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Global output-feedback stabilization for a class of stochastic non-minimum-phase nonlinear systems ☆

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Abstract

In this paper, the problem of output-feedback stabilization is investigated for the first time for a class of stochastic nonlinear systems whose zero dynamics may be unstable. Under the assumption that the inverse dynamics of the system is stochastic input-to-state stabilizable, a stabilizing output-feedback controller is constructively designed by the integrator backstepping method together with a new reduced-order observer design and the technique of changing supply functions. It is shown that, under small-gain type conditions for small signals, the resulting closed-loop system is globally asymptotically stable in probability. The obtained results extend the existing methodology from deterministic systems to stochastic systems. An example is given to demonstrate the main features and effectiveness of the proposed output-feedback control scheme.

Keywords: Stochastic nonlinear systems; Integrator backstepping; Stochastic input-to-state stability; Non-minimum phase; Output-feedback stabilization

1. Introduction

Global output-feedback control for non-minimum-phase nonlinear systems is a challenging problem in control theory. It is known that stabilizable and detectable linear systems are globally output-feedback stabilizable, but only some classes of minimum-phase nonlinear systems are known to be globally stabilizable by output feedback (see Battilotti, 1997; Huang, 2004; Krishnamurthy, Khorrami, & Jiang, 2002; Krstić, Kanellakopoulos, & Kokotović, 1995; Marino & Tomei, 1991; Praly & Jiang, 1993; Praly, 2003, and the references therein). Semiglobal output-feedback stabilization can be achieved for nonlinear systems which are globally state-feedback stabilizable and uniformly observable (Teel & Praly, 1994). No such general results are so far available on the global output-feedback stabilization of non-minimum-phase nonlinear systems. Nevertheless, some interesting results have been proposed for some classes of non-minimum-phase nonlinear systems; see, for instance, Andrieu and Praly (2005), Isidori (2000), Karagiannis, Jiang, Ortega, and Astolfi (2005), Marino and Tomei (2005).

Global stabilization for stochastic nonlinear systems has been an active area of research in recent years (Arslan & Başar, 2002; Deng & Krstić, 1999, 2000; Deng, Krstić, & Williams, 2001; Florchinger, 1995; Krstić & Deng, 1998; Liu, Zhang, & Jiang, 2007; Liu, Pan, & Shi, 2003; Liu, Zhang, & Pan, 2003; Liu & Zhang, 2006; Pan & Başar, 1999; Pan, Liu, & Shi, 2001). The design tool used in these recent works is based on the famous integrator backstepping method, which has been widely used to solve numerous control problems of both theoretic and practical importance for deterministic nonlinear systems; see Isidori (1999), Jiang (1999), Jiang and Praly (1998), Kokotović and Arcak (2001), Krstić et al. (1995), Pan and Başar (1998), and the numerous references therein. In Pan (2002), the author examines three canonical forms of stochastic nonlinear systems, namely the strict-feedback form, observer canonical form and zero dynamics canonical form. The early work

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focused on stabilization and risk-sensitive control for systems in the strict-feedback form (Arslan & Başar, 2002; Deng & Krstić, 1999, 2000; Deng et al., 2001; Liu, Pan, et al. 2003; Liu, Zhang, et al. 2003; Pan & Başar, 1999). For systems in the observer canonical form, which can be transformed into the ones with linear zero dynamics by coordinate transformation, the problems of output-feedback stabilization and practical output-feedback risk-sensitive control are investigated in Pan et al. (2001), and Liu and Zhang (2006), respectively. For systems with stochastic nonlinear zero dynamics, in Liu et al. (2007) a novel systematic procedure was given to design decentralized, adaptive, output-feedback controllers achieving practical and asymptotic stabilization. The design techniques rely upon the concept of stochastic input-to-state stability and the method of changing supply functions. However, these results are limited to the systems without zero dynamics or with stable zero dynamics. To the best of our knowledge, for stochastic nonlinear systems with unstable zero dynamics, there is only few published work. In Pan and Başar (1999), the problem of full state-feedback risk-sensitive control was studied for a class of nonlinear systems with strongly stabilizable zero dynamics. But, when the states of the zero dynamics are not measurable, even for this class of systems with strongly stabilizable zero dynamics, it remains a difficult issue how to stabilize the systems by output feedback.

Inspired by the recent papers Andrieu and Praly (2005), Karagiannis, Ortega, and Astolfi (2003), and Karagiannis et al. (2005) on the stabilizer design for deterministic non-minimum-phase systems and our work Liu et al. (2007), in this paper, we consider the output-feedback stabilization problem for a class of stochastic nonlinear systems with unstable zero dynamics. It is shown that under the assumption that the inverse dynamics is stochastic input-to-state stabilizable, the systems can be stabilized by an output-feedback controller which is designed based on the integrator backstepping method and the techniques of novel reduced-order observer design and changing supply functions. The obtained results extend the existing methodology from deterministic systems to stochastic systems.

The remainder of the paper is organized as follows. Section 2 provides some notations. Section 3 describes the problem to be investigated. Section 4 presents the design of reduced-order observer. The output-feedback control design procedure is given in Section 5. Stability analysis of the closed-loop system in question is given in Section 6. The main conditions and the systems satisfying these conditions are discussed in Section 7. Section 8 gives a numerical example to illustrate the effectiveness of our methods. Section 9 contains some concluding remarks. Mathematical preliminaries on the stability of stochastic nonlinear systems are given in Appendix A.

2. Notation

The following notations will be used throughout this paper. $\mathbb{R}_+$ denotes the set of all nonnegative real numbers; $\mathbb{R}^n$ denotes the real n-dimensional space; $\mathbb{R}^{n\times r}$ denotes the real $n\times r$ matrix space. For a given vector or matrix X, X^{T} denotes its transpose; $\mathrm{Tr}(X)$ denotes its trace when X is square; $|X|$ denotes the Euclidean norm of a vector X; $\|X\|$ denotes the Frobenius norm of the matrix X defined by $\|X\| = \sqrt{\mathrm{Tr}(X^{\mathrm{T}}X)}$; $\lambda_{\min}(X)$ denotes the minimal eigenvalue of symmetric real matrix X; $\mathcal{C}^i$ denotes the set of all functions with continuous ith partial derivatives; $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanish at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded; $\mathcal{KL}$ denotes the set of all functions $\beta(s,t)$: $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which are of class $\mathcal{K}$ for each fixed t, and decrease to zero as $t\to\infty$ for each fixed s.

For a given stochastic system

$$\mathrm{d}x = (f(x) + g(x)u)\mathrm{d}t + h(x)\mathrm{d}w,$$

define a differential operator $\mathcal{L}$ as follows:

$$\mathcal{L}V = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u + \frac{1}{2}\mathrm{Tr}\left\{h^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}h\right\},$$

where $V(x)\in\mathcal{C}^2$; $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}$ is the control input; $f\in\mathbb{R}^n$, $g\in\mathbb{R}^n$ and $h\in\mathbb{R}^{n\times r}$ are $\mathcal{C}^1$ functions; w is an r-dimensional standard Brownian motion.

3. Problem formulation

Consider the system described by the following Itô stochastic differential equations

$$\begin{aligned}\mathrm{d}x_z &= [F_0(y)x_z + H(y) + f_0(x_z,y)]\mathrm{d}t \\ &\quad + g_0(x_z,y)\mathrm{d}w, \end{aligned} \tag{1}$$

$$\begin{aligned}\mathrm{d}x_1 &= (x_2 + h_1^{\mathrm{T}}(y)x_z + f_1(x_z,y))\mathrm{d}t + g_1(x_z,y)\mathrm{d}w,\\ &\vdots \\ \mathrm{d}x_{n-1} &= (x_n + h_{n-1}^{\mathrm{T}}(y)x_z + f_{n-1}(x_z,y))\mathrm{d}t \\ &\quad + g_{n-1}(x_z,y)\mathrm{d}w,\\ \mathrm{d}x_n &= (u + h_n^{\mathrm{T}}(y)x_z + f_n(x_z,y))\mathrm{d}t + g_n(x_z,y)\mathrm{d}w,\end{aligned} \tag{2}$$

$$y = x_1 \tag{3}$$

where $x = [x_z^{\mathrm{T}}, x_1, \ldots, x_n]^{\mathrm{T}}$, $u\in\mathbb{R}$, $y\in\mathbb{R}$ represent the state, the control input, the measured output, respectively; $x_z\in\mathbb{R}^m$ is referred to the state of the stochastic inverse dynamics; the initial value $x(0) = x_0$ is fixed; $F_0\in\mathbb{R}^{m\times m}$, $H\in\mathbb{R}^m$ and $h_i\in\mathbb{R}^m$, $i=1,\ldots,n$, are known smooth functions; $f_0\in\mathbb{R}^m$, $g_0\in\mathbb{R}^{m\times r}$, $f_i\in\mathbb{R}$ and $g_i^{\mathrm{T}}\in\mathbb{R}^r$, $i=1,\ldots,n$, are uncertain locally Lipschitz functions; w is an r-dimensional standard Brownian motion defined on the complete probability space $(\Omega,\mathcal{F},P)$ with Ω being a sample space, $\mathcal{F}$ being a σ-field, and P being the probability measure.

Remark 1. Similar to the deterministic case (see, e.g., Krstić et al. (1995) and Praly and Jiang (1993)), the dynamics (1) is said to be the inverse dynamics of the system (1)–(2), since the input of the subsystem (1) is the output of the whole system (1)–(2). When its input equals zero, the inverse dynamics is said to be zero dynamics. When its zero dynamics is not asymptotically stable in probability, the system composed of (1)–(3) is said to be of non-minimum phase.

The main results of this paper are based on the following assumptions:

Assumption 1. $H(0) = 0$, and there are known smooth nonnegative functions $\varphi_{i1}(\cdot)$, $\varphi_{i2}(\cdot)$, $\psi_{i1}(\cdot)$ and $\psi_{i2}(\cdot)$ with $\varphi_{i1}(0) = \varphi_{i2}(0) = \psi_{i1}(0) = \psi_{i2}(0) = 0$, such that for $i = 0, 1, \ldots, n$,

$$|f_i(x_z, y)|^4 \le \varphi_{i1}(|x_z|) + \varphi_{i2}(|y|),$$
$$\|g_i(x_z, y)\|^4 \le \psi_{i1}(|x_z|) + \psi_{i2}(|y|).$$

Assumption 2. There exists a smooth function $\Xi(\cdot)$ such that for the dynamics

$$\begin{aligned} \mathrm{d}x_z = {} & [F_0(\Xi(x_z + d_1) + d_2)x_z + H(\Xi(x_z + d_1) + d_2) \\ & + f_0(x_z, \Xi(x_z + d_1) + d_2)]\mathrm{d}t \\ & + g_0(x_z, \Xi(x_z + d_1) + d_2)\mathrm{d}w, \end{aligned} \tag{4}$$

there exist a function $V_z(x_z) \in \mathcal{C}^2$ and $\alpha_{z1}, \alpha_{z2}, \alpha_{01}, \alpha_{02}, \alpha_{03} \in \mathcal{K}_\infty$ such that

$$\alpha_{z1}(|x_z|) \le V_z(x_z) \le \alpha_{z2}(|x_z|), \tag{5}$$
$$\mathcal{L}V_z \le -\alpha_{01}(|x_z|) + \alpha_{02}(|d_1|) + \alpha_{03}(|d_2|). \tag{6}$$

Assumption 3. There exist known smooth nonnegative functions ψ_z and ψ_0 with $\psi_z(0) = \psi_0(0) = 0$, such that $|\frac{\partial V_z}{\partial x_z}| \le \psi_z(|x_z|)$ and $\|g_0(x_z, y)\| \le \psi_0(|x_z|)$.

Remark 2. According to Assumption 1, the nonlinear terms of system (1)–(2) depend on both the measured output y and the unmeasurable state x_z. In this paper, by the concept of stochastic input-to-state stabilizability (see Assumption 2), the global stabilization via output feedback is solved for a class of stochastic non-minimum-phase nonlinear systems. For the deterministic systems, in Mazenc, Praly, and Dayawansa (1994) counterexamples were given indicating that the global stabilization of nonlinear systems in general low-triangular form via output feedback is usually impossible without introducing extra growth conditions on the unmeasurable states of the system. For the stochastic nonlinear systems in which the nonlinear terms depended on general unmeasurable states, see our work Liu and Zhang (2008).

Remark 3. Assumption 2 is a stochastic input-to-state stabilizability condition on the stochastic inverse dynamics. Indeed, from (5) and (6), it follows that the dynamics (4) is stochastic input-to-state stable (SISS) (see Definition A.2 in Appendix A) with respect to the inputs d_1 and d_2 by Theorem A.2. In the case of $\Xi \equiv 0$, the dynamics (1) is SISS with respect to y, and the system is of minimum phase.

Remark 4. As a simple example, any system with the following inverse dynamics is of non-minimum phase:

$$\mathrm{d}x = (ax + y)\mathrm{d}t + \sigma x \mathrm{d}w, \quad \text{where } a > \frac{\sigma^2}{2}. \tag{7}$$

Indeed, its zero dynamics is $\mathrm{d}x = ax\mathrm{d}t + \sigma x \mathrm{d}w$. When $a > \frac{\sigma^2}{2}$, the zero-dynamics is not asymptotically stable in probability (Has'minskii, 1980). In other words, any system with (7) as the inverse dynamics is of non-minimum phase. For such systems, take $\Xi = -(a + c + \frac{\sigma^2}{2})(x + d_1)$ with $c > 1$ being a constant. Then, along the trajectory of $\mathrm{d}x = (ax + \Xi + d_2)\mathrm{d}t + \sigma x \mathrm{d}w$, we have $\mathcal{L}(x^2) \le -(2c - 2)x^2 + (a + c + \frac{\sigma^2}{2})^2 d_1^2 + d_2^2$. Thus, Assumption 2 holds.

Remark 5. In Assumption 3, $|\frac{\partial V_z}{\partial x_z}| \le \psi_z(|x_z|)$ is a general assumption and easy to be satisfied. $\|g_0(x_z, y)\| \le \psi_0(|x_z|)$ is a constraint on the diffusion vector field of inverse dynamics (1), which implies that the diffusion vector field of inverse dynamics (1) is confined by the dynamics itself, and that the effect of the subsystem (2) can be viewed as bounded. The zero dynamics canonical form studied in Pan (2002) belongs to this class. And moreover, for the deterministic cases, Assumption 3 is satisfied trivially.

The control objective in this paper is to design a smooth output-feedback control law of the form

$$\begin{cases} \dot{\chi} = \varpi(\chi, y), \\ u = \mu(\chi, y) \end{cases} \tag{8}$$

such that the zero solution of the closed-loop system composed of (1), (2) and (8) is globally asymptotically stable in probability.

4. Reduced-order observer design

We first design a reduced-order observer for the unmeasurable states $x_z, x_2, \ldots, x_n$.

Define the error variables

$$\tilde{x}_z = \hat{x}_z - x_z + \beta_0(y), \tilde{x}_i = \hat{x}_i - x_i + \beta_i(y), i = 2, \ldots, n,$$
$$\tilde{x} = [\tilde{x}_z^{\mathrm{T}}, \tilde{x}_2, \ldots, \tilde{x}_n]^{\mathrm{T}},$$

where $\beta_i(y), i = 0, 2, \ldots, n$, are $\mathcal{C}^2$ functions yet to be defined, introduce the following observer

$$\begin{aligned} \dot{\hat{x}}_z = {} & F_0(y)(\hat{x}_z + \beta_0(y)) + H(y) \\ & - \frac{\partial \beta_0}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y))], \end{aligned} \tag{9}$$

$$\begin{aligned} \dot{\hat{x}}_i = {} & \hat{x}_{i+1} + \beta_{i+1}(y) + h_i^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\ & - \frac{\partial \beta_i}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y))], \end{aligned} \tag{10}$$

$i = 2, \ldots, n$,

with $\hat{x}_{n+1} = u$, $\beta_{n+1}(y) \equiv 0$, and let

$$A_0(y) = \begin{bmatrix} F_0(y) & 0 & 0 & \ldots & 0 \\ h_2^{\mathrm{T}}(y) & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ h_{n-1}^{\mathrm{T}}(y) & 0 & 0 & \ldots & 1 \\ h_n^{\mathrm{T}}(y) & 0 & 0 & \ldots & 0 \end{bmatrix},$$

$$\beta(y) = \begin{bmatrix} \beta_0^{\mathrm{T}}(y) & \beta_2(y) & \ldots & \beta_n(y) \end{bmatrix}^{\mathrm{T}},$$
$$C_0(y) = \begin{bmatrix} h_1^{\mathrm{T}}(y) & 1 & \ldots & 0 \end{bmatrix},$$

$$A(y) = A_0(y) - \frac{\partial \beta(y)}{\partial y} C_0(y),$$
$$F(x_z, y) = \left[f_0^{\mathrm{T}}(x_z, y) \quad f_2(x_z, y) \quad \ldots \quad f_n(x_z, y)\right]^{\mathrm{T}},$$
$$G(x_z, y) = \left[g_0^{\mathrm{T}}(x_z, y) \quad g_2^{\mathrm{T}}(x_z, y) \quad \ldots \quad g_n^{\mathrm{T}}(x_z, y)\right]^{\mathrm{T}},$$
$$\tilde{F}(x_z, y) = -F(x_z, y) + \frac{\partial \beta(y)}{\partial y} f_1 + \frac{1}{2}\frac{\partial^2 \beta(y)}{\partial y^2}\|g_1\|^2, \tag{11}$$
$$\tilde{G}(x_z, y) = \frac{\partial \beta(y)}{\partial y} g_1 - G(x_z, y).$$

Then we obtain the observation error dynamics

$$\mathrm{d}\tilde{x} = A(y)\tilde{x}\mathrm{d}t + \tilde{F}(x_z, y)\mathrm{d}t + \tilde{G}(x_z, y)\mathrm{d}w. \tag{12}$$

Hence, the complete system can be expressed as

$$\mathrm{d}x_z = [F_0(y)x_z + H(y) + f_0(x_z, y)]\mathrm{d}t + g_0(x_z, y)\mathrm{d}w,$$
$$\mathrm{d}\tilde{x} = A(y)\tilde{x}\mathrm{d}t + \tilde{F}(x_z, y)\mathrm{d}t + \tilde{G}(x_z, y)\mathrm{d}w,$$
$$\begin{cases} \mathrm{d}y = (x_2 + h_1^{\mathrm{T}}(y)x_z + f_1(x_z, y))\mathrm{d}t + g_1(x_z, y)\mathrm{d}w, \\ \mathrm{d}\hat{x}_z = \Big(F_0(y)(\hat{x}_z + \beta_0(y)) + H(y) \\ \quad - \frac{\partial \beta_0}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y))]\Big)\mathrm{d}t, \\ \mathrm{d}\hat{x}_i = \Big(\hat{x}_{i+1} + \beta_{i+1}(y) + h_i^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\ \quad - \frac{\partial \beta_i}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y))]\Big)\mathrm{d}t, \\ i = 2, \ldots, n. \end{cases} \tag{13}$$

Define the output error $\tilde{y} = y - \Xi$ with the function $\Xi = \Xi(\hat{x}_z + \beta_0(y)) = \Xi(x_z + \tilde{x}_z)$ given by Assumption 2. Consider the quartic function

$$V_e(\tilde{x}) = \frac{\delta_1}{2}(\tilde{x}^{\mathrm{T}} P \tilde{x})^2,$$

where P is a positive-definite constant matrix and $\delta_1 > 0$ is to be determined. Then, by Itô formula, we have

$$\mathcal{L}V_e = \delta_1\{\tilde{x}^{\mathrm{T}} P\tilde{x}[\tilde{x}^{\mathrm{T}}(A(y)^{\mathrm{T}} P + PA(y))\tilde{x}] + 2\tilde{x}^{\mathrm{T}} P\tilde{x}(\tilde{F}^{\mathrm{T}} P\tilde{x}) + \mathrm{Tr}\{\tilde{G}^{\mathrm{T}}(2P\tilde{x}\tilde{x}^{\mathrm{T}} P + \tilde{x}^{\mathrm{T}} P\tilde{x}P)\tilde{G}\}\}, \tag{14}$$

and for any function $r(y) > 0$, by Young's inequality,

$$\begin{aligned} 2\tilde{x}^{\mathrm{T}} P\tilde{x}(\tilde{F}^{\mathrm{T}} P\tilde{x}) &= 2\tilde{x}^{\mathrm{T}} P\tilde{x} \\ &\times \left(-F^{\mathrm{T}} P\tilde{x} + \left(\frac{\partial\beta}{\partial y} f_1\right)^{\mathrm{T}} P\tilde{x} + \frac{1}{2}\left(\frac{\partial^2\beta}{\partial y^2}\right)^{\mathrm{T}}\|g_1\|^2 P\tilde{x}\right) \\ &\le \tilde{x}^{\mathrm{T}} P\tilde{x}\left[\frac{\|P\tilde{x}\|^2}{r(y)} + r(y)\|F\|^2 + \frac{\|(\frac{\partial\beta}{\partial y})^{\mathrm{T}} P\tilde{x}\|^2}{r(y)} \right. \\ &\quad \left. + r(y)\|f_1\|^2 + \frac{\|(\frac{\partial^2\beta}{\partial y^2})^{\mathrm{T}} P\tilde{x}\|^2}{2r(y)} + \frac{r(y)}{2}\|g_1\|^4\right] \\ &\le \tilde{x}^{\mathrm{T}} P\tilde{x}\left[\frac{\|P\tilde{x}\|^2}{r(y)} + \frac{\|(\frac{\partial\beta}{\partial y})^{\mathrm{T}} P\tilde{x}\|^2}{r(y)} + \frac{\|(\frac{\partial^2\beta}{\partial y^2})^{\mathrm{T}} P\tilde{x}\|^2}{2r(y)}\right] \\ &\quad + \frac{(\tilde{x}^{\mathrm{T}} P\tilde{x})^2}{r(y)} + \frac{r^3(y)}{4}\left[\|F\|^2 + \|f_1\|^2 + \frac{\|g_1\|^4}{2}\right]^2, \end{aligned} \tag{15}$$

$$\begin{aligned} &\mathrm{Tr}\{\tilde{G}^{\mathrm{T}}(2P\tilde{x}\tilde{x}^{\mathrm{T}} P + \tilde{x}^{\mathrm{T}} P\tilde{x}P)\tilde{G}\} \\ &\le 2\|\tilde{G}^{\mathrm{T}} P\tilde{x}\|^2 + \tilde{x}^{\mathrm{T}} P\tilde{x}\lambda_{\max}(P)\|\tilde{G}\|^2 \\ &\le 4\left\|\left(\frac{\partial\beta}{\partial y} g_1\right)^{\mathrm{T}} P\tilde{x}\right\|^2 + 4\|GP\tilde{x}\|^2 \\ &\quad + \tilde{x}^{\mathrm{T}} P\tilde{x}\lambda_{\max}(P)\left[2\left\|\frac{\partial\beta}{\partial y} g_1\right\|^2 + 2\|G^{\mathrm{T}}\|^2\right] \\ &\le 2\frac{\|(\frac{\partial\beta}{\partial y})^{\mathrm{T}} P\tilde{x}\|^4}{r(y)} + 2r(y)\|g_1\|^4 + 2\frac{\|P\tilde{x}\|^4}{r(y)} + 2r(y)\|G\|^4 \\ &\quad + \frac{(\tilde{x}^{\mathrm{T}} P\tilde{x}\lambda_{\max}(P))^2\|\frac{\partial\beta}{\partial y}\|^4}{r(y)} + r(y)\|g_1\|^4 \\ &\quad + \frac{(\tilde{x}^{\mathrm{T}} P\tilde{x}\lambda_{\max}(P))^2}{r(y)} + r(y)\|G\|^4 \\ &\le \left[2\frac{\|(\frac{\partial\beta}{\partial y})^{\mathrm{T}} P\tilde{x}\|^4}{r(y)} + 2\frac{\|P\tilde{x}\|^4}{r(y)} + \frac{(\tilde{x}^{\mathrm{T}} P\tilde{x}\lambda_{\max}(P))^2\|\frac{\partial\beta}{\partial y}\|^4}{r(y)} \right. \\ &\quad \left. + \frac{(\tilde{x}^{\mathrm{T}} P\tilde{x}\lambda_{\max}(P))^2}{r(y)}\right] + r(y)[3\|g_1\|^4 + 3\|G\|^4]. \end{aligned} \tag{16}$$

From Assumption 1, there exist smooth nonnegative functions $r_1(\cdot)$ and $r_2(\cdot)$ which vanish at zero, such that

$$\frac{r^3(y)}{4}\left[\|F\|^2 + \|f_1\|^2 + \frac{\|g_1\|^4}{2}\right]^2 + r(y)[3\|g_1\|^4 + 3\|G\|^4] \le r_1(|y|) + r_2(|x_z|). \tag{17}$$

This together with $y = \tilde{y} + \Xi(x_z + \tilde{x}_z)$ implies that there exist $\mathcal{K}_\infty$ functions $\kappa_{e1}(\cdot)$, $\kappa_{e2}(\cdot)$ and $\kappa_{e3}(\cdot)$ such that

$$\frac{r^3(y)}{4}\left[\|F\|^2 + \|f_1\|^2 + \frac{\|g_1\|^4}{2}\right]^2 + r(y)[3\|g_1\|^4 + 3\|G\|^4] \le \kappa_{e1}(|\tilde{y}|) + \kappa_{e2}(|x_z|) + \kappa_{e3}(|\tilde{x}_z|). \tag{18}$$

Thus, by (14)–(16) and (18), we have

$$\begin{aligned} \mathcal{L}V_e &\le \delta_1\tilde{x}^{\mathrm{T}} P\tilde{x}\Bigg\{\tilde{x}^{\mathrm{T}}(A(y)^{\mathrm{T}} P + PA(y))\tilde{x} \\ &\quad + \frac{1}{r(y)}\left[\|P\tilde{x}\|^2\left(1 + \frac{2\|P\tilde{x}\|^2}{\tilde{x}^{\mathrm{T}} P\tilde{x}}\right) \right. \\ &\quad + \left\|\left(\frac{\partial\beta}{\partial y}\right)^{\mathrm{T}} P\tilde{x}\right\|^2\left(1 + \frac{2\|\frac{\partial\beta}{\partial y}\|^2\|P\tilde{x}\|^2}{\tilde{x}^{\mathrm{T}} P\tilde{x}}\right) \\ &\quad + \frac{1}{2}\left\|\left(\frac{\partial^2\beta}{\partial y^2}\right)^{\mathrm{T}} P\tilde{x}\right\|^2 + \tilde{x}^{\mathrm{T}} P\tilde{x}(\lambda_{\max}(P))^2\left\|\frac{\partial\beta}{\partial y}\right\|^4 \\ &\quad \left. + \tilde{x}^{\mathrm{T}} P\tilde{x}(\lambda_{\max}(P))^2 + \tilde{x}^{\mathrm{T}} P\tilde{x}\right]\Bigg\} + \delta_1\kappa_{e1}(|\tilde{y}|) \end{aligned}$$

$$+\delta_1\kappa_{e2}(|x_z|)+\delta_1\kappa_{e3}(|\tilde{x}_z|). \quad (19)$$

Let

$$Q(y)=P^2\left(1+\frac{2\lambda_{\max}(P^2)}{\lambda_{\min}(P)}\right)+P(\lambda_{\max}(P))^2+P+P\left(\frac{\partial\beta}{\partial y}\right)\left(\frac{\partial\beta}{\partial y}\right)^{\mathrm{T}}P\left(1+\frac{2\|\frac{\partial\beta}{\partial y}\|^2\lambda_{\max}(P^2)}{\lambda_{\min}(P)}\right)+P\left(\frac{\partial^2\beta}{\partial y^2}\right)\left(\frac{\partial^2\beta}{\partial y^2}\right)^{\mathrm{T}}P+P(\lambda_{\max}(P))^2\left\|\frac{\partial\beta}{\partial y}\right\|^4. \quad (20)$$

If there exist functions $\beta(y)$, $r(y)$, a positive-definite matrix P, and $\mathcal{K}_\infty$ functions $\kappa_{e0}(\cdot)$, $\kappa_{ez}(\cdot)$ such that

$$\tilde{x}^{\mathrm{T}}P\tilde{x}\left\{\tilde{x}^{\mathrm{T}}\left[A(y)^{\mathrm{T}}P+PA(y)+\frac{Q(y)}{r(y)}\right]\tilde{x}\right\}+\kappa_{e3}(|\tilde{x}_z|)\le-\kappa_{e0}(|\tilde{x}|)-\kappa_{ez}(|\tilde{x}_z|),\quad\forall y\in\mathbb{R},$$

then, from (19) and (20) one can get

$$\mathcal{L}V_e\le-\delta_1\kappa_{e0}(|\tilde{x}|)-\delta_1\kappa_{ez}(|\tilde{x}_z|)+\delta_1\kappa_{e1}(|\tilde{y}|)+\delta_1\kappa_{e2}(|x_z|). \quad (21)$$

This inequality is the key to designing the stabilizing controller and analyzing the stability of the closed-loop system. Therefore, for convenience of citation, we give the following assumption.

Assumption 4. There exist functions $\beta(y)$, $r(y)$, a positive-definite matrix P, and $\mathcal{K}_\infty$ functions $\kappa_{e0}(\cdot)$, $\kappa_{ez}(\cdot)$ such that for any y,

$$\tilde{x}^{\mathrm{T}}P\tilde{x}\left\{\tilde{x}^{\mathrm{T}}\left[A(y)^{\mathrm{T}}P+PA(y)+\frac{Q(y)}{r(y)}\right]\tilde{x}\right\}+\kappa_{e3}(|\tilde{x}_z|)\le-\kappa_{e0}(|\tilde{x}|)-\kappa_{ez}(|\tilde{x}_z|).$$

Remark 6. At first glance, Assumption 4 may appear restrictive. In fact, it is somewhat related to the detectability condition which has been used in the past literature of output-feedback control. More precisely,

(i) Assumption 4 can be viewed as a robust detectability condition on system (1)–(2). In the linear deterministic case, it is a necessary and sufficient condition for detectability when $f_i=g_i=0$ (Karagiannis et al., 2005).

(ii) When nonlinear functions f_i, g_i, $i=0,\dots,n$, are bounded with respect to y (in the sense that there exist smooth functions $\varphi_{i1}(|x_z|)$ and $\psi_{i1}(|x_z|)$ such that $|f_i(x_z,y)|^4\le\varphi_{i1}(|x_z|)$, $\|g_i(x_z,y)\|^4\le\psi_{i1}(|x_z|)$, $i=0,\dots,n$), $\beta(y)$ can be selected to be linear. Thus, from (20), $Q(y)=Q$ is a constant positive-definite matrix. In this case, $r(y)=r$ can be chosen to be a constant and sufficiently large such that $\frac{Q}{r}\le I$. Then, by (17) and (18) we have $\kappa_{e3}(\tilde{x}_z)=0$. Thus, Assumption 4 is now reduced to the following condition:

There exist a vector-valued function $\beta(y)$, a positive-definite matrix P, and $\mathcal{K}_\infty$ functions $\kappa_{e0}(\cdot)$, $\kappa_{ez}(\cdot)$ such that

$$\tilde{x}^{\mathrm{T}}P\tilde{x}[\tilde{x}^{\mathrm{T}}(A(y)^{\mathrm{T}}P+PA(y)+I)\tilde{x}]\le-\kappa_{e0}(|\tilde{x}|)-\kappa_{ez}(|\tilde{x}_z|),\quad\forall y\in\mathbb{R}.$$

This is nothing but a detectability condition of the systems without disturbances.

(iii) For system (1)–(2), Assumption 4 is true if (a) for any $d>0$, there exist a function $\beta(y)=Ly$ and a positive-definite matrix P such that $A(y)^{\mathrm{T}}P+PA(y)\le-(1+d)I$, $\forall y\in\mathbb{R}$; and (b) for the $\Xi(\cdot)$ in Assumption 2, there exist a constant $\sigma>0$ and sufficiently small constants $\varrho_1>0$, $\varrho_2>0$ such that

$$\sum_{i=0}^{n}[\varphi_{i1}(|x_z|)+\psi_{i1}(|x_z|)+\varphi_{i2}(|\Xi(x_z+\tilde{x}_z)+\tilde{y}|)+\psi_{i2}(|\Xi(x_z+\tilde{x}_z)+\tilde{y}|)]\le\varrho_1|\tilde{x}_z|^4+\varrho_2|x_z|^4+\sigma|\tilde{y}|^4.$$

In fact, choose $r(y)=r$ independent of ϱ_1 and sufficiently large such that $Q(y)=Q\le rI$. Then, noticing that ϱ_1 is sufficiently small, from (18) we see that $\kappa_{e3}(|\tilde{x}_z|)=\kappa_{e3}|\tilde{x}_z|^4$ with a sufficiently small κ_{e3}. Thus, Assumption 4 is true.

(iv) To design an asymptotically stabilizing control law for deterministic non-minimum-phase systems, some conditions on the observer design and small-gain conditions seem necessary (Andrieu & Praly, 2005; Karagiannis et al., 2003, 2005). For stochastic non-minimum-phase nonlinear systems, such conditions are first introduced here, and will be investigated in detail in Section 7.

5. Control law design

We now use the backstepping method to design an output-feedback control law with a to-be-determined gain function such that the subsystem (13) is SISS with respect to x_z and $\tilde{x}$.

In the following, let $\beta(y)$ be a known function obtained according to Assumption 4.

Step 1. Set $z_1=\tilde{y}=y-\Xi(\hat{x}_z+\beta_0(y))$. Then, it follows from (9) that

$$\mathrm{d}z_1=\Big\{\hat{x}_2+\beta_2(y)+h_1^{\mathrm{T}}(y)(\hat{x}_z+\beta_0(y))-(\tilde{x}_2+h_1^{\mathrm{T}}(y)\tilde{x}_z-f_1)-\frac{\partial\Xi}{\partial(\hat{x}_z+\beta_0(y))}\Big[F_0(y)(\hat{x}_z+\beta_0(y))+H(y)-\frac{\partial\beta_0}{\partial y}(\tilde{x}_2+h_1^{\mathrm{T}}(y)\tilde{x}_z-f_1)\Big]-\frac{1}{2}\frac{\partial^2\Xi}{\partial y^2}\|g_1\|^2\Big\}\mathrm{d}t+\left(g_1-\frac{\partial\Xi}{\partial y}g_1\right)\mathrm{d}w.$$

Define the virtual control as

$$\phi_1(y,\hat{x}_z)=\lambda_1(y,\hat{x}_z)-\beta_2(y)-\nu(z_1^2)z_1-h_1^{\mathrm{T}}(y)(\hat{x}_z+\beta_0(y))+\frac{\partial\Xi}{\partial(\hat{x}_z+\beta_0(y))}[F_0(y)(\hat{x}_z+\beta_0(y))+H(y)] \quad (22)$$

with $\lambda_1(\cdot,\cdot)$ being a smooth function to be designed later and the gain function $\nu(\cdot)\ge0$ to be determined in the next section.

Step i. Set $z_i = \hat{x}_i - \phi_{i-1}(y, \hat{x}_z, \hat{x}_2, \ldots, \hat{x}_{i-1}), i = 2, \ldots n-1$. Then, by (9) and (10) we have

$$
\begin{aligned}
\mathrm{d}z_i = \Bigg\{ & \hat{x}_{i+1} + \beta_{i+1}(y) + h_i^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\
& - \frac{\partial \beta_i}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y))] \\
& - \frac{\partial \phi_{i-1}}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\
& - (\tilde{x}_2 + h_1^{\mathrm{T}}(y)\tilde{x}_z - f_1)] \\
& - \frac{\partial \phi_{i-1}}{\partial \hat{x}_z}\Bigg[F_0(y)(\hat{x}_z + \beta_0(y)) + H(y) \\
& - \frac{\partial \beta_0}{\partial y}(\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)))\Bigg] \\
& + \sum_{j=2}^{i-1} \frac{\partial \phi_{i-1}}{\partial \hat{x}_j}\Bigg[\hat{x}_{j+1} + \beta_{j+1}(y) + h_j^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\
& - \frac{\partial \beta_j}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y))]\Bigg] \\
& - \frac{1}{2}\frac{\partial^2 \phi_{i-1}}{\partial y^2}\|g_1\|^2 \Bigg\} \mathrm{d}t - \frac{\partial \phi_{i-1}}{\partial y} g_1 \mathrm{d}w.
\end{aligned}
$$

Define the virtual control as

$$
\begin{aligned}
& \phi_i(y, \hat{x}_z, \hat{x}_2, \ldots, \hat{x}_i) \\
& \quad = \lambda_i(y, \hat{x}_z, \hat{x}_2, \ldots, \hat{x}_i) - \beta_{i+1}(y) - h_i^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\
& \qquad + \frac{\partial \beta_i}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y))] \\
& \qquad + \frac{\partial \phi_{i-1}}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y))] \\
& \qquad + \frac{\partial \phi_{i-1}}{\partial \hat{x}_z}\Bigg[F_0(y)(\hat{x}_z + \beta_0(y)) + H(y) \\
& \qquad - \frac{\partial \beta_0}{\partial y}(\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)))\Bigg] \\
& \qquad + \sum_{j=2}^{i-1} \frac{\partial \phi_{i-1}}{\partial \hat{x}_j}\Bigg[\hat{x}_{j+1} + \beta_{j+1}(y) + h_j^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\
& \qquad - \frac{\partial \beta_j}{\partial y}(\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)))\Bigg]
\end{aligned} \tag{23}
$$

with $\lambda_i(\cdot, \ldots, \cdot)$ being a smooth function to be designed later.

Step n. Set $z_n = \hat{x}_n - \phi_{n-1}(y, \hat{x}_z, \hat{x}_2, \ldots, \hat{x}_{n-1})$. Then, by (9) and (10) we have

$$
\begin{aligned}
\mathrm{d}z_n = \Bigg\{ & u + h_n^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\
& - \frac{\partial \beta_n}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y))] \\
& - \frac{\partial \phi_{n-1}}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\
& - (\tilde{x}_2 + h_1^{\mathrm{T}}(y)\tilde{x}_z - f_1)] \\
& - \frac{\partial \phi_{n-1}}{\partial \hat{x}_z}\Bigg[F_0(y)(\hat{x}_z + \beta_0(y)) + H(y) \\
& - \frac{\partial \beta_0}{\partial y}(\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)))\Bigg] \\
& + \sum_{j=2}^{n-1} \frac{\partial \phi_{n-1}}{\partial \hat{x}_j}\Bigg[\hat{x}_{j+1} + \beta_{j+1}(y) + h_j^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\
& - \frac{\partial \beta_j}{\partial y}(\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)))\Bigg] \\
& - \frac{1}{2}\frac{\partial^2 \phi_{n-1}}{\partial y^2}\|g_1\|^2 \Bigg\} \mathrm{d}t - \frac{\partial \phi_{n-1}}{\partial y} g_1 \mathrm{d}w.
\end{aligned}
$$

Design the control law as

$$
\begin{aligned}
u = {} & \lambda_n(y, \hat{x}_z, \hat{x}_2, \ldots, \hat{x}_n) - h_n^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\
& + \frac{\partial \beta_n}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y))] \\
& + \frac{\partial \phi_{n-1}}{\partial y}[\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y))] \\
& + \frac{\partial \phi_{n-1}}{\partial \hat{x}_z}\Bigg[F_0(y)(\hat{x}_z + \beta_0(y)) + H(y) \\
& - \frac{\partial \beta_0}{\partial y}(\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)))\Bigg] \\
& + \sum_{j=2}^{n-1} \frac{\partial \phi_{n-1}}{\partial \hat{x}_j}\Bigg[\hat{x}_{j+1} + \beta_{j+1}(y) + h_j^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)) \\
& - \frac{\partial \beta_j}{\partial y}(\hat{x}_2 + \beta_2(y) + h_1^{\mathrm{T}}(y)(\hat{x}_z + \beta_0(y)))\Bigg].
\end{aligned} \tag{24}
$$

Then, we obtain the following dynamics:

$$
\begin{aligned}
\mathrm{d}z_1 &= \Bigg[-\nu(z_1^2)z_1 + \lambda_1(y, \hat{x}_z) + z_2 \\
&\quad + \left(\frac{\partial \Xi}{\partial y} - 1\right)(\tilde{x}_2 + h_1^{\mathrm{T}}(y)\tilde{x}_z - f_1) \\
&\quad - \frac{1}{2}\frac{\partial^2 \Xi}{\partial y^2}\|g_1\|^2\Bigg]\mathrm{d}t + \left(g_1 - \frac{\partial \Xi}{\partial y} g_1\right)\mathrm{d}w, \\
\mathrm{d}z_i &= \Bigg[\lambda_i(y, \hat{x}_z, \hat{x}_2, \ldots, \hat{x}_i) + z_{i+1} \\
&\quad + \frac{\partial \phi_{i-1}}{\partial y}(\tilde{x}_2 - f_1 + h_1^{\mathrm{T}}(y)\tilde{x}_z) \\
&\quad - \frac{1}{2}\frac{\partial^2 \phi_{i-1}}{\partial y^2}\|g_1\|^2\Bigg]\mathrm{d}t - \frac{\partial \phi_{i-1}}{\partial y} g_1 \mathrm{d}w, \\
\mathrm{d}z_n &= \Bigg[\lambda_n(y, \hat{x}_z, \hat{x}_2, \ldots, \hat{x}_n) \\
&\quad + \frac{\partial \phi_{n-1}}{\partial y}(\tilde{x}_2 - f_1 + h_1^{\mathrm{T}}(y)\tilde{x}_z) \\
&\quad - \frac{1}{2}\frac{\partial^2 \phi_{n-1}}{\partial y^2}\|g_1\|^2\Bigg]\mathrm{d}t - \frac{\partial \phi_{n-1}}{\partial y} g_1 \mathrm{d}w.
\end{aligned} \tag{25}
$$

Consider the following Lyapunov function candidate

$$
V(z) = \frac{1}{4}\sum_{i=1}^{n} z_i^4, \quad z = [z_1, \ldots, z_n]^{\mathrm{T}}.
$$

From (25), we obtain

$$\begin{aligned}\mathcal{L}V &= -\nu(z_1^2)z_1^4 + \sum_{i=1}^{n}\lambda_i z_i^3 + \sum_{i=1}^{n-1} z_i^3 z_{i+1}\\ &+ z_1^3\left[\left(\frac{\partial \Xi}{\partial y} - 1\right)(\tilde{x}_2 + h_1^{\mathrm{T}}(y)\tilde{x}_z - f_1) - \frac{1}{2}\frac{\partial^2 \Xi}{\partial y^2}\|g_1\|^2\right]\\ &+ \frac{3}{2}z_1^2\left\|g_1 - \frac{\partial \Xi}{\partial y}g_1\right\|^2\\ &+ \sum_{i=2}^{n} z_i^3\left[\frac{\partial \phi_{i-1}}{\partial y}(\tilde{x}_2 + h_1^{\mathrm{T}}(y)\tilde{x}_z - f_1)\right.\\ &\left.- \frac{1}{2}\frac{\partial^2 \phi_{i-1}}{\partial y^2}\|g_1\|^2\right] + \frac{3}{2}\sum_{i=2}^{n} z_i^2\left\|\frac{\partial \phi_{i-1}}{\partial y}g_1\right\|^2.\end{aligned}$$

Noticing that Ξ depends on the disturbances x_z and $\tilde{x}$, we select the functions $\lambda_i(\cdot,\ldots,\cdot)$, $i = 1,\ldots,n$, such that for a constant $c > 0$ and a nonnegative function $\delta(\cdot)$,

$$\mathcal{L}V \le -cV - \nu(\tilde{y}^2)\tilde{y}^4 + \delta(|(x_z,\tilde{x})|), \tag{26}$$

where the nonnegative function $\nu(\cdot)$ is to be designed in the next section.

6. Stability analysis of the closed-loop system

In this section, we use the method of changing supply functions to design the gain function $\nu(\cdot)$ and analyze the stability of the closed-loop system, which is made up of three dynamics: inverse dynamics (1), error dynamics (12) and the controlled dynamics (13). Firstly, we consider the cascade of the inverse dynamics and error dynamics.

For $y = \tilde{y} + \Xi$, the inverse dynamics (1) can be rewritten as

$$\begin{aligned}\mathrm{d}x_z &= \left[F_0(\Xi(x_z+\tilde{x}_z)+\tilde{y})x_z + H(\Xi(x_z+\tilde{x}_z)+\tilde{y})\right.\\ &\quad\left.+ f_0(x_z,\Xi(x_z+\tilde{x}_z)+\tilde{y})\right]\mathrm{d}t\\ &\quad+ g_0(x_z,\Xi(x_z+\tilde{x}_z)+\tilde{y})\mathrm{d}w.\end{aligned} \tag{27}$$

Then, from Assumption 2 we have

$$\mathcal{L}V_z \le -\alpha_{01}(|x_z|) + \alpha_{02}(|\tilde{x}_z|) + \alpha_{03}(|\tilde{y}|). \tag{28}$$

Lemma 1. *For the functions $\alpha_{z1}(\cdot)$, $\kappa_{e2}(\cdot)$, $\alpha_{01}(\cdot)$, $\psi_z(\cdot)$ and $\psi_0(\cdot)$ given by (5), (21) and (28) and Assumption 3, if*

$$\limsup_{s\to 0+}\frac{\kappa_{e2}(s)}{\alpha_{01}(s)} < \infty, \qquad \limsup_{s\to 0+}\frac{\psi_z^2(s)\psi_0^2(s)}{\alpha_{01}(s)} < \infty, \tag{29}$$

$$\int_0^\infty e^{-\int_0^u \frac{1}{\zeta(\alpha_{z1}^{-1}(\tau))}\mathrm{d}\tau}[\xi(\alpha_{z1}^{-1}(u))]'\mathrm{d}u < \infty, \tag{30}$$

where $\xi(\cdot)$ and $\zeta(\cdot)$ are continuous increasing positive functions satisfying

$$\xi(s) \ge \frac{4\delta_1\kappa_{e2}(s)}{\alpha_{01}(s)}, \qquad \zeta(s) \ge \frac{2\psi_z^2(s)\psi_0^2(s)}{\alpha_{01}(s)}, \qquad \forall s > 0,$$

then there exists a nondecreasing positive function $\rho_1(\cdot) \in \mathcal{C}^1[0,\infty)$ such that

$$\begin{aligned}\rho_1(V_z(x))\alpha_{01}(|x|) &\ge 2\rho_1'(V_z(x))\psi_z^2(|x|)\psi_0^2(|x|)\\ &\quad+ 4\delta_1\kappa_{e2}(|x|), \quad \forall x \in \mathbb{R}^m.\end{aligned} \tag{31}$$

Proof. See Appendix B.

Remark 7. Different from that in the deterministic case due to the appearance of the second-order differential term of Itô formula, the condition (30) is needed. If the inverse dynamics of the system degenerates to $g_0(x_z,y) = 0$, then $\psi_0(|x_z|)$ can be simply taken as 0. In this case, (31) becomes $\rho_1(V_z(x))\alpha_{01}(|x|) \ge 4\delta_1\kappa_{e2}(|x|)$, $\forall x \in \mathbb{R}^m$. To get a ρ_1 satisfying the above inequality, condition (30) is not needed (see Liu et al., 2007).

Theorem 1. *Suppose that Assumptions 1–4 and the conditions of Lemma 1 hold. For $\delta_1 > 0$, $\rho_1(\cdot)$ given by Lemma 1, if*

$$\delta_1\kappa_{ez}(s) \ge \rho_1(\eta_1(s))\alpha_{02}(s), \tag{32}$$

where $\eta_1(\cdot) = \alpha_{z2}(\alpha_{01}^{-1}(4\alpha_{02}(\cdot)))$, then the system composed of (12) and (27) is SISS with respect to the output error $\tilde{y}$.

Proof. Let

$$V_{ze}(x_z,\tilde{x}) = \int_0^{V_z(x_z)} \rho_1(t)\mathrm{d}t + V_e(\tilde{x}). \tag{33}$$

Then, by (21) and (28), Assumption 3 and Lemma 1 we have

$$\begin{aligned}\mathcal{L}V_{ze} &\le \rho_1(V_z)\mathcal{L}V_z + \frac{1}{2}\rho_1'(V_z)\|\nabla V_z^{\mathrm{T}} g_0\|^2 - \delta_1\kappa_{e0}(|\tilde{x}|)\\ &\quad- \delta_1\kappa_{ez}(|\tilde{x}_z|) + \delta_1\kappa_{e1}(|\tilde{y}|) + \delta_1\kappa_{e2}(|x_z|)\\ &\le \rho_1(V_z)[-\alpha_{01}(|x_z|) + \alpha_{02}(|\tilde{x}_z|) + \alpha_{03}(|\tilde{y}|)]\\ &\quad+ \frac{1}{2}\rho_1'(V_z)\psi_z^2(|x_z|)\psi_0^2(|x_z|) - \delta_1\kappa_{e0}(|\tilde{x}|)\\ &\quad- \delta_1\kappa_{ez}(|\tilde{x}_z|) + \delta_1\kappa_{e1}(|\tilde{y}|) + \delta_1\kappa_{e2}(|x_z|)\\ &\le \rho_1(\eta_1(|\tilde{x}_z|))\alpha_{02}(|\tilde{x}_z|) + \rho_1(\eta_2(|\tilde{y}|))\alpha_{03}(|\tilde{y}|)\\ &\quad- \frac{1}{2}\rho_1(V_z)\alpha_{01}(|x_z|) + \frac{1}{2}\rho_1'(V_z)\psi_z^2(|x_z|)\psi_0^2(|x_z|)\\ &\quad- \delta_1\kappa_{e0}(|\tilde{x}|) - \delta_1\kappa_{ez}(|\tilde{x}_z|) + \delta_1\kappa_{e1}(|\tilde{y}|)\\ &\quad+ \delta_1\kappa_{e2}(|x_z|)\\ &\le \rho_1(\eta_2(|\tilde{y}|))\alpha_{03}(|\tilde{y}|) - \frac{1}{4}\rho_1(V_z)\alpha_{01}(|x_z|)\\ &\quad- \delta_1\kappa_{e0}(|\tilde{x}|) + \delta_1\kappa_{e1}(|\tilde{y}|),\end{aligned} \tag{34}$$

where $\eta_2(\cdot) = \alpha_{z2}(\alpha_{01}^{-1}(4\alpha_{03}(\cdot))) \in \mathcal{K}_\infty$. Let

$$\alpha_{ze}(r) = \inf_{|(x_z,\tilde{x})|\ge r}\{\delta_1\kappa_{e0}(|\tilde{x}|) + \frac{1}{4}\rho_1(0)\alpha_{01}(|x_z|)\},$$

$$\gamma_1(r) = \rho_1(\eta_2(r))\alpha_{03}(r) + \delta_1\kappa_{e1}(r).$$

Then, by (34) we have that

$$\mathcal{L}V_{ze} \le -\alpha_{ze}(|(x_z,\tilde{x})|) + \gamma_1(|\tilde{y}|). \tag{35}$$

Therefore, by Theorem A.2, the system composed of (12) and (27) is SISS with respect to $\tilde{y}$. □

Remark 8. It is worth noticing that the technical inequalities in (29) are reminiscent of, but are different from, the (local) small-gain conditions in the setting of deterministic controller design (Isidori, 1999; Jiang & Praly, 1998). For both settings, these conditions are required to hold only for *small* signals. As to condition (30), it represents the main difference between the stochastic systems and deterministic ones. In the work Liu, Zhang, and Jiang (2008), detailed analysis and some classes of systems with this condition satisfied are given.

Remark 9. To obtain that the system composed of (12) and (27) is SISS with respect to $\tilde{y}$, in addition to the small-gain type conditions (29) and (30), the condition (32) is needed, which is a main constraint on the system (1)–(2) and different from that on the systems with SISS inverse dynamics. Since it is hard to illustrate the structure of the general nonlinear function $\Xi(\cdot)$, the condition (32) is hard to verify. But for locally Lipschitz and linear growth systems (e.g. globally Lipschitz systems), the condition (32) can be verified, which will be given in the next section. Besides, if $\kappa_{ez}(\tilde{x}_z) \leq a|\tilde{x}_z|^4$ for some constant $a > 0$, ρ_1 can be chosen as a constant (e.g. linear systems) and $\alpha_{02}(|\tilde{x}_z|) \leq \varepsilon|\tilde{x}_z|^4$ for some sufficiently small ε, then (32) is true.

In the following, we will design the function $\nu(\cdot)$ and state the main results of the paper.

By the definition of V_{ze}, there exist $\alpha_{ze1}, \alpha_{ze2} \in \mathcal{K}_\infty$ such that

$$\alpha_{ze1}(|(x_z, \tilde{x})|) \leq V_{ze} \leq \alpha_{ze2}(|(x_z, \tilde{x})|). \tag{36}$$

By $y = \tilde{y} + \Xi(x_z + \tilde{x}_z)$, there exist smooth nonnegative functions ψ_{z1} and ψ_{01} such that $\|\frac{\partial V_{ze}}{\partial(x_z,\tilde{x})}\| \leq \psi_{z1}(|(x_z, \tilde{x})|)$ and $\|g_{ze}\| \leq \psi_{01}(|(x_z, \tilde{x})|)$, where

$$g_{ze} = \begin{bmatrix} g_0(x_z, y) \\ \tilde{G}(x_z, y) \end{bmatrix}.$$

With these notations, we have the following lemma.

Lemma 2. *For the functions $\alpha_{ze1}(\cdot)$, $\delta(\cdot)$, $\alpha_{ze}(\cdot)$ given by (26), (35) and (36), if*

$$\limsup_{s\to 0+} \frac{\delta(s)}{\alpha_{ze}(s)} < \infty, \qquad \limsup_{s\to 0+} \frac{\psi_{z1}^2(s)\psi_{01}^2(s)}{\alpha_{ze}(s)} < \infty, \tag{37}$$

$$\int_0^\infty e^{-\int_0^u \frac{1}{\zeta_1(\alpha_{ze1}^{-1}(\tau))}\mathrm{d}\tau} [\xi_1(\alpha_{ze1}^{-1}(u))]'\mathrm{d}u < \infty, \tag{38}$$

where $\xi_1(\cdot)$ and $\zeta_1(\cdot)$ are continuous increasing positive functions satisfying

$$\xi_1(s) \geq \frac{4\delta(s)}{\alpha_{ze}(s)}, \quad \zeta_1(s) \geq \frac{2\psi_{z1}^2(s)\psi_{01}^2(s)}{\alpha_{ze}(s)}, \quad \forall s > 0,$$

then there exists a nondecreasing positive function $\rho_2(\cdot) \in \mathcal{C}^1[0,\infty)$ such that

$$\rho_2(V_{ze}(x))\alpha_{ze}(|x|) \geq 2\rho_2'(V_{ze}(x))\psi_{z1}^2(|x|)\psi_{01}^2(|x|) + 4\delta(|x|), \quad \forall x \in \mathbb{R}^{2m+n-1}. \tag{39}$$

Proof. The proof is similar to that of Lemma 1, and so, omitted. □

Theorem 2. *Suppose that Assumptions 1–4 and the conditions of Lemmas 1 and 2 hold. For $\gamma_1(\cdot)$ given by (35), if*

$$\limsup_{s\to 0+} \frac{\gamma_1(s)}{s^4} < \infty, \tag{40}$$

then, under the control law (22)–(24), the closed-loop system has an almost surely unique strong solution on $[0,\infty)$, and its equilibrium is globally asymptotically stable in probability.

Proof. Let

$$W(x_z, \tilde{x}, z) = \int_0^{V_{ze}(x_z,\tilde{x})} \rho_2(t)\mathrm{d}t + V(z), \tag{41}$$

where $\rho_2(\cdot)$ is given by Lemma 2. Then, by (26), (35) and Lemma 2 we have

$$\begin{aligned}
\mathcal{L}W &\leq \rho_2(V_{ze})\mathcal{L}V_{ze} + \frac{1}{2}\rho_2'(V_{ze})\|\nabla V_{ze}^{\mathrm{T}} g_{ze}\|^2 - cV \\
&\quad - \nu(\tilde{y}^2)\tilde{y}^4 + \delta(|(x_z,\tilde{x})|) \\
&\leq \rho_2(V_{ze})[-\alpha_{ze}(|(x_z,\tilde{x})|) + \gamma_1(|\tilde{y}|)] \\
&\quad + \frac{1}{2}\rho_2'(V_{ze})\psi_{z1}^2(|(x_z,\tilde{x})|)\psi_{01}^2(|(x_z,\tilde{x})|) \\
&\quad - cV - \nu(\tilde{y}^2)\tilde{y}^4 + \delta(|(x_z,\tilde{x})|) \\
&\leq \rho_2(\eta_{z1}(|\tilde{y}|))\gamma_1(|\tilde{y}|) - \frac{1}{2}\rho_2(V_{ze})\alpha_{ze}(|(x_z,\tilde{x})|) \\
&\quad + \frac{1}{2}\rho_2'(V_{ze})\psi_{z1}^2(|(x_z,\tilde{x})|)\psi_{01}^2(|(x_z,\tilde{x})|) \\
&\quad - cV - \nu(\tilde{y}^2)\tilde{y}^4 + \delta(|(x_z,\tilde{x})|) \\
&\leq \rho_2(\eta_{z1}(|\tilde{y}|))\gamma_1(|\tilde{y}|) - \frac{1}{4}\rho_2(V_{ze})\alpha_{ze}(|(x_z,\tilde{x})|) \\
&\quad - cV - \nu(\tilde{y}^2)\tilde{y}^4,
\end{aligned} \tag{42}$$

where $\eta_{z1}(\cdot) = \alpha_{ze2}(\alpha_{ze}^{-1}(2\gamma_1(\cdot))) \in \mathcal{K}_\infty$.

From (40) we can construct a smooth function $\nu \in \mathcal{K}_\infty$ such that

$$\nu(s^2) \geq \rho_2(\eta_{z1}(s)) \sup_{t\in(0,s]} \frac{\gamma_1(t)}{t^4},$$

and hence,

$$\rho_2(\eta_{z1}(|\tilde{y}|))\gamma_1(|\tilde{y}|) \leq \nu(\tilde{y}^2)\tilde{y}^4.$$

This together with (42) gives

$$\mathcal{L}W \leq -\frac{1}{4}\rho_2(0)\alpha_{ze}(|(x_z,\tilde{x})|) - cV.$$

Thus, by Theorem A.1 the closed-loop system has an almost surely unique strong solution on $[0,\infty)$ and the equilibrium of the closed-loop system is globally asymptotically stable in probability. □

7. On Assumption 4 and the small-gain type conditions

In this section, we will investigate what kind of systems of the form (1)–(2) satisfy the main conditions of Theorems 1 and 2, that is, Assumption 4, and the small-gain type conditions (29)–(30), (32) and (37)–(38). It is found that the following four classes of often-encountered systems satisfy these conditions.

Class 1. Systems without inverse dynamics

In this case, x_z does not exist. $A(y)$ is reduced to a $(n-1)\times(n-1)$ Hurwitz matrix by properly selecting $\beta_i(y) = L_i y$, $i = 2\ldots, n$. Thus, Assumption 4 is satisfied. Since the nonlinear terms are uncertain and with known upper bounds, this paper can be viewed as a robust version of Deng and Krstić (1999).

Class 2. *Systems with SISS inverse dynamics*

In this case, Assumption 2 holds for $\Xi = 0$, and the condition (6) is reduced to

$$\mathcal{L}V_z \le -\alpha_{01}(|x_z|) + \alpha_{03}(|y|).$$

Thus, the condition (32) is satisfied for $\alpha_{02}(|\tilde{x}_z|) = 0$. Select a function $\beta(y)$ to yield a constant Hurwitz matrix A. Then, Assumption 4 is satisfied. Under small-gain type conditions similar to (29)–(30) and (37)–(38), a stabilizing controller can be constructively designed in a similar way as in the case $N = 1$ of Liu et al. (2007) without parametric uncertainties.

Class 3. *Systems without disturbance*

In this case, $f_i = g_i = 0, i = 0, 1, \ldots, n$. This means that the system is deterministic. Assumption 4 and conditions (30), (32) and (38) can be relaxed as stated by the following corollary:

Corollary 1 (*Karagiannis et al., 2005*). *Consider system* (1) *and* (2) *with $f_i = g_i = 0$, $i = 0, 1, \ldots, n$, with Assumptions* 1 *and* 2 *satisfied. Suppose that there exist functions $\beta_i(y)$, $i = 0, 2, \ldots, n$, and a positive-definitive matrix P such that*

$$\tilde{x}^{\mathrm{T}}(A(y)^{\mathrm{T}}P + PA(y))\tilde{x} \le -\kappa_{e0}(|\tilde{x}|), \quad \forall y \in \mathbb{R}. \tag{43}$$

Then, there exists a dynamic output-feedback control law of the form (8) *such that the closed-loop system is globally asymptotically stable.*

Class 4. *Systems with locally Lipschitz and linear growth conditions*

Consider the system of the following form

$$\mathrm{d}x_z = [F_0x_z + Hy + f_0(x_z, y)]\mathrm{d}t + g_0(x_z, y)\mathrm{d}w, \tag{44}$$

$$\begin{aligned}
\mathrm{d}x_1 &= [x_2 + h_1^{\mathrm{T}}x_z + f_1(x_z, y)]\mathrm{d}t + g_1(x_z, y)\mathrm{d}w,\\
&\;\;\vdots\\
\mathrm{d}x_{n-1} &= [x_n + h_{n-1}^{\mathrm{T}}x_z + f_{n-1}(x_z, y)]\mathrm{d}t + g_{n-1}(x_z, y)\mathrm{d}w,\\
\mathrm{d}x_n &= [u + h_n^{\mathrm{T}}x_z + f_n(x_z, y)]\mathrm{d}t + g_n(x_z, y)\mathrm{d}w,\\
y &= x_1,
\end{aligned} \tag{45}$$

where $f_i, g_i, i = 0, 1, \ldots, n$, are uncertain, vanish at zero and satisfy the locally Lipschitz and linear growth conditions, that is, for each $k = 1, 2, \ldots,$ there exists $L_k > 0$ such that

$$|f_i(x_1, y_1) - f_i(x_2, y_2)| \vee \|g_i(x_1, y_1) - g_i(x_2, y_2)\| \le L_k(|x_1 - x_2| + |y_1 - y_2|),$$

for $(x_1, x_2, y_1, y_2) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \le k$, and moreover, there exists a constant $c > 0$ such that

$$|f_i(x, y)| \vee \|g_i(x, y)\| \le c(1 + |x| + |y|)$$

for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}$. This implies that there are positive constants φ_{ij}, ψ_{ij}, $j = 1, 2$, $i = 0, 1, \ldots, n$, such that Assumption 1 is satisfied with quartic functions $\varphi_{ij}(s) = \varphi_{ij}s^4$, $\psi_{ij}(s) = \psi_{ij}s^4$. Suppose that Assumption 2 holds for a linear function $\Xi = -b(x_z + \tilde{x}_z)$ with $b > 0$. Then, (29)–(30) and (37)–(38) are satisfied, and the functions $\rho_1(\cdot)$, $\rho_2(\cdot)$ in (31) and (39) can be chosen as constants.

System (44)–(45) can be written in matrix form as

$$\begin{bmatrix}\mathrm{d}\eta\\ \mathrm{d}x_1\end{bmatrix} = \begin{bmatrix} & H\\ A_0 & \vdots\\ & 0\\ C_0 & 0\end{bmatrix}\begin{bmatrix}\eta\\ x_1\end{bmatrix}\mathrm{d}t + \begin{bmatrix}0\\ \vdots\\ 1\\ 0\end{bmatrix}u\mathrm{d}t + F_1(x_z, x_1)\mathrm{d}t + G_1(x_z, x_1)\mathrm{d}w,$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\eta\\ x_1\end{bmatrix},$$

where $\eta = [x_z^{\mathrm{T}}, x_2, \ldots, x_n]^{\mathrm{T}}$, $F_1(x_z, x_1) = [f_0^{\mathrm{T}}, f_2, \ldots, f_n, f_1]^{\mathrm{T}}$, $G_1(x_z, x_1) = [g_0^{\mathrm{T}}, g_2^{\mathrm{T}}, \ldots, g_n^{\mathrm{T}}, g_1^{\mathrm{T}}]^{\mathrm{T}}$,

$$A_0 = \begin{bmatrix} F_0 & 0 & 0 & \ldots & 0\\ h_2^{\mathrm{T}} & 0 & 1 & \ldots & 0\\ \vdots & \vdots & & \ddots & \\ h_{n-1}^{\mathrm{T}} & 0 & 0 & \ldots & 1\\ h_n^{\mathrm{T}} & 0 & 0 & \ldots & 0\end{bmatrix},$$

$$C_0 = \begin{bmatrix}h_1^{\mathrm{T}} & 1 & 0 & \ldots & 0\end{bmatrix}.$$

Define the function $\beta(y) = Ly$ with L being a constant vector. Then, the matrix A in (12) is $A_0 - LC_0$. Let

$$\begin{aligned}
Q = {}& P^2\left(1 + \frac{2\lambda_{\max}(P^2)}{\lambda_{\min}(P)}\right) + P(\lambda_{\max}(P))^2 + P\\
&+ PLL^{\mathrm{T}}P\left(1 + \frac{2\|L\|^2\lambda_{\max}(P^2)}{\lambda_{\min}(P)}\right)\\
&+ P(\lambda_{\max}(P))^2\|L\|^4
\end{aligned}$$

and choose constant r sufficiently large such that $Q \le rI$. Then, Assumption 4 is reduced to the following condition:

There exist a vector L and two constants κ_{e0}, κ_{ez} such that

$$\begin{aligned}
&\tilde{x}^{\mathrm{T}}P\tilde{x}\{\tilde{x}^{\mathrm{T}}[A^{\mathrm{T}}P + PA + I]\tilde{x}\} + \kappa_{e3}|\tilde{x}_z|^4\\
&\quad \le -\kappa_{e0}|\tilde{x}|^4 - \kappa_{ez}|\tilde{x}_z|^4.
\end{aligned} \tag{46}$$

If system (44)–(45) with $F_1 = G_1 = 0$ is detectable, then the pair (A_0, C_0) is detectable, too. Hence, there exists a positive-definite matrix P such that the matrix $A^{\mathrm{T}}P + PA + I$ is negative-definite. Since κ_{e3} depends on r, Assumption 4 does not hold naturally. We now discuss the following two cases:

Case (i). $\kappa_{e3} = 0$.

When locally Lipschitz and linear growth functions f_i, g_i, $i = 0, \ldots, n$, are bounded with respect to y, there are positive constants φ_{i1}, ψ_{i1} such that $|f_i(x_z, y)|^4 \le \varphi_{i1}|x_z|^4$, $\|g_i(x_z, y)\|^4 \le \psi_{i1}|x_z|^4$. In this case, we have $\kappa_{e3} = 0$, and hence, from (46) it follows that Assumption 4 is satisfied if (A_0, C_0) is detectable.

Case (ii). $\kappa_{e3} \ne 0$.

It can be verified that the following functions satisfy (18):

$$\kappa_{e1}(|\tilde{y}|) = \kappa_{e1}|\tilde{y}|^4 = (8r^3\Phi_2 + 8r\Psi_2)|\tilde{y}|^4,$$

$$\begin{aligned}
\kappa_{e2}(|x_z|) &= \kappa_{e2}|x_z|^4\\
&= [(8r^3\Phi_2 + 8r\Psi_2)b^4 + r^3\Phi_1 + r\Psi_1]|x_z|^4,
\end{aligned}$$

$$\kappa_{e3}(|\tilde{x}_z|) = \kappa_{e3}|\tilde{x}_z|^4 = (8r^3\Phi_2 + 8r\Psi_2)b^4|\tilde{x}_z|^4,$$

where for $j = 1, 2$,

$$\Phi_j = \frac{n+1}{4}\sum_{i=0}^{n}\varphi_{ij}, \qquad \Psi_j = 3\psi_{1j} + 3n\sum_{i=0,i\neq 1}^{n}\psi_{ij}.$$

In this case, if Φ_2 and Ψ_2 are sufficiently small, then the κ_{e3} in (46) is sufficiently small. Hence, from (46) it follows that Assumption 4 is satisfied if (A_0, C_0) is detectable.

Finally, for the error dynamics and inverse dynamics, we have the following dissipative inequalities

$$\mathcal{L}V_e \le -\delta_1\kappa_{e0}|\tilde{x}|^4 - \delta_1\kappa_{ez}|\tilde{x}_z|^4 + \delta_1\kappa_{e1}|\tilde{y}|^4 + \delta_1\kappa_{e2}|x_z|^4,$$
$$\mathcal{L}V_z \le -\alpha_{01}|x_z|^4 + \alpha_{02}|\tilde{x}_z|^4 + \alpha_{03}|\tilde{y}|^4.$$

Choose $\rho_2 = \frac{\delta_1\kappa_{e2}+q_1}{\alpha_{01}}$. Then, the condition (32) is reduced to

$$\delta_1\kappa_{ez}s^4 \ge \frac{(\delta_1\kappa_{e2}+q_1)\alpha_{02}}{\alpha_{01}}s^4,$$

or equivalently, $\delta_1(\kappa_{ez}\alpha_{01} - \kappa_{e2}\alpha_{02}) \ge q_1\alpha_{02}$ for some $\delta_1 > 0$.

Let

$$B = \left\{h \ge 0 : |h| < \frac{\kappa_{ez}\alpha_{01}}{\alpha_{02}}\right\}.$$

Then, for sufficiently small $\Phi_j, \Psi_j, j = 1, 2$, we have $\kappa_{e2} \in B$, that is, the condition (32) is satisfied with a sufficiently large δ_1.

In summary, for the non-minimum-phase stochastic system (44)–(45) with the locally Lipschitz and linear growth conditions, if its linear growth bounds $\varphi_{ij}, \psi_{ij}, i = 0, 1, \ldots, n$, $j = 1, 2$, are sufficiently small, then an output-feedback controller can be designed to make the closed-loop system globally asymptotically stable in probability.

8. A numerical example

In the following we give an example to illustrate the design method of this paper.

Consider the following system

$$\begin{cases} dx_z = (x_z + x_1)dt + \frac{\sqrt{2}}{20}x_z\sin x_1 dw, \\ dx_1 = (x_2 + x_z)dt, \\ dx_2 = (u + 2x_z)dt + \frac{\sqrt{2}}{20}x_z\frac{x_1^2}{1+x_1^2}dw, \\ y = x_1. \end{cases} \tag{47}$$

When $y = 0$, the zero-dynamics $dx_z = x_z dt$ is clearly unstable. Thus, the system (47) is of non-minimum phase. But the inverse dynamics is SISS stabilizable by smooth function $\Xi = -11(x_z + d_1)$ in the sense of Assumption 2. That is,

$$\mathcal{L}V_z = \mathcal{L}\left(\frac{1}{4}x_z^4\right) \le -\left(1 - \frac{3}{400}\right)x_z^4 + \frac{1}{4}d_2^4 + 2\frac{3}{4}d_1^4.$$

Consider a reduced-order observer in the form of (9) and (10):

$$\dot{\hat{x}}_z = \hat{x}_z + \beta_0(y) + x_1 - \frac{\partial\beta_0}{\partial y}[\hat{x}_2 + \beta_2(y) + \hat{x}_z + \beta_0(y)],$$

$$\dot{\hat{x}}_2 = u + 2(\hat{x}_z + \beta_0(y)) - \frac{\partial\beta_2}{\partial y}[\hat{x}_2 + \beta_2(y) + \hat{x}_z + \beta_0(y)].$$

Choose $\beta_0(y) = 1.5y$, $\beta_2(y) = 0.5y$. Then we have

$$\dot{\hat{x}}_z = -0.5\hat{x}_z - 1.5\hat{x}_2 - 0.5y,$$
$$\dot{\hat{x}}_2 = u + 1.5\hat{x}_z - 0.5\hat{x}_2 + 2y.$$

In this case, we have

$$A(y) = \begin{bmatrix} -0.5 & -1.5 \\ 1.5 & -0.5 \end{bmatrix},$$

and for any given constant $d > 0$,

$$P = \begin{bmatrix} 1+d & 0 \\ 0 & 1+d \end{bmatrix}$$

is a positive-definite matrix solution of $A(y)^{\mathrm{T}}P + PA(y) = -(1+d)I$.

Since $f_i = 0$, $g_1 = 0$, by (15), (16) and (19) we have

$$Q(y) = P^2\frac{2\lambda_{\max}(P^2)}{\lambda_{\min}(P)} + P\lambda_{\max}^2(P) = \begin{bmatrix} 3(1+d)^3 & 0 \\ 0 & 3(1+d)^3 \end{bmatrix}.$$

Let $r(y) = \lambda_{\max}(Q(y)) = 3(1+d)^3$. Then

$$\mathcal{L}V_e \le -\delta_1(1+d)d\tilde{x}^4 + \delta_1 3r(y)\frac{1}{10^4}x_z^4.$$

Let $z_1 = \tilde{y}$, $z_2 = \hat{x}_2 - \phi_1$, $V = \frac{1}{4}(z_1^4 + z_2^4)$. Then, by (22) and (24) we have

$$\phi_1(y, \hat{x}_z) = \lambda_1(y, \hat{x}_z) - \nu(z_1^2)z_1 - 12\hat{x}_z - 29.5y,$$
$$u = \lambda_2(y, \hat{x}_z, \hat{x}_2) - 25\hat{x}_z - 11\hat{x}_2 - 55y.$$

Select

$$\lambda_1(y, \hat{x}_z) = -27.25z_1, \qquad \lambda_2(y, \hat{x}_z, \hat{x}_2) = -44.75z_2,$$

where $\nu(\cdot)$ is a function to be specified. Then, (26) becomes

$$\mathcal{L}V \le -0.25z_1^4 - 0.25z_2^4 - \nu(z_1^2)z_1^4 + 11.75\tilde{x}_2^4 + 11.75x_z^4.$$

As in (33), define

$$V_{ze} = 1\frac{3}{397}\left(\delta_1 3r(y)\frac{1}{10^4} + q_1\right)V_z + V_e,$$

where $q_1 > 0$. Then

$$\begin{aligned} \mathcal{L}V_{ze} \le{}& -q_1x_z^4 + \frac{100}{397}\left(\delta_1 3r(y)\frac{1}{10^4} + q_1\right)\tilde{y}^4 \\ &+ 11\frac{100}{397}\left(\delta_1 3r(y)\frac{1}{10^4} + q_1\right)\tilde{x}_z^4 \\ &- \delta_1\frac{(1+d)d}{2}|\tilde{x}|^4 - \delta_1\frac{(1+d)d}{2}\tilde{x}_z^4. \end{aligned} \tag{48}$$

Choose $d = 1$, $q_1 = 0.5$, $\delta_1 = 2$. Then,

$$\delta_1\frac{(1+d)d}{2} \ge 11\frac{100}{397}\left(\delta_1 3r(y)\frac{1}{10^4} + q_1\right).$$

This together with (48) implies that

$$\mathcal{L}V_{ze} \le -0.5x_z^4 - 2|\tilde{x}|^4 + 0.1296\tilde{y}^4.$$

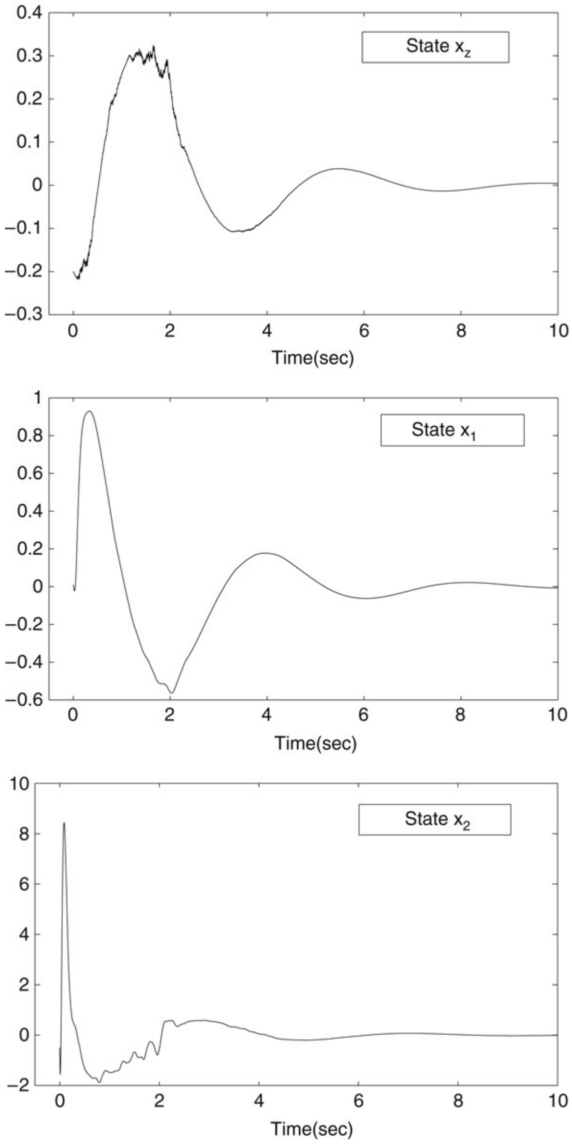

Fig. 1. States of the closed-loop system.

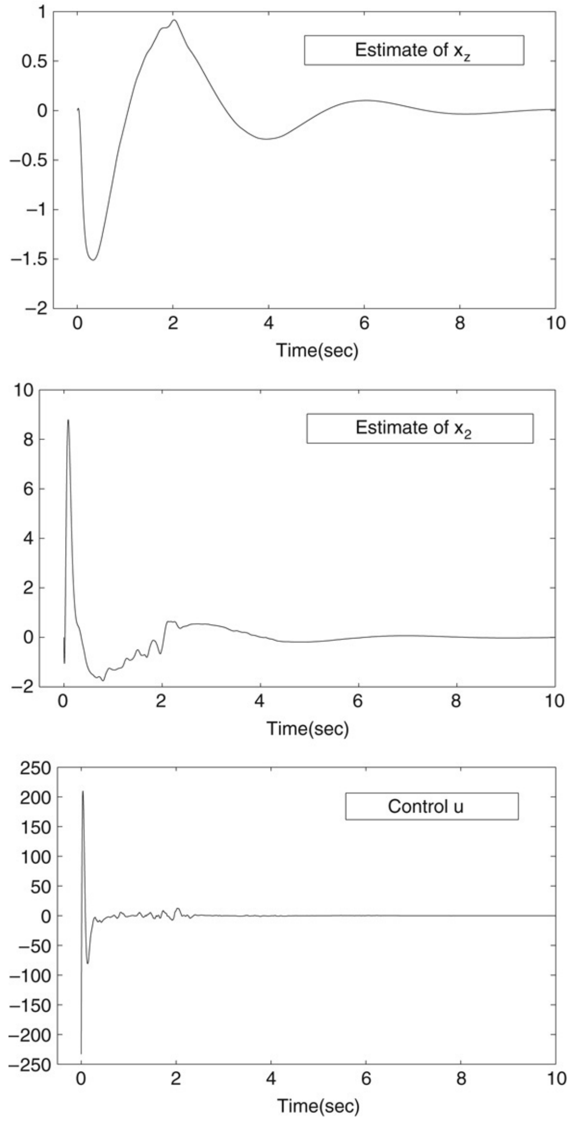

Fig. 2. Estimate of states and control of the closed-loop system.

Let $W = 6V_{ze} + V$ as in (41), and take $\nu(z_1^2) = 6 \times 0.13$. Then, we have

$$\begin{aligned}\mathcal{L}W &\leq -3x_z^4 + 6 \times 0.13\tilde{y}^4 - 12|\tilde{x}|^4 - 0.25z_1^4 - 0.25z_2^4 \\ &\quad - 24 \times 0.1296z_1^4 + 11.75\tilde{x}_2^4 + 11.75\tilde{x}_z^4 \\ &\leq -3x_z^4 - 0.25|\tilde{x}|^4 - 0.25z_1^4 - 0.25z_2^4.\end{aligned}$$

Figs. 1 and 2 are the simulation results with initial values $x_z(0) = -0.2$, $x_1(0) = 0.01$, $x_2(0) = -0.5$, $\hat{x}_z(0) = 0$, $\hat{x}_2(0) = 0$. From Figs. 1 and 2, we can see that the controller renders the resulting closed-loop system globally asymptotically stable in probability.

9. Concluding remarks

In this paper, the problem of global output-feedback stabilization has been studied for a class of stochastic nonlinear systems with unstable zero dynamics. It has been shown that using the integrator backstepping method together with the techniques of novel reduced-order observer design and changing supply functions, it is possible to obtain a global output-feedback stabilizing control law. It is the first attempt to address the constructive design of output-feedback controllers for stochastic non-minimum-phase nonlinear systems. By the proposed control schemes, the output-feedback stabilization problem was solved for some types of non-minimum-phase systems, at least including locally Lipschitz and linear growth (e.g. globally Lipschitz) stochastic non-minimum-phase nonlinear systems and some nonlinear systems in special output-feedback form.

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Appendix A. Stability of stochastic control systems

Consider the following stochastic nonlinear systems:

$$dx = f(x)dt + h(x)dw, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{A.1}$$

where $x \in \mathbb{R}^n$ is the state; the Borel measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are locally Lipschitz; w is an r-dimensional standard Brownian motion.

The following stability notion introduced in Deng et al. (2001) is used throughout this paper.

Definition A.1. For the system (A.1) with $f(0) = 0, h(0) = 0$, the equilibrium $x(t) = 0$ is said to be globally asymptotically stable in probability, if for any given $\varepsilon > 0$, there exists $\gamma(\cdot) \in \mathcal{K}$ such that

$$P\{|x(t)| < \gamma(|x_0|)\} \geq 1 - \varepsilon, \quad \forall t \geq 0, \quad x_0 \in \mathbb{R}^n \setminus \{0\},$$

and for any given initial condition x_0,

$$P\left\{\lim_{t\to\infty} x(t) = 0\right\} = 1.$$

The following theorem gives sufficient conditions on the stability introduced above.

Theorem A.1 (*Deng et al., 2001*). *For the system (A.1) with locally Lipschitz functions $f(x)$ and $h(x)$, if there exist a positive-definite and radially unbounded function $V(x) \in \mathcal{C}^2$ and a positive-definite function $W(x)$ such that $\mathcal{L}V \leq -W(x)$, then*

(a) *for (A.1) there exists an almost surely unique strong solution on $[0, \infty)$ for each $x_0 \in \mathbb{R}^n$;*
(b) *the equilibrium $x = 0$ of the system (A.1) is globally asymptotically stable in probability, when $f(0) = 0, h(0) = 0$.*

To introduce the concept of SISS, we consider the following system:

$$dx = f(x, v)dt + g(x, v)dw, \tag{A.2}$$

where $x \in \mathbb{R}^n$ is the state, $v = v(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$ is the input, w is an r-dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\{\mathcal{F}_t\}_{t\geq 0}$ being a filtration, and P being a probability measure; $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times r}$ are assumed to be locally Lipschitz in their arguments. Assume that for every initial condition x_0, each essentially bounded measurable input v, the system (A.2) has an almost surely unique strong solution[1] $x(t)$ on $[0, \infty)$ which is $\mathcal{F}_t$-adapted, t-continuous, and measurable with respect to $\mathcal{B} \times \mathcal{F}$, where $\mathcal{B}$ denotes the Borel σ-algebra of $\mathbb{R}$ (see Has'minskii, 1980). Then we have the following definition.

[1] For notational simplicity, this solution process $x(\omega, t)(\omega \in \Omega)$ is often abbreviated as $x(t)$.

Definition A.2 (*Liu et al., 2008*). The system (A.2) is SISS if for any given $\epsilon > 0$, there exist a $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$P\{|x(t)| < \beta(|x_0|, t) + \gamma(\sup_{0\leq s\leq t} \|v_s\|)\} \geq 1 - \epsilon,$$
$$\forall t \geq 0, \quad \forall x_0 \in \mathbb{R}^n \setminus \{0\},$$

where $\|v_s\| = \inf_{\mathcal{A}\subset\Omega, P(\mathcal{A})=0} \sup\{|v(x(\omega, s), s)| : \omega \in \Omega\backslash\mathcal{A}\}$.

The following theorem provides a sufficient condition for SISS.

Theorem A.2 (*Liu et al., 2008*). *The system (A.2) is SISS if there exist a $\mathcal{C}^2$ function $V(x)$ and $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha, \chi$ such that*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \qquad \mathcal{L}V \leq \chi(|v|) - \alpha(|x|).$$

Appendix B. Proof of Lemma 1

Proof. Let

$$q_1(s) = \frac{1}{\zeta(\alpha_{z1}^{-1}(s))}, \qquad q_2(s) = \frac{\xi(\alpha_{z1}^{-1}(s))}{\zeta(\alpha_{z1}^{-1}(s))},$$

and

$$\rho_1(s) = e^{\int_0^s q_1(\tau)d\tau}\left[\rho_1(0) - \int_0^s q_2(u)e^{-\int_0^u q_1(\tau)d\tau}du\right]$$

with $\rho_1(0)$ being an arbitrary positive number satisfying

$$\rho_1(0) \geq \xi(0) + \int_0^\infty [\xi(\alpha_{z1}^{-1}(s))]' e^{-\int_0^s q_1(\tau)d\tau}ds.$$

Then, it is easy to see that

$$\rho_1(s) = \rho_1'(s)\zeta(\alpha_{z1}^{-1}(s)) + \xi(\alpha_{z1}^{-1}(s)), \quad s \geq 0. \tag{B.1}$$

Noticing that

$$\begin{aligned}
&\int_0^s q_2(u)e^{-\int_0^u q_1(\tau)d\tau}du + \frac{q_2(s)}{q_1(s)}e^{-\int_0^s q_1(\tau)d\tau} \\
&\quad = -\int_0^s \frac{q_2(u)}{q_1(u)}de^{-\int_0^u q_1(\tau)d\tau} + \frac{q_2(s)}{q_1(s)}e^{-\int_0^s q_1(\tau)d\tau} \\
&\quad = -\frac{q_2(u)}{q_1(u)}e^{-\int_0^u q_1(\tau)d\tau}\Big|_0^s + \int_0^s \left[\frac{q_2(u)}{q_1(u)}\right]' e^{-\int_0^u q_1(\tau)d\tau}du \\
&\qquad + \frac{q_2(s)}{q_1(s)}e^{-\int_0^s q_1(\tau)d\tau} \\
&\quad = \xi(0) + \int_0^s [\xi(\alpha_{z1}^{-1}(u))]' e^{-\int_0^u q_1(\tau)d\tau}du \\
&\quad \leq \rho_1(0), \quad \forall s \geq 0,
\end{aligned}$$

we have

$$\rho_1'(s) = e^{\int_0^s q_1(\tau)\mathrm{d}\tau}\left[\rho_1(0) - \int_0^s q_2(u)e^{-\int_0^u q_1(\tau)\mathrm{d}\tau}\mathrm{d}u - \frac{q_2(s)}{q_1(s)}e^{-\int_0^s q_1(\tau)\mathrm{d}\tau}\right] \geq 0, \quad \forall s \geq 0.$$

This together with (B.1) leads to

$$\begin{aligned}\rho_1(V_z(x)) &= \rho_1'(V_z(x))\zeta(\alpha_{z1}^{-1}(V_z(x))) + \xi(\alpha_{z1}^{-1}(V_z(x)))\\ &\geq \rho_1'(V_z(x))\zeta(|x|) + \xi(|x|)\\ &\geq \rho_1'(V_z(x))\frac{2\psi_z^2(|x|)\psi_0^2(|x|)}{\alpha_{01}(|x|)} + \frac{4\delta_1\kappa_{e2}(|x|)}{\alpha_{01}(|x|)}.\end{aligned}$$

Multiplying both sides of the above inequality by $\alpha_{01}(|x|)$ gives (31).

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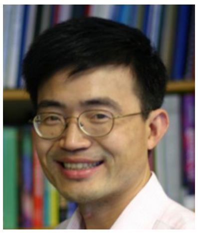

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