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Input-to-state stability of switched nonlinear systems

FENG Wei[1,2†] & ZHANG JiFeng[2]

[1] College of Information Sciences and Engineering, Shandong Agricultural University, Tai'an 271018, China;
[2] Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China

The input-to-state stability (ISS) problem is studied for switched systems with infinite subsystems. By using multiple Lyapunov function method, a sufficient ISS condition is given based on a quantitative relation of the control and the values of the Lyapunov functions of the subsystems before and after the switching instants. In terms of the average dwell-time of the switching laws, some sufficient ISS conditions are obtained for switched nonlinear systems and switched linear systems, respectively.

switched system, input-to-state stability, Lyapunov function, dwell-time

1 Introduction

Since the performance of a real control system is affected more or less by uncertainties, such as unmodelled dynamics, parameter perturbations, exogenous disturbances, measurement errors, etc., the research on robustness of control systems do always have a vital status in the development of control theory and technology. Aiming at robustness analysis of nonlinear control systems, Sontag, Wang and Lin[1−11] developed a new method from the point of view of input-to-state stability (ISS), input-to-output stability (IOS) and integral input-to-state stability (iISS), and obtained a series of fundamental results by utilizing ISS-, IOS-Lyapunov functions. Recently, Mancilla-Aguilar and García[12] applied the idea to study the robustness of switched nonlinear systems of the form $\dot{x}(t) = f_i(x(t), u(t))$ ($i \in \Lambda$, where Λ is the index set).

For switched systems, although lots of results have been presented, they mainly focus on the problems of stability, controllability, observability and stabilization control[13−21]. For the robustness study of such systems, the relevant literature is not rich, and ref. [12] seems the only one on the ISS of switched nonlinear systems, to our knowledge.

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†Corresponding author (email: fwsnakefa@126.com)
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In this paper, we investigate the ISS of general switched nonlinear systems (including the case where there is no common Lyapunov function). Unlike the existing results, which mainly focus on establishing ISS converse theorems for nonlinear systems[1−11], by opening out the characteristic of switched nonlinear (SNL) systems, we aim at presenting some sufficient ISS conditions for SNL systems, including for instance, the relation of the ISS and the average dwell-time of the switching law. Precisely, we will investigate SNL time-varying systems, which may involve in infinite subsystems. In this case, switching among different subsystems may lead to discontinuity of the system function, and dissatisfies the continuity assumption required by refs. [1—11]. Thus, the results in refs. [1—11] cannot be generalized to general SNL systems directly. In some special cases, for instance, where there exists a common ISS-Lyapunov function (CISSLF), a sufficient and necessary ISS condition of SNL systems with arbitrary switching laws is given[12] under the assumption that $f_i(x,u)$ is uniformly (with respect to i) locally Lipschitz continuous on x,u. Here, by using the methods of multiple Lyapunov function and average dwell-time, some sufficient ISS conditions are given for general SNL systems, which may have no CISSLF. The ISS-Lyapunov functions of the subsystems are allowed to be different from each other rather than simply assuming the existence of a CISSLF. Besides, the uniformity assumption on the local Lipschitz continuity of $f_i(x,u)$ with respect to i is not required. Thus, our framework is more general than that in ref. [12].

The remainder of this paper is organized as follows. Section 2 describes the problem to be investigated and introduces some notations and definitions. In section 3, by using the multiple Lyapunov function method, a sufficient ISS condition is given for general switched nonlinear systems. In section 4, by using the average dwell-time method, some sufficient ISS conditions are presented for SNL systems and switched linear systems, respectively. In section 5, some concluding remarks are given.

2 Notations and problem formulation

Consider the following switched nonlinear system

$$\dot{x}(t) = f_{\sigma(t,x(t))}(t, x(t), u(t)), \quad x(t_0) = x_0, \tag{1}$$

where $x(\cdot) \in \mathbb{R}^n$ and $u(\cdot) \in \mathbb{R}^m$ are the system state and input, respectively; and $\sigma(\cdot,\cdot)$: $[t_0,\infty) \times \mathbb{R}^n \to \mathcal{I}$ ($\mathcal{I}$ is the index set, maybe infinite) is the switching law and is right-hand continuous and piecewise constant on t; for any $i \in \mathcal{I}$, function $f_i(t,x,u)$: $[t_0,\infty) \times \mathbb{R}^{n+m}$ $\to \mathbb{R}^n$ is continuous with respect to t,x,u, uniformly locally Lipschitz continuous with respect to x,u, and satisfies $f(\cdot,0,0) \equiv 0$.

Here, being different from ref. [12], $f_i(t,x,u)$ is time varying, and the uniformity of the local Lipschitz continuity of $f_i(t,x,u)$ is with respect to t rather than i.

Throughout the paper, $\mathbb{R}^+$ denotes the real number set $[0,\infty)$; for a function $\gamma(t): \mathbb{R}^+ \to \mathbb{R}^+$, $\gamma \in \mathcal{K}$ means that γ is continuous and strictly increasing, and satisfies $\gamma(0) = 0$; $\gamma \in \mathcal{K}_\infty$ means that $\gamma \in \mathcal{K}$ and γ are unbounded; for a function $\beta(t,s): \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $\beta \in \mathcal{KL}$

means that for any fixed s, $\beta(t,s)\in\mathcal{K}$, and for any fixed t, $\beta(t,s)$ is continuous and decreases to zero as $s\to\infty$; $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ and the corresponding induced matrix norm, and for a nonempty subset $\mathcal{M}\subset\mathbb{R}^n$, $|x|_{\mathcal{M}}\triangleq\inf_{\eta\in\mathcal{M}}|x-\eta|$ (obviously, it holds $|x|_{\{0\}}=|x|$ when $\mathcal{M}=\{0\}$); L_∞^m denotes the set of all the measurable and locally essentially bounded input $u(\cdot)\in\mathbb{R}^m$ on $[t_0,\infty)$ under the following norm,

$$\|u\|=\sup\{|u(t)|,t\geqslant t_0\}<\infty; \tag{2}$$

for two functions $\varphi(\cdot)$ and $\chi(\cdot)$, symbol $\varphi\circ\chi(\cdot)$ denotes the composite function $\varphi(\chi(\cdot))$; ∇ is the gradient operator.

For any given switching law $\sigma(\cdot,\cdot)$, initial condition $x_0\in\mathbb{R}^n$, $u(\cdot)\in L_\infty^m$, $x(t)\triangleq x_\sigma(t;t_0,x_0,u)$ denotes the state trajectory of system (1) with the maximal existing interval $[t_0,T_\sigma)$, where the constant $T_\sigma\triangleq T_\sigma(t_0,x_0,u)\leqslant\infty$.

Definition 1. Consider the following general nonlinear system

$$\dot{\omega}(t)=g(t,\omega(t),v(t)),\quad \omega(t_0)=\omega_0, \tag{3}$$

where function $g(\cdot,\cdot,\cdot):[t_0,\infty)\times\mathbb{R}^{n+m}\to\mathbb{R}^n$ satisfies $g(\cdot,0,0)\equiv 0$. For any $\omega_0\in\mathbb{R}^n$, $v\in L_\infty^m$, if the trajectory $\omega(t)\triangleq\omega(t;t_0,\omega_0,v)$ of (3) is defined well on $[t_0,\infty)$, then the system is called forward complete. For a closed set $\mathcal{M}\subset\mathbb{R}^n$, if system (3) is forward complete for any $\omega_0\in\mathcal{M}$, $v\in L_\infty^m$, and $\omega(t)\in\mathcal{M},\forall t\geqslant t_0$, then $\mathcal{M}$ is called a closed invariant set of system (3).

Remark 1. By Definition 1, if system (1) is forward complete for any $\sigma(t,x)$, then all of the subsystems are forward complete.

Remark 2. Obviously, if $\mathcal{M}$ is a closed invariant set of all subsystems of system (1), then it is also a closed invariant set of system (1).

Definition 2[1]. For the forward complete system (3) and its closed invariant set $\mathcal{M}\subset\mathbb{R}^n$, the system (3) is called (globally) input-to-state stable (ISS) with respect to $\mathcal{M}$, if there exist two functions $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}$ such that for any $\omega_0\in\mathbb{R}^n\setminus\mathcal{M}$ and $v\in L_\infty^m$,

$$|\omega(t;t_0,\omega_0,v)|_{\mathcal{M}}\leqslant\beta(|\omega_0|_{\mathcal{M}},t-t_0)+\gamma(\|v\|),\quad \forall t\geqslant t_0. \tag{4}$$

Definition 3[3]. For the forward complete system (3) and its closed invariant set $\mathcal{M}\subset\mathbb{R}^n$, a smooth function $V_g(\xi,t):\mathbb{R}^n\times[t_0,\infty)\to\mathbb{R}^+$ is called an ISS-Lyapunov function of the system (3) with respect to $\mathcal{M}\subset\mathbb{R}^n$, if there exist functions $\underline{\alpha},\overline{\alpha}\in\mathcal{K}_\infty$, $\alpha,\chi\in\mathcal{K}$ such that for any $\xi\in\mathbb{R}^n\setminus\mathcal{M}$, $\mu\in\mathbb{R}^m$ and $t\geqslant t_0$,

$$\underline{\alpha}(|\xi|_{\mathcal{M}})\leqslant V_g(\xi,t)\leqslant\overline{\alpha}(|\xi|_{\mathcal{M}}), \tag{5}$$

$$|\xi|_{\mathcal{M}}\geqslant\chi(|\mu|)\Rightarrow\tfrac{\partial}{\partial t}V_g(\xi,t)+\nabla V_g(\xi,t)\cdot g(t,\xi,\mu)\leqslant-\alpha(|\xi|_{\mathcal{M}}). \tag{6}$$

For short, they will be denoted as $(V_g;\underline{\alpha},\overline{\alpha},\alpha,\chi)$ in the sequel.

3 ISS conditions based on multiple Lyapunov functions

In this section, by using the multiple Lyapunov function method, sufficient ISS conditions are explored for switched nonlinear systems. To this end, we need the following lemmas.

Lemma 1[3]**.** For the forward complete system (3), assume $\mathcal{M} \subset \mathbb{R}^n$ is its closed invariant set. If system (3) has an ISS-Lyapunov function $(V_g; \underline{\alpha}, \overline{\alpha}, \alpha, \chi)$ such that (5) and (6) hold for any $\xi \in \mathbb{R}^n \setminus \mathcal{M}$, $\mu \in \mathbb{R}^m$ and $t \geqslant t_0$, then there exists $t'_g \triangleq t'_g(t_0, \omega_0, v)$ satisfying $t_0 \leqslant t'_g \leqslant \infty$ such that the solution $\omega(t) \triangleq \omega(t; t_0, \omega_0, v)$ of system (3) has the following property: $(\omega(t), t) \in S_g$ for any $t \geqslant t'_g$, and $(\omega(t), t) \notin S_g$ for any $t < t'_g$. Here,

$$S_g = \{(\xi, t): V_g(\xi, t) \leqslant \overline{\alpha} \circ \chi(\| v \|)\}.$$

Lemma 2[6]**.** For any $\kappa \in \mathcal{K}$, there exists a C^1 function $\rho \in \mathcal{K}_\infty$ such that

$$\dot{\rho}(r)\, \kappa(r) \geqslant \rho(r), \quad \forall r \geqslant 0.$$

Lemma 3. For the forward complete system (3) and its closed invariant set $\mathcal{M} \subset \mathbb{R}^n$, if system (3) has an ISS-Lyapunov function $(V_g; \underline{\alpha}, \overline{\alpha}, \alpha, \chi)$ such that (5) and (6) hold for any $\xi \in \mathbb{R}^n \setminus \mathcal{M}$, $\mu \in \mathbb{R}^m$ and $t \geqslant t_0$, then there exists a C^1 function $\rho \in \mathcal{K}_\infty$ depending only on $\overline{\alpha}$ and α such that

$$W_g(\xi, t) \geqslant \overline{\chi}(|\mu|) \Rightarrow \frac{\partial}{\partial t} W_g(\xi, t) + \nabla W_g(\xi, t) \cdot g(t, \xi, \mu) \leqslant -W_g(\xi, t),$$

where $W_g(\xi, t) = \rho \circ V_g(\xi, t)$ and $\overline{\chi}(\cdot) = \rho \circ \overline{\alpha} \circ \chi(\cdot) \in \mathcal{K}$.

Proof. For $\kappa(\cdot) \triangleq \alpha \circ \overline{\alpha}^{-1}(\cdot) \in \mathcal{K}$, by Lemma 2, $\overline{\alpha}$ and $\alpha \in \mathcal{K}_\infty$ can determine a C^1 function $\rho \in \mathcal{K}_\infty$ such that $\dot{\rho}(r)\, \kappa(r) \geqslant \rho(r), \forall r \geqslant 0$. Then, by testifying directly for $W_g(\xi, t) = \rho \circ V_g(\xi, t)$, one can obtain the conclusion.

Lemma 4. For the forward complete system (1), suppose that $\mathcal{M} \subset \mathbb{R}^n$ is a closed invariant set of system (1), and the switching instants of switching law $\sigma(t, x)$ are $t_1 < t_2 < \cdots < t_k < \cdots$. If for any given $i \in \mathcal{I}$, subsystem $f_i(t, x, u)$ has an ISS-Lyapunov function $(V_i; \underline{\alpha}_i, \overline{\alpha}_i, \alpha_i, \chi_i)$ such that $\overline{\alpha}(\cdot) \triangleq \sup_{i \in \mathcal{I}} \overline{\alpha}_i(\cdot) \in K_\infty$ and

$$\max\{V_{\sigma(t_{l-1}, x(t_{l-1}))}(x(t_l), t_l), \overline{\alpha} \circ \chi(\| u \|)\} \geqslant V_{\sigma(t_l, x(t_l))}(x(t_l), t_l), \tag{7}$$

then for set $S_i = \{(\xi, t): V_i(\xi, t) \leqslant \overline{\alpha} \circ \chi(\| v \|)\}\ (i \in \mathcal{I})$, there exists a common time-instant $t^*_\sigma \triangleq t^*_\sigma(t_0, x_0, u(t))$ such that

$$(x(t), t) \notin S_{\sigma(t, x(t))}, \ \forall t \in [t_0, t^*_\sigma), \tag{8}$$

$$(x(t), t) \leqslant S_{\sigma(t, x(t))}, \ \forall t \in [t^*_\sigma, +\infty). \tag{9}$$

Proof. For $l = 0, 1, 2, \cdots$, let $\sigma(t_l, x(t_l)) = i_l$. Then, by Lemma 1, for the subsystem $f_{i_l}(t, x, u)$ on the interval $[t_l, t_{l+1})$, there exists

$$t'_{i_l} \triangleq t'_{i_l}(t_l, x(t_l), u) \geqslant t_l, \tag{10}$$

such that for any $t \geqslant t'_{i_l}$, the state trajectory $x(t) \triangleq x_\sigma(t; t_0, x_0, u) = x_\sigma(t; t_{l-1}, x(t_{l-1}), u)$ satisfies $\big(x(t), t\big) \in S_{i_l}$ (that is, $V_{i_l}(x(t), t) \leqslant \overline{\alpha} \circ \chi(\| u \|)$); and for any $t < t'_{i_l}$, $\big(x(t), t\big) \notin S_{i_l}$.

If $t_{l+1} < t'_{i_l}$ for any $l = 0, 1, 2, \cdots$, then $\big(x(t), t\big) \notin S_{i_l}$ $(l = 0, 1, 2, \cdots)$ for any $t \geqslant t_0$. In this case, set $t^*_\sigma = \infty$. Otherwise, there exists a nonnegative integer l_0 such that $t'_{i_{l_0}} \leqslant t_{l_0+1}$. Let

$$l* = \min_{0 \leqslant l \leqslant l_0} \left\{ l : \ t'_{i_l} \leqslant t_{l+1} \right\}, \quad t^*_\sigma = t'_{i_{l*}}.$$

Then, by Lemma 1, we have (8), and $\big(x(t), t\big) \in S_{i_{l*}}$ for $\forall t \in [t^*_\sigma, t_{l*+1})$. Particularly, $(x(t_{l*+1}), t_{l*+1}) \in S_{i_{l*}}$, that is, $V_{i_{l*}}(x(t_{l*+1}), t_{l*+1}) \leqslant \overline{\alpha} \circ \chi(\| u \|)$. Thus, from (7), it follows that

$$V_{i_{l*+1}}(x(t_{l*+1}), t_{l*+1}) \leqslant \max\{V_{i_{l*}}(x(t_{l*+1}), t_{l*+1}), \overline{\alpha} \circ \chi(\| u \|)\} \leqslant \overline{\alpha} \circ \chi(\| u \|).$$

This implies that $t'_{i_{l*+1}} \triangleq t'_{i_{l*+1}}(t_{l*+1}, x(t_{l*+1}), u) \leqslant t_{l*+1}$. Therefore,

$$V_{i_{l*+1}}(x(t), t) \leqslant \overline{\alpha} \circ \chi(\| u \|), \quad \forall t \in [t_{l*+1}, t_{l*+2}).$$

Repeating the above process for $l = l*+2, l*+3, \cdots$, one can obtain (9).

Based on the method of multiple Lyapunov function, we have the following theorem.

Theorem 1. Consider the forward complete system (1). Suppose that $\mathcal{M} \subset \mathbb{R}^n$ is its closed invariant set, and the switching instants of the switching law $\sigma(t, x)$ are $t_1 < \cdots < t_k < \cdots$. If there exists ISS-Lyapunov function $(V_i; \underline{\alpha}_i, \overline{\alpha}_i, \alpha_i, \chi_i)$ of subsystem $f_i(t, x, u)$ ($i \in \mathcal{I}$) such that

(i) $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty, \alpha, \chi \in \mathcal{K}$, where $\underline{\alpha}(\cdot) \triangleq \inf_{i \in \mathcal{I}} \underline{\alpha}_i(\cdot)$, $\overline{\alpha}(\cdot) \triangleq \sup_{i \in \mathcal{I}} \overline{\alpha}_i(\cdot)$, $\alpha(\cdot) \triangleq \inf_{i \in \mathcal{I}} \alpha_i(\cdot)$ and $\chi(\cdot) \triangleq \sup_{i \in \mathcal{I}} \chi_i(\cdot)$;

(ii) (7) holds at each switching instant t_l ($l = 0, 1, 2, \cdots$),

then system (1) is input-to-state stable.

Proof. First, by Definition 3 and (i), for any $\xi \in \mathbb{R}^n \setminus \mathcal{M}$, $\mu \in \mathbb{R}^m$ and $i \in \mathcal{I}$, we have

$$\underline{\alpha}(|\xi|_\mathcal{M}) \leqslant V_i(\xi, t) \leqslant \overline{\alpha}(|\xi|_\mathcal{M}), \ \forall t \geqslant t_0,$$

$$|\xi|_\mathcal{M} \geqslant \chi(|\mu|) \Rightarrow \frac{\partial}{\partial t} V_i(\xi, t) + \nabla V_i(\xi, t) \cdot f_i(t, \xi, \mu) \leqslant -\alpha(|\xi|_\mathcal{M}), \ \forall t \geqslant t_0; \tag{11}$$

and by Lemma 2, we know that there exists a C^1 function $\rho \in \mathcal{K}_\infty$ depending only on $\overline{\alpha}$ and α such that $\dot{\rho}(r)\, \kappa(r) \geqslant \rho(r), \forall r \geqslant 0$, where $\kappa(\cdot) \triangleq \alpha \circ \overline{\alpha}^{-1}(\cdot)$. Let $W_i(\xi, t) = \rho \circ V_i(\xi, t)$ and $\overline{\chi}(\cdot) = \rho \circ \overline{\alpha} \circ \chi(\cdot)$. Then, by Lemma 3 we have

$$\rho \circ \underline{\alpha}(|\xi|_\mathcal{M}) \leqslant W_i(\xi, t) \leqslant \rho \circ \overline{\alpha}(|\xi|_\mathcal{M}), \quad \forall i \in \mathcal{I}, \tag{12}$$

$$W_i(\xi, t) \geqslant \overline{\chi}(|\mu|) \Rightarrow \frac{\partial}{\partial t} W_i(\xi, t) + \nabla W_i(\xi, t) \cdot f_i(t, \xi, \mu) \leqslant -W_i(\xi, t), \quad \forall i \in \mathcal{I}. \tag{13}$$

By Lemma 4, there exists t^*_σ such that (8) and (9) hold. Let $l*$ be the largest integer l such that $t_l \leqslant t^*_\sigma$. Then, from (8) and (9) and the definition of $W_i(\xi, t)$, it follows that

$$W_{\sigma(t_l, x(t_l))}(x(t), t) > \overline{\chi}(\| u \|), \quad \forall t \in [t_{l*}, t^*_\sigma) \ \text{ or } \ [t_l, t_{l+1}), \quad l = 0, 1, \cdots, l*-1, \tag{14}$$

$$W_{\sigma(t_l,x(t_l))}(x(t),t)\leqslant \bar{\chi}(\|u\|),\quad \forall t\in[t_\sigma^*,t_{l*+1})\ \text{ or }\ [t_l,t_{l+1}),\quad l=l*+1,\cdots \tag{15}$$

This together with (13) and (14) gives

$$\frac{d}{dt}W_{\sigma(t_l,x(t_l))}(x(t),t)<-W_{\sigma(t_l,x(t_l))}(x(t),t),\quad \forall t\in[t_{l*},t_\sigma^*)\ \text{ or }\ [t_l,t_{l+1}),\quad l=0,1,\cdots,l*-1. \tag{16}$$

Hence, we have

$$\begin{cases} W_{\sigma(t_{l*},x(t_{l*}))}(x(t),t)\leqslant W_{\sigma(t_{l*},x(t_{l*}))}(x(t_{l*}),t_{l*})e^{-(t-t_{l*})}, & \forall t\in[t_{l*},t_\sigma^*),\\ W_{\sigma(t_l,x(t_l))}(x(t_{l+1}),t_{l+1})\leqslant W_{\sigma(t_l,x(t_l))}(x(t_l),t_l)e^{-(t_{l+1}-t_l)}, & l=0,1,\cdots,l*-1. \end{cases} \tag{17}$$

From (7) and (8), the definition of $W_i(\xi,t)$, (14), and (13) it is easy to see

$$W_{\sigma(t_l,x(t_l))}(x(t_l),t_l)\leqslant W_{\sigma(t_{l-1},x(t_{l-1}))}(x(t_l),t_l),\quad l=0,1,\cdots,l*. \tag{18}$$

Then, for any $t\in[t_{l*},t_\sigma^*)$ or $[t_l,t_{l+1})$, $l=0,1,\cdots,l*-1,$ by (17) and (18) we have

$$\begin{aligned} W_{\sigma(t_l,x(t_l))}(x(t),t)&\leqslant W_{\sigma(t_{l-1},x(t_{l-1}))}(x(t_l),t_l)e^{-(t-t_l)}\\ &\leqslant W_{\sigma(t_{l-2},x(t_{l-2}))}(x(t_{l-1}),t_{l-1})e^{-(t-t_{l-1})}\leqslant\cdots\leqslant W_{\sigma(t_0,x(t_0))}(x(t_0),t_0)e^{-(t-t_0)}. \end{aligned}$$

This together with (12) and (15) leads to

$$\rho\circ\underline{\alpha}(|x(t)|_{\mathcal{M}})\leqslant\max\{\rho\circ\bar{\alpha}(|x_0|_{\mathcal{M}})e^{-(t-t_0)},\ \bar{\chi}(\|u\|)\}.$$

Let $\beta(t,s)=\underline{\alpha}^{-1}\circ\rho^{-1}(\rho(\bar{\alpha}(r))e^{-s})$ and $\gamma(r)=\underline{\alpha}^{-1}\circ\bar{\alpha}\circ\chi(r)$. Then, $\beta(t,s)\in\mathcal{KL},\gamma\in\mathcal{K}$ and

$$|x(t)|_{\mathcal{M}}=|x_\sigma(t;t_0,x_0,u)|_{\mathcal{M}}\leqslant\beta(|x_0|_{\mathcal{M}},t-t_0)+\gamma(\|u\|),\quad\forall t\geqslant t_0.$$

Thus, system (1) is input-to-state stable.

Remark 3. Condition (ii) of Theorem 1 says that the energy of the system should not be increasing at switching instants. This is because that the ISS is a global property holding for all $t\geqslant t_0$ with respect to $x(t_0)=x_0$ and $u(t)$, rather than a limit-sup property. Otherwise, for instance, if $\limsup_{t\to\infty}|x(t)|_{\mathcal{M}}$ is considered, then the condition can be relaxed to that (7) holds after finite switching instants.

Remark 4. From the proof of Theorem 1, we see that $\beta(t,s)\in\mathcal{KL}$ and $\gamma\in\mathcal{K}$ are independent of the concrete choice of $\sigma(\cdot,\cdot)$. In other words, the switched nonlinear system (1) is ISS for all $\sigma(\cdot,\cdot)$ satisfying (7).

Remark 5. In Theorem 1, instead of condition (ii), if we assume that there exists an integer l_0 such that $t_{l_0+1}\geqslant t'_{i_{l_0}}\triangleq t'_{i_{l_0}}(t_{l_0},x(t_{l_0}),u),$ and

$$\begin{cases} W_{\sigma(t_l,x(t_l))}(x(t_l),t_l)\leqslant W_{\sigma(t_{l-1},x(t_{l-1}))}(x(t_l),t_l), & l=0,1,\cdots,l_0,\\ W_{\sigma(t_l,x(t_l))}(x(t_l),t_l)\leqslant\alpha\circ\chi(\|u\|), & l=l_0+1,l_0+2,\cdots, \end{cases} \tag{19}$$

then system (1) is input-to-state stable.

In the sequel, we will provide some sufficient ISS conditions from another point of view by employing the concept of dwell-time of a switching law.

Definition 4. For a switching law $\sigma(t,x),$ suppose its switching instants are $t_0<t_1<\cdots<t_k<\cdots.$ Then, $\tau=\inf_{k\geqslant 0}(t_{k+1}-t_k)$ is called its dwell-time.

Corollary 1. For the forward complete system (1), suppose that $\mathcal{M} \subset \mathbb{R}^n$ is its closed invariant set, and the switching instants of switching law $\sigma(t,x)$ are $t_0 < t_1 < \cdots < t_k < \cdots$. Under the conditions and notations of Theorem 1, instead of condition (ii), if we assume that there exists an integer l_0 such that $\tau \geqslant t'_{i_{l_0}} - t_{l_0}$ and (19) holds, then system (1) is input-to-state stable. Here, t'_{i_l} is given by (10).

Proof. By Definition 4, we see $t_{l_0+1} \geqslant t_{l_0} + \tau$. Hence, $t_{l_0+1} \geqslant t'_{i_{l_0}} \triangleq t'_{i_{l_0}}(t_{l_0}, x(t_{l_0}), u)$. This together with Remark 5 leads to the conclusion.

Corollary 2. For the forward complete system (1), suppose that $\mathcal{M} \subset \mathbb{R}^n$ is its closed invariant set, and the switching instants of $\sigma(t,x)$ are $t_0 < t_1 < \cdots < t_k < \cdots$, and the index set $\mathcal{I}$ is finite and denoted with $\{1,2,\cdots,N\}(N<\infty)$. If there exits ISS-Lyapunov function $(V_i; \underline{\alpha}_i, \overline{\alpha}_i, \alpha_i, \chi_i)$ $(i \in \mathcal{I})$ such that

$$\max\{V_{\sigma(t_{l-1},x(t_{l-1}))}(x(t_l),t_l), \overline{\alpha}_1 \circ \chi(\|u\|), \cdots, \overline{\alpha}_N \circ \chi(\|u\|)\} \geqslant V_{\sigma(t_l,x(t_l))}(x(t_l),t_l),$$

for $l = 0,1,\cdots,$ then system (1) is input-to-state stable.

Its proof is easy and omitted.

4 ISS conditions based on the average dwell-time

In this section, we will use the concept of average dwell-time to obtain some sufficient ISS conditions for both SNL systems and switched linear systems.

Definition 5[17]**.** For any given constants $\tau^* > 0$ and N_0, let $N_\sigma(s,t)$ denote the switch number of $\sigma(t,x)$ in $[s,t), \forall t > s \geqslant t_0,$ and let

$$\mathcal{S}[\tau^*, N_0] = \left\{\sigma(\cdot,\cdot): N_\sigma(s,t) \leqslant N_0 + \frac{t-s}{\tau^*}, \ \ \forall t > s \geqslant t_0\right\}.$$

Then, τ^* is called the average dwell-time of $\mathcal{S}[\tau^*, N_0]$, and

$$\tau_\sigma \triangleq \sup_{t \geqslant t_0} \sup_{t > s \geqslant t_0} \frac{t-s}{N_\sigma(s,t) - N_0}$$

is called the average dwell-time of $\sigma(t,x)$.

4.1 ISS analysis of switched nonlinear systems

Theorem 2. For the forward complete system (1), suppose that $\mathcal{M} \subset \mathbb{R}^n$ is its closed invariant set, and switching instants of switching law $\sigma(t,x)$ are $t_0 < t_1 < \cdots < t_k < \cdots$. If there are ISS-Lyapunov functions $(V_i; \underline{\alpha}_i, \overline{\alpha}_i, \alpha_i, \chi_i)$ $(i \in \mathcal{I})$ and constants $c > 0$, $\eta_0 \geqslant 1$, such that for any $\xi \in \mathbb{R}^n \setminus \mathcal{M}$, $\mu \in \mathbb{R}^m$ and $i \in \mathcal{I}$,

$$\underline{\alpha}(|\xi|_{\mathcal{M}}) \leqslant V_i(\xi,t) \leqslant \overline{\alpha}(|\xi|_{\mathcal{M}}), \quad \forall t \geqslant t_0, \tag{20}$$

$$|\xi|_{\mathcal{M}} \geqslant \chi(|\mu|) \Rightarrow \frac{\partial}{\partial t}V_i(\xi,t) + \nabla V_i(\xi,t) \cdot f_i(t,\xi,\mu) \leqslant -cV_i(\xi,t), \quad \forall t \geqslant t_0, \tag{21}$$

$$\max\{\eta_0 V_{\sigma(t_{l-1},x(t_{l-1}))}(x(t_l),t_l), \overline{\alpha} \circ \chi(\|u\|)\} \geqslant V_{\sigma(t_l,x(t_l))}(x(t_l),t_l), \quad l = 0,1,\cdots, \tag{22}$$

then system (1) is input-to-state stable for any $\sigma(\cdot,\cdot)\in\mathcal{S}_0\left[\frac{\ln\eta_0}{c},N_0\right]$, where

$$\mathcal{S}_0\left[\frac{\ln\eta_0}{c},N_0\right]=\left\{\sigma(\cdot,\cdot)\in\mathcal{S}[\tau^*,N_0]:\tau^*>\frac{\ln\eta_0}{c}\right\}.$$

Proof. For a given time instant $t\geqslant t_0$, assume that system (1) has j switching instants in $[t_0,t)$, and denote them as $t_1<t_2<\cdots<t_j$. Let $\sigma(t_l,x(t_l))=i_l\ (l=0,1,\cdots,j)$ and $t'_{i_l}\ (l=0,1,\cdots)$ be the time instant defined according to (10). If $t\geqslant t'_{i_j}$ on $[t_j,t)$, then by Lemma 1,

$$V_{i_j}(x(t),t)\leqslant\bar{\alpha}\circ\chi(\|u\|);\tag{23}$$

if $t<t'_{i_j}$, then $V_{i_j}(x(s),s)>\bar{\alpha}\circ\chi(\|u\|)$ for any $s\in[t_j,t)$, and hence, by (20) we have

$$|x(s)|_{\mathcal{M}}>\chi(\|u\|),\quad\forall s\in[t_j,t).\tag{24}$$

This together with (21) gives

$$\frac{d}{ds}V_{i_j}(x(s),s)\leqslant -cV_{i_j}(x(s),s),\quad\forall s\in[t_j,t).\tag{25}$$

Thus, when $t<t'_{i_j}$, we have

$$V_{i_j}(x(s),s)\leqslant V_{i_j}(x(t_j),t_j)e^{-c(s-t_j)},\quad\forall s\in[t_j,t).\tag{26}$$

Now, let us consider the interval $[t_{l-1},t_l)\ (l=1,2,\cdots,j)$. When $t_l\geqslant t'_{i_{l-1}}$, we have

$$V_{i_{l-1}}(x(t_l),t_l)\leqslant\bar{\alpha}\circ\chi(\|u\|);\tag{27}$$

while when $t_l<t'_{i_{l-1}}$, we have $V_{i_{l-1}}(x(s),s)>\bar{\alpha}\circ\chi(\|u\|),\forall s\in[t_{l-1},t_l)$, which together with (20) leads to

$$|x(s)|_{\mathcal{M}}>\chi(\|u\|),\quad\forall s\in[t_{l-1},t_l).\tag{28}$$

Then, by (21) we have

$$\frac{d}{ds}V_{i_{l-1}}(x(s),s)\leqslant -cV_{i_{l-1}}(x(s),s),\quad\forall s\in[t_{l-1},t_l).\tag{29}$$

This implies that

$$V_{i_{l-1}}(x(s),s)\leqslant V_{i_{l-1}}(x(t_{l-1}),t_{l-1})e^{-c(s-t_{l-1})},\quad\forall s\in[t_{l-1},t_l).\tag{30}$$

In particular,

$$V_{i_{l-1}}(x(t_l),t_l)\leqslant V_{i_{l-1}}(x(t_{l-1}),t_{l-1})e^{-c(t_l-t_{l-1})}.\tag{31}$$

Let $\pi=\bar{\alpha}\circ\chi(\|u\|)$. Then, by (23), (26), (27), and (31), we have

$$V_{i_j}(x(t),t)\leqslant\max\left\{V_{i_j}(x(t_j),t_j)e^{-c(t-t_j)},\pi\right\},\tag{32}$$

$$V_{i_{l-1}}(x(t_l),t_l)\leqslant\max\left\{V_{i_{l-1}}(x(t_{l-1}),t_{l-1})e^{-c(t_l-t_{l-1})},\pi\right\},\quad l=1,2,\cdots,j.\tag{33}$$

Recalling (22) and substituting (33) into (32) sequentially, we obtain

$$V_{i_j}(x(t),t)\leqslant\max\Big\{\eta_0^jV_{i_0}(x(t_0),t_0)e^{-c(t_1-t_0)}\cdots e^{-c(t_j-t_{j-1})}e^{-c(t-t_j)},$$
$$\eta_0^{j-1}\pi e^{-c(t_2-t_1)}\cdots e^{-c(t_j-t_{j-1})}e^{-c(t-t_j)},\cdots,\pi e^{-c(t-t_j)},\pi\Big\}.$$

Notice that $N_\sigma(t_l,t)=j-l\ (l=0,1,2,\cdots,j)$. Then, we have $\eta_0^{j-l}=e^{N_\sigma(t_l,t)\ln\eta_0}$. Thus,

$$V_{i_j}(x(t),t)\leqslant\max\Big\{\bar\alpha(|x(t_0)|_{\mathcal{M}})e^{N_\sigma(t_0,t)\ln\eta_0-c(t-t_0)},\pi e^{N_\sigma(t_1,t)\ln\eta_0-c(t-t_1)},\\ \cdots,\pi e^{N_\sigma(t_{j-1},t)\ln\eta_0-c(t_{j-1}-t)},\pi e^{-c(t-t_j)},\pi\Big\}.$$

Let $a=c-\dfrac{\ln\eta_0}{\tau^*}$. Then, $a>0$. By Definition 5, for any given $\sigma(\cdot,\cdot)\in\mathcal{S}_0\left[\dfrac{\ln\eta_0}{c},N_0\right]$, we have $N_\sigma(s,t)\leqslant N_0+\dfrac{t-s}{\tau^*}$, $\forall t>s\geqslant t_0$. This results in $N_\sigma(s,t)\ln\eta_0-c(t-s)\leqslant N_0\ln\eta_0-a(t-s)$. Therefore, by (20) we have

$$\underline{\alpha}(|x(t_0)|_{\mathcal{M}})\leqslant V_{i_j}(x(t),t)\leqslant\max\Big\{\bar\alpha(|x(t_0)|_{\mathcal{M}})e^{N_0\ln\eta_0-a(t-t_0)},\ \bar\alpha\circ\chi(\|u\|)e^{N_0\ln\eta_0}\Big\}.$$

Let $\beta(r,s)=\underline{\alpha}^{-1}(\bar\alpha(r)e^{N_0\ln\eta_0-as})$, $\gamma(r)=\underline{\alpha}^{-1}(\bar\alpha\circ\chi(r)e^{N_0\ln\eta_0})$, $\forall r\geqslant 0,\forall s\geqslant t_0$. Then $\beta\in\mathcal{KL}$, $\gamma\in\mathcal{K}$ and $|x(t)|_{\mathcal{M}}=|x_\sigma(t;t_0,x_0,u)|_{\mathcal{M}}\leqslant\beta(|x_0|_{\mathcal{M}},t-t_0)+\gamma(\|u\|)$, $\forall t\geqslant t_0$. Thus, system (1) is input-to-state stable.

Remark 6. By $\eta_0\geqslant 1$, the condition (22) in Theorem 2 is obviously weaker than the condition (7) in Theorem 1. This includes the case where the "energy" of the subsystem after a switching instant is greater than that of the subsystem before the switching instant.

4.2 ISS analysis of switched linear systems

In this subsection, we will investigate switched linear multi-variable systems of the form

$$\dot x(t)=A_{\sigma(t,x(t))}x(t)+B_{\sigma(t,x(t))}u(t),\ \ x(t_0)=x_0,\tag{34}$$

where $A_i\in\mathbb{R}^{n\times n}$, $B_i\in\mathbb{R}^{n\times m}$ are constant matrices for any $i\in\mathcal{I}$, respectively.

In the sequel, for any $n\times n$ matrix A, J_A denotes the Jordanian normal form of A, $\eta(A)$ is the largest real part of the eigenvalues of A, and $\Delta_M(A)$ and $\Delta_m(A)$ are the largest and smallest singular values of matrix A, respectively.

For a given matrix set $\mathcal{A}\in\mathbb{R}^{n\times n}$, $\mathcal{A}_1$ denotes the set of all stable matrices of $\mathcal{A}$, and $\mathcal{A}_2$ denotes $\mathcal{A}\backslash\mathcal{A}_1$. Noticing that $\eta(A)$ depends continuously upon the parameter of A, when $\mathcal{A}_1$ is compact, we have $\max_{A\in\mathcal{A}_1}\eta(A)<0$ and $|\max_{A\in\mathcal{A}_1}\eta(A)|=\min_{A\in\mathcal{A}_1}|\eta(A)|$. In particular, when the number of all stable matrices in $\mathcal{A}$ is finite and greater than zero, we have $\max_{A\in\mathcal{A}_1}\eta(A)<0$.

Lemma 5. For any given matrix set $\mathcal{A}\subset\mathbb{R}^{n\times n}$, if $\mathcal{A}$ and $\mathcal{A}_1$ are compact, and $\mathcal{A}_1$ is nonempty, then for any $\varepsilon\in(0,\min_{A\in\mathcal{A}_1}|\eta(A)|)$, there exists a constant $M(\varepsilon)>0$ such that

$$|e^{At}|\leqslant\lambda_\varepsilon(A)e^{a_\varepsilon(A)t},\quad\forall A\in\mathcal{A},\forall t\geqslant 0,\tag{35}$$

$$\lambda_\varepsilon(\mathcal{A})\triangleq\max_{A\in\mathcal{A}}\lambda_\varepsilon(A)<\infty,\tag{36}$$

where $a_\varepsilon(A)=\eta(A)+\varepsilon$, $\lambda_\varepsilon(A)\triangleq\sqrt{M(\varepsilon)}\dfrac{\Delta_M(P(A))}{\Delta_m(P(A))}>0$, and $P(A)$ is $n\times n$ nonsingular matrix satisfying $A = P(A) J_A P(A)^{-1}$.

Proof. For any given $A \in \mathcal{A}$, assume that it has p Jordan blocks with dimensions of $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_p$, and the real parts of the corresponding eigenvalues are $\tau_1, \tau_2, \cdots, \tau_p$, respectively. Then, $n_1 + n_2 + \cdots + n_p = n$ and $\eta(A) = \max_{1 \leqslant j \leqslant p} \tau_j$.

Noticing that $e^{At} = P(A) e^{J_A t} P(A)^{-1}$, we have

$$|e^{At}| = |P(A) e^{J_A t} P(A)^{-1}| \leqslant \frac{\Delta_M(P(A))}{\Delta_m(P(A))} |e^{J_A t}|, \quad \forall t > 0.$$

Further, it is easy to see that $|e^{J_A t}| = (\Sigma(t))^{1/2}$, where

$$\Sigma(t) = e^{2\tau_1 t}\left[n_1 + (n_1 - 1)t^2 + (n_1 - 2)\frac{t^4}{(2!)^2} + \cdots + \frac{t^{2(n_1 - 1)}}{((n_1 - 1)!)^2}\right] + e^{2\tau_2 t}\left[n_2 + (n_2 - 1)t^2 + (n_2 - 2)\frac{t^4}{(2!)^2}\right.$$
$$\left. + \cdots + \frac{t^{2(n_2 - 1)}}{((n_2 - 1)!)^2}\right] + \cdots + e^{2\tau_p t}\left[n_p + (n_p - 1)t^2 + (n_p - 2)\frac{t^4}{(2!)^2} + \cdots + \frac{t^{2(n_p - 1)}}{((n_p - 1)!)^2}\right].$$

Since for $\forall \varepsilon \in (0, \min_{A \in \mathcal{A}_1} |\eta(A)|)$,

$$\frac{\Sigma(t)}{e^{2a_\varepsilon(A)t}} \leqslant \left[n + (n - p)t^2 + (n - 2p)\frac{t^4}{(2!)^2} + \cdots + \frac{t^{2(n_p - 1)}}{((n_p - 1)!)^2}\right] e^{-2\varepsilon t},$$

we have $\lim_{t \to \infty} \frac{\Sigma(t)}{e^{2a_\varepsilon(A)t}} = 0$, and hence, $M(\varepsilon) \triangleq \sup_{t \in [0,\infty)} \frac{\Sigma(t)}{e^{2a_\varepsilon(A)t}} < \infty$, and

$$\frac{|e^{At}|}{e^{a_\varepsilon(A)t}} \leqslant \frac{\Delta_M(P(A))}{\Delta_m(P(A))} \frac{|e^{J_A t}|}{e^{a_\varepsilon(A)t}} = \frac{\Delta_M(P(A))}{\Delta_m(P(A))} \left(\frac{\Sigma(t)}{e^{2a_\varepsilon(A)t}}\right)^{\frac{1}{2}} \leqslant \lambda_\varepsilon(A),$$

i.e., (35) holds.

Further, by the compactness of $\mathcal{A}$, nonsingularity of $P(A)$ on $\mathcal{A}$, and the continuity of $\Delta_M(P(A))$ and $\Delta_m(P(A))$ on $\mathcal{A}$, $\lambda_\varepsilon(A)$ is continuous on the compact set $\mathcal{A}$ with respect to A. Thus, (36) holds.

Now, we study the ISS property of the switched linear system (34).

For system (34), let $\mathcal{A} = \{A_i : i \in \mathcal{I}\} \subset \mathbb{R}^{n \times n}$ and $\mathcal{B} = \{B_i : i \in \mathcal{I}\} \subset \mathbb{R}^{n \times n}$, assume that $\mathcal{A}$ and $\mathcal{B}$ are compact, and the subset $\mathcal{A}_1$ consisting of all the stable matrices of $\mathcal{A}$ is nonempty and compact. For any given $\varepsilon \in (0, \min_{A \in \mathcal{A}_1} |\eta(A)|)$, define $a_\varepsilon(A)$ and $\lambda_\varepsilon(A)$ as in Lemma 5, and set

$$a_\varepsilon^-(A) = \min_{A \in \mathcal{A}_1} |a_\varepsilon(A)|, \quad a_\varepsilon^+(A) = \max\{0, \max_{A \in \mathcal{A}} a_\varepsilon(A)\}, \tag{37}$$

$$b_0(\mathcal{B}) = \max_{B_i \in \mathcal{B}} |B_i|, \quad M_\varepsilon = e^{(1+N_0)\ln \lambda_\varepsilon(\mathcal{A})}. \tag{38}$$

For a given switching law $\sigma(t, x)$ and a time interval $[s, t)$, let $T_\sigma^+(s, t)$ and $T_\sigma^-(s, t)$ be the total time of system (34) running on stable subsystems and unstable subsystems in $[s, t)$, respectively; and for any $a^* \in (0, a_\varepsilon^-(\mathcal{A})]$ and $\tau^* > 0$, define

$$\mathcal{S}[a^*,\tau^*;\mathcal{A}]=\left\{\sigma(\cdot,\cdot)\in\mathcal{S}[\tau^*,N_0]:\sup_{t>s\geqslant t_0}\frac{T_\sigma^+(s,t)}{T_\sigma^-(s,t)}\leqslant\frac{a_\varepsilon^-(A)-a^*}{a_\varepsilon^+(A)+a^*},\ \ \tau_\sigma>\tau^*\right\},$$

where the average dwell-time τ_σ of $\sigma(t,x)$ is given by Definition 5.

In the sequel, for simplicity of expression, we will drop the arguments of $\lambda_\varepsilon(\mathcal{A})$, $a_\varepsilon^-(A)$, $a_\varepsilon^+(A)$ and $b_0(\mathcal{B})$ and denote them by $\lambda_\varepsilon, a_\varepsilon^-, a_\varepsilon^+$ and b_0, respectively.

Theorem 3. For switched linear system (34), assume that $\mathcal{A}$ and $\mathcal{B}$ are compact, and the subset $\mathcal{A}_1$ consisting of all the stable matrices of $\mathcal{A}$ is nonempty and compact. Then for any given $\varepsilon\in(0,\min_{A\in\mathcal{A}_1}|\eta(A)|)$, and $a^*\in(0,a_\varepsilon^-]$, there exists $\tau^*\geqslant\frac{1}{a^*}\ln\lambda_\varepsilon$ such that

(i) for any $\sigma(\cdot,\cdot)\in\mathcal{S}[a^*,\tau^*;\mathcal{A}]$, system (34) is forward complete, and

(ii) system (34) is ISS if and only if the control-free system $\dot{x}(t)=A_{\sigma(t,x(t))}x(t)$ is asymptotically stable.

Proof. Part (i) and the necessity of part (ii) are obvious. Thus, below we need only to show the sufficiency.

For any given time instant $t\geqslant t_0$, assume that in the time interval $[t_0,t)$, system (34) has j switching instants: $t_1<t_2<\cdots<t_j$. Let $\sigma(t_l,x(t_l))=i_l\ (l=0,1,\cdots,j)$. Then, the solution of system (34) can be expressed as

$$x(t)=x_\sigma(t;t_0,x_0,u)=\Phi(t,t_0)x_0+\int_{t_0}^{t_1}\Phi(t,s)B_{\sigma(t_0,x(t_0))}u(s)ds+\cdots+\int_{t_j}^{t}\Phi(t,s)B_{\sigma(t_j,x(t_j))}u(s)ds, \tag{39}$$

where

$$\Phi(t,s)=e^{A_{\sigma(t_j,x(t_j))}(t-t_j)}e^{A_{\sigma(t_{j-1},x(t_{j-1}))}(t_j-t_{j-1})}\cdots e^{A_{\sigma(t_l,x(t_l))}(t_{l+1}-s)},s\in[t_l,t_{l+1}).$$

We first show that for any given constant $a^*\in(0,a_\varepsilon^-]$, $\tau^*\geqslant\frac{1}{a^*}\ln\lambda_\varepsilon$, and switching law $\sigma(\cdot,\cdot)\in\mathcal{S}[a^*,\tau^*;\mathcal{A}]$, there are $a>0$ and $M>0$ such that

$$|\Phi(t,s)|\leqslant Me^{-a(t-s)},\quad\forall\, t\geqslant s\geqslant t_0. \tag{40}$$

Noticing that

$$N_\sigma(s,t)=\begin{cases}j-l, & t_l\leqslant s\leqslant t_{l+1},\quad l=0,1,2,\cdots,j-1;\\ 0, & t_j\leqslant s\leqslant t,\end{cases}$$

we have $\lambda_\varepsilon=e^{[1+N_\sigma(s,t)]\ln\lambda_\varepsilon}$ for $s\in[t_j,t)$; and $\lambda_\varepsilon^{j-l+1}=e^{[1+N_\sigma(s,t)]\ln\lambda_\varepsilon}$ for $s\in[t_l,t_{l+1})$ $(l=0,1,\cdots,j)$. In particular, $N_\sigma(t_0,t)=j$ and $\lambda_\varepsilon^{j+1}=e^{[1+N_\sigma(t_0,t)]\ln\lambda_\varepsilon}$. Thus,

$$\begin{aligned}|\Phi(t,s)|&\leqslant\lambda_\varepsilon^{j-l+1}e^{a_\varepsilon(A_{\sigma(t_j,x(t_j))})(t-t_j)}e^{a_\varepsilon(A_{\sigma(t_{j-1},x(t_{j-1}))})(t_j-t_{j-1})}\cdots e^{a_\varepsilon(A_{\sigma(t_l,x(t_l))})(t_{l+1}-s)}\\ &\leqslant e^{[1+N_\sigma(s,t)]\ln\lambda_\varepsilon}e^{a_\varepsilon^+T_\sigma^+(s,t)-a_\varepsilon^-T_\sigma^-(s,t)}.\end{aligned}$$

By the definition of $\mathcal{S}[a^*,\tau^*;\mathcal{A}]$, for any given $\sigma(\cdot,\cdot)\in\mathcal{S}[a^*,\tau^*;\mathcal{A}]$, we have $a_\varepsilon^+T_\sigma^+(s,t)-a_\varepsilon^-T_\sigma^-(s,t)\leqslant-a^*(t-s)$. This together with $N_\sigma(s,t)\leqslant N_0+\frac{t-s}{\tau_\sigma}$ and $\tau_\sigma>\tau^*\geqslant\frac{1}{a^*}\ln\lambda_\varepsilon$

implies that $a \triangleq a* - \frac{1}{\tau^*}\ln\lambda_\varepsilon > 0$. Then, by some straightforward calculations, we have

$$|\Phi(t,s)| \leqslant Me^{-a(t-s)}, \quad \forall\, s \in [t_l, t_{l+1}) \cup [t_j, t), \quad l = 0,1,\cdots, j-1,$$

where $M = e^{(1+N_0)\ln\lambda_\varepsilon}$. Thus, (40) is true, which together with (39) gives

$$\begin{aligned}|x_\sigma(t;t_0,x_0,u)| &\leqslant Me^{-a(t-t_0)}|x_0| + M\left(\int_{t_0}^{t_1} e^{-a(t-s)}b_0\|u\|ds + \cdots + \int_{t_j}^{t} e^{-a(t-s)}b_0\|u\|ds\right)\\ &\leqslant Me^{-a(t-t_0)}|x_0| + M\int_{t_0}^{t} e^{-a(t-s)}b_0\|u\|ds \leqslant Me^{-a(t-t_0)}|x_0| + \frac{Mb_0}{a}\|u\|.\end{aligned}$$

Let $\beta(r,s) = Me^{-as}r$ and $\gamma(r) = \frac{Mb_0}{a}r$. Then, $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and

$$|x_\sigma(t;t_0,x_0,u)| \leqslant \beta(|x_0|, t-t_0) + \gamma(\|u\|),$$

i.e., system (34) is input-to-state stable.

Remark 7. Comparing Theorem 3 with Theorem 2, one can see that for switched linear system case, some of the subsystems of system (34) are allowed to be unstable. However, for switched nonlinear systems, all of its subsystems are required to be stable, since the degree of instability of nonlinear systems is hard to be characterized.

Remark 8. By Theorem 3, the ISS of switched linear system (34) is independent of the concrete choice of $\sigma(\cdot,\cdot)$ in $\mathcal{S}[a*,\tau*;\mathcal{A}]$.

5 Conclusion

In this paper, the ISS of switched nonlinear system and switched linear system are investigated, respectively. The main results can roughly be divided into two classes. One is based on multiple Lyapunov function method, and the other is based on (average) dwell-time method. Firstly, by using the method of multiple Lyapunov function, a sufficient ISS condition is given for general SNL systems based on a quantitative relation of the control and the values of the Lyapunov functions of the subsystems before and after the switching instants. Here, the ISS-Lyapunov functions of the subsystems are allowed to be different from each other rather than simply assuming the existence of a CISSLF. Thus, the condition is sufficient not only for the switched systems possessing a CISSLF, but also sufficient for the switched systems without any CISSLF. Secondly, by employing the method of the average dwell-time, some ISS sufficient conditions are given for switched nonlinear systems and switched linear systems, respectively. Among others, the condition on switched nonlinear systems is characterized by the size of the dwell-time, and that on switched linear systems is characterized by the average dwell-time and the ratio of the total time that the system runs on unstable subsystems to the total time that the system runs on stable subsystems.

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