([33]) we know that there is a random variable $\tilde{V}^*$, such that $\lim_{t\to\infty}\tilde{V}(t) = \tilde{V}^*$ a.s., which gives $\lim_{t\to\infty}\hat{V}(t) = \tilde{V}^* + \int_0^\infty b^2(t)tr(\Sigma^{(m_t)T}\Sigma^{(m_t)})dt$ a.s.. Furthermore, by (21) we have $\lim_{t\to\infty}\hat{V}(t)=0$ a.s., which implies Theorem 4.2. □

Appendix F
Proof of Theorem 4.3

Proof of Theorem 4.3: We rewrite algorithm (22) in the following stochastic differential equation associated with the parameter estimation error $\delta(t)$:

$$\begin{aligned} d\delta(t) =& -\bar{b}(t)\left[\left(\mathcal{L}_{\mathcal{G}}^{(m_t)}\otimes I_n\right)+\bar{C}^T\bar{C}\right]\delta(t)dt \\ &+(\Delta(t)\otimes I_n)\left[\left(\mathcal{L}_{\mathcal{G}}^{(m_t)}\otimes I_n\right)+\bar{C}^T\bar{C}\right]\delta(t)dt \\ &+(B(t)\otimes I_n)\Sigma^{(m_t)}dW(t) \end{aligned} \tag{F.1}$$

where $\bar{b}(t) = (1/N)\sum_{i=1}^N b_i(t)$, $\Delta(t) = diag\{\Delta_1(t),\cdots,\Delta_N(t)\}$, $\Delta_i(t)=\bar{b}(t)-b_i(t)$.

By (F.1), similar to the derivation of (D.2), we have $dV_i(t)/dt = -\bar{b}(t)(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n+\bar{C}^T\bar{C})V_i(t)-\bar{b}(t)V_i(t)(\mathcal{L}_{\mathcal{G}}^{(i)T}\otimes I_n+\bar{C}^T\bar{C})(\Delta(t)\otimes I_n)(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n+\bar{C}^T\bar{C})V_i(t)+V_i(t)(\mathcal{L}_{\mathcal{G}}^{(i)T}\otimes I_n+\bar{C}^T\bar{C})(\Delta(t)\otimes I_n)+(B(t)\otimes I_n)\Sigma^{(i)}\Sigma^{(i)T}(B(t)\otimes I_n)p_i(t)+\sum_{j=1}^s q_{ji}V_j(t)$, which implies

$$\begin{aligned} \frac{d\varphi(V_i(t))}{dt} =& -\bar{b}(t)\left[\left(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n+\bar{C}^T\bar{C}\right)\right. \\ &\left.\oplus\left(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n+\bar{C}^T\bar{C}\right)\right]\varphi(V_i(t)) \\ &+\left[(\Delta(t)\otimes I_n)\left(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n+\bar{C}^T\bar{C}\right)\right] \\ &\oplus\left[(\Delta(t)\otimes I_n)\left(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n+\bar{C}^T\bar{C}\right)\right]\varphi(V_i(t)) \\ &+(B(t)\otimes I_n)\otimes(B(t)\otimes I_n)\varphi(R_i(t)) \\ &+\left(Q_i^T\otimes I_{(nN)^2}\right)\hat{\varphi}\left(\bar{V}(t)\right). \end{aligned} \tag{F.2}$$

Let

$$\begin{aligned} \Gamma_\Delta^{(i)} =& \left[(\Delta(t)\otimes I_n)\left(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n+\bar{C}^T\bar{C}\right)\right] \\ &\oplus\left[(\Delta(t)\otimes I_n)\left(\mathcal{L}_{\mathcal{G}}^{(i)}\otimes I_n+\bar{C}^T\bar{C}\right)\right], \\ \Gamma_\Delta =& diag\left\{\Gamma_\Delta^{(1)},\cdots,\Gamma_\Delta^{(s)}\right\}. \end{aligned} \tag{F.3}$$

Then, by (D.4) we can transform (F.2) into the following compact form:

$$\begin{aligned} \frac{d\hat{\varphi}\left(\bar{V}(t)\right)}{dt} =& \left(-\bar{b}(t)\Gamma_L-\bar{b}(t)\Gamma_C+Q^T\otimes I_{(nN)^2}\right)\hat{\varphi}\left(\bar{V}(t)\right) \\ &+\Gamma_\Delta\hat{\varphi}\left(\bar{V}(t)\right)+(B(t)\otimes I_n)\otimes(B(t)\otimes I_{ns})\hat{\varphi}\left(\bar{R}(t)\right). \end{aligned} \tag{F.4}$$

Noticing that $\{\mathcal{G}^{(i)}, i\in\mathcal{S}\}$ is a set of balanced graphs, by (F.4) we have

$$\begin{aligned} &\frac{d\hat{\varphi}^T\left(\bar{V}(t)\right)\hat{\varphi}\left(\bar{V}(t)\right)}{dt} \\ &=2\hat{\varphi}^T\left(\bar{V}(t)\right)\left(-\bar{b}(t)\Gamma_{\hat{L}}-\bar{b}(t)\Gamma_C+\hat{Q}\otimes I_{(nN)^2}\right)\hat{\varphi}\left(\bar{V}(t)\right) \\ &\quad+2\hat{\varphi}^T\left(\bar{V}(t)\right)\left[(B(t)\otimes I_n)\otimes(B(t)\otimes I_{ns})\right]\hat{\varphi}\left(\bar{R}(t)\right) \\ &\quad+2\hat{\varphi}^T\left(\bar{V}(t)\right)\Gamma_\Delta^T\hat{\varphi}\left(\bar{V}(t)\right) \end{aligned} \tag{F.5}$$

where $\Gamma_{\hat{L}}$ is the Γ_L with $\mathcal{L}_{\mathcal{G}}^{(i)}$ replaced by $\mathcal{L}_{\hat{\mathcal{G}}}^{(i)}$, $\hat{\mathcal{G}}^{(i)}$ is the mirror graph of $\mathcal{G}^{(i)}$, $\mathcal{L}_{\hat{\mathcal{G}}}^{(i)}$ is the Laplacian matrix of $\hat{\mathcal{G}}^{(i)}$. As the transition rate matrix Q is doubly stochastic, we know that $\hat{Q} = (Q+Q^T)/2$ corresponds to a symmetric irreducible transition rate matrix.

By (25) we have $\Delta_i(t) = o(\bar{b}(t))$, and hence, by the definition of Γ_Δ in (F.3), $\|\Gamma_\Delta\| = o(\bar{b}(t))$. From the definition of Kronecker product, $(B(t)\otimes I_n)\otimes(B(t)\otimes I_{ns})$ can be expressed by $(B(t)\otimes I_n)\otimes(B(t)\otimes I_{ns}) = \bar{b}^2(t)I_{s(nN)^2}-\bar{b}(t)(I_{nN}\otimes(\Delta(t)\otimes I_{ns}))-\bar{b}(t)(\Delta(t)\otimes I_{n^2Ns})+(\Delta(t)\otimes I_n)\otimes(\Delta(t)\otimes I_{ns})$. From this, noticing that $\hat{Q}$ is negative semi-definite, $\Gamma_{\hat{L}}+\Gamma_C-\hat{Q}\otimes I_{(nN)^2}$ is positive definite and $\bar{b}(t)\underset{t\to\infty}{\longrightarrow}0$, by the inequality $2x^Ty\le\epsilon x^Tx+\epsilon^{-1}y^Ty, \forall x,y\in\mathcal{R}^{s(nN)^2}$, $\epsilon>0$, and (F.5), we know that there is $t_0\ge 0$ such that for any $t\ge t_0$

$$\begin{aligned} &\frac{d\hat{\varphi}^T\left(\bar{V}(t)\right)\hat{\varphi}\left(\bar{V}(t)\right)}{dt} \\ &\le -2\bar{b}(t)\hat{\varphi}^T\left(\bar{V}(t)\right)\left(\Gamma_{\hat{L}}+\hat{Q}\otimes I_{(nN)^2}-\Gamma_C\right)\hat{\varphi}\left(\bar{V}(t)\right) \\ &\quad+2\|\Gamma_\Delta\|\cdot\left\|\hat{\varphi}\left(\bar{V}(t)\right)\right\|^2-2\bar{b}(t)\hat{\varphi}^T\left(\bar{V}(t)\right) \\ &\quad\times\left(\Delta(t)\otimes I_{n^2Ns}+I_{nN}\otimes(\Delta(t)\otimes I_{ns})\right)\hat{\varphi}\left(\bar{R}(t)\right) \\ &\quad+2\bar{b}^2(t)\hat{\varphi}^T\left(\bar{V}(t)\right)\hat{\varphi}\left(\bar{R}(t)\right) \\ &\quad+2\hat{\varphi}^T\left(\bar{V}(t)\right)\left[(\Delta(t)\otimes I_n)\otimes(\Delta(t)\otimes I_{ns})\right]\hat{\varphi}\left(\bar{R}(t)\right) \\ &\le -\lambda\bar{b}(t)\left\|\hat{\varphi}\left(\bar{V}(t)\right)\right\|^2+2\bar{b}^2(t)\left\|\hat{\varphi}\left(\bar{R}(t)\right)\right\|^2. \end{aligned} \tag{F.6}$$

where $\lambda > 0$ is the minimal eigenvalue of $\Gamma_{\hat{L}}+\Gamma_C-\hat{Q}\otimes I_{(nN)^2}$. Then, similar to (D.6), by the comparison theorem ([32]), Assumption B1), (23), (24), and (F.6), we have $\lim_{t\to\infty}\|\hat{\varphi}(\bar{V}(t))\|=0$, which implies $\lim_{t\to\infty}\|V(t)\|=0$. Hence, we have $\lim_{t\to\infty}E[\|\delta(t)\|^2]=\lim_{t\to\infty}E[tr(V(t))]=0$.

Below, we prove the almost sure convergence of algorithm (22). By (F.1) we have $d\hat{V}(t) = -\delta^T(t)[(\mathcal{L}_{\mathcal{G}}^{(m_t)T}\otimes I_n+\bar{C}^T\bar{C})(B(t)\otimes I_n)+(B(t)\otimes I_n)(\mathcal{L}_{\mathcal{G}}^{(m_t)}\otimes I_n+\bar{C}^T\bar{C})]\delta(t)dt+tr[(B^2(t)\otimes I_n)\Sigma^{(m_t)}\Sigma^{(m_t)T}]dt+2\delta^T(t)(B(t)\otimes I_n)\Sigma^{(m_t)}dW(t)$. Taking integral from 0 to t on both sides of the above equation leads to

$$\begin{aligned} \hat{V}(t)=&\hat{V}(0) \\ &-\int_0^t\delta^T(s)\Big[\left(\mathcal{L}_{\mathcal{G}}^{(m_s)T}\otimes I_n+\bar{C}^T\bar{C}\right)(B(s)\otimes I_n) \\ &\qquad+(B(s)\otimes I_n)\left(\mathcal{L}_{\mathcal{G}}^{(m_s)}\otimes I_n+\bar{C}^T\bar{C}\right)\Big]\delta(s)ds \\ &+\int_0^t tr\Big[\left(B^2(s)\otimes I_n\right)\Sigma^{(m_s)}\Sigma^{(m_s)T}\Big]ds \\ &+2\int_0^t\delta^T(s)(B(s)\otimes I_n)\Sigma^{(m_s)}dW(s). \end{aligned} \tag{F.7}$$

By [26], there exists a unique solution $\delta(t)$ for the Markovian switching stochastic differential equation (F.1), which satisfies

$$E\left[\sup_{0\le s\le t}\|\delta(s)\|^2\right]<\infty,\quad t\ge 0. \tag{F.8}$$

Thus, $\{\int_0^t \delta^T(s)(B(s) \otimes I_n)\Sigma^{(m_s)}dW(s), \mathcal{F}_t\}$ is a martingale, i.e., $E[\int_s^t \delta^T(u)(B(u) \otimes I_n)\Sigma^{(m_u)}dW(u)|\mathcal{F}_s] = 0$, $\forall s \leq t$. Denote $\tilde{V}(t) = \hat{V}(t) - \int_0^t tr[(B^2(s) \otimes I_n)\Sigma^{(m_s)}\Sigma^{(m_s)T}]ds - \int_0^t \delta^T(s)[(\mathcal{L}_{\mathcal{G}}^{(m_s)T} \otimes I_n + \bar{C}^T\bar{C})(\Delta(s) \otimes I_n) + (\Delta(s) \otimes I_n)(\mathcal{L}_{\mathcal{G}}^{(m_s)} \otimes I_n + \bar{C}^T\bar{C})]\delta(s)ds$. Then, by (F.7) and (F.8) we have $E[\tilde{V}(t) - \tilde{V}(s)|\mathcal{F}_s] = E[-2\int_s^t \bar{b}(u)\delta^T(u)(\mathcal{L}_{\hat{\mathcal{G}}}^{(m_u)} \otimes I_n + \bar{C}^T\bar{C})\delta(u)du|\mathcal{F}_s] \leq 0$. Therefore, $\{\tilde{V}(t)\mathcal{F}(t), t \geq 0\}$ is a continuous supermartingale. Here, we have used the positive semi-definiteness of matrix $\mathcal{L}_{\hat{\mathcal{G}}}^{(m_t)} \otimes I_n + \bar{C}^T\bar{C}$, $t \geq 0$. Noticing that $\hat{V}(t) \geq 0$, by (24), (26), (F.8) and Lyapunov inequality ([24]) we have $\sup_{t\geq 0} E[\tilde{V}(t)^-] \leq \int_0^\infty b^2(t)\max_{1\leq i\leq s} tr[(B^2(t) \otimes I_n)\Sigma^{(i)}\Sigma^{(i)T}]dt + 2\max_{1\leq i\leq s}\|\mathcal{L}_{\mathcal{G}}^{(i)} \otimes I_n + \bar{C}^T\bar{C}\|\sup_{t\geq 0}(E[\|\delta(t)\|^2])^{1/2}\int_0^\infty \max_{1\leq i,j\leq N}|b_i(t) - b_j(t)|dt < \infty$. Then, by continuous supermartingale convergence theorem ([33]), we know that there exists a random variable $\tilde{V}^*$ such that $\lim_{t\to\infty}\tilde{V}(t) = \tilde{V}^*$ a.s., which gives $\lim_{t\to\infty}\hat{V}(t) = \tilde{V}^* + \int_0^\infty tr[(B^2(t) \otimes I_n)\Sigma^{(m_t)}\Sigma^{(m_t)T}]dt + \int_0^\infty \delta^T(t)[(\mathcal{L}_{\mathcal{G}}^{(m_t)T} \otimes I_n + \bar{C}^T\bar{C})(\Delta(t) \otimes I_n) + (\Delta(t) \otimes I_n)(\mathcal{L}_{\mathcal{G}}^{(m_t)} \otimes I_n + \bar{C}^T\bar{C})]\delta(t)dt$. In addition, by (23) one can see that (26) implies (25), which together with (23) and (24) gives $\lim_{t\to\infty} E[\hat{V}(t)] = 0$. Thus, we have $\lim_{t\to\infty}\hat{V}(t) = 0$ a.s., which implies Theorem 4.3. □

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