J Syst Sci Complex (2012) 25: 13–29

ON FORMABILITY OF LINEAR CONTINUOUS-TIME MULTI-AGENT SYSTEMS*

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DOI: 10.1007/s11424-012-0108-3
Received: 11 May 2010 / Revised: 12 July 2010

Abstract This paper is focused on formability of multi-agent systems (MASs). The problem is concerned with the existence of a protocol that has the ability to drive the MAS involved to the desired formation, and thus, is of essential importance in designing formation protocols. Formability of an MAS depends on several key factors: agents' dynamic structures, connectivity topology, properties of the desired formation and the admissible control set. Agents of the MASs considered here are described by a general continuous linear time-invariant (LTI) model. By using the matrix analysis and algebraic graph theory, some necessary and sufficient conditions on formability of LTI-MASs are obtained. These conditions characterize in some sense the relationship of formability, connectivity topology, formation properties and agent dynamics with respect to some typical and widely used admissible protocol sets.

Key words Formability, formability condition, formation protocol, linear time-invariant system, multi-agent systems.

1 Introduction

As one of the most important and fundamental problems of coordinated control, formation control of multi-agent systems (MASs) has recently attracted much attention from the control community[1−4]. For multi-robot systems, Das, et al.[1] used a leader-following approach to design controllers and estimators such that robots can maintain a specified formation while following a given trajectory. Beard, et al.[2] embedded the leader-following, behavioral, and virtual-structure approaches to a unified framework, and investigated the spacecraft formation control problem.

These works on formation control mainly focused on the design of formation protocols and the closed-loop analysis. However, when starting to design a formation protocol, one should first be clear about a problem: Whether or not there exists such a protocol that can drive the

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*This work was supported by the National Nature Science Foundation of China under Grants Nos. 60934006 and 61104136, the Shandong Provincial Natural Science Foundation under Grant No. ZR2010FQ002, the School Foundation of Qufu Normal University under Grant No. XJ200913, and the Scientific Research Foundation of Qufu Normal University.
◇ *This paper was recommended for publication by Editor Jing HAN.*

MAS involved to the desired formation. We call this problem formability of MASs. It is of essential importance in both synthesis and implementation of formation protocols, and thus, worth investigating and solving.

When the dynamic structures of the agents are fixed (e.g., the first-order integral MASs, the second-order integral MASs, etc.), lots of literature[4−6] investigated the impact of connectivity among agents on formation control of MASs. These results, however, all emphasized on the relationship between formability and the connectivity topology. This naturally leads to a question: whether formability of MASs is only affected by the connectivity topology. From the examples given below, it can be seen that with respect to (w.r.t.) a given admissible control set, formability of MASs depends on three key factors: The connectivity among agents, agents' dynamic structures and properties of the desired formation. Thus, it is worth analyzing the joint impact of the above three factors on formability. This paper makes a first step towards this direction.

Linear time-invariant MASs (LTI-MASs) whose agents are described by LTI systems are a class of basic MASs, since in some practical situations, the main features of the agents can be approximately characterized by a linear model. For example, in [7], when injecting the formation of the micro-satellites in the Libration orbit, close to this orbit, a time-dependent linear model is used to describe the dynamics of the micro-satellites. With these observations, we take LTI-MASs as the proper access to the research of formability. Considering that linear state feedback control and output feedback control are two well-known control protocols in liner system theory, we will investigate formability of LTI-MASs w.r.t. the state feedback based and output feedback based protocols, respectively.

Within the LTI-MASs, the connectivity topology is characterized by a digraph. Different from the undirected graph case, here the eigenvalues of the Laplacian matrix may all be complex, since generally speaking, the Laplacian matrix of a digraph is non-symmetric. This brings difficulties to derive formability criteria. To overcome these difficulties, we transfer the formability problem of LTI-MASs to stability analysis of a closed-loop system with the help of some linear transformations. By using the properties of the solutions of Riccati equation, matrix analysis and algebraic graph theory, some necessary and sufficient conditions on formability of LTI-MASs are obtained. These conditions characterize in some sense the relationship of formability, connectivity topology, formation properties and agent dynamics w.r.t. some typical and widely used admissible protocol sets, including linear state feedback based and output feedback based protocols.

The rest of the paper is arranged as follows. In Section 2, some preliminary results on digraphs are reviewed, and a detailed description on MAS formation is given. In Section 3, the problem to be investigated is formulated; conditions on formability of LTI-MASs are presented and used to analyze formability of two typical and widely studied MASs. In Section 4, some concluding remarks and further research topics are discussed.

The following notations will be used throughout this paper. $\mathbb{R}^{m\times n}$ denotes the family of $m\times n$ dimensional real matrices; I_m denotes the m-dimensional identity matrix; $\mathbb{C}$ denotes the field of complex numbers. For a given vector or matrix X, $\|X\|$ denotes its Euclidean norm and $X^{\rm T}$ denotes its transpose; X^{-1} denotes the inverse of an invertible matrix X.

2 Preliminaries

For convenience of description, we introduce the following terms[8]. Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ be a weighted digraph with the vertex set $\mathcal{V}=\{1,2,\cdots,N\}$ and the edge set $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$. In $\mathcal{G}$, the ith vertex represents the ith agent, and a directed edge from i to j is denoted as an ordered pair $(i, j) \in \mathcal{E}$, which means that agent j can directly receive information from agent i. In this case, the vertex i is called the parent vertex and the vertex j is called the child vertex. The set of neighbors of the ith agent is denoted by $N_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements, and $a_{ii} = 0$, $a_{ij} > 0 \Leftrightarrow j \in N_i$. The in-degree of vertex i is defined as $\deg_{in}(i) = \sum_{j=1}^{N} a_{ij}$ and the Laplacian matrix of the weighted digraph $\mathcal{G}$ is defined as $L_{\mathcal{G}} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}\,(\deg_{in}(1), \deg_{in}(2), \cdots, \deg_{in}(N))$.

A directed tree is such a directed graph whose every vertex except the root, which has only children but no parent, has exactly one parent. A spanning tree of a digraph is a directed tree that contains all the vertices of the digraph.

A formation of MASs is a geometric pattern satisfying some predefined inter-agent geometric constraints which the MAS is required to achieve and maintain. MAS formations have broad applications in commercial and military fields. For example, in automated highway systems, to increase the transportation efficiency, the automated vehicles are required to form a line at a desired velocity while keeping a specified distance between vehicles[9]. To provide high resolution images of the earth's environment, man-made satellites and landsats are asked to fly with a specified space formation in low earth orbit[7].

Formation representations of MASs are various. A formation of an MAS can be represented by the distances between agents, the relative vectors between agents and so on. Here, we describe a formation of MASs based on the relative vectors between agents. Explicitly, we use $H = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_N^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{nN}$ to describe a formation consisting of N agents in $\mathbb{R}^n$, where $h_i \in \mathbb{R}^n$ is a formation variable, which can be either the state of the ith agent or output projection onto some space of interest, etc. Consider an MAS consisting of four agents in the plane as an example. As shown in Figures 1–3, $H = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, h_3^{\mathrm{T}}, h_4^{\mathrm{T}})^{\mathrm{T}}$ represents a point, a straight line and a parallelogram in the x-y plane, respectively.

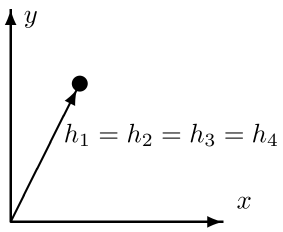

Figure 1 A point formation

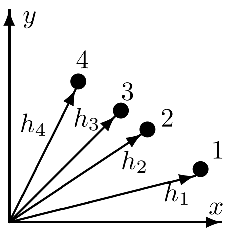

Figure 2 A straight line formation

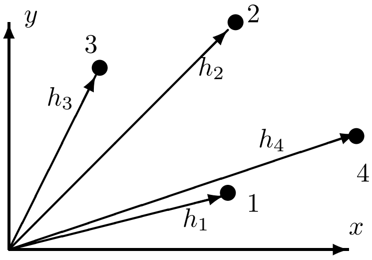

Figure 3 A parallelogram formation

It is worth pointing out that the formation variable might represent physical quantities including position, velocity of an agent or pressure, temperature, altitude, and so on. Particularly, when formation variable represents the position, and $H = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_N^{\mathrm{T}})^{\mathrm{T}}$ represents the desired formation, the formation information $h_i - h_j$ not only provides a desired relative distance between agents i and j but also gives the desired relative orientation, since it has direction as well as magnitude. The formation of multi-robot systems in [1] is denoted by the desired relative distance and orientation† between the following robot and its leader. Therefore, the formation in [1] can be represented by H.

In the following, we will analyze formability of MASs under this representation framework.

3 Formability and Formable Conditions

Consider a system consisting of N agents indexed by $1, 2, \cdots, N$, respectively. The dynamics of the ith agent is described as follows:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \\ y_i(t) = Cx_i(t), \end{cases} \qquad i = 1, 2, \cdots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^r$ and $y_i(t) \in \mathbb{R}^m$ are the state, control input and output of the ith agent, respectively; $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$ and $C \in \mathbb{R}^{m\times n}$ are constant matrices.

The digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is used to describe the topology relationship among the above N agents with

$$\mathcal{V} = \{1, 2, \cdots, N\}, \quad \mathcal{A} = (a_{ij}) \in \mathbb{R}^{N\times N}. \tag{2}$$

The formation protocol for each agent in MASs is distributed[8] and only depends on the information of the agent itself and its neighbors, since each agent has limited capability of collecting information.

Roughly speaking, formability of an MAS is concerned with the existence of a set of distributed formation protocols such that all the states of the MAS asymptotically achieve a pre-specified formation.

Since linear state feedback control and output feedback control are two well-known control protocols in liner system theory, we will investigate in detail formability of LTI-MASs w.r.t. the linear state-feedback-based and output-feedback-based formation protocols, respectively. For the linear state-feedback-based case, we get a necessary and sufficient formability condition, while for the output-feedback-based case, we get a necessary one.

3.1 State-Feedback-Based Formation Protocols

To get a desired formation, the following protocol is often used[10–12]:

$$u_i(t) = \sum_{j=1}^{N} a_{ij} \left[x_j(t) - x_i(t) - (h_j - h_i)\right], \ t \geq 0, \ i = 1, 2, \cdots, N, \tag{3}$$

where $a_{ij} \geq 0$ is defined in (2), and $H = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_N^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{nN}$ represents the desired formation.

In fact, (3) is a consensus-based protocol[11]. This is because when $h_j - h_i = 0$, $\forall i, j = 1, 2, \cdots, N$, H is nothing but a point formation. This implies that the state difference between each pair of agents is required to be zero, or the MASs are required to be consensus.

Protocol (3) is not only distributed but also utilizes the state difference information between an agent and its neighbors. When agent j is a neighbor of agent i, agent i would use the state difference $x_j(t) - x_i(t)$ and the desired state difference $h_j - h_i$ to adjust its state, such that the

†The relative orientation denotes the heading direction of the following robot w.r.t. its leader.

error between these two state differences tends to zero, and hence, MASs achieve the desired formation H.

Now, we analyze formability of MASs w.r.t. the formation protocols of the following form:

$$u_i(t) = G\sum_{j=1}^{N} a_{ij}\left[x_j(t) - x_i(t) - (h_j - h_i)\right],\ t \geq 0,\ i = 1, 2, \cdots, N, \tag{4}$$

where $G \in \mathbb{R}^{r\times n}$ is a weighted constant matrix.

Let $u(t) = (u_1^{\mathrm{T}}(t), u_2^{\mathrm{T}}(t), \cdots, u_N^{\mathrm{T}}(t))^{\mathrm{T}}$. We consider the following admissible protocol set

$$\begin{aligned}\mathcal{U}_s = \Bigg\{ u(t) : [0,\infty) \to \mathbb{R}^{rN} \mid u_i(t) = G\sum_{j=1}^{N} a_{ij}\left[x_j(t) - x_i(t) - (h_j - h_i)\right], \forall\, t \geq 0,\\ i = 1, 2, \cdots, N, G \in \mathbb{R}^{r\times n}, H = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_N^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{nN}\\ \text{denotes a pre-specified formation in } \mathbb{R}^n \Bigg\}.\end{aligned} \tag{5}$$

Remark 1 Obviously, when $G = I_n$, the formation protocol in (5) degenerates to the well-known protocol (3).

The admissible protocol set $\mathcal{U}_s$ covers a class of linear state feedback distributed formation protocols. It naturally arises a question: Under what conditions, the MAS is formable w.r.t. such kind of admissible protocol set. To answer this question, we first give a definition of formability of an MAS w.r.t. $\mathcal{U}_s$.

Definition 1 Let $H = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_N^{\mathrm{T}})^{\mathrm{T}}$ be a specified formation. For the system (1), if there exists a $u(t) \in \mathcal{U}_s$ such that for any initial value $x_i(0)$,

$$\lim_{t\to\infty} \|x_j(t) - x_i(t) - (h_j - h_i)\| = 0, \quad i, j = 1, 2, \cdots, N,$$

then we say that the system (1) is formable w.r.t. $\mathcal{U}_s$.

Remark 2 Unlike the Definition 2.2 in [10] where, to achieve the specified formation, all the agents are required to move along the same trajectory, here only the state difference between each pair of agents is required to converge to the desired values, no matter how it converges.

Remark 3 In [13], the MAS is said to achieve a given formation if the error between the formation and its desired value is below some upper bound. But here the formation error is required to tend to zero as time goes on, in other words, the MAS can achieve the given formation precisely.

Up to now, many existing works[5,7] focus on design of formation protocols of the form (4) for MASs with specific dynamic structures. They are limited to investigating the connectivity topology conditions ensuring that the MAS considered can achieve the desired formation. However, is formability of an MAS w.r.t. $\mathcal{U}_s$ only affected by the connectivity topology? To make this clear, let's first look at the following two examples.

Example 1 Suppose S_1 and S_2 are two systems both consisting of three agents in the plane. $1, 2, 3$ and 1^*, 2^*, 3^* are used to label the agents in S_1 and S_2, respectively. The dynamics of the ith agent is described as follows:

$$\begin{aligned}&S_1: \ \dot{x}_i(t) = u_i(t), \quad i = 1, 2, 3,\\ &S_2: \ \dot{x}_i(t) = (0, 1)^{\mathrm{T}} u_i(t), \quad i = 1^*, 2^*, 3^*.\end{aligned}$$

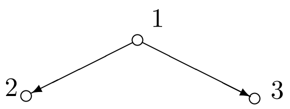

Figure 4 Topology relationship of $\mathcal{G}_1$

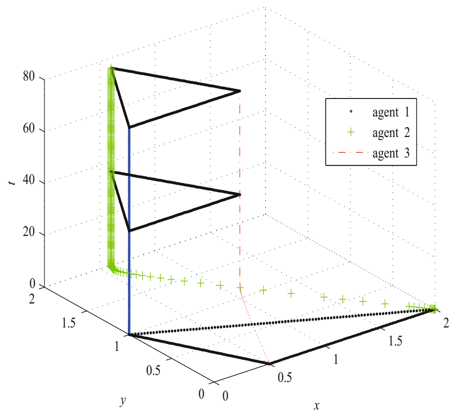

Figure 5 Agents inS_1 evolves to H_1

Assume S_1 and S_2 have the same topology relationship. For simplicity, it is denoted as the digraph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1)$ with $\mathcal{V}_1 = \{1, 2, 3\}$ and $\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. See Figure 4.

$H_1 = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, h_3^{\mathrm{T}})^{\mathrm{T}}$ is used to represent an equilateral triangle with edge length 1, where $h_1 = (-\frac{1}{2}, 0)^{\mathrm{T}}$, $h_2 = (0, \frac{\sqrt{3}}{2})^{\mathrm{T}}$ and $h_3 = (\frac{1}{2}, 0)^{\mathrm{T}}$.

Next, we will choose formation protocols from $\mathcal{U}_s$ for S_1 and S_2, respectively, such that they achieve formation H_1.

For S_1, take $G = I_2$ in $\mathcal{U}_s$. Then, for all initial values, the agents in S_1 achieve asymptotically the desired triangle H_1. As shown in Figure 5, the x-axis and y-axis form the plane and the axis perpendicular to the x-y plane represents time. As time goes on, the simulation state trajectories of the agents in S_1 are presented in Figure 5, which shows that agents in S_1 eventually achieve the equilateral triangle H_1.

However, for S_2, no matter how to choose the gain matrix $G = (g_1, g_2) \in \mathbb{R}^{1\times 2}$, the distances between any two different agents will either diverge or converge to a constant value different from 1. In other words, S_2 cannot achieve formation H_1. See Figure 6.

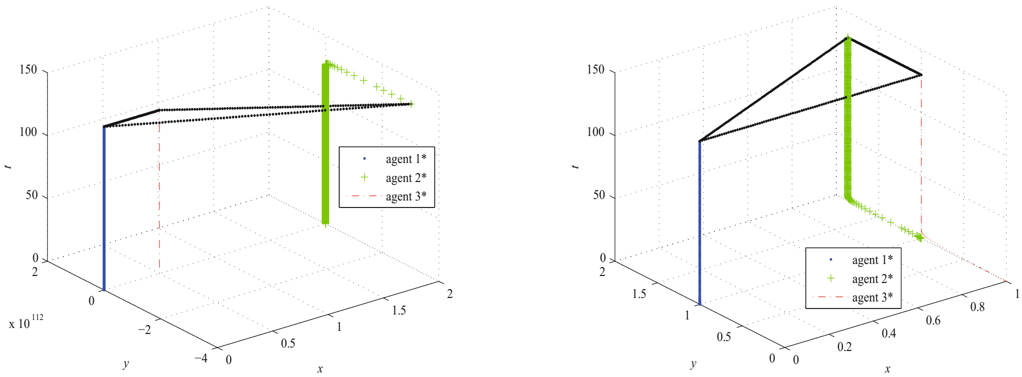

Figure 6 The state trajectories of agents in S_2

From Example 1 it can be seen that even under the same connectivity topology, because of the difference of the dynamic structures, the formabilities of S_1 and S_2 are different. Thus, formability of MASs w.r.t. $\mathcal{U}_s$ is affected not only by the connectivity topology but also by the dynamic properties of the agents.

Example 2 Consider a system consisting of five agents indexed by $1, 2, \cdots, 5$, respectively. The dynamics of the ith agent is described as follows:

$$\dot{x}_i(t) = -x_i(t) + (0,1)^{\mathrm{T}} u_i(t),\ i = 1, 2, \cdots, 5, \tag{6}$$

where $x_i(t) \in \mathbb{R}^2$ and $u_i(t) \in \mathbb{R}$ are the position and control input of the ith agent, respectively.

The topology relationship among the above agents is characterized by the digraph $\mathcal{G}^* = (\mathcal{V}^*, \mathcal{E}^*, \mathcal{A}^*)$ as shown in Figure 7, with $\mathcal{V}^* = \{1, 2, \cdots, 5\}$, $\mathcal{E}^* = \{(2,1),\ (2,5),\ (3,2), (3,4), (5,3)\}$, and

$$\mathcal{A}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}.$$

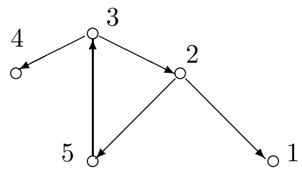

Figure 7 Topology relationship of digraph $\mathcal{G}^*$

Our task is to choose two protocols from $\mathcal{U}_s$ such that one makes the system (6) achieve a point formation, and the other makes the system achieve formation $H_2 = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_5^{\mathrm{T}})^{\mathrm{T}}$, where $h_1 = (0,1)^{\mathrm{T}}$, $h_2 = (1,1)^{\mathrm{T}}$, $h_3 = (2,1)^{\mathrm{T}}$, $h_4 = (3,1)^{\mathrm{T}}$ and $h_5 = (4,1)^{\mathrm{T}}$. Obviously, H_2 is a straight line parallel to the x-axis. See Figure 8.

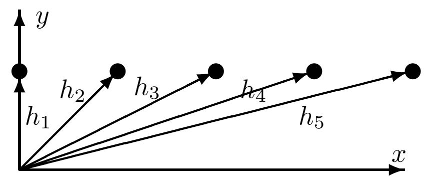

Figure 8 A straight line formation $H_2 = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_5^{\mathrm{T}})^{\mathrm{T}}$

To achieve the point formation, it suffices to take $G = (0, \sqrt{2} - 1) \in \mathbb{R}^{1\times 2}$ in $\mathcal{U}_s$. Then, for all initial positions, the five agents achieve the point formation. The simulation trajectories of the five agents are shown in Figure 9.

For formation H_2, it can be found that no matter how to choose $G = (g_1, g_2) \in \mathbb{R}^{1\times 2}$, the distances between each pair of the five agents are either divergent to infinity or convergent to different values as time goes on. In other words, formation H_2 is not achievable. See Figure 10.

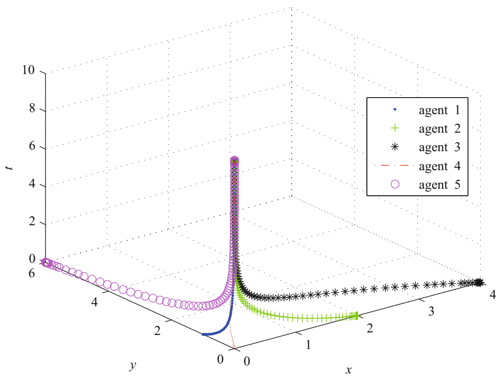

Figure 9 Five agents evolve to the point formation

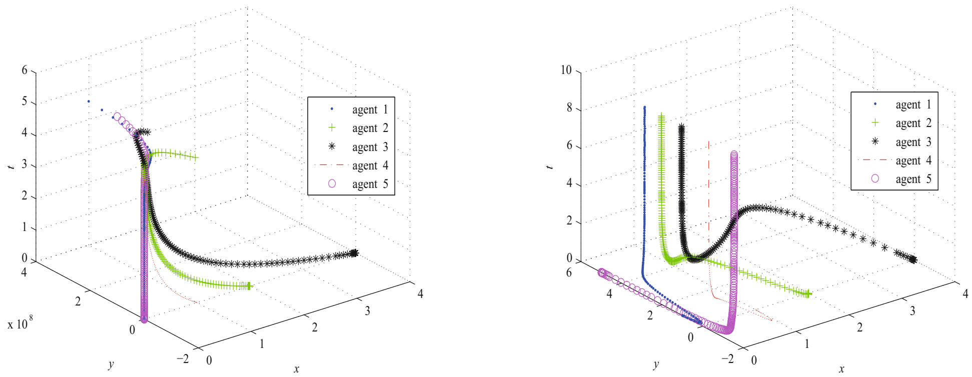

Figure 10 The state trajectories of five agents in the x-y plane

From Example 2, one can see that under the same connectivity conditions, the system (6) can achieve the point formation, but cannot achieve formation H_2. Hence, formability of MASs w.r.t. $\mathcal{U}_s$ depends on the properties of the desired formation, too.

In summary, formability of MASs w.r.t. $\mathcal{U}_s$ is influenced by not only the connectivity topology but also the agents' dynamic structures and the properties of the formations to achieve. A natural question is: What is the relationship between formability and the connectivity topology, the properties of the formation and the agents' dynamics? The following theorem will give an answer. Unless otherwise stated, in this paper, formation $H = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_N^{\mathrm{T}})^{\mathrm{T}}$ does not represent a point formation.

Theorem 1 *System* (1) *is formable w.r.t.* $\mathcal{U}_s$ *if and only if* (A, B) *is stabilizable,* $A(h_i - h_j) = 0$, $\forall i, j = 1, 2, \cdots, N$, *and the topology* $\mathcal{G}$ *must have a spanning tree.*

Proof See the Appendix. ▮

Remark 4 From Theorem 1 one can see, to ensure formability of MASs, not only the connectivity topology of the MAS is required to have a spanning tree, but also the agents are required to be stabilizable and the formation must satisfy some constraints: $A(h_i - h_j) = 0$, $\forall i, j = 1, 2, \cdots, N$. Apparently, the former is a requirement on connectivity topology, while the latter is on the properties of the agents' dynamics and the desired formation.

By the property of the point formation, when $H = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_N^{\mathrm{T}})^{\mathrm{T}}$ represents a point, we have $h_i - h_j = 0$, $\forall i, j = 1, 2, \cdots, N$. Hence, similar to Theorem 1 we have the following result.

Corollary 1 *System* (1) *is formable to a point formation w.r.t.* $\mathcal{U}_s$ *if and only if* (A, B) *is stabilizable, and the topology* $\mathcal{G}$ *must have a spanning tree when the eigenvalues of* A *are not all in the open left half plane.*

Proof The proof is similar to that of Theorem 1. The only thing we need to note is that in the proof of the sufficiency when the eigenvalues of A are all in the open left half plane, the system (1) is naturally formable w.r.t. $\mathcal{U}_s$, since in this case, by simply taking $G = 0$ in (5), one can get that x_i $(i = 1, 2, \cdots, N)$ converges to zero exponentially, and hence, $\lim\limits_{t\to\infty} \|x_j(t) - x_i(t)\| = 0$, $i, j = 1, 2, \cdots, N$. ∎

Remark 5 Corollary 1 is consistent with Theorem 2 in [14]. Actually, by Definition 1, one can see that the state differences between different agents are required to be zero while the system (1) is said to be formable to a point formation. This is equivalent to requiring the system (1) is consensusable. More details on consensusability of MASs are referred to [14].

From the above analysis, one can see that agents' dynamic structures, connectivity among agents and the properties of the desired formations are key to formability of LTI-MASs. Thus, the research on formability of LTI-MASs essentially comes down to the study on the above three factors.

3.2 Formability of Two Typical MASs

In this section, we use the conditions obtained in Section 3.1 to analyze formability of two typical MASs widely studied in the literature. It can be seen that the conclusions in the literature are special cases of our results.

3.2.1 First-Order Integral MASs

Consider the first-order integral MASs consisting of N agents. The dynamics of the ith agent is described as follows:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \cdots, N, \tag{7}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ are the state and control input of the ith agent, respectively.

There are lots of works on formation control of the model (7)[6,11−12]. To get a desired formation $H = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_N^{\mathrm{T}})^{\mathrm{T}}$, [11] and [12] used the following formation protocol

$$u_i(t) = \sum_{j=1}^{N} a_{ij} \left[x_j(t) - x_i(t) - (h_j - h_i)\right], \ t \geq 0, \ i = 1, 2, \cdots, N.$$

In fact, the system (7) can be regarded as a special case of the system (1) with $A = 0$ and $B = C = I_n$. Obviously, $A(h_i - h_j) = 0$, $\forall i, j = 1, 2, \cdots, N$, and (A, B) is stabilizable. Thus, by Theorem 1 we have the following result.

Corollary 2 *System* (7) *is formable w.r.t.* $\mathcal{U}_s$ *if and only if the topology* $\mathcal{G}$ *has a spanning tree.*

From this one can see that for the first-order integral MAS (7), formability w.r.t. $\mathcal{U}_s$ can be described only by the connectivity topology. Thus, to ensure the existence of a formation protocol driving the system to a desired formation, it is sufficient to verify whether or not the topology has a spanning tree.

3.2.2 Generalized Second-Order Integral MASs

Consider the generalized second-order integral MASs. The dynamics of the ith agent is described as:

$$\dot{x}_i(t) = A_{veh}x_i(t) + B_{veh}u_i(t), \quad i = 1, 2, \cdots, N, \tag{8}$$

where $x_i(t) \in \mathbb{R}^{2n}$ represents n position variables for agent i and their derivatives, $u_i(t) \in \mathbb{R}^n$ represents the control input, and

$$A_{veh} = \text{diag}\left(\begin{pmatrix} 0 & 1 \\ 0 & a_{22}^1 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ 0 & a_{22}^n \end{pmatrix}\right), \quad B_{veh} = I_n \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{9}$$

where $\otimes$ denotes the Kronecker product.

The above model is widely studied[10,15]. Lafferriere, et al.[10] pointed out forming a given formation was equivalent to making the position vector achieve the pre-specified value, and the velocity of all the agents be the same. With respect to formation $H_s = (h_{s1}^{\mathrm{T}}, h_{s2}^{\mathrm{T}}, \cdots, h_{sN}^{\mathrm{T}})^{\mathrm{T}} \otimes (1,0)^{\mathrm{T}} \in \mathbb{R}^{2nN}$, the formation protocol of the following form was proposed

$$u_i(t) = F_{veh} \sum_{j \in N_i} \left\{\left[x_j(t) - h_{sj} \otimes (1,0)^{\mathrm{T}}\right] - \left[x_i(t) - h_{si} \otimes (1,0)^{\mathrm{T}}\right]\right\}, \quad i = 1, 2, \cdots, N, \tag{10}$$

where $F_{veh} \in \mathbb{R}^{n \times 2n}$ was a constant matrix. Lafferriere, et al.[10] investigated the influence of connectivity topology on the existence of a gain matrix F_{veh}. They proved that under the protocol (10), the connectivity topology having a spanning tree is the necessary and sufficient condition for existing a matrix $F_{veh} \in \mathbb{R}^{n \times 2n}$ making the system (8) achieve the specified formation H_s.

In fact, by (9) we know that (A_{veh}, B_{veh}) is stabilizable, and $A_{veh}\left[(h_{si} - h_{sj}) \otimes (1,0)^{\mathrm{T}}\right] = 0$, $\forall i, j = 1, 2, \cdots, N$. Thus, by Theorem 1 we can get the following similar result as in [10].

Corollary 3[10] *System* (8) *is formable w.r.t.* $\mathcal{U}_s$ *if and only if the topology has a spanning tree.*

From the above analysis, one can see that for the system (8) to be formable w.r.t. $\mathcal{U}_s$, the requirements on the agents' dynamics and formation properties are naturally met. Thus, connectivity among agents becomes key to formability.

3.3 Output-Feedback-Based Formation Protocols

In practice, due to economic costs or constraints on measurement, only the output (rather than state) information of the agents is available. Thus, it would be more meaningful to consider the formation protocol based on the output feedback.

We consider the following output-feedback-based admissible protocol set

$$\mathcal{U}_o = \Big\{u(t) : [0, \infty) \to \mathbb{R}^{rN} \mid u_i(t) = K \sum_{j=1}^{N} a_{ij}\left[y_j(t) - y_i(t) - C\left(h_j - h_i\right)\right], \forall\, t \geq 0,$$
$$i = 1, 2, \cdots, N, K \in \mathbb{R}^{r \times m}, H = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_N^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{nN}$$
$$\text{denotes a pre-specified formation in } \mathbb{R}^n\Big\}. \tag{11}$$

Similarly to the state feedback case, we first give the definition of formability of an MAS w.r.t. $\mathcal{U}_o$.

Definition 2 Let $H = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \cdots, h_N^{\mathrm{T}})^{\mathrm{T}}$ be a specified formation. For the system (1), if there exists a $u(t) \in \mathcal{U}_o$ such that for any initial value $x_i(0)$,

$$\lim_{t\to\infty} \|x_j(t) - x_i(t) - (h_j - h_i)\| = 0, \ i, j = 1, 2, \cdots, N,$$

then we say the system (1) is formable w.r.t. $\mathcal{U}_o$.

Next, we investigate the formable conditions of MASs w.r.t. $\mathcal{U}_o$. Recalling the analysis in the state feedback case, we know that formability of MASs w.r.t. $\mathcal{U}_s$ depends on agents' dynamic structures, connectivity topology and the properties of the desired formation. A natural question is, in the output feedback case, does formability of MASs w.r.t. $\mathcal{U}_o$ still depend on the above three factors? To answer this, let us first consider the following example.

Example 3 Consider two MASs, S_3 and S_4, both consisting of four agents in the plane indexed by $1, 2, 3, 4$, respectively. The dynamics of the agent are described as

$$\begin{aligned} S_3: \ \dot{x}_i(t) &= -\, u_i(t) \\ y_i(t) &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x_i(t), \qquad i = 1, 2, 3, 4, \\ S_4: \ \dot{x}_j(t) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x_j(t) + u_j(t) \\ y_j(t) &= (0, 1) x_j(t), \qquad j = 1, 2, 3, 4. \end{aligned}$$

Suppose S_3 and S_4 have the same topology relationship. It is described by the digraph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ with $\bar{\mathcal{V}} = \{1, 2, 3, 4\}$ and $\bar{\mathcal{A}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & 0 \end{pmatrix}$. See Figure 11.

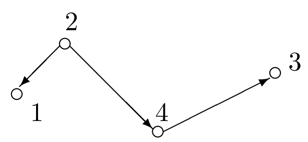

Figure 11 Topology relationship of $\bar{\mathcal{G}}$

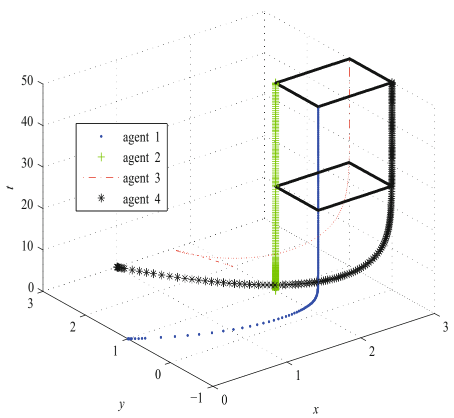

Figure 12 Agents in S_3 evolve to H_3

$H_3 = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, h_3^{\mathrm{T}}, h_4^{\mathrm{T}})^{\mathrm{T}}$ is used to describe a square with edge length 1, where $h_1 = (1, 1)^{\mathrm{T}}$, $h_2 = (1, 2)^{\mathrm{T}}$, $h_3 = (2, 2)^{\mathrm{T}}$ and $h_4 = (2, 1)^{\mathrm{T}}$.

Our objective is to find formation protocols from $\mathcal{U}_o$ for S_3 and S_4, respectively, such that they achieve formation H_3.

For S_3, take $K = -I_2$ in $\mathcal{U}_o$. Then, under this control, the state trajectories of four agents in S_3 are shown in Figure 12. We can clearly see that agents in S_3 in the x-y plane achieve formation H_3 as time goes on.

However, for S_4, no matter how to choose $K = (k_1, k_2)^{\mathrm{T}} \in \mathbb{R}^{2\times 1}$, distances between any two different agents will either diverge or converge to a constant value different from 1. The simulation state trajectories of agents in S_4 are shown in Figure 13. Namely, no control protocols in $\mathcal{U}_o$ can drive S_4 to formation H_3.

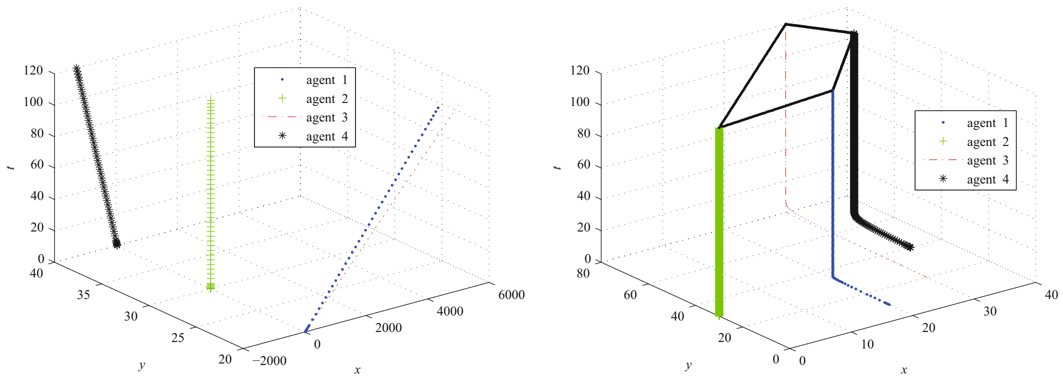

Figure 13 The state trajectories of agents in S_4

From the analysis above, it can be seen that due to the difference of dynamic structures, even under the same connectivity topology, the formabilities of S_3 and S_4 w.r.t. $\mathcal{U}_o$ are different. Therefore, besides the connectivity topology, formability of MASs w.r.t. $\mathcal{U}_o$ is related to agents' dynamic structures.

However, whether formability of MASs w.r.t. $\mathcal{U}_o$ is only dependent on connectivity topology and agents' dynamic structures. To clarify this, we investigate the following case.

Consider formation $H_4 = (h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, h_3^{\mathrm{T}}, h_4^{\mathrm{T}})^{\mathrm{T}}$ in the plane, where $h_1 = (1,1)^{\mathrm{T}}$, $h_2 = (1,2)^{\mathrm{T}}$, $h_3 = (1,3)^{\mathrm{T}}$ and $h_4 = (1,4)^{\mathrm{T}}$. Obviously, it is a straight line parallel to the y-axis. See Figure 14.

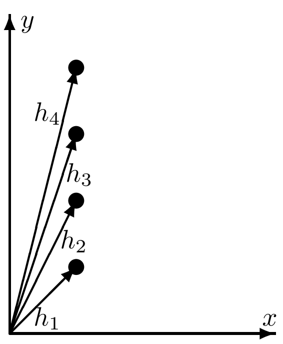

Figure 14 A straight line formation H_4

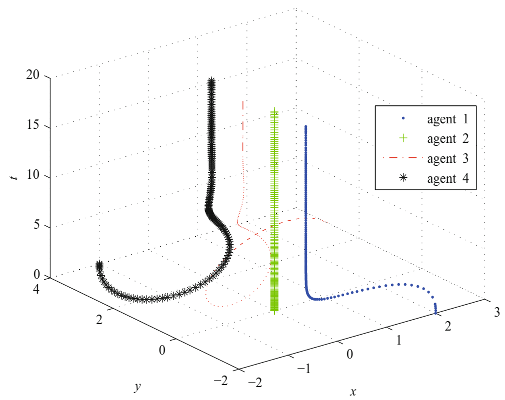

Figure 15 Agents in S_4 evolve to H_4

For S_4, the topology digraph is the same as the one in Figure 11. Take $K = (4, 4)^{\mathrm{T}} \in \mathbb{R}^{2\times 1}$ in $\mathcal{U}_o$. Then, under this control, agents in S_4 achieve the straight line formation H_4 as time goes on. In other words, S_4 can achieve formation H_4 based on the formation protocols in $\mathcal{U}_o$. The simulation trajectories of agents in S_4 are shown in Figure 15.

Noticing that under the same connectivity topology, S_4 cannot achieve the square formation H_3. Consequently, formability of MASs w.r.t. $\mathcal{U}_o$ depends on not only connectivity topology and agents' dynamic structures but also the properties of the desired formations.

Below, we give a necessary formability condition w.r.t. $\mathcal{U}_o$, which depends on agents' dynamic structures, connectivity topology and the formation's properties.

Theorem 2 *If the system* (1) *is formable w.r.t* $\mathcal{U}_o$*, then* (A, B, C) *is stabilizable and detectable,* $A(h_i - h_j) = 0, \forall i, j = 1, 2, \cdots, N$*, and the topology* $\mathcal{G}$ *must have a spanning tree.*

Proof By Definition 2, if the system (1) is formable w.r.t. $\mathcal{U}_o$, then there exist a matrix $K \in \mathbb{R}^{r\times m}$ and formation protocols

$$u_i(t) = K\sum_{j=1}^{N} a_{ij}\left[y_j(t) - y_i(t) - C\left(h_j - h_i\right)\right],\ i = 1, 2, \cdots, N,$$

such that for any $i, j = 1, 2, \cdots, N$,

$$\|x_j(t) - x_i(t) - (h_j - h_i)\| \to 0,\ t \to \infty.$$

Let $\delta_i(t) = x_i(t) - x_1(t) - (h_i - h_1)$, $\forall i = 2, 3, \cdots, N$. Then, the system (1) is formable w.r.t. $\mathcal{U}_o$ is equivalent to

$$\|\delta_i(t)\| \to 0,\ t \to \infty,\ \forall i = 2, 3, \cdots, N. \tag{12}$$

Notice that

$$\dot{\delta}_i(t) = A\delta_i(t) + BKC\left[\sum_{j=1}^{N}(a_{ij} - a_{1j})\delta_j(t) - deg_{in}(i)\delta_i(t)\right] + A(h_i - h_1),\ i = 2, 3, \cdots, N.$$

Let $\delta(t) = \left(\delta_2^{\mathrm{T}}(t), \delta_3^{\mathrm{T}}(t), \cdots, \delta_N^{\mathrm{T}}(t)\right)^{\mathrm{T}}$. Then,

$$\dot{\delta}(t) = \left[I_{N-1} \otimes A - \left(L_{22} + \mathbf{1}_{N-1}\cdot\alpha^{\mathrm{T}}\right)\otimes BKC\right]\delta(t) + (I_{N-1}\otimes A)\begin{pmatrix} h_2 - h_1 \\ h_3 - h_1 \\ \vdots \\ h_N - h_1 \end{pmatrix}, \tag{13}$$

where $\mathbf{1}_{N-1}$ denotes an $N-1$ dimensional column vector with all components 1,

$$\alpha = (a_{12}, a_{13}, \cdots, a_{1N})^{\mathrm{T}},$$

$$L_{22} = \begin{pmatrix} deg_{in}(2) & -a_{23} & \cdots & -a_{2N} \\ -a_{32} & deg_{in}(3) & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N2} & -a_{N3} & \cdots & deg_{in}(N) \end{pmatrix}.$$

Thus, by (12), all the eigenvalues of $I_{N-1}\otimes A - \left(L_{22} + \mathbf{1}_{N-1}\cdot\alpha^{\mathrm{T}}\right)\otimes BKC$ are in the open left half plane, and

$$(I_{N-1}\otimes A)\begin{pmatrix} h_2 - h_1 \\ h_3 - h_1 \\ \vdots \\ h_N - h_1 \end{pmatrix} = 0,$$

i.e., $A(h_i - h_j) = 0$, $\forall i, j = 1, 2, \cdots, N$.

Take $S = \left[\begin{smallmatrix} 1 & 0 \\ \mathbf{1}_{N-1} & I_{N-1} \end{smallmatrix}\right]$. By the definition of the Laplacian matrix we have

$$S^{-1} L_{\mathcal{G}} S = \left[\begin{array}{cc} 0 & -\alpha^{\mathrm{T}} \\ 0 & L_{22} + \mathbf{1}_{N-1} \cdot \alpha^{\mathrm{T}} \end{array}\right].$$

Assume that $\lambda_1 = 0, \lambda_2, \cdots, \lambda_N$ are the eigenvalues of Laplacian matrix (where $\lambda_2, \lambda_3, \cdots, \lambda_N$ may be not different). Then, $\lambda_2, \lambda_3, \cdots, \lambda_N$ are all the eigenvalues of $L_{22} + \mathbf{1}_{N-1} \cdot \alpha^{\mathrm{T}}$, and there exists an invertible matrix T such that $L_{22} + \mathbf{1}_{N-1} \cdot \alpha^{\mathrm{T}}$ is similar to a Jordan canonical matrix, i.e.

$$T^{-1}\left(L_{22} + \mathbf{1}_{N-1} \cdot \alpha^{\mathrm{T}}\right) T = J = \mathrm{diag}\left(J_1, J_2, \ \cdots \ , J_s\right),$$

where $J_1, J_2, \ \cdots, J_s$ are upper triangular Jordan blocks, whose principal diagonal elements consist of $\lambda_2, \lambda_3, \cdots, \lambda_N$. Therefore,

$$(T \otimes I_n)^{-1}\left[I_{N-1} \otimes A - \left(L_{22} + \mathbf{1}_{N-1} \cdot \alpha^{\mathrm{T}}\right) \otimes BKC\right](T \otimes I_n) = I_{N-1} \otimes A - J \otimes BKC. \quad (14)$$

This together with the properties of Kronecker product[16] implies that all the eigenvalues of $A - \lambda_i BKC$, $i = 2, 3, \cdots, N$ are in the open left half plane.

Now, we prove (A, B, C) is stabilizable and detectable.

In fact, if at least one of λ_i, $i = 2, 3, \cdots, N$, is real, say λ_2, then (A, B, C) is stabilizable and detectable since all the eigenvalues of $A - \lambda_2 BKC$ are in the open left half plane. If all λ_i, $i = 2, 3, \cdots, N$ are complex numbers, that is, none of their imaginary parts are zero, then noticing that $L_{22} + \mathbf{1}_{N-1} \cdot \alpha^{\mathrm{T}}$ is a real matrix, the eigenvalues will appear in conjugate pair form. Without loss of generality, we assume that λ_2 and λ_3 are a pair of conjugate roots with $\lambda_2 = e + \jmath d$ and $\lambda_3 = e - \jmath d$, $(\jmath^2 = -1)$. Noticing that

$$\begin{aligned} & \left|\begin{array}{cc} \lambda I_n - (A - eBKC) & -dBKC \\ dBKC & \lambda I_n - (A - eBKC) \end{array}\right| \\ = & \ |\ \lambda I_n - (A - \lambda_2 BKC)\ | \cdot |\ \lambda I_n - (A - \lambda_3 BKC)\ |, \quad \forall\ \lambda \in \mathbb{C}, \end{aligned}$$

we can see that all the eigenvalues of

$$\left[\begin{array}{cc} A - eBKC & dBKC \\ -dBKC & A - eBKC \end{array}\right]$$

are in the open left half plane, since all the eigenvalues of $A - \lambda_2 BKC$ and $A - \lambda_3 BKC$ are in the open left half plane. This together with

$$\begin{aligned} \left[\begin{array}{cc} A - eBKC & dBKC \\ -dBKC & A - eBKC \end{array}\right] &= \left[\begin{array}{cc} A & 0 \\ 0 & A \end{array}\right] + \left[\begin{array}{cc} B & 0 \\ 0 & B \end{array}\right]\left[\begin{array}{cc} -eKC & dKC \\ -dKC & -eKC \end{array}\right] \\ &= \left[\begin{array}{cc} A & 0 \\ 0 & A \end{array}\right] + \left[\begin{array}{cc} -eBK & dBK \\ -dBK & -eBK \end{array}\right]\left[\begin{array}{cc} C & 0 \\ 0 & C \end{array}\right] \end{aligned}$$

implies that $(\left[\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right], \left[\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right])$ is stabilizable and $(\left[\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right], \left[\begin{smallmatrix} C & 0 \\ 0 & C \end{smallmatrix}\right])$ is detectable. Thus,

$$\mathrm{Rank}\left(\begin{array}{cccc} sI_n - A & 0 & B & 0 \\ 0 & sI_n - A & 0 & B \end{array}\right) = 2n, \ \forall\ s \in \mathbb{C}, \mathrm{Re}s \geq 0,$$

where $\mathrm{Rank}(X)$ denotes the rank of matrix X, and $\mathrm{Re}s$ denotes the real part of s. Combining this with

$$\text{Rank}\begin{pmatrix} sI_n - A & 0 & B & 0 \\ 0 & sI_n - A & 0 & B \end{pmatrix} = 2\text{Rank}(sI_n - A \quad B),\ \forall\ s \in \mathbb{C}$$

gives

$$\text{Rank}(sI_n - A \quad B) = n, \forall\ s \in \mathbb{C},\ \text{Re}s \geq 0.$$

Or equivalently, (A, B) is stabilizable.

Similarly, we can show the detectability of (A, C). Thus, (A, B, C) is stabilizable and detectable.

Below we only need to prove that the topology $\mathcal{G}$ must have a spanning tree. In fact, from Corollary 1 of [17], we know that for all $i = 2, 3, \cdots, N$, $\lambda_i = 0$ or $\text{Re}\lambda_i > 0$. Since $H = (h_1^{\text{T}}, h_2^{\text{T}}, \cdots, h_N^{\text{T}})^{\text{T}}$ is not a point formation, not all $h_i - h_j = 0$, and hence, $A(h_i - h_j) = 0$ implies that A must have a zero eigenvalue. Then, for all $i = 2, 3, \cdots, N$, $\lambda_i \neq 0$ must be true, since otherwise, there would be an $i \in \{2, 3, \cdots, N\}$ such that $\lambda_i = 0$, which in turn implies that all the eigenvalues of $A = A - \lambda_i BKC$ are in the open left half plane. This is a contradiction. Thus, by Corollary 2 of [17], $\mathcal{G}$ must have a spanning tree. ▮

Remark 6 From the proof of Theorem 2, it can be seen that by introducing a linear transformation, the formability w.r.t. $\mathcal{U}_o$ of the system (1) can be converted into a stability problem of the error Equation (13), which actually is equivalent to whether there exists a gain matrix $K \in \mathbb{R}^{r \times m}$ such that all the eigenvalues of $A - \lambda_i BKC$, $i = 2, 3, \cdots, N$ are in the open left half plane. The latter is essentially a static output feedback stability problem. Usually, only numerical solution to this problem can be given, and the analytical solution is involved in a Lapunov inequality and algebraic Riccati inequality[18].

When $H = (h_1^{\text{T}}, h_2^{\text{T}}, \cdots, h_N^{\text{T}})^{\text{T}}$ represents a point, similar to Theorem 2, we have the following result.

Corollary 4 *If the system* (1) *is formable to a point formation w.r.t.* $\mathcal{U}_o$*, then* (A, B, C) *is stabilizable and detectable, and if in addition, the eigenvalues of* A *are not all in the open left half plane, then the topology* $\mathcal{G}$ *must have a spanning tree.*

From the above analysis, one can see that agent dynamic structures, connectivity among agents and the desired formation properties are also crucial to formability of LTI-MASs w.r.t $\mathcal{U}_o$. To study formability of LTI-MASs, we need to consider the joint impact of the above three aspects.

4 Conclusions

This paper studies formability of LTI-MASs. Different from the existing works, here the joint impact of the connectivity topology, the dynamics of the agents and the properties of the desired formations on formability is considered. By using matrix and graph theory as well as the linear system theory, some necessary and sufficient conditions on formability are provided.

It is worth pointing out that this paper is only a first step on formability of LTI-MASs. Many important issues are still open and worth investigating, including the impact of admissible protocol sets on formability, necessary and sufficient formability conditions in the case where the connectivity topology is time-varying, or the dynamic structures of the agents are different or nonlinear.

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Appendix: Proof of Theorem 1

Proof Necessity was shown in Theorem 2. Next, we only need to prove the sufficiency.

Since the connectivity topology $\mathcal{G}$ must have a spanning tree, by Corollary 1 and Corollary 2 of [17], we get $\mathrm{Re}\{\lambda_i\} > 0$, $i = 2, 3, \cdots, N$.

Denote

$$\delta \triangleq \min_{2\le i\le N}\{\mathrm{Re}(\lambda_i)\}, \tag{15}$$

where $\min\{S\}$ denotes the minimum one in S.

From the stabilization of (A, B) we know that the following Riccati equation

$$A^{\mathrm{T}}P + PA - PBB^{\mathrm{T}}P + I_n = 0 \tag{16}$$

has a unique nonnegative definite solution $P = P^{\mathrm{T}}$.

Take $G = \max\{1, \delta^{-1}\}B^{\mathrm{T}}P$, where $\max\{S\}$ denotes the maximum one in S. Then, by (15) and

$$A - \lambda_i BG = A - \lambda_i \max\{1, \delta^{-1}\}BB^{\mathrm{T}}P, \quad i = 2, 3, \cdots, N,$$

we know that all the eigenvalues of $A - \lambda_i BG$, $i = 2, 3, \cdots, N$, are in the open left half plane, since for any $\sigma \ge 1$ and $\omega \in \mathbb{R}$, all the eigenvalues of $A - (\sigma + \jmath\omega)BB^{\mathrm{T}}P$ ($\jmath^2 = -1$) are in the open left half plane[19]. Noticing that when $C = I_n$ the admissible protocol set (11) degenerates to the state feedback admissible protocol set (5), by (13) and (14) one can get for all $i = 2, 3, \cdots, N$, $\|\delta_i(t)\| \to 0$, $t \to 0$. Thus, by Definition 1, the system (1) is formable w.r.t. $\mathcal{U}_s$. ∎