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Sampled-data based average consensus with measurement noises: convergence analysis and uncertainty principle

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In this paper, sampled-data based average-consensus control is considered for networks consisting of continuous-time first-order integrator agents in a noisy distributed communication environment. The impact of the sampling size and the number of network nodes on the system performances is analyzed. The control input of each agent can only use information measured at the sampling instants from its neighborhood rather than the complete continuous process, and the measurements of its neighbors' states are corrupted by random noises. By probability limit theory and the property of graph Laplacian matrix, it is shown that for a connected network, the static mean square error between the individual state and the average of the initial states of all agents can be made arbitrarily small, provided the sampling size is sufficiently small. Furthermore, by properly choosing the consensus gains, almost sure consensus can be achieved. It is worth pointing out that an uncertainty principle of Gaussian networks is obtained, which implies that in the case of white Gaussian noises, no matter what the sampling size is, the product of the steady-state and transient performance indices is always equal to or larger than a constant depending on the noise intensity, network topology and the number of network nodes.

multi-agent systems, average consensus, stochastic systems, sampled-data based control, distributed stochastic approximation, uncertainty principle

1 Introduction

In recent years, distributed coordination and self-organization of multi-agent systems have attracted a lot of attention from the researchers in the control community. On the one hand, due to the quick development of technologies of communication, robots and micro sensors, many problems have emerged in the control and filtering of distributed systems, such as swarming[1], flocking[2], formation control[3], distributed computation[4] and distributed filtering and information fusion of sensor networks[5,6]. Other than centralized systems, control of distributed systems comes down to the design of network topologies and communication protocols, which needs a compact combination of the control and communication theory. On the other hand, many self-organizing phenomena revealed in biological and social economic

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systems[7−10] need a rigorous theoretical analysis. In self-organization, a stable collective behavior often emerges from local interaction of individuals. At present, distributed coordination and self-organization have become two closely related popular topics in the research of complex systems. In fact, they are two sides of the same coin, which are from the perspectives of control and modeling, respectively.

For distributed coordination, it is a fundamental requirement that without central control stations, the whole group can also achieve consensus on the shared data only through local communication. And for self-organization, network synchronization is the most elementary phenomenon. Therefore, consensus or agreement has become a common problem for distributed coordination and self-organization of multi-agent systems. Consensus control generally means to design a network protocol for the communication of the group, such that as time goes on, all agents asymptotically reach an agreement on their information states.

Jadbabaie et al.[11] considered the angle synchronization of networks of integrator agents with switching undirected network topology and gave a theoretical explanation for the synchronization phenomenon observed by Vicsek et al.[7] They proved that if the network topology is jointly-connected, then synchronization will be achieved asymptotically. Then Liu and Guo[12] gave a sufficient condition on the system parameters to ensure the connectivity of the network topology for the first time. Ren and Beard[13] extended the convergence analysis to the case with directed topology. They proved that if the network topology is jointly-containing-spanning tree, then consensus will be achieved asymptotically. Wang and Guo[14] showed that the system considered in ref. [13] is input to state stable (ISS) from the input noises to consensus error if and only if the network topology is jointly-containing-spanning-tree. For some consensus problems, the common value to which all the states converge is also required to be a function of the initial states of the group. It is often called χ-consensus. One of its most common form is the average-consensus. How to design χ-consensus protocols is a challenging problem. Olfati-Saber and Murray[15] considered average-consensus problems of networks of first-order integrator agents with fixed and switching topologies. They proved that if the network is an instantaneous balanced and strongly connected digraph, then average-consensus can be achieved. Kingston and Beard[16] extended the results of ref. [15] to the discrete-time models and weakened the condition of instantaneous strong connectivity to being jointly-containing-spanning-tree.

Many researchers have considered various consensus problems with different agent dynamics and applications, such as high-order models[17−19], leader-follower systems[20], nonlinear protocols[21,22], asynchronous protocols[23] and random topology graphs[24,25]. For the state of art of consensus research, we refer the readers to refs. [26, 27].

Most research on consensus in the above literature assumes an exact data exchange between agents, i.e., each agent measures its neighbors' states accurately. Obviously, this assumption is only an ideal approximation for real communication channels, since real networks are often in uncertain communication environment with various channel noises, source noises and sink noises. Recently, consensus problems with random measurement noises have attracted the attention of some researchers[28−31]. Kingston et al.[28] and Ren et al.[29] introduced time-varying consensus gains and designed consensus protocols based on a Kalman filter structure. They proved that the closed-loop system is input-to-state stable (ISS) from measurement noises to consensus errors, and when there is no noise, the protocols designed ensure consensus to be achieved asymptotically. Huang and Manton[30] introduced decreasing consensus gains to attenuate the measurement noises. They proved that for a connected undirected network topology, the states of all agents converge to a common random variable in mean square; while, for a strongly connected circulant network topology, the static mean square error between the individual state and the average of the initial states of all agents is in the same order as that of the variance of the measurement noises. Li and Zhang[31] considered the first-order continuous-time average-consensus control with fixed topologies and Gaussian communication noises. They gave a necessary and sufficient condition for asymptotically unbiased mean square average-consensus.

According to the dynamics of agents, the consensus problems can be divided into two categories: discrete-time systems and continuous-time systems. Up to now, the analysis for these two kinds of systems are almost mutually independent: for discrete-time agents described by difference equations, the control inputs are also discrete sequences; and for continuous-time agents described by differential equations, the control inputs are also continuous processes. However, in many cases, though the system itself is a continuous process, only sampled-data (SD) at discrete sampling instants is available for control synthesis, due to the application of digital sensors and controllers. This naturally leads to the following questions: for a given control objective, whether or not an SD-based control can match a full-state based continuous control; if not, how to characterize the difference quantitatively by the sampling size and system parameters[32]. Different from discrete-time systems, SD-based control systems are hybrid and with both continuous-time and discrete-time signals. For an SD-based control system, not only the states at sampling instants, but also the final effect of the control law on the original continuous system should be considered, which is what the designers are really concerned about. In addition, for discrete-time systems, the sampling size is often assumed to be 1 and its impact on system performances is neglected. While, for SD-based control systems, the impact of the sampling size is particularly paid attention to. Up to now, centralized SD-based control systems have been widely addressed[33]. In distributed networks, transmitting full-state information may result in high communication traffic. Considering the wide application of digital communication and control in distributed systems, communication between adjacent nodes at only sampling instants will save the communication cost effectively. So it is of great significance to pursue the research on SD-based controls of distributed systems[27].

In this paper, SD-based average-consensus control is considered for networks of continuous-time first-order integrator agents with undirected topology. The control input of each agent is based only on the information measured at the sampling instants from itself and its neighborhood rather than the complete continuous process. In addition, the measurements of its neighbors' states are corrupted by random measurement noises. At each sampling instant, a decreasing consensus gain is used. Different from the consensus problem with full-state information, here, the closed-loop system is a hybrid system with the coupling of the continuous-time and discrete-time states. The deviation between the states at continuous time instants and those at the sampling instants need to be estimated. It is shown that for a connected network, when the sampling size h is sufficiently small, the static mean square error between the individual state and the average of the initial states of all agents can be made arbitrarily small. Furthermore, by properly choosing the consensus gains, almost sure consensus can be achieved, and as time goes on, all agents' states will converge to a common random variable with probability 1. The mathematical expectation of the random variable is just the average of the initial states of the group, and the variation of the random variable vanishes as the sampling size h decreases to zero. By analyzing the impact of the number of network nodes N and the sampling size h on the system performances, it is shown that the choice of N is a trade-off between the steady-state performance and the cost of the network. In particular, for regular networks, the steady-state performance can be improved at the cost of an increase of N. It is also shown that the choice of h is a trade-off between the steady-state and transient performances. In the case of white Gaussian noises, reducing h can not optimize the static and transient performances simultaneously, since no matter what the sampling size is, the product of the steady-state and transient performance indices is always equal to or larger than a positive constant depending on the noise intensity, network topology and the number of network nodes. This constant can be viewed as an integrated index reflecting the static and transient performances. We call this phenomenon the uncertainty principle of Gaussian networks.

The remainder of this paper is organized as follows. In section 2, some concepts in graph theory are described, and the problem to be investigated is formulated. In section 3, the protocol designed is proved to be an asymptotic mean square average-consensus protocol. In section 4, the consensus gains are properly chosen such that almost sure consensus can be achieved, and the impact of the number of network nodes and the sampling size on the system performances is analyzed. In section 5, two numerical examples are given to illustrate our results. In section 6, some concluding remarks and further research topics are discussed.

The following notations will be used throughout this paper: $\mathbf{1}$ denotes the N-dimensional column vector with all ones. I_N denotes the N-dimensional identity matrix. For a given vector or matrix A, A^{T} denotes its transpose; $\|A\|_\infty$ denotes its infinity-norm; $\|A\|_2$ denotes its 2-norm; $\rho(A)$ denotes its spectral radius. For a given real number x, $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to it. For a given random variable X, $E(X)$ denotes its mathematical expectation; $\mathrm{Var}(X)$ denotes its variance.

2 Preliminaries and problem formulation

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$ be a weighted graph, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of nodes, node i represents the ith agents; $\mathcal{E}$ is the set of edges, an edge in $\mathcal{G}$ is denoted by an unordered pair (j, i). $(j, i) \in \mathcal{E}$ if and only if information can be exchanged between the ith and the jth agent, and $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. The neighborhood of the ith agent is denoted by $N_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$, the cardinal number of N_i is called the degree of node i, denoted by d_i. $d_{\mathcal{G}} \triangleq \max_{1 \leqslant i \leqslant N} d_i$ is called the degree of $\mathcal{G}$. $\mathcal{G}$ is called a d-regular graph, if $d_i \equiv d > 0$, $i = 1, 2, \ldots, N$.

We suppose that there is no isolated node in $\mathcal{G}$, that is, $d_i > 0$, $i = 1, 2, \ldots, N$.

$\mathcal{W} = [w_{ij}] \in \mathbb{R}^{N \times N}$ is called the weighted adjacency matrix of $\mathcal{G}$, for any $i, j \in \mathcal{V}$, $w_{ij} \geqslant 0$, $w_{ij} = w_{ji}$. $w_{ij} > 0 \Leftrightarrow j \in N_i$. $L_{\mathcal{G}} = \mathcal{D} - \mathcal{W}$ is called the Laplacian matrix of $\mathcal{G}$, where $\mathcal{D} = \mathrm{diag}(w_1, \ldots, w_N)$, $w_i = \sum_{j=1}^N w_{ij}$.

A sequence of (i_1, i_2), $(i_2, i_3), \ldots, (i_{k-1}, i_k)$ is called a path from node i_1 to node i_k. If for any $i, j \in \mathcal{V}$, there is a path from i to j, then $\mathcal{G}$ is called a connected graph.

Below is a well-known theorem for connected graphs.

Theorem 2.1[34]. If $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$ is a connected graph, then $L_{\mathcal{G}}$ is a symmetric matrix, $\mathbf{1}^{\mathrm{T}} L_{\mathcal{G}} = L_{\mathcal{G}} \mathbf{1} = 0$, and $L_{\mathcal{G}}$ has N real eigenvalues, in an ascending order:

$$0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_N \leqslant 2\Delta,$$

where $\Delta = \max_{1 \leqslant i \leqslant N} w_i$. λ_2 is called the algebraic connectivity.

In this paper, we consider the average-consensus control for a network of continuous-time first-order integrator agents with the dynamics

$$\frac{dx_i(t)}{dt} = u_i(t), \quad i = 1, 2, \ldots, N, \tag{1}$$

where x_i is the state of the ith agent, u_i is the control input. Here, for simplicity, we suppose that both $x_i(t)$ and $u_i(t)$ are scalars, generalization of the results to the vector case is straightforward.

Denote $X(t) = [x_1(t), \ldots, x_N(t)]^{\mathrm{T}}$.

The ith agent can receive information from its neighbors:

$$y_{ji}(t) = x_j(t) + n_{ji}(t), \quad j \in N_i, \tag{2}$$

where $y_{ji}(t)$ denotes the measurement of the jth agent's state $x_j(t)$ by the ith agent. $\{n_{ji}(t), \mathcal{F}_{ji}(t), t \geqslant 0\}$ is a stochastic process on the probability space $\{\Omega, \mathcal{F}, P\}$, representing the measurement noises, $\{\mathcal{F}_{ji}(t) = \sigma(n_{ji}(s), 0 \leqslant s \leqslant t), t \geqslant 0\}$ is a sequence of nondecreasing σ-algebras. Therefore, the graph $\mathcal{G}$ shows the structure of information flow in the system (1), called the information flow graph or network topology graph of the system (1). $(\mathcal{G}, X)$ is usually called a dynamic network[15].

We call the group of controls

$$\mathcal{U} = \{u_i, i = 1, 2, \ldots, N\}$$

a measurement-based distributed protocol, if

$$u_i(t) \in \sigma(x_i(s),\ y_{ji}(s),\ j \in N_i,\ 0 \leqslant s \leqslant t), \quad \forall\ t \geqslant 0, i = 1, 2, \ldots, N. \tag{3}$$

The so-called consensus control means designing a measurement-based distributed protocol for the dynamic network $(\mathcal{G}, X)$, such that all agents achieve an agreement on their information states in some sense, when $t \to \infty$.

Below we introduce two definitions about consensus protocols. Considering the impact of random measurement noises, these definitions are both in the probabilistic sense.

Definition 2.1[30]. A distributed protocol $\mathcal{U}$ is called a mean square weak consensus protocol if under $\mathcal{U}$, system (1) has the following properties: $\lim_{t\to\infty} E(x_i(t) - x_j(t))^2 = 0$, $i, j = 1, 2, \ldots, N$, $i \neq j$.

Definition 2.2[30]. A distributed protocol $\mathcal{U}$ is called an almost sure strong consensus protocol if under $\mathcal{U}$, system (1) has the following properties: $\lim_{t\to\infty} x_i(t) = x^*$ a.s., $i = 1, 2, \ldots, N$, where x^* is a finite random variable.

Let $\chi : \mathbb{R}^N \to \mathbb{R}$ be a function of N variables, the so-called χ-consensus control means designing a distributed protocol for the dynamic network $(\mathcal{G}, X)$ such that the states of all the agents achieve $\chi(X(0))$ when $t \to \infty$, that is, computing $\chi(X(0))$ in a distributed way. $\chi(X(0))$ is called the group decision value[15]. The special case of $\chi(x) = \mathrm{Ave}(x) \triangleq \frac{1}{N}\sum_{j=1}^N x_j$ is called average-consensus control. In this case, the group decision value is the average of the initial states of the group, namely, $\chi(X(0)) = \frac{1}{N}\sum_{j=1}^N x_j(0)$.

Remark 1. When there is no measurement noise, the states of all agents are deterministic processes, Definitions 2.1 and 2.2 both have the same meaning as the consensus definition of deterministic process in refs. [13, 15]. However, when there are random measurement noises, these two concepts are, generally speaking, not equivalent any more.

It can be seen that, if the mean square weak consensus is achieved, then for any given $\epsilon, \delta > 0$, there is always $T > 0$ such that $P\{\max_{1\leqslant i<j\leqslant N} |x_j(t) - x_i(t)| > \epsilon\} < \delta$, $\forall t \geqslant T$. This tells us that the probability of the static difference between agents' states to be larger than any pre-given positive constant can be made arbitrarily small.

Almost sure strong consensus implies that the closed-loop system achieves consensus with probability 1, and the states of the group are bounded with probability 1: $\sup_{1\leqslant i\leqslant N}\sup_{t\geqslant 0} |x_i(t)| < \infty$ a.s.

In this paper, we assume that for each agent, only the information at the sampling instants from its neighbors is available. For the dynamic network $(\mathcal{G}, X)$, we propose the distributed protocol as

$$u_i(t) = a(k)\sum_{j\in N_i} w_{ij}(y_{ji}(kh) - x_i(kh)),$$

$$\forall\ t \in [kh, (k+1)h), k = 0, 1, \ldots, i = 1, \ldots, N, \quad (4)$$

where h is the sampling size; $\{a(k), k = 0, 1, \ldots\}$ is usually called the sequence of consensus gains[29].

By (2) and the definition of $L_{\mathcal{G}}$, protocol (4) can be written in the vector form

$$U(t) = -a\left(\frac{t'}{h}\right)L_{\mathcal{G}}X(t') + a\left(\frac{t'}{h}\right)N(t'),\ \ t \geqslant 0, \quad (5)$$

where $n_i(t) = \sum_{j\in N_i} w_{ij}n_{ji}(t)$, $t' = \lfloor \frac{t}{h} \rfloor h$ and

$$U(t) = [u_1(t), \ldots, u_N(t)]^{\mathrm{T}},$$

$$N(t) = [n_1(t), \ldots, n_N(t)]^{\mathrm{T}}.$$

In this paper, we will find proper sequences of consensus gains, such that under protocol (4), the static mean square error between the individual state $x_i(t)$ and the group decision value $\frac{1}{N}\sum_{j=1}^N x_j(0)$ can be made arbitrarily small, provided the sampling period h is sufficiently small.

3 Asymptotic mean square average consensus

The discrete-time model with zero-order holder of (1) is

$$x_i((k+1)h) = x_i(kh) + hu_i(kh),\ k = 0, 1, \ldots, \quad (6)$$

which can be re-written in the vector form

$$X((k+1)h) = X(kh) + hU(kh). \quad (7)$$

Substituting the protocol (5) into (7) leads to

$$X((k+1)h) = P_h(k)X(kh) + ha(k)N(kh), \quad (8)$$

where $P_h(k) = I_N - a(k)hL_{\mathcal{G}}$.

Denote

$$\begin{cases} \Phi_h(n, k-1) = P_h(n)P_h(n-1)\cdots P_h(k), \\ \qquad\qquad \forall\ n = 0, 1, \ldots;\ k = 0, 1, \ldots, n, \\ \Phi_h(k, k) = I_N, \quad \forall\ k = 0, 1, \ldots \end{cases} \quad (9)$$

$$\begin{cases} \Psi_h(n,k-1) = \Big(P_h(n) - \frac{1}{N}11^{\mathrm{T}}\Big) \times \cdots \\ \qquad \times \Big(P_h(k) - \frac{1}{N}11^{\mathrm{T}}\Big), \\ \qquad \forall\ n = 0,1,\ldots;\ k = 0,1,\ldots,n, \\ \Psi_h(k,k) = I_N, \quad \forall\ k = 0,1,\ldots \end{cases} \tag{10}$$

Then, we have the following lemma.

Lemma 3.1. If $\mathcal{G}$ is connected, and the sampling size $h < \frac{1}{2\Delta \sup_{k\geqslant 0} a(k)}$, then

$$\begin{aligned} &\|\Psi_h(n,k-1)\|_2 \\ &= \left\| \Phi_h(n,k-1) - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}} \right\|_2 \\ &= \prod_{i=k}^{n}(1 - ha(i)\lambda_2), \\ &\forall\ n = 0,1,\ldots;\ k = 0,1,\ldots,n, \end{aligned} \tag{11}$$

where λ_2 is given in Theorem 2.1.

Proof. By Theorem 2.1, one can get $\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}P_h(k) = P_h(k)\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}} = \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}$, $\forall\ k = 0,1,\ldots$, and $\Psi_h(n,k-1) = \Phi_h(n,k-1) - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}$, $\forall\ n = 0,1,\ldots$; $k = 0,1,\ldots,n$.

Notice that $L_{\mathcal{G}}$ is symmetric, and has a single eigenvalue 0 with right eigenvector $\mathbf{1}$, and left eigenvector $\mathbf{1}^{\mathrm{T}}$. Then, there exists a standard orthogonal matrix $T = (t_1,\ldots,t_N)$ with $t_1 = \frac{1}{\sqrt{N}}\mathbf{1}$, such that $L_{\mathcal{G}} = T\mathrm{diag}(0,\lambda_2,\ldots,\lambda_N)T^{-1}$, and hence

$$\begin{aligned} P_h(k) = T\mathrm{diag}(1, 1 - ha(k)\lambda_2, \ldots, \\ 1 - ha(k)\lambda_N)T^{-1}, \forall\ k = 0,1,\ldots \end{aligned}$$

This together with $T\mathrm{diag}(1,0,\ldots,0)T^{-1} = t_1 t_1^{\mathrm{T}} = \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}$ leads to

$$\begin{aligned} &P_h(k) - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}} \\ &= T\mathrm{diag}(0, 1 - ha(k)\lambda_2, \ldots, \\ &\quad 1 - ha(k)\lambda_N)T^{-1}, \forall\ k = 0,1,\ldots \end{aligned}$$

Furthermore, by $2\Delta h \sup_{k\geqslant 0} a(k) < 1$, $0 < \lambda_i \leqslant 2\Delta$, $i = 2,\ldots,N$, we have

$$\begin{aligned} &\|\Psi_h(n,k-1)\|_2 \\ &= \rho\Big(T\mathrm{diag}\Big(0, \prod_{i=k}^{n}(1 - ha(i)\lambda_2), \\ &\quad \ldots, \prod_{i=k}^{n}(1 - ha(i)\lambda_N)\Big)T^{-1}\Big) \\ &= \prod_{i=k}^{n}(1 - ha(i)\lambda_2), \\ &\forall\ n = 0,1,\ldots; k = 0,1,\ldots,n. \end{aligned}$$

Thus, (11) holds.

To get the main result, we need the following assumptions on the network topology, the measurement noises and the sequence of consensus gains $\{a(k), k = 0,1,\ldots\}$.

(**H1**) $\mathcal{G}$ is a connected graph.

(**H2**) $\{n_{ji}(kh),\ k = 0,1,\ldots,\ i,\ j = 1,2,\ldots,N,\ (j,i) \in \mathcal{E}\}$ are independent sequences of martingale differences, satisfying $\max_{(j,i)\in\mathcal{E}} \sup_{k\geqslant 0} En_{ji}^2(kh) \leqslant \sigma_h$, where $\sigma_h = o(h^{-2})$, $h \to 0$.

(**H3**) $a(k) > 0$, $k = 0,1,\ldots$;

$$\sum_{k=0}^{\infty} a(k) = \infty; \quad \sum_{k=0}^{\infty} a^2(k) < \infty.$$

Remark 2. Here we consider only the networks with time-invariant topology. This is because on the one hand, research on time-invariant topology is the foundation of that on various kinds of time-varying topologies; on the other hand, time-invariant topology is suitable for many applications in distributed computation, formation control and data fusion of sensor networks[5, 6, 35, 36].

Remark 3. Assumption (H2) includes bounded noises and white Gaussian noises as its special cases:

(**H2.A**) $\{n_{ji}(kh), k = 0,1,\ldots,\ i, j = 1,2,\ldots,N,\ (j,i) \in \mathcal{E}\}$ are independent i.i.d sequences, each with a uniform distribution on $[-\delta_{ij},\delta_{ij}]$, $\delta_{ij} \in (0,\infty)$.

In this case, $\sigma_h = \frac{\max_{(j,i)\in\mathcal{E}} \delta_{ij}^2}{3} = O(1)$, $h \to 0$.

(**H2.B**) $n_{ji}(kh) = \sigma_{ij}\frac{B_{ji}((k+1)h) - B_{ji}(kh)}{h}$, $\{B_{ji}, i = 1,2,\ldots,N,\ j \in N_i\}$ are independent standard Brownian motions, $\sigma_{ij} \in (0,\infty)$.

In this case, $\sigma_h = \frac{\max_{(j,i)\in\mathcal{E}} \sigma_{ij}^2}{h} = O(h^{-1})$, $h \to 0$.

Theorem 3.1. For system (1) with protocol (4), if the sampling size h satisfies $h < \frac{1}{2\Delta \sup_{k\geqslant 0} a(k)}$, then under Assumptions (H1)–(H3), the closed-loop system has the following properties:

$$\limsup_{t\to\infty} E\bigg(x_i(t)-\frac{1}{N}\sum_{j=1}^{N}x_j(0)\bigg)^2=o(1),$$

$$h\to 0,\quad i=1,2,\ldots,N. \tag{12}$$

Proof. From (8), we have

$$X((n+1)h)=\Phi_h(n,-1)X(0)+h\sum_{k=0}^{n}\Phi_h(n,k)a(k)N(kh), \tag{13}$$

where $\Phi_h(n,k)$ is given by (9). From $2\Delta h \sup_{k\geqslant 0} a(k)<1$, it follows that $1-ha(k)w_i\geqslant 0$, $\forall\ k=0,1,\ldots,i=1,\ldots,N$. Therefore, for any $k=0,1,\ldots,$ $P_h(k)$ is a row stochastic matrix[1)]. Then, by the property of the row stochastic matrices, we have $\|\Phi_h(n,k-1)\|_\infty=1$, $\forall\ n=0,1,\ldots;$ $k=0,1,\ldots,n$.

From (13) and Assumption (H2), we have

$$\begin{aligned}
&E\bigg\|X((n+1)h)-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}X(0)\bigg\|_2^2\\
&\leqslant 2\bigg\|\Phi_h(n,-1)-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\bigg\|_2^2\|X(0)\|_2^2\\
&\quad+2h^2E\bigg\|\sum_{k=0}^{n}\Phi_h(n,k)a(k)N(kh)\bigg\|_2^2\\
&=2\bigg\|\Phi_h(n,-1)-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\bigg\|_2^2\|X(0)\|_2^2\\
&\quad+2h^2\sum_{k=0}^{n}E\|\Phi_h(n,k)a(k)N(kh)\|_2^2.
\end{aligned} \tag{14}$$

By the equivalence of matrix norms, there exist $C_1>0$, $C_2>0$ (which may depend on N) such that $\frac{1}{C_2}\|\cdot\|_\infty\leqslant\|\cdot\|_2\leqslant C_1\|\cdot\|_\infty$. Thus, by (14) and noticing that $\|\Phi_h(n,k-1)\|_\infty=1$, we get

$$\begin{aligned}
&E\bigg\|X((n+1)h)-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}X(0)\bigg\|_2^2\\
&\leqslant 2\bigg\|\Phi_h(n,-1)-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\bigg\|_2^2\|X(0)\|_2^2\\
&\quad+2h^2C_1^2\sum_{k=0}^{n}\|\Phi_h(n,k)\|_\infty^2a^2(k)E\|N(kh)\|_\infty^2\\
&\leqslant 2\bigg\|\Phi_h(n,-1)-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\bigg\|_2^2\|X(0)\|_2^2\\
&\quad+2h^2C_1^2C_2^2\sum_{k=0}^{n}a^2(k)E\|N(kh)\|_2^2\\
&\leqslant 2\bigg\|\Phi_h(n,-1)-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\bigg\|_2^2\|X(0)\|_2^2\\
&\quad+2\sigma_h h^2\bigg(\sum_{i=1}^{N}\sum_{j\in N_i}w_{ij}^2\bigg)C_1^2C_2^2\sum_{k=0}^{n}a^2(k).
\end{aligned} \tag{15}$$

From Lemma 3.1 and $1-x\leqslant e^{-x}, x\geqslant 0$ it follows that

$$\bigg\|\Phi_h(n,-1)-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\bigg\|_2\leqslant\exp\bigg\{-h\lambda_2\sum_{k=0}^{n}a(k)\bigg\},\quad n=0,1,\ldots,$$

which together with Assumption (H3) gives

$$\lim_{n\to\infty}\bigg\|\Phi_h(n,-1)-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\bigg\|_2=0. \tag{16}$$

Thus, by (15) and Assumptions (H2)–(H3), we have

$$\begin{aligned}
&\limsup_{n\to\infty}E\bigg\|X(nh)-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}X(0)\bigg\|_2^2\\
&\leqslant 2\sigma_h h^2\bigg(\sum_{i=1}^{N}\sum_{j\in N_i}w_{ij}^2\bigg)C_1^2C_2^2\sum_{k=0}^{\infty}a^2(k)\\
&=o(1),\quad h\to 0.
\end{aligned} \tag{17}$$

Notice that from (1) and (5), we have

$$\frac{dX(t)}{dt}=-a\Big(\frac{t'}{h}\Big)L_{\mathcal{G}}X(t')+a\Big(\frac{t'}{h}\Big)N(t').$$

Then, integrating both sides of the above equation from t' to t leads to

$$\begin{aligned}
&X(t)-X(t')\\
&=-a\Big(\frac{t'}{h}\Big)L_{\mathcal{G}}X(t')(t-t')\\
&\quad+a\Big(\frac{t'}{h}\Big)N(t')(t-t'),\quad\forall\ t\geqslant 0.
\end{aligned} \tag{18}$$

1) A row stochastic matrix is a square matrix with all elements nonnegative and all rows summed to 1. A products of two row stochastic matrices is also a row stochastic matrix[37].

This together with Assumption (H2) implies that

$$\sup_{t'\leqslant s\leqslant t} E\|X(s)-X(t')\|_2^2$$
$$\leqslant 2a\left(\frac{t'}{h}\right)^2 h^2\|L_{\mathcal{G}}\|_2^2 E\|X(t')\|_2^2$$
$$+2a\left(\frac{t'}{h}\right)^2 h^2 E\|N(t')\|_2^2$$
$$\leqslant 2a\left(\frac{t'}{h}\right)^2 h^2\|L_{\mathcal{G}}\|_2^2 E\|X(t')\|_2^2$$
$$+2a\left(\frac{t'}{h}\right)^2\left(\sum_{i=1}^{N}\sum_{j\in N_i} w_{ij}^2\right)\sigma_h h^2. \quad (19)$$

From Assumption (H3) it is obvious that $\lim_{t\to\infty} a(\frac{t'}{h})=0$. Then, by (17) we get

$$\sup_{t\geqslant 0} E\|X(t')\|_2^2<\infty,$$

and by (19),

$$\lim_{t\to\infty} E\|X(t)-X(t')\|_2^2=0, \quad (20)$$

which together with (17) gives

$$\limsup_{t\to\infty} E\left\|X(t)-\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}X(0)\right\|_2^2=o(1), \quad h\to 0.$$

Thus, (12) holds.

If Assumption (H2) in Theorem 3.1 is weakened to the following assumption (H2′), then the mean square weak consensus can be obtained.

(H2′) $\{n_{ji}(kh),\ k=0,1,\ldots,i,j=1,2,\ldots,N,\ (j,i)\in\mathcal{E}\}$ are independent sequences of martingale differences satisfying $\max_{(j,i)\in\mathcal{E}}\sup_{k\geqslant 0} En_{ji}^2(kh)<\infty$.

Theorem 3.2. For system (1), if protocol (4) is applied, then under Assumptions (H1), (H2′) and (H3), the closed-loop system has the following properties:

$$\lim_{t\to\infty} E(x_i(t)-x_j(t))^2=0,\ \ i,j=1,2,\ldots,N. \quad (21)$$

Proof. Since $\sum_{k=0}^{\infty} a^2(k)<\infty$, for any given $\epsilon>0$, there is $r>0$ such that

$$4h^2\left(\sum_{i=1}^{N}\sum_{j\in N_i} w_{ij}^2\right)C_1^2C_2^2$$
$$\times\max_{(j,i)\in\mathcal{E}}\sup_{k\geqslant r} En_{ji}^2(kh)\sum_{k=r}^{\infty} a^2(k)<\frac{\epsilon}{2}, \quad (22)$$

$$1-ha(k)w_i\geqslant 0,\ \ k=r,r+1,\ldots,\ \ i=1,2,\ldots,N.$$

Let $\boldsymbol{e}_{ij}$ denote the N-dimensional column vector with the ith and jth components equal to 1, and others zero. Then, similarly to (15), by (13) and (22) we have

$$E(x_i((n+1)h)-x_j((n+1)h))^2$$
$$=E\left\|\boldsymbol{e}_{ij}^{\mathrm{T}}\Phi_h(n,r-1)X(r)\right.$$
$$\left.+h\boldsymbol{e}_{ij}^{\mathrm{T}}\sum_{k=r}^{n}\Phi_h(n,k)a(k)N(kh)\right\|_2^2$$
$$\leqslant 2\|\boldsymbol{e}_{ij}^{\mathrm{T}}\Phi_h(n,r-1)\|_2^2E(\|X(r)\|_2^2)$$
$$+2h^2\|\boldsymbol{e}_{ij}^{\mathrm{T}}\|_2^2E\left\|\sum_{k=r}^{n}\Phi_h(n,k)a(k)N(kh)\right\|_2^2$$
$$\leqslant 2\|\boldsymbol{e}_{ij}^{\mathrm{T}}\Phi_h(n,r-1)\|_2^2E(\|X(r)\|_2^2)$$
$$+4h^2\left(\sum_{i=1}^{N}\sum_{j\in N_i} w_{ij}^2\right)C_1^2C_2^2$$
$$\times\max_{(j,i)\in\mathcal{E}}\sup_{k\geqslant r} En_{ji}^2(kh)\sum_{k=r}^{n} a^2(k)$$
$$\leqslant 2\|\boldsymbol{e}_{ij}^{\mathrm{T}}\Phi_h(n,r-1)\|_2^2E(\|X(r)\|_2^2)+\frac{\epsilon}{2},$$
$$n=r,r+1,\ldots \quad (23)$$

Similarly to (16), by Lemma 3.1 and Assumption (H3), we have

$$\lim_{n\to\infty}\|\boldsymbol{e}_{ij}^{\mathrm{T}}\Phi_h(n,r-1)\|_2=\left\|\boldsymbol{e}_{ij}\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\right\|_2=0,$$

which together with (23) and the arbitrariness of ϵ results in

$$\lim_{n\to\infty} E(x_i(nh)-x_j(nh))^2=0,$$
$$i,j=1,2,\ldots,N. \quad (24)$$

Noticing that for any $i,j=1,2,\ldots,N$,

$$E(x_i(t)-x_j(t))^2$$
$$=E(x_i(t)-x_i(t')+x_i(t')$$
$$-x_j(t')+x_j(t')-x_j(t))^2$$
$$\leqslant 3E(x_i(t)-x_i(t'))^2+3E(x_i(t')-x_j(t'))^2$$
$$+3E(x_j(t')-x_j(t))^2,$$

by (20) and (24) we obtain (21).

4 Almost sure average consensus

For a stochastic system, what can be really observed (in an experiment) is always a sample path. Hence, for consensus problems of stochastic systems, we are naturally concerned with whether or not almost sure consensus can be achieved. In this section, we will properly choose the sequence of consensus gains such that protocol (4) can ensure almost sure strong consensus.

To this end, we need the following assumptions on the measurement noises.

(H2″) $\{n_{ji}(kh), k = 0, 1, \dots, i, j = 1, 2, \dots, N, (j,i) \in \mathcal{E}\}$ are i.i.d. sequences with zero mean and satisfy $\max_{(j,i)\in\mathcal{E}} \sup_{k\geqslant 0} En_{ji}^2(kh) \leqslant \sigma_h$, where $\sigma_h = o(h^{-2})$, $h \to 0$.

Below is the main result of this section.

Theorem 4.1. For system (1), if protocol (4) is applied, and $a(k) = \frac{\log(k+2)}{k+2}$, $k = 0, 1, \dots$, then under Assumptions (H1), (H2″), the closed-loop system has the following properties:

(i) there exists a random variable x_h^* such that

$$\lim_{t\to\infty} x_i(t) = x_h^* \text{ a.s.}, \quad i = 1, 2, \dots, N;$$

(ii) x_h^* has finite expectation and finite variation:

$$E(x_h^*) = \frac{1}{N}\sum_{j=1}^{N} x_j(0);$$

$$\mathrm{Var}(x_h^*) = h^2\sigma_h \frac{\sum_{i=1}^{N}\sum_{j\in N_i} w_{ij}^2}{N^2} \sum_{k=0}^{\infty} a^2(k) = o(1),\ h \to 0. \tag{25}$$

The proof of this theorem is put into the appendix.

Remark 4. When there is no measurement noises, i.e., $n_{ji}(t) \equiv 0$, we can take $a(k) \equiv 1$. In this case, $\{u_i(kh), k = 0, 1, \dots\}$, $i = 1, 2, \dots, N$, is just Algorithm (A1) for discrete-time models in ref. [15]. However, when the noises are not zeros, if we still take $a(k) \equiv 1$ (or any other positive constant), then from (A1) and the law of iterated logarithm, it can be seen that even if the noises $\{n_{ji}(kh), k = 0, 1, \dots, i, j = 1, 2, \dots, N, (j,i) \in \mathcal{E}\}$ are uniformly bounded, the centroid of the system states will diverge to infinity with probability 1, and $P\{\lim_{k\to\infty}(x_i(kh) - x_j(kh)) = 0, \sup_k |x_i(kh)| < \infty, i, j = 1, 2, \dots, N\} = 0$. This means that the consensus of the closed-loop system cannot be achieved. That is why decreasing consensus gains are used when there are measurement noises. Actually, it can be seen that for consensus problems of dynamic networks, the impact of measurement noises on the design of protocols is essential, even if the topology is time-invariant.

In some engineering application of the information fusion of wireless sensor networks, the number of network nodes N is usually very large. This gives rise to investigating the impact of N on the information fusion and the asymptotic property of the system when N increases to infinity. By (25) one can get the following corollaries.

Corollary 1. For system (1) with protocol (4), if $\sup_{N\geqslant 1}\sup_{1\leqslant i<j\leqslant N} w_{ij} < \infty$ and the increasing rate of the network degree satisfies $d_{\mathcal{G}} = o(N)$, then under the condition of Theorem 4.1, $\mathrm{Var}(x_h^*) = o(1)$ as $N \to \infty$.

Corollary 2. For system (1) with protocol (4), if $w_{ij} \equiv w > 0$, $i = 1, 2, \dots, N$, $j \in N_i$, $d_i \equiv d > 0$, $i = 1, 2, \dots, N$, then under the condition of Theorem 4.1, $\mathrm{Var}(x_h^*) = O(N^{-1})$ as $N \to \infty$.

The x_h^* in Corollaries 1 and 2 has the same meaning as in Theorem 4.1.

Remark 5. From Corollaries 1 and 2, it can be seen that, under the conditions of Theorem 4.1, if the weighted adjacency matrix and the network degree do not diverge too fast with respect to N, then the more the network nodes are, the better the effect of the information fusion is. Especially, for an equally weighted regular network, the steady-state error of the information fusion is inversely proportional to the number of network nodes. Though more nodes can be added, a large number of nodes will result in a large cost of running and maintaining the whole network, so the choice of N is a trade-off between the fusion accuracy and the cost.

By (25) we know that under the condition of Theorem 4.1, a small sampling size gives a small steady-state error. However, it is not the case that the smaller the h is, the better the performance of the system is. This can actually be seen from the following theorem and Example 2 given below.

Theorem 4.2. For system (1) with protocol (4), if $n_{ji}(kh) = \sigma\frac{B_{ji}((k+1)h)-B_{ji}(kh)}{h}$, $k = 0,1,\ldots$, $i = 1,2,\ldots,N$, $j \in N_i$, where $\{B_{ji}, i = 1,2,\ldots,N, j \in N_i\}$ are independent standard Brownian motions, $\sigma \in (0,\infty)$, $a(k) = \frac{\log(k+2)}{k+2}$, $k = 0,1,\ldots$, and $h < \frac{\ln 2}{4\Delta \sup_{k\geqslant 0} a(k)}$, then under Assumptions (H1), the closed-loop system has the following properties:

$$s_h t_h \geqslant \frac{\sigma \sum_{i=1}^N \sum_{j\in N_i} w_{ij}^2}{2\lambda_2 N^2}, \tag{26}$$

where

$$s_h = \frac{\mathrm{Var}(x_h^*)}{\sum_{k=0}^\infty a^2(k)},$$

$$t_h = \frac{1}{|\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^n a(k)} \log \|\Psi_h(n,-1)\|_2|}.$$

Proof. By the condition of the theorem, we know that Assumption (H2$''$) holds. Thus, by Theorem 4.1,

$$s_h = \frac{h\sigma \sum_{i=1}^N \sum_{j\in N_i} w_{ij}^2}{N^2}. \tag{27}$$

Noticing that $h < \frac{\ln 2}{4\Delta \sup_{k\geqslant 0} a(k)}$ and $1 - x \geqslant e^{-2x}$, $\forall x \in [0, \frac{\ln 2}{2})$, by Lemma 3.1, we have

$$\limsup_{n\to\infty} \frac{\log \|\Psi_h(n,-1)\|_2}{\sum_{k=0}^n a(k)} \geqslant \limsup_{n\to\infty} \frac{\log(\exp\{-2h\lambda_2 \sum_{k=0}^n a(k)\})}{\sum_{k=0}^n a(k)} = -2h\lambda_2,$$

which together with $\limsup_{n\to\infty} \frac{\log \|\Psi_h(n,-1)\|_2}{\sum_{k=0}^n a(k)} \leqslant -h\lambda_2$ gives $t_h \geqslant \frac{1}{2h\lambda_2}$. Hence, by (27) we have (26).

Remark 6. The s_h in Theorem 4.2 represents the normalized steady-state error with respect to the sequence of consensus gains, which can be viewed as a measure of the steady-state performance of the closed-loop systems. The smaller the s_h is, the smaller the steady-state error between each agent's state and the average of the initial states of the group is, in other words, the better the steady-state performance is.

$$\left\| X(kh) - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}} X(kh) \right\|_2^2 = \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} (x_i(kh) - x_j(kh))^2$$

represents the derivation between different agents, and so, called consensus error. While, $\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^n a(k)} \log \| \varPsi_h(n,-1)\|_2$ represents the normalized Lyapunov index of the consensus error equation (A3) with respect to the sequence of consensus gains. This index is a negative real number and the smaller it is, the faster the state transfer matrix $\varPsi_h(n,-1)$ converges to zero. Therefore, t_h can be viewed as a measure of the transient performance of the system. The smaller the t_h is, the faster the states converge to consensus, that is, the better the transient performance becomes.

Under the condition of Theorem 4.2, the sampling size h has dual impacts on the steady-state and transient performances. Making s_h smaller implies making t_h larger, and vice versa. No matter what the sampling size is, the integrated performance $s_h t_h$ is always equal to or larger than a positive infimum. We call this phenomenon uncertainty principle of Gaussian networks. As is well-known, uncertainty principle is also called Gabor inequality; and such principles play important roles in modern physics, biology and information science. By (26), the infimum of $s_h t_h$ is proportional to $(\lambda_2 N^2)^{-1}\sigma \sum_{i=1}^N \sum_{j\in N_i} w_{ij}^2$, which is a constant characterizing the intrinsic property of the communication network. It can be seen that, stronger connected topology (larger λ_2) together with smaller noises intensity will lead to a smaller value of this constant. Therefore, $(\lambda_2 N^2)^{-1}\sigma \sum_{i=1}^N \sum_{j\in N_i} w_{ij}^2$ can be viewed as a measure of the intrinsic performance of the communication network. We call this constant the Gabor constant of Gaussian networks.

5 Numerical examples

Example 1. Consider a dynamic network of three agents with the topology graph $\mathcal{G}_1 = \{1,2,3, \{(1,2), (2,1), (2,3), (3,2), (3,1), (1,3)\}, \mathcal{W}_1 = [w_{ij}]_{3\times 3}\}$, where $w_{12} = w_{21} = w_{23} = w_{32} = w_{31} = w_{13} = 1$, and other elements are all zeros. The measurement noises are independent standard white Gaussian sequences. The initial states of the three agents are $x_1(0) = -2$, $x_2(0) = -4$, $x_3(0) = 6$, respectively. Take the sequence of consensus gains as $a(k) = \frac{\log(k+2)}{k+2}$, $k = 0,1,\ldots$, and the sampling size $h = 0.5$ s. Then, under the control of protocol (4), the states of the closed-loop system are shown in Figure 1. It can be seen that as time goes on, the states of the group asymptotically achieve consensus.

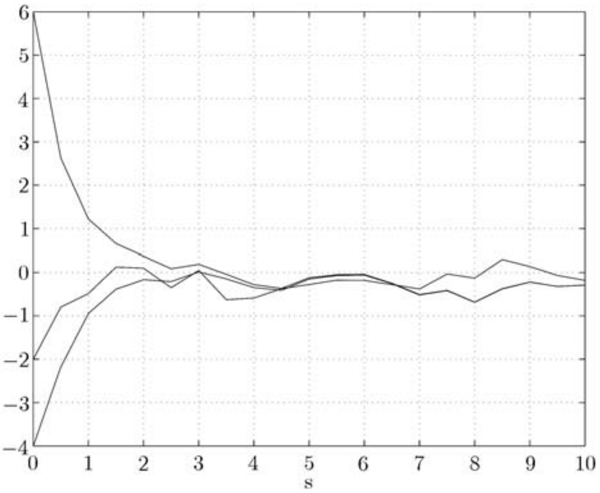

Figure 1 State curves of Example 1.

Example 2. By this example we investigate the impact of the sampling size on both static and transient performances by the simplest network: two agent interacting system with i.i.d. standard white Gaussian measurement noises. The initial states of the two agents are $x_1(0) = 5, x_2(0) = -5$, respectively. The sequence of consensus gain is taken as $a(k) = \frac{\log(k+2)}{k+2}$, $k = 0, 1, \ldots$, and the sampling sizes are taken as 1 s, 0.5 s, 0.1 s and 0.06 s, respectively. Then, under the control of protocol

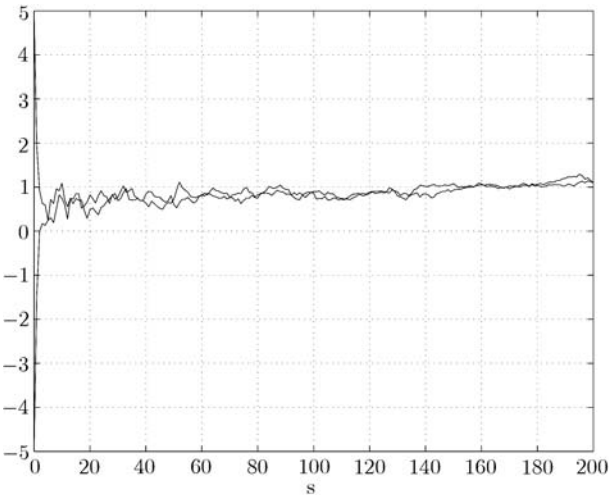

Figure 2 State curves of Example 2 with $h = 1$ s.

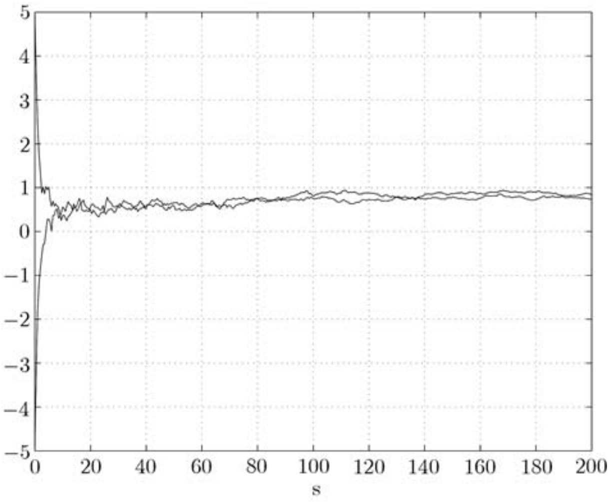

Figure 3 State curves of Example 2 with $h = 0.5$ s.

(4), the states of the closed loop system are shown in Figures 2–4, respectively, where the curves for $h = 0.1$ s and $h = 0.06$ s are drawn in one figure to exhibit the impact of the sampling size on the convergence rate. From Figures 2–4, it can be seen that reducing the sampling size does improve the static performance, but may slow down the convergence rate of the closed-loop system to the static state.

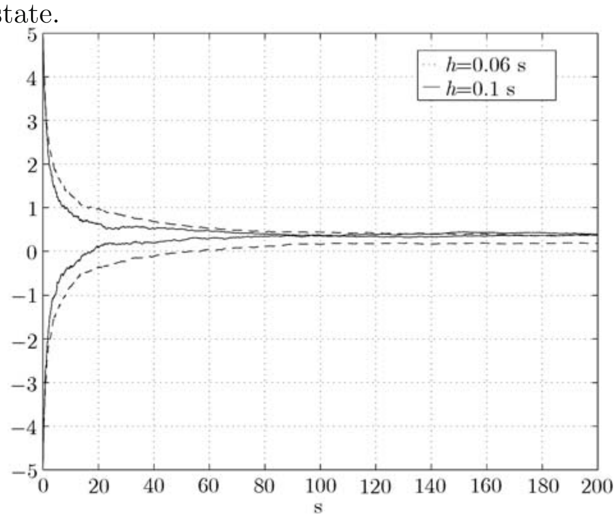

Figure 4 State curves of Example 2 with $h = 0.1$ s and $h = 0.06$ s.

6 Concluding remarks

In this paper, SD-based average-consensus control is considered for networks of continuous-time first-order integrator agents with random measurement noises. Different from previous research, though each agent itself is a continuous-time process, its control input is based only on the information obtained at the discrete sampling instants. Under mild conditions on network topology and measurement noises, by the theory of probability and the graph Laplacian matrix, we have proved that, if the sequence of consensus gains satisfies the step rules of the classical stochastic approximation, then the control law designed is an asymptotic mean square average-consensus protocol. By choosing the consensus gains properly, we obtain the almost sure consensus and an analytic expression of the steady-state error with respect to the sampling size h, noise intensity, network parameters and the number of nodes N. It is shown that for networks with regular topology, increasing N will decrease the steady-state error. Besides, an uncertainty principle is found for Gaussian networks, and a Gabor constant is given, which can be viewed as an integrated index reflecting the steady-state and transient performances.

It should be mentioned that what we have considered here is only the agents of first-order integrators and the networks with time-invariant topology. Extending the results to the high order and time-varying topology case is an important issue for future research.

The performance index t_h in Theorem 4.2 is for the discrete-time system. Whether or not there is a counterpart for the original continuous system is another issue worthwhile to study.

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Appendix

Lemma A.1[38]**.** Let $\{z_i, i \geqslant 1\}$ be an i.i.d. random variable sequence with zero mean, $\{a_{ki}, i = 1, 2, \ldots, k, k = 1, 2, \ldots\}$ be a double array of constants. If there exists $p \in [0, 2)$ such that $E|z_1|^p < \infty$ and $\max_{1\leqslant i\leqslant k} |a_{ki}| = O(\frac{1}{k^{1/p}\log k})$, then

$$\lim_{k\to\infty} \sum_{i=1}^{k} a_{ki} z_i = 0 \text{ a.s.}$$

Proof of Theorem 4.1. By (8) and the fact that $\mathbf{1}^{\mathrm{T}} P_h(k) = \mathbf{1}^{\mathrm{T}}$, we have

$$\frac{1}{N}\sum_{j=1}^{N} x_j((k+1)h) = \frac{1}{N}\sum_{j=1}^{N} x_j(kh) + ha(k)\frac{1}{N}\sum_{j=1}^{N} n_j(kh).$$

Summing up both sides of the above equation from $k = 0$ to $k = n$ gives

$$\frac{1}{N}\sum_{j=1}^{N} x_j((n+1)h) = h\sum_{k=0}^{n} a(k)\frac{1}{N}\sum_{j=1}^{N} n_j(kh) + \frac{1}{N}\sum_{j=1}^{N} x_j(0). \quad \text{(A1)}$$

Notice that $\sum_{k=1}^{\infty}\frac{(\log(k+1))^2}{(k+1)^2} < \infty$. Then, by the Khintchine-Kolmogorov convergence theorem[39] we know that $\sum_{k=0}^{n} a(k)\frac{1}{N}\sum_{j=1}^{N} n_j(kh)$ converges both almost surely and in mean square. This together with (A1) leads to

$$\lim_{n\to\infty}\frac{1}{N}\sum_{j=1}^{N} x_j(nh) = x_h^*, \quad \text{(A2)}$$

where

$$x_h^* \triangleq \frac{1}{N}\sum_{j=1}^{N} x_j(0) + \lim_{n\to\infty} h\sum_{k=0}^{n} a(k)\frac{1}{N}\sum_{j=1}^{N} n_i(kh).$$

From $\lim_{k\to\infty} a(k) = 0$, we know that for any given $M_0 \geqslant 2$, there is $k_0(h) > 0$, such that: $2h\Delta \sup_{k\geqslant k_0} a(k) < 1$, $\forall k = k_0, k_0+1, \ldots$; $\frac{\log(x+2)}{x+2}$ is strictly decreasing on $[k_0, \infty)$; and $h\lambda_2 \log[(n+3)(k+3)] \geqslant M_0$, $\forall n \geqslant k \geqslant k_0$.

By (8) and the fact that $\frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}} P_h(k) = \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}$, we have that for $n = k_0, k_0+1, \ldots$,

$$X((k+1)h) - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}} X((k+1)h)$$

$$= \left(P_h(k) - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\right)\left(X(kh) - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}X(kh)\right) + ha(k)\left(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\right)N(kh)$$

$$= \Psi_h(n, k_0 - 1)\left(X(k_0) - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}X(k_0)\right) + h\sum_{k=k_0}^{n}\Psi_h(n,k)a(k)\left(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\right)N(kh), \quad \text{(A3)}$$

where $\Psi_h(n,k)$ is given by (10).

We now prove that

$$\sum_{k=k_0}^{n}\Psi_h(n,k)a(k)\left(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\right)N(kh) = o(1), \quad n \to \infty \text{ a.s.} \quad \text{(A4)}$$

Denote by $\Psi_h(n,k)_{ij}$ the element of matrix $\Psi_h(n,k)$ in the ith row and the jth column; by s_{jl} the element of matrix $(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}})$ in the jth row and the lth column. Noticing that $\sum_{k=k_0}^{n}\Psi_h(n,k)a(k)(I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}})N(kh)$ is an N-dimensional column vector, with the ith component $\sum_{j=1}^{N}\sum_{k=k_0}^{n}\Psi_h(n,k)_{ij}a(k)(\sum_{l=1}^{N}s_{jl}n_l(kh))$, then, to verify (A4), it suffices to show

$$\lim_{n\to\infty}\sum_{k=k_0}^{n}\Psi_h(n,k)_{ij}a(k)n_l(kh) = 0 \text{ a.s.}, \quad \forall\ i,\ j,\ l = 1,2,\ldots,N. \quad \text{(A5)}$$

Similarly to Lemma 3.1, by $2\Delta h\sup_{k\geqslant k_0}a(k) < 1$ we have

$$\|\Psi_h(n,k)\|_2 \leqslant \prod_{i=k+1}^{n}\exp\{-ha(i)\lambda_2\} \leqslant \exp\left\{-h\lambda_2\sum_{i=k+1}^{n}\frac{\log(i+2)}{i+2}\right\}, \quad \forall\ n = k_0,\ldots; \ k = k_0 - 1,\ldots,n-1. \quad \text{(A6)}$$

Since $\frac{\log(x+2)}{x+2}$ is strictly decreasing on $[k_0,\infty)$, we have

$$\sum_{i=k+1}^{n}\frac{\log(i+2)}{i+2} \geqslant \int_{k+1}^{n+1}\frac{\log(x+2)}{(x+2)}dx, \quad \forall\ n = k_0+1,\ldots; k = k_0,\ldots,n-1. \quad \text{(A7)}$$

Noticing that

$$\int_{k+1}^{n+1}\frac{\log(x+2)}{x+2}dx = (\log(n+3))^2 - (\log(k+3))^2 - \int_{k+1}^{n+1}\frac{\log(x+2)}{x+2}dx,$$

one can get

$$\int_{k+1}^{n+1}\frac{\log(x+2)}{x+2}dx = \frac{1}{2}(\log[(n+3)(k+3)])\left(\log\frac{n+3}{k+3}\right) \geqslant \frac{M_0}{2h\lambda_2}\left(\log\frac{n+3}{k+3}\right), \quad \forall\ n = k_0+1,\ldots; k = k_0,\ldots,n-1.$$

This together with (A6) and (A7) gives

$$\|\Psi_h(n,k)\|_2 a(k) \leqslant \left(\frac{k+3}{n+3}\right)^{M_0/2}\frac{\log(k+2)}{k+2} \leqslant C_3\frac{1}{(n+2)^{M_0/2}}\frac{\log(k+2)}{(k+2)^{1-M_0/2}}, \quad \forall\ n = k_0+1,\ldots; k = k_0,\ldots,n-1,$$

where $C_3 = \sup_{k\geqslant k_0}(\frac{k+3}{k+2})^{M_0/2}$. Thus, by $M_0 \geqslant 2$ we have

$$\max_{k_0\leqslant k<n}\|\Psi_h(n,k)\|_2 a(k) \leqslant \frac{C_3\log(n+2)}{n+2} = O\left(\frac{\log n}{n}\right), \quad \forall\ n = k_0+1,\ldots,$$

and hence, by

$$\max_{1\leqslant i,j\leqslant N}|\Psi_h(n,k)_{ij}| \leqslant \sqrt{N}\|\Psi_h(n,k)\|_2,$$

we have

$$\max_{k_0\leqslant k<n}|\Psi_h(n,k)_{ij}|a(k) = O\left(\frac{\log n}{n}\right), \quad \forall\ n = k_0+1,\ldots;\ i,j = 1,2,\ldots,N.$$

This together with Assumption (H2″) and Lemma A1 leads to (A5). Therefore, (A4) holds.

From (A6) it follows that

$$\lim_{n\to\infty}\|\Psi_h(n,k_0-1)\|_2 = 0,$$

which together with (A3) and (A4) gives

$$\lim_{n\to\infty}\left(x_i(nh)-\frac{1}{N}\sum_{j=1}^{N}x_j(nh)\right)=0 \text{ a.s.},$$

$$i=1,2,\ldots,N.$$

Thus, by (A2) we have

$$\lim_{k\to\infty} x_i(kh)=x_h^* \text{ a.s.},\quad i=1,2,\ldots,N, \tag{A8}$$

$$\sup_{t\geqslant 0}\|X(t')\|_2<\infty \text{ a.s.} \tag{A9}$$

Noticing that $\sum_{k=0}^{\infty}a(k)N(kh)$ converges almost surely, we know that

$$\lim_{t\to\infty} a\left(\frac{t'}{h}\right)N(t')=0 \text{ a.s.}, \tag{A10}$$

which together with (18), (A8) and (A9) leads to

$$\lim_{t\to\infty} x_i(t)=x_h^* \text{ a.s.},\quad i=1,2,\ldots,N.$$

Thus, (i) is true.

Recalling that $\sum_{k=0}^{n}a(k)\frac{1}{N}\sum_{j=1}^{N}n_j(kh)$ converges in mean square, by the dominated convergence theorem[39], we can get that $E(x_h^*)=\frac{1}{N}\sum_{j=1}^{N}x_j(0)$ and

$$\begin{aligned}\mathrm{Var}(x_h^*)&=E\left(\lim_{n\to\infty}h\sum_{k=0}^{n}a(k)\frac{1}{N}\sum_{j=1}^{N}n_j(kh)\right)^2\\&=\lim_{n\to\infty}E\left[\left(h\sum_{k=0}^{n}a(k)\frac{1}{N}\sum_{j=1}^{N}n_j(kh)\right)^2\right]\\&=h^2\sigma_h\frac{\sum_{i=1}^{N}\sum_{j\in N_i}w_{ij}^2}{N^2}\sum_{k=0}^{\infty}a^2(k)\\&=o(1),\quad h\to 0.\end{aligned}$$

Thus, (ii) is true.