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A Notion of Stochastic Input-to-State Stability and Its Application to Stability of Cascaded Stochastic Nonlinear Systems

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Abstract In this paper, the property of practical input-to-state stability and its application to stability of cascaded nonlinear systems are investigated in the stochastic framework. Firstly, the notion of (practical) stochastic input-to-state stability with respect to a stochastic input is introduced, and then by the method of changing supply functions, (a) an (practical) SISS-Lyapunov function for the overall system is obtained from the corresponding Lyapunov functions for cascaded (practical) SISS subsystems.

Keywords Stochastic nonlinear systems, Stochastic input-to-state stability (SISS), SISS-Lyapunov function
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1 Introduction

The notion of deterministic input-to-state stability (ISS) introduced in Sontag's seminal paper (see [15]) plays an important role in nonlinear system analysis and synthesis (see [7,8,11,16–19]). In view of its importance, the similar property in the stochastic setting has attracted intensive attention (see [20–23]). The notion of γ-input-to-state stability (γ-ISS) was introduced in [21], in which the problems of linear partial state-feedback and output-feedback stabilization were considered for a class of stochastic systems with γ-ISS subsystem under the assumption that nonlinear functions satisfy certain kind of global linear growth conditions. Following this work, the notion of exponential input-to-state stability for stochastic systems was then introduced in [22], in which the problems of static and dynamic output-feedback control were studied for certain class of stochastic nonlinear system with global linear growth vector fields. Different from the γ-ISS, the notion of SISS with respect to a deterministic input, which is a generalization of the notion of noise-to-state stability (NSS) (see [10]), was introduced in [20] to analyze the stochastic stability of singularly perturbed nonlinear systems. Recently, a gain-function-based stochastic nonlinear small-gain theorem was given for stochastic input-to-state practically stable (with respect to a deterministic input) subsystems in [23], in which the same small-gain condition as the deterministic case was established. It is well known that small-gain theorem based on Lyapunov functions is more convenient to design the small-gain controllers using backstepping technique. But, how to generalize the deterministic Lyapunov-based (SISS) nonlinear small-gain theorem to stochastic case is a challenging and meaningful issue (see [23]).

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Since the development of differential geometric theory for nonlinear systems, it is known that under some conditions a system can be transformed to the system of some normal form, which is closely related to a class of cascaded nonlinear systems. Recently, much attention has been paid to some control problems of cascaded nonlinear systems (see [3]).

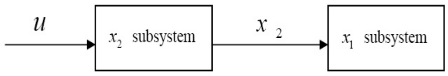

Figure 1. cascaded system

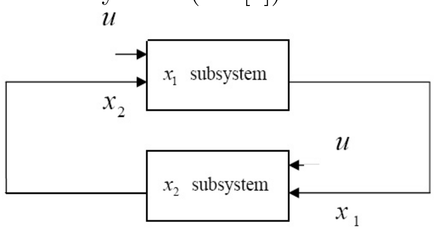

Figure 2. interconnected system

Notice that the cascaded system like Figure 1 is a special form of the interconnected system like Figure 2, which is the class of systems studied for the SISS nonlinear small-gain theorem. Thus, for the study of Lyapunov-based SISS nonlinear small-gain theorem, it is meaningful and helpful to study the stochastic input-to-state stability of this class of cascaded systems.

The purpose of this paper is to prepare for deriving a Lyapunov-type SISS nonlinear small-gain theorem. Different from all the existing notions characterizing the SISS behavior, a more practical notion of SISS with respect to a stochastic input is introduced so as to describe the stochastic stability under stochastic inputs. The key technique adopted for stability analysis is the method of changing supply function. The initial idea of this method came from the deterministic work [17], but its extension to the stochastic setting will be more difficult and challenging. It is worth pointing out that new small-gain type conditions for small signals are obtained in the construction of Lyapunov functions for cascaded SISS systems.

The organization of this paper is as follows. Section 2 gives some notations. Section 3 illustrates the property of stochastic input-to-state stability. Section 4 presents the stability analysis of cascaded SISS systems. Section 5 gives some conclusions.

2 Notation

The following notations will be used throughout this paper. $\mathbb{R}_+$ denotes the set of all nonnegative real numbers. $\mathbb{R}^n$ denotes the real n-dimensional space. For a given vector or matrix X, X^T denotes its transpose; $\mathrm{Tr}(X)$ denotes its trace when X is a square matrix; $|X|$ denotes the Euclidean norm of a vector X and the corresponding induced norm for matrices is denoted by $\|X\|$; $\|X\|_F$ denotes the Frobenius norm of X defined by $\|X\|_F = \sqrt{Tr(X^TX)}$; $\mathcal{C}^i$ denotes the set of all functions with continuous ith partial derivative; $\mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$ denotes the family of all nonnegative functions $V(x)$ on $\mathbb{R}^n$ which are $\mathcal{C}^2$ in x; $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanish at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded; $\mathcal{KL}$ denotes the set of all functions $\beta(s,t)$: $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which is of class $\mathcal{K}$ for each fixed t, and decreases to zero as $t \to \infty$ for each fixed s; $I_{\mathcal{A}}$ denotes the indicator function of set $\mathcal{A}$; $\mathcal{A}^c$ denotes the complementary set of set $\mathcal{A}$.

Consider the following stochastic system

$$dx = (f(x) + g(x)u)dt + h(x)dw, \tag{1}$$

where w is an r-dimensional standard Brownian motion, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are assumed to be locally Lipschitz in their arguments.

Definition 1. *For any given $V(x) \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$, associated with the stochastic differential equation (1) we define the differential operator $\mathcal{L}$ as follows:*

$$\mathcal{L}V = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u + \frac{1}{2}\mathrm{Tr}\Big\{h^T \frac{\partial^2 V}{\partial x^2} h\Big\}.$$

3 Stochastic Input-to-State Stability

Before stating our main results in next section, we first present some notions and results about stochastic input-to-state stability.

For control-free stochastic nonlinear systems of the form:

$$dx = f(x)dt + h(x)dw, \tag{2}$$

the following stability notion introduced in [2] is often used in the controller design for stochastic nonlinear systems (see [12,14] and the references therein).

Definition 2. *The solution process $\{x(t), t \geq 0\}$ of the stochastic system (2) is said to be bounded in probability, if*

$$\lim_{c\to\infty} \sup_{0\leq t<\infty} P\{|x(t)| > c\} = 0.$$

To introduce the notion of SISS, consider the following system:

$$dx = f(x, v)dt + g(x, v)dw, \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $v = v(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$ is the input, w is an r-dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\{\mathcal{F}_t\}_{t\geq 0}$ being a filtration, and P being a probability measure; $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times r}$ are assumed to be locally Lipschitz in their arguments. Assume that for every initial condition x_0, each essentially bounded measurable input v, the system (3) has a unique solution[1] $x(t)$ on $[0, \infty)$ which is $\mathcal{F}_t$-adapted, t-continuous, and measurable with respect to $\mathcal{B} \times \mathcal{F}$, where $\mathcal{B}$ denotes the Borel $\sigma-$algebra of $\mathbb{R}$ (see [2]). Then we have the following definition.

Definition 3. *The system (3) is practically SISS if for any given $\varepsilon > 0$, there exist a $\mathcal{KL}$ function $\beta(\cdot,\cdot)$, a $\mathcal{K}$ function $\gamma(\cdot)$ and a constant $d \geq 0$ such that*

$$P\Big\{|x(t)| < \beta(|x_0|, t) + \gamma(\sup_{0\leq s\leq t} \|v_s\|) + d\Big\} \geq 1 - \varepsilon, \qquad \forall t \geq 0, \quad \forall x_0 \in \mathbb{R}^n \backslash \{0\}, \tag{4}$$

where $\|v_s\| = \inf_{\mathcal{A}\subset\Omega, P(\mathcal{A})=0} \sup\{|v(x(\omega, s), s)| : \omega \in \Omega\backslash\mathcal{A}\}$. When $d = 0$ in (4), the system (3) is said to be SISS.

Remark 1. (i) Different from all the existing notions characterizing the SISS behavior for a locally Lipschitz system without growth restriction, here the input v in the system (3) is assumed to be a function of x and t, precisely, $v = v(x(\omega, t), t)$, and can be regarded as a Markov control input, which ensures the corresponding solution process $x(\omega, t)$ is an Itô diffusion, and hence, a Markov process (see [13]). This kind of input is the most general one for the systems described by Itô diffusion stochastic differential equations, for which controller

[1]For simplicity, the solution process $x(\omega, t)$ is abbreviated as $x(t)$ through out of the paper when there is no confusion caused.

design has been an active area of research in recent years (see [12,14] and the references therein). When $v(x,t) = v(t)$ is deterministic and $d = 0$, the above definition is the SISS given in [20], and a generalization of NSS (see [10]).

(ii) When $d = 0$, if we view the input as the random perturbation, from (4) we can conclude that the system (3) is stable under small random perturbations in the following sense (see [2]): for any given $\varepsilon > 0$ and $\Delta > 0$, there exists a $\kappa > 0$ such that whenever $|x_0| + \sup_{x,t} |v(x,t)| < \kappa$, the following inequality holds:

$$P\{|x(t)| > \Delta\} < \varepsilon, \qquad \forall t > 0.$$

In fact, for given $\varepsilon > 0$ and $\Delta > 0$, if (4) holds, then by letting $\gamma_1(\cdot) = \beta(\cdot, 0)$ and $\kappa = \min\{\gamma_1^{-1}(\frac{\Delta}{2}), \gamma^{-1}(\frac{\Delta}{2})\}$, from $|x_0| + \sup_{x,t} |v(x,t)| < \kappa$, one can get $|x_0| < \kappa$ and $\sup_{x,t} |v(x,t)| < \kappa$, which ensures $\beta(|x_0|, 0) < \frac{\Delta}{2}$, $\gamma(\sup_{x,t} |v(x,t)|) < \frac{\Delta}{2}$. Hence, $P\{|x(t)| \le \Delta\} \ge P\{|x(t)| \le \beta(|x_0|, 0) + \gamma(\sup_{x,t} |v(x,t)|)\} \ge P\{|x(t)| < \beta(|x_0|, t) + \gamma(\sup_{0 \le s \le t} \|v_s\|)\} \ge 1 - \varepsilon$, or equivalently, $P\{|x(t)| > \Delta\} < \varepsilon$.

(iii) The above definition is also different from the notion of the γ-ISS (see [21]). Actually, they address two different aspects. γ-ISS addresses whether the state $x(t)$ can stay in a small neighborhood of the equilibrium when $t \to \infty$, or when the initial condition $x_0 \to 0$ and the input is bounded by a function of the state. While SISS given by Definition 3 addresses whether the state $x(t)$ can be bounded by a function of the initial conditions x_0, the time t, and the input $\{v(s) : 0 \le s \le t\}$. The former is a kind of qualitative description, while the later makes the role of the initial conditions and the input explicit.

(iv) The property (4) ensures that when the input $v \equiv 0$ and $d = 0$, the system (3) is *globally asymptotically stable in probability* (see [10]). It also ensures that the solution process of the system (3) is bounded in probability when the input v is bounded almost surely (in other words, there exists a constant $M > 0$ such that $P\{\sup_{t \ge 0} |v(x(\omega,t),t)| \le M\} = 1$). In fact, let $\mathcal{A} = \{|x(t)| < \beta(|x_0|, t) + \gamma(\sup_{0 \le s \le t} \|v_s\|) + d\}$, $\mathcal{B} = \{\sup_{t \ge 0} |v(x(\omega,t),t)| \le M\}$, $\mathcal{A}^c = \Omega - \mathcal{A}$ and $\mathcal{B}^c = \Omega - \mathcal{B}$. Then, we have $P(\mathcal{B}^c) = 1 - P(\mathcal{B}) = 0$, and so,

$$P(\mathcal{A}^c) \le P(\mathcal{A}^c \cup \mathcal{B}^c) \le P(\mathcal{A}^c) + P(\mathcal{B}^c) = P(\mathcal{A}^c).$$

This leads to $P(\mathcal{A}^c \cup \mathcal{B}^c) = P(\mathcal{A}^c) = 1 - P(\mathcal{A}) \le \varepsilon$, and

$$P(\mathcal{A} \cap \mathcal{B}) = 1 - P((\mathcal{A} \cap \mathcal{B})^c) = 1 - P(\mathcal{A}^c \cup \mathcal{B}^c) \ge 1 - \varepsilon.$$

Thus, $P\{|x(t)| < \beta(|x_0|, t) + \gamma(M) + d\} \ge 1 - \varepsilon$, $\forall t \ge 0$, or equivalently, $P\{|x(t)| \ge \beta(|x_0|, t) + \gamma(M) + d\} < \varepsilon$, $\forall t \ge 0$. This together with the property of the function β gives

$$P\{|x(t)| \ge \beta(|x_0|, 0) + \gamma(M) + d\} \le P\{|x(t)| \ge \beta(|x_0|, t) + \gamma(M) + d\} < \varepsilon.$$

Let $C = \beta(|x_0|, 0) + \gamma(M) + d$. Then, $\sup_{0 \le t < \infty} P\{|x(t)| > c\} < \varepsilon$, $\forall c > C$. Hence, the solution process of the system (3) is bounded in probability by Definition 2.

Definition 4. *A $\mathcal{C}^2$ function $V(x)$ is said to be a practical SISS-Lyapunov function for the system (3) if there exist $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha, \chi$ and a constant $d \ge 0$ such that*

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|), \tag{5}$$

$$\mathcal{L}V \le \chi(|v|) - \alpha(|x|) + d. \tag{6}$$

When $d = 0$ in (6), the function V is said to be an SISS-Lyapunov function for the system (3).

The following theorem provides a sufficient condition for (practically) SISS.

Theorem 1. *The system (3) is practically SISS (resp. SISS) if there exists a practical SISS- (resp. SISS-) Lyapunov function.*

Proof. See the Appendix.

Remark 2. From Theorem 1 one can see that SISS is endowed with the property of dissipation. The combination of the functions χ and α serves as one characterization of the input-to-state gain of the system. Here as the deterministic case in [17], the function $V(x)$ is called storage function for the system (3), and the pair of the $\mathcal{K}_\infty$ functions (χ, α) is a supply pair for the system (3). In the next section, we will explore the possible supply pair for a given dissipative system so as to design an SISS-Lyapunov function for the whole system.

Remark 3. In [23], under the conditions (5)–(6), the following result is given: for any given $\varepsilon > 0$, there exists a $\mathcal{KL}$ function $\beta(\cdot, \cdot)$, a $\mathcal{K}$ function $\gamma(\cdot)$, and a constant $\widetilde{d} \geq 0$ such that

$$P\big\{|x(t)| < \beta(|x_0|, t) + \gamma(\sup_{0 \leq s \leq t} |v(s)|) + \widetilde{d}\big\} \geq 1 - \varepsilon, \qquad \forall t \geq 0, \quad \forall x_0 \in \mathbb{R}^n \backslash \{0\}.$$

It should be pointed out that this result holds only for the deterministic input from the idea of the proof presented in [23]. Here we consider the stochastic input case and obtain a general and more reasonable stability result (4) as Remark 1 states.

From Theorem 1, we can obtain the following useful criterion on the boundedness of the solution.

Corollary 1. *For the system (2), assume that $f(x)$ and $h(x)$ are locally Lipschitz. If there exists a positive-definite, radially unbounded, twice continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, a constant $c \geq 0$, and a positive-definite and radially unbounded function $W(x)$ such that*

$$\mathcal{L}V \leq -W(x) + c, \tag{7}$$

then the solution process is bounded in probability.

Proof. According to Theorem 4.1 of [2], the system (2) has a unique solution $x(t)$ on $[0, \infty)$. Noticing that V is positive-definite, radially unbounded and continuous, we have (5). Let $\alpha(r) = \inf_{|x| \geq r} W(x)$. Then, $\alpha(r) \in \mathcal{K}_\infty$ and $\mathcal{L}V \leq -\alpha(|x|) + c$. In the proof of Theorem 1 (see Appendix), let $v = 0$ and $d = c$. Then we can conclude that for any given $\varepsilon > 0$, there exists $\beta(\cdot, \cdot) \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$P\{|x(t)| < \beta(|x_0|, t) + \gamma(c)\} \geq 1 - \varepsilon, \qquad \forall t \geq 0, \quad \forall x_0 \in \mathbb{R}^n \backslash \{0\}.$$

By Remark 1(iv), the solution of the system (2) is bounded in probability.

It is known that in the deterministic case, finite-dimensional stable linear systems are ISS. In the following, we will illustrate the SISS property by three classes of stochastic systems including the linear and bilinear ones.

1) Consider the following stochastic linear system with input and additive noise:

$$dx = (Ax + Bv)dt + C^T dw, \tag{8}$$

where $x \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}^m$ is the input, w is an r-dimensional standard Brownian motion, and A, B, C are constant matrices with appropriate dimensions.

When A is asymptotically stable, $A^TP+PA=-2I$ has a positive-definite solution P. Let $V=x^TPx$. Then, by simple computation we have

$$\mathcal{L}V \le -|x|^2+\|PB\|^2|v|^2+Tr(P)\|C\|_F^2. \tag{9}$$

By Theorem 1, the system (8) is practically SISS when v is regarded as the input.

2) Consider the following stochastic bilinear system with multiplicative noise:

$$dx=(Ax+Bv)dt+xC^Tdw, \tag{10}$$

where x,v,w are the same as in (8), and A,B,C are constant matrices with appropriate dimensions.

When $A+\frac{C^TC}{2}I$ is asymptotically stable, there exists a positive-definite matrix P satisfying

$$A^TP+PA+C^TCP=\Big(A+\frac{C^TC}{2}I\Big)^TP+P\Big(A+\frac{C^TC}{2}I\Big)=-2I.$$

Let $V=x^TPx$. Then we have

$$\mathcal{L}V \le -|x|^2+\|PB\|^2|v|^2. \tag{11}$$

By Theorem 1, the system (10) is SISS with respect to the input v.

3) Consider the following bilinear system with input in diffusion term:

$$dx=(Ax+Bv)dt+(Cx+Dv)dw, \tag{12}$$

where x,v are the same as in (8), w is a 1-dimensional standard Brownian motion, A,B,C,D are constant matrices with appropriate dimensions.

When the following Riccati equation

$$A^TP+PA+C^TPC=-3I$$

has a positive-definite solution P, let $V=x^TPx$. Then we have

$$\mathcal{L}V \le -|x|^2+(\|PB\|^2+\|C^TPD\|^2+\|D^TPD\|)|v|^2. \tag{13}$$

According to Theorem 1, the system (12) is SISS with respect to the input v.

4 Stability of Cascaded SISS Stochastic Nonlinear Systems

In this section, we are to study the cascade of the two SISS systems.

Consider the system in cascade form:

$$dx=f_1(x,z)dt+g_1(x,z)dw, \tag{14}$$
$$dz=f_2(z,u)dt+g_2(z,u)dw, \tag{15}$$

where $x\in\mathbb{R}^n$, $z\in\mathbb{R}^m$ are the states, $u\in\mathbb{R}^k$ is the input; f_i,g_i are locally Lipschitz functions; there exist two positive-definite and radically unbounded functions $V_1(x)\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R}_+)$, $V_2(z)\in\mathcal{C}^2(\mathbb{R}^m,\mathbb{R}_+)$, $\mathcal{K}_\infty$ functions $\alpha_x,\alpha_z,\gamma_1$ and γ_2, and constants $d_1\ge 0$, $d_2\ge 0$ such that

$$\mathcal{L}V_1(x) \le -\alpha_x(|x|)+\gamma_1(|z|)+d_1, \tag{16}$$
$$\mathcal{L}V_2(z) \le -\alpha_z(|z|)+\gamma_2(|u|)+d_2. \tag{17}$$

In the following, we study the stochastic input-to-state stability of the system (14)–(15) by changing supply functions of the subsystem (15).

Consider the following condition:

Assumption 1. *There exist known smooth nonnegative functions ψ_z and ψ_2 satisfying $|\nabla V_2| \le \psi_z(|z|)$ and $\|g_2(z,u)\| \le \psi_2(|z|)$, where and whereafter $\nabla V_2 = \frac{\partial V_2}{\partial z}$.*

Remark 4. In the above assumption, $|\nabla V_2| \le \psi_z(|z|)$ is a general assumption and easy to verify; while $\|g_2(z,u)\| \le \psi_2(|z|)$ is a constraint on the diffusion term, which reflects that the diffusion term of the subsystem is confined by the dynamics itself and that the effect of the input can be viewed as bounded.

For $V_2(z)$ is positive-definite, radially unbounded and continuous, there exist $\alpha_{1z}, \alpha_{2z} \in \mathcal{K}_\infty$ such that $\alpha_{1z}(|z|) \le V_2(z) \le \alpha_{2z}(|z|)$.

With these notations, we have the following results.

Lemma 1. *If*

$$\limsup_{s\to 0+} \frac{\gamma_1(s)}{\alpha_z(s)} < \infty, \qquad \limsup_{s\to 0+} \frac{\psi_z^2(s)\psi_2^2(s)}{\alpha_z(s)} < \infty, \tag{18}$$

and

$$\int_0^\infty [\xi(\alpha_{1z}^{-1}(s))]' e^{-\int_0^s [\zeta(\alpha_{1z}^{-1}(\tau))]^{-1} d\tau} ds < \infty, \tag{19}$$

where continuous increasing functions $\xi(s) \ge 0$ and $\zeta(s) > 0$ defined on $[0,\infty)$ satisfy that

$$\xi(s)\alpha_z(s) \ge 8\gamma_1(s), \qquad \zeta(s)\alpha_z(s) \ge 4\psi_z^2(s)\psi_2^2(s), \tag{20}$$

then there exists a nondecreasing positive function $\rho(\cdot) \in \mathcal{C}^1[0,\infty)$ such that [2]

$$\rho(V_2(z))\alpha_z(|z|) \ge 4\rho'(V_2(z))\psi_z^2(|z|)\psi_2^2(|z|) + 8\gamma_1(|z|), \qquad \forall z \in \mathbb{R}^m. \tag{23}$$

Proof. Let

$$q_1(s) = \frac{1}{\zeta(\alpha_{1z}^{-1}(s))}, \qquad q_2(s) = \frac{\xi(\alpha_{1z}^{-1}(s))}{\zeta(\alpha_{1z}^{-1}(s))},$$

and

$$\rho(s) = e^{\int_0^s q_1(\tau)d\tau}\Big[\rho(0) - \int_0^s q_2(u) e^{-\int_0^u q_1(\tau)d\tau} du\Big] \tag{24}$$

with $\rho(0)$ being an arbitrary positive number satisfying

$$\rho(0) \ge \xi(0) + \int_0^\infty [\xi(\alpha_{1z}^{-1}(s))]' e^{-\int_0^s q_1(\tau)d\tau} ds.$$

Then, it is easy to see that

$$\rho(s) = \rho'(s)\zeta(\alpha_{1z}^{-1}(s)) + \xi(\alpha_{1z}^{-1}(s)), \qquad s \ge 0. \tag{25}$$

[2] When $d_1 = d_2 = 0$, accordingly, (20) and (23) change to the following, respectively,

$$\xi(s)\alpha_z(s) \ge 4\gamma_1(s), \qquad \zeta(s)\alpha_z(s) \ge 2\psi_z^2(s)\psi_2^2(s), \tag{21}$$

$$\rho(V_2(z))\alpha_z(|z|) \ge 2\rho'(V_2(z))\psi_z^2(|z|)\psi_2^2(|z|) + 4\gamma_1(|z|), \qquad \forall z \in \mathbb{R}^m. \tag{22}$$

Noticing that

$$\begin{aligned}&\int_0^s q_2(u)e^{-\int_0^u q_1(\tau)d\tau}du+\frac{q_2(s)}{q_1(s)}e^{-\int_0^s q_1(\tau)d\tau}\\=&-\int_0^s \frac{q_2(u)}{q_1(u)}de^{-\int_0^u q_1(\tau)d\tau}+\frac{q_2(s)}{q_1(s)}e^{-\int_0^s q_1(\tau)d\tau}\\=&-\frac{q_2(u)}{q_1(u)}e^{-\int_0^u q_1(\tau)d\tau}\Big|_0^s+\int_0^s\Big[\frac{q_2(u)}{q_1(u)}\Big]'e^{-\int_0^u q_1(\tau)d\tau}du+\frac{q_2(s)}{q_1(s)}e^{-\int_0^s q_1(\tau)d\tau}\\=&\xi(0)+\int_0^s[\xi(\alpha_{1z}^{-1}(u))]'e^{-\int_0^u q_1(\tau)d\tau}du\\\le&\rho(0),\qquad \forall s\ge 0,\end{aligned}$$

we have

$$\rho'(s)=e^{\int_0^s q_1(\tau)d\tau}\Big[\rho(0)-\int_0^s q_2(u)e^{-\int_0^u q_1(\tau)d\tau}du-\frac{q_2(s)}{q_1(s)}e^{-\int_0^s q_1(\tau)d\tau}\Big]\ge 0,\qquad \forall s\ge 0.$$

This together with (25) leads to

$$\begin{aligned}\rho(V_2(z))=&\rho'(V_2(z))\zeta(\alpha_{1z}^{-1}(V_2(z)))+\xi(\alpha_{1z}^{-1}(V_2(z)))\ge\rho'(V_2(z))\zeta(|z|)+\xi(|z|)\\\ge&\rho'(V_2(z))\frac{4\psi_z^2(|z|)\psi_2^2(|z|)}{\alpha_z(|z|)}+\frac{8\gamma_1(|z|)}{\alpha_z(|z|)}.\end{aligned}$$

Multiplying both sides of the above inequality by $\alpha_z(|z|)$ gives (23).

Thus, we obtain the stability of cascaded SISS nonlinear systems.

Theorem 2. *For the system (14)–(15) with (16)–(17) satisfied, suppose that the small-gain type conditions (18)–(19) hold. Then, the system (14)–(15) is practically SISS with respect to the input u.*

Proof. For the nondecreasing positive function $\rho(\cdot)\in\mathcal{C}^1[0,\infty)$ defined in Lemma 1, let

$$U(z)=\int_0^{V_2(z)}\rho(t)dt.$$

Denote $\eta(\cdot)=\alpha_{2z}(\alpha_z^{-1}(2\gamma_2(\cdot)))\in\mathcal{K}_\infty$ and notice that when $\frac{1}{2}\alpha_z(|z|)\ge\gamma_2(|u|)$,

$$\rho(V_2(z))[\gamma_2(|u|)-\alpha_z(|z|)]\le\rho(\eta(|u|))\gamma_2(|u|)-\frac{1}{2}\rho(V_2(z))\alpha_z(|z|);\tag{26}$$

when $\frac{1}{2}\alpha_z(|z|)<\gamma_2(|u|)$, $|z|<\alpha_z^{-1}(2\gamma_2(|u|))$ and $V_2(z)<\alpha_{2z}(\alpha_z^{-1}(2\gamma_2(|u|)))$, and hence,

$$\rho(V_2(z))[\gamma_2(|u|)-\alpha_z(|z|)]<\rho(\eta(|u|))\gamma_2(|u|)-\frac{1}{2}\rho(V_2(z))\alpha_z(|z|).\tag{27}$$

Similarly, considering the cases where $\frac{1}{4}\alpha_z(|z|)\ge d_2$ and $\frac{1}{4}\alpha_z(|z|)<d_2$, we have

$$\rho(V_2(z))[d_2-\frac{1}{2}\alpha_z(|z|)]<\rho(\alpha_{2z}(\alpha_z^{-1}(4d_2)))d_2-\frac{1}{4}\rho(V_2(z))\alpha_z(|z|).$$

Combining with (26)–(27), we have

$$\rho(V_2(z))[d_2+\gamma_2(|u|)-\alpha_z(|z|)]<\rho\big(\alpha_{2z}(\alpha_z^{-1}(4d_2))\big)d_2+\rho\big(\eta(|u|)\big)\gamma_2(|u|)-\frac{1}{4}\rho(V_2(z))\alpha_z(|z|).$$

Then, by Itô formula and (23) we have[3]

$$\begin{aligned}\mathcal{L}U =&\rho(V_2(z))\mathcal{L}V_2 + \frac{1}{2}\rho'(V_2(z))\|\nabla V_2^T g_2\|^2\\ \leq&\rho(V_2(z))[d_2 + \gamma_2(|u|) - \alpha_z(|z|)] + \frac{1}{2}\rho'(V_2(z))\psi_z^2(|z|)\psi_2^2(|z|)\\ \leq&\rho\big(\alpha_{2z}(\alpha_z^{-1}(4d_2))\big)d_2 + \rho\big(\eta(|u|)\big)\gamma_2(|u|) - \frac{1}{4}\rho(V_2(z))\alpha_z(|z|) + \frac{1}{2}\rho'(V_2(z))\psi_z^2(|z|)\psi_2^2(|z|)\\ \leq&\rho\big(\alpha_{2z}(\alpha_z^{-1}(4d_2))\big)d_2 + \rho(\eta(|u|))\gamma_2(|u|) - \frac{1}{8}\rho\big(V_2(z)\big)\alpha_z(|z|) - \gamma_1(|z|).\end{aligned} \tag{29}$$

Let

$$W(x,z) = V_1(x) + U(z).$$

Then, by (16) and (29) we obtain

$$\begin{aligned}\mathcal{L}W =&\mathcal{L}V_1 + \mathcal{L}U\\ \leq&\rho(\alpha_{2z}(\alpha_z^{-1}(4d_2)))d_2 + \rho(\eta(|u|))\gamma_2(|u|) - \frac{1}{8}\rho(V_2(z))\alpha_z(|z|) - \alpha_x(|x|) + d_1\\ \leq& -\alpha(|(x,z)|) + \gamma(|u|) + d,\end{aligned} \tag{30}$$

where $\alpha(r) = \inf_{|(x,z)|\geq r}\{\frac{1}{8}\rho(0)\alpha_z(|z|) + \alpha_x(|x|)\}$, $\gamma(r) = \rho(\eta(r))\gamma_2(r), d = \rho(\alpha_{2z}(\alpha_z^{-1}(4d_2)))d_2 + d_1$.

Thus, by Theorem 1, the system (14)–(15) is practically SISS with respect to the input u. The following corollary is a generalization of Theorem 2 in [17].

Corollary 2. *For the system (15) with the supply pair (γ_2, α_z) and $d_2 = 0$, suppose that Assumption 1 and $\limsup_{s\to 0+}\frac{\psi_z^2(s)\psi_2^2(s)}{\alpha_z(s)} < \infty$ hold. If there is a $\mathcal{K}_\infty$ function $\widetilde{\alpha}(\cdot)$ such that*

$$\limsup_{s\to 0+}\frac{\widetilde{\alpha}(s)}{\alpha_z(s)} < \infty, \tag{31}$$

and

$$\int_0^\infty [\xi(\alpha_{1z}^{-1}(s))]' e^{-\int_0^s [\zeta(\alpha_{1z}^{-1}(\tau))]^{-1}d\tau} ds < \infty, \tag{32}$$

where $\xi(s) \geq 0$ and $\zeta(s) > 0$ are continuous increasing functions defined on $[0,\infty)$ satisfying

$$\xi(s)\alpha_z(s) \geq 4\widetilde{\alpha}(s), \qquad \zeta(s)\alpha_z(s) \geq 2\psi_z^2(s)\psi_2^2(s), \tag{33}$$

then there exists a function $\widetilde{\gamma} \in \mathcal{K}_\infty$ such that $(\widetilde{\gamma}, \widetilde{\alpha})$ is a supply pair.

Proof. In the proof of Theorem 2, let $\widetilde{\alpha}(s) = \gamma_1(s)$ and $\widetilde{\gamma}(s) = \rho(\eta(s))\gamma_2(s)$. Then, by (28) we obtain a new supply pair $(\widetilde{\gamma}, \widetilde{\alpha})$ and a new storage function $U(\cdot)$ satisfying

$$\mathcal{L}U \leq \widetilde{\gamma}(|u|) - \widetilde{\alpha}(|z|).$$

Remark 5. It is worth noticing that the technical inequalities in (18) are reminiscent of, but are different from, the (local) small-gain conditions in the setting of deterministic controller design (see [4,6]). For both settings, these conditions are required to hold only for *small* signals.

[3] When $d_1 = d_2 = 0$, accordingly, (29) changes to

$$\mathcal{L}U \leq \rho(\eta(|u|))\gamma_2(|u|) - \frac{1}{4}\rho(V_2(z))\alpha_z(|z|) - \gamma_1(|z|). \tag{28}$$

Remark 6. To ensure the existence of the $\rho(s)$ satisfying (23), the condition (18) seems necessary; however, the condition (19) is not. In fact, let $\zeta_1(s) = \psi_z^2(s)\psi_2^2(s)/\alpha_z(s)$. Then, when $\limsup_{s\to 0+} \zeta_1(s) = \infty$, $\zeta_1(s)$ is decreasing, $\limsup_{s\to 0^+} \gamma_1(s)/\alpha_z(s) < \infty$ and

$$\int_0^\infty e^{-\int_0^s \frac{1}{\zeta_1(\alpha_{2z}^{-1}(\tau))}d\tau}[\xi(\alpha_{1z}^{-1}(s))]'ds < \infty, \tag{34}$$

we can obtain a function $\rho(s)$ similarly:

$$\rho(s) = e^{\int_0^s \frac{1}{\zeta_1(\alpha_{2z}^{-1}(\tau))}d\tau}\Big[\rho(0) - \int_0^s \frac{\xi(\alpha_{1z}^{-1}(u))}{\zeta_1(\alpha_{2z}^{-1}(u))}e^{-\int_0^u \frac{1}{\zeta_1(\alpha_{2z}^{-1}(\tau))}d\tau}du\Big],$$

where $\rho(0)$ can be any constant satisfying

$$\rho(0) \geq \xi(0) + \int_0^\infty e^{-\int_0^u \frac{1}{\zeta_1(\alpha_{2z}^{-1}(\tau))}d\tau}[\xi(\alpha_{1z}^{-1}(u))]'du.$$

It should be pointed out that if $\zeta_1(\alpha_{2z}^{-1}(s))$ has zeros in $(0,\infty)$, then $\zeta_1(\alpha_{2z}^{-1}(s)) + a$ can be used to replace $\zeta(\alpha_{2z}^{-1}(s))$ with a being an arbitrary positive constant.

Remark 7. We now give some classes of systems with (19) or (34) satisfied[4].

Class 1. There is a known positive constant M such that $4\gamma_1(s)/\alpha_z(s) \leq M$, $\forall s > 0$.

In this case, the condition (21) holds for $\xi(s) = M$, which implies (19). $\rho(s) = M$ or a larger constant can satisfy (22). Stochastic linear systems are within this class. By (9), (11), (13), we can see $\alpha_z(s) = c_1s^2, \gamma_1 = c_2s^2$, which always satisfy Class 1.

Class 2. There are known constants $a > 0$, $b \geq 0$, $c \in (0,1)$ and $d \geq 0$ such that $4\gamma_1(s)/\alpha_z(s) \leq a\alpha_{1z}(s) + b$, $2\psi_z^2(s)\psi_2^2(s)/\alpha_z(s) \leq c\alpha_{1z}(s) + d$, $\forall s > 0$.

In this case, by taking $\xi(s) = a\alpha_{1z}(s) + b$ and $\zeta(s) = c\alpha_{1z}(s) + d$ one can verify (19).

Below is an example of such systems.

Example 1. Consider the following system:

$$\begin{cases} dx = (-x^3 - xz^2)dt + zxdw, \\ dz = (-4z + \dfrac{1}{2}u^2)dt + \dfrac{1}{\sqrt{2}}z\sin u dw. \end{cases} \tag{35}$$

Let $V_1 = \frac{1}{2}x^2$, $V_2 = \frac{1}{2}z^2$, $\psi_z(s) = s$, $\psi_2(s) = \frac{1}{\sqrt{2}}s$. Then we have

$$\begin{aligned} &|g_2(z,u)| \leq \psi_2(|z|), && |\nabla V_2| \leq \psi_z(|z|), \\ &\mathcal{L}V_1 \leq -\frac{3}{4}x^4 + \frac{1}{4}z^4, && \mathcal{L}V_2 \leq -\frac{7}{2}z^2 + \frac{1}{4}u^4, \end{aligned}$$

and $4\gamma_1(s)/\alpha_z(s) \leq \frac{2}{7}s^2$, $2\psi_z^2(s)\psi_2^2(s)/\alpha_z(s) \leq \frac{2}{7}s^2$.

By Lemma 1, we can choose $\rho(s) = \frac{4}{3}s + \frac{16}{21}$.

Class 3. There is a constant $M > 0$ and a nonnegative polynomial function $P_n(s)$ such that $2\psi_z^2(s)\psi_2^2(s)/\alpha_z(s) \leq M$ and $4\gamma_1(\alpha_{1z}^{-1}(s))/\alpha_z(\alpha_{1z}^{-1}(s)) \leq P_n(s)$.

In this case, by taking $\zeta(s) = M$ and $\xi(s) = P_n^*(\alpha_1(s))$ one can verify (19). Here $P_n^*(s)$ is an increasing nonnegative polynomial function satisfying $P_n(s) \leq P_n^*(s)$.

Below are two examples of such systems.

[4]Without loss of generality, here the conditions with respect to SISS (i.e. $d_1 = d_2 = 0$) are discussed.

Example 2. When the second subsystem is noise-free, i.e. $g_2 \equiv 0$, we have $\psi_2 = 0$ and $\zeta(s)$ can be any small positive constant. In this case, the condition (18) comes to be $\limsup_{s\to 0+} \gamma_1(s)/\alpha_z(s) < \infty$. And an increasing positive function $\rho(\cdot)$ satisfying (22) can easily be found.

Example 3. Consider the z-dynamics

$$dz = (-6z^3 + zu^2)dt + z^2 \frac{u}{1+u^2} dw.$$

Let $V_2(z) = \ln(1+z^2)$, $\psi_2(s) = \frac{1}{2}s^2$, $\psi_z(s) = \frac{2s}{1+s^2}$. Then, we have

$$|g_2(z,u)| \le \psi_2(|z|), \qquad |\nabla V_2| = |\frac{2z}{1+z^2}| \le \psi_z(|z|), \qquad \mathcal{L}V_2 \le -\frac{10z^4}{1+z^2} + u^4.$$

If the x-dynamics satisfies that

$$\mathcal{L}V_1 \le -\alpha_x(|x|) + \gamma_1(|z|)$$

with $\gamma_1(s) = cs^4$, then $4\gamma_1(s)/\alpha_z(s) \le \frac{2}{5}(1+s^2)$, $2\psi_z^2(s)\psi_2^2(s)/\alpha_z(s) \le \frac{1}{5}$.

By Lemma 1, we can design $\rho(s) = c_3 e^s$ for some positive constant c_3.

Class 4. $\psi_z^2(s)\psi_2^2(s)/\alpha_z(s)$ is decreasing, $\limsup_{s\to 0+} \psi_z^2(s)\psi_2^2(s)/\alpha_z(s) = \infty$, and there is a nonnegative polynomial function $P_n(s)$ such that $4\gamma_1(\alpha_{1z}^{-1}(s))/\alpha_z(\alpha_{1z}^{-1}(s)) \le P_n(s)$.

In this case, one can choose an increasing nonnegative polynomial function $P_n^*(s)$ satisfying $P_n^*(s) \ge P_n(s)$, and $\xi(s) = P_n^*(\alpha_{1z}(s))$. Thus, (34) holds.

Example 4. Suppose the z-dynamics is of the form

$$dz = \Big(-z^5 - \frac{1}{2}z + z^2u^2\Big)dt + z\sin u dw.$$

Let $V_2 = \frac{1}{2}z^2$, $\alpha_{1z}(s) = \frac{1}{2}s^2$, $\psi_2(s) = s$, $\psi_z(s) = s$. Then, we have

$$\mathcal{L}V_2 \le -\alpha_z(|z|) + \gamma_2(|u|),$$
$$2\psi_z^2(s)\psi_2^2(s)/\alpha_z(s) = \frac{2}{s^2},$$

where $\alpha_z(s) = \frac{1}{2}s^6$, $\gamma_2(s) = \frac{1}{2}s^4$. The x-dynamics can be any system such that $\frac{\gamma_1(\alpha_{1z}^{-1}(s))}{\alpha_z(\alpha_{1z}^{-1}(s))} = \frac{\gamma_1(\sqrt{2s})}{4s^3}$ can be dominated by a nonnegative polynomial function.

5 Conclusions

The stability of cascaded stochastic input-to-state stable stochastic nonlinear systems was investigated. Based on a general and more reasonable (practical) SISS notion introduced, (a) an (practical) SISS-Lyapunov function for the overall system was constructed from the corresponding (practically) SISS-Lyapunov functions of the subsystems. It has been shown that the cascade of two (practical) SISS systems is again (a) an (practical) SISS system if the new small-gain type conditions disclosed in this paper are satisfied.

Appendix: Proof of Theorem 1

Let

$$\mathcal{B} = \{x : |x| < \alpha^{-1}(q\chi(\|v\|_\infty) + qd)\}, \qquad \mathcal{B}^c = \mathbb{R}^n \backslash \mathcal{B},$$

where $\|v\|_\infty = \sup_{t\geq 0} \|v_t\| = \sup_{t\geq 0} \inf_{\mathcal{A}\subset\Omega, P(\mathcal{A})=0} \sup\{|v(x(\omega,t),t)| : \omega \in \Omega\backslash\mathcal{A}\}$, and $q \geq 1$ is a constant. Define a sequence of stopping times $\{\tau_i\}_{i\geq 0}$:

$$\tau_0 = 0,$$
$$\tau_1 = \begin{cases} \inf\{t > \tau_0 : x(t) \in \mathcal{B}\}, & \text{if } \{t > \tau_0 : x(t) \in \mathcal{B}\} \neq \emptyset; \\ \infty, & \text{otherwise;} \end{cases}$$
$$\tau_{2i} = \begin{cases} \inf\{t > \tau_{2i-1} : x(t) \in \mathcal{B}^c\}, & \text{if } \{t > \tau_{2i-1} : x(t) \in \mathcal{B}^c\} \neq \emptyset; \\ \infty, & \text{otherwise;} \end{cases}$$
$$\tau_{2i+1} = \begin{cases} \inf\{t > \tau_{2i} : x(t) \in \mathcal{B}\}, & \text{if } \{t > \tau_{2i} : x(t) \in \mathcal{B}\} \neq \emptyset; \\ \infty, & \text{otherwise,} \end{cases}$$

where $i = 1, 2, \cdots$. Note that $\mathcal{B}^c$ is a closed set, for any $t \geq 0$ and any $i = 1, 2, \cdots$. Then, if $t \in [\tau_{2i}, \tau_{2i+1}]$, then $x(t) \in \mathcal{B}^c$, and if $t \in (\tau_{2i+1}, \tau_{2i+2})$, then $x(t) \in \mathcal{B}$.

We now complete the proof by considering the following two cases: $x_0 \in \mathcal{B}^c\backslash\{0\}$ and $x_0 \in \mathcal{B}\backslash\{0\}$, respectively.

Case 1. $x_0 \in \mathcal{B}^c\backslash\{0\}$. In this case, for any $t \in [0, \tau_1]$, $x(t) \in \mathcal{B}^c$.

By the definitions of τ_{2i} and τ_{2i+1}, for any $t \in [\tau_{2i}, \tau_{2i+1}]$, $i = 0, 1, 2, \cdots$,

$$|x(t)| \geq \alpha^{-1}(q\chi(\|v\|_\infty) + qd) \geq \alpha^{-1}(q\chi(|v|) + qd) \quad a.s.$$

which together with (6) leads to

$$\mathcal{L}V(x(t)) \leq -\Big(1 - \frac{1}{q}\Big)\alpha(|x(t)|), \quad a.s. \tag{A.1}$$

By (3) and Itô formula we have

$$V\big(x(t)\big) = V\big(x(0)\big) + \int_0^t \mathcal{L}V\big(x(s)\big)ds + \int_0^t \frac{\partial V\big(x(s)\big)}{\partial x} g\big(x(s), v(x(s),s)\big)dw(s), \tag{A.2}$$

and by [1,p.72], for any $t \geq 0$, $i = 0, 1, 2, \cdots$,

$$V\big(x(t\wedge\tau_{2i})\big) = V\big(x(0)\big) + \int_0^{t\wedge\tau_{2i}} \mathcal{L}V\big(x(s)\big)ds + \int_0^{t\wedge\tau_{2i}} \frac{\partial V\big(x(s)\big)}{\partial x} g\big(x(s), v\big(x(s),s\big)\big)dw(s),$$
$$V\big(x(t\wedge\tau_{2i+1})\big) = V\big(x(0)\big) \int_0^{t\wedge\tau_{2i+1}} \mathcal{L}V\big(x(s)\big)ds + \int_0^{t\wedge\tau_{2i+1}} \frac{\partial V\big(x(s)\big)}{\partial x} g\big(x(s), v\big(x(s),s\big)\big)dw(s).$$

From the above two equalities and Lemma 4.1 of Chapter 4 in [1] it follows that

$$\begin{aligned} &V\big(x(t\wedge\tau_{2i+1})\big) - V\big(x(t\wedge\tau_{2i})\big) \\ &= \int_{t\wedge\tau_{2i}}^{t\wedge\tau_{2i+1}} \mathcal{L}V\big(x(s)\big)ds + \int_{t\wedge\tau_{2i}}^{t\wedge\tau_{2i+1}} \frac{\partial V\big(x(s)\big)}{\partial x} g\big(x(s), v(x(s),s)\big)dw(s). \end{aligned} \tag{A.3}$$

By Lemma 4.1 and Theorem 4.7 of Chapter 4 in [1] we obtain that

$$\begin{aligned} &\int_{t\wedge\tau_{2i}}^{t\wedge\tau_{2i+1}} \frac{\partial V\big(x(s)\big)}{\partial x} g\big(x(s), v(x(s),s)\big)dw(s) \\ &= \int_{\tau_{2i}}^{(t\vee\tau_{2i})\wedge\tau_{2i+1}} \frac{\partial V\big(x(s)\big)}{\partial x} g\big(x(s), v(x(s),s)\big)dw(s) \quad a.s. \end{aligned} \tag{A.4}$$

Noticing that

$$V\big(x(t\wedge\tau_{2i+1})\big)-V\big(x(t\wedge\tau_{2i})\big)=V\big(x\big((t\vee\tau_{2i})\wedge\tau_{2i+1}\big)\big)-V\big(x(\tau_{2i})\big),$$

and

$$\int_{t\wedge\tau_{2i}}^{t\wedge\tau_{2i+1}}\mathcal{L}V\big(x(s)\big)ds=\int_{\tau_{2i}}^{(t\vee\tau_{2i})\wedge\tau_{2i+1}}\mathcal{L}V\big(x(s)\big)ds,$$

by (A.3) and (A.4) we have

$$\begin{aligned}V\big(x((t\vee\tau_{2i})\wedge\tau_{2i+1})\big)=&V(x(\tau_{2i}))+\int_{\tau_{2i}}^{(t\vee\tau_{2i})\wedge\tau_{2i+1}}\mathcal{L}V\big(x(s)\big)ds\\&+\int_{\tau_{2i}}^{(t\vee\tau_{2i})\wedge\tau_{2i+1}}\frac{\partial V\big(x(s)\big)}{\partial x}g\big(x(s),v(x(s),s)\big)dw(s)a.s.\end{aligned}$$

According to the above equality and A.1, noticing that $q\geq 1$, we obtain that the process $V_t^i:=V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}))$ is a supermartingale.

Thus, by following the proof of Theorem 3.3 in [10], we obtain that for any ε', there exists a class $\mathcal{KL}$-function $\beta_i(\cdot,\cdot)$ such that

$$P\{|x((t\vee\tau_{2i})\wedge\tau_{2i+1})|<\beta_i(|x_{\tau_{2i}}|,t)\}\geq 1-\varepsilon',\qquad\forall t\geq 0,\quad x(\tau_{2i})\in\mathbb{R}^n\backslash\{0\}.$$

In particular, for $i=0$, if we write β_0 as β, then

$$P\{|x(t\wedge\tau_1)|<\beta(|x_0|,t)\}\geq 1-\varepsilon',\qquad\forall t\geq 0,\quad x_0\in\mathbb{R}^n\backslash\{0\}.\tag{A.5}$$

Now let us pay attention to $x(t\vee\tau_1)$. Define

$$\mathcal{A}=\bigcup_{i=0}^{\infty}(\tau_{2i+1},\tau_{2i+2}),\quad\mathcal{C}=\bigcup_{i=1}^{\infty}[\tau_{2i},\tau_{2i+1}].$$

Then, $\mathcal{A}\cap\mathcal{C}=\emptyset$ and $(\tau_1,\infty)=\mathcal{A}\cup\mathcal{C}$, and hence,

$$\begin{aligned}&E[V(x(t\vee\tau_1))]\\=&E[V(x(t\vee\tau_1))I_{\{t\in[0,\tau_1]\}}]+E[V(x(t\vee\tau_1))I_{\{t\in(\tau_1,\infty)\}}]\\=&E[V(x(\tau_1))I_{\{t\in[0,\tau_1]\}}]+E[V(x(t\vee\tau_1))I_{\{t\in\mathcal{A}\}}]+E[V(x(t\vee\tau_1))I_{\{t\in\mathcal{C}\}}]\\=&E[V(x(\tau_1))I_{\{t\in[0,\tau_1]\}}]+\sum_{i=0}^{\infty}E[V(x(t))I_{\{t\in(\tau_{2i+1},\tau_{2i+2})\}}]\\&+\sum_{i=1}^{\infty}E[V(x(t))I_{\{t\in[\tau_{2i},\tau_{2i+1}]\}}\ ].\end{aligned}\tag{A.6}$$

Since the process $V_t^i:=V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}))$ is a supermartingale, we have

$$E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}))]\leq E[V(x(\tau_{2i}))].\tag{A.7}$$

By the continuity of the trajectory, $x(\tau_{2i})$ and $x(\tau_{2i+1})$ lie on the boundary of the set $\mathcal{B}$, i.e. $x(\tau_{2i})=x(\tau_{2i+1})=\alpha^{-1}(q\chi(\|v\|_\infty)+qd)$ is a constant. Hence, we have

$$E[V(x(\tau_1))I_{\{t\in[0,\tau_1]\}}]\leq P\{t\in[0,\tau_1]\}\cdot[\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd))],\tag{A.8}$$

and by (A.7),

$$\begin{aligned}
&E[V(x(t))I_{\{t\in[\tau_{2i},\tau_{2i+1}]\}}]\\
=&E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}))I_{\{t\in[\tau_{2i},\tau_{2i+1}]\}}]\\
=&E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}))]-E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}))I_{\{t<\tau_{2i}\}}]\\
&-E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}))I_{\{t>\tau_{2i+1}\}}]\\
=&E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}))]-E[V(x(\tau_{2i})I_{\{t<\tau_{2i}\}}]-E[V(x(\tau_{2i+1})I_{\{t>\tau_{2i+1}\}}]\\
=&E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}))]-E[V(x(\tau_{2i}))]\cdot P\{t<\tau_{2i}\text{ or }t>\tau_{2i+1}\}\\
=&E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}))]-E[V(x(\tau_{2i}))]\cdot(1-P\{t\in[\tau_{2i},\tau_{2i+1}]\})\\
\le&E[V(x(\tau_{2i}))]-E[V(x(\tau_{2i}))]\cdot(1-P\{t\in[\tau_{2i},\tau_{2i+1}]\})\\
=&P\{t\in[\tau_{2i},\tau_{2i+1}]\}\cdot E[V(x(\tau_{2i}))]\\
\le&P\{t\in[\tau_{2i},\tau_{2i+1}]\}\cdot[\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd))].
\end{aligned}\tag{A.9}$$

Noticing that $t\in(\tau_{2i+1},\tau_{2i+2})$ implies $x(t)\in\mathcal{B}$, we have

$$\begin{aligned}
&\sum_{i=0}^{\infty}E[V(x(t))I_{\{t\in(\tau_{2i+1},\tau_{2i+2})\}}]\\
\le&\sum_{i=0}^{\infty}P\{t\in(\tau_{2i+1},\tau_{2i+2})\}\cdot[\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd))].
\end{aligned}\tag{A.10}$$

Thus, by (A.6), (A.8), (A.9) and (A.10) one can get

$$E[V(x(t\vee\tau_1))]\le\alpha_2\big(\alpha^{-1}(q\chi(\|v\|_\infty)+qd)\big).\tag{A.11}$$

Recalling that $V(x)$ is nonnegative, we have

$$\begin{aligned}
&E[V(x(t\vee\tau_1))]\\
\ge&E[V(x(t\vee\tau_1))I_{\{V(x(t\vee\tau_1))\ge\delta(\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd)))\}}]\\
\ge&[\delta(\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd)))]P\big\{V\big(x(t\vee\tau_1)\big)\ge\delta(\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd)))\big\}.
\end{aligned}\tag{A.12}$$

This together with (A.11) gives

$$P\{V(x(t\vee\tau_1))\ge\delta(\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd)))\}\le\frac{\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd))}{\delta(\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd)))}\le\varepsilon'',\tag{A.13}$$

where ε'' can be made arbitrarily small by an appropriate choice of $\delta\in\mathcal{K}_\infty$. Thus, by (5) and (A.13) we have

$$P\{|x(t\vee\tau_1)|<\alpha_1^{-1}(\delta(\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd))))\}\ge1-\varepsilon''.\tag{A.14}$$

Let $\gamma(s)=\alpha_1^{-1}(\delta(\alpha_2(\alpha^{-1}(2q\chi(s)))))$ and $\gamma_d=\alpha_1^{-1}(\delta(\alpha_2(\alpha^{-1}(2qd))))$. Then, by simple calculation it can be verified that for any $t\ge0,x_0\in\mathcal{B}^c\backslash\{0\}$,

$$\begin{aligned}
&P\{|x(t)|<\beta(|x_0|,t)+\gamma(\|v\|_\infty)+\gamma_d\}\\
\ge&P\{|x(t)|<\beta(|x_0|,t)+\alpha_1^{-1}(\delta(\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd))))\}\\
\ge&P\{\{|x(t\wedge\tau_1)|<\beta(|x_0|,t)\}\cup\{|x(t\vee\tau_1)|<\alpha_1^{-1}(\delta(\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty)+qd))))\}\}.
\end{aligned}$$

Combining this with (A.5) and (A.14) leads to

$$\begin{aligned} &P\{|x(t)| < \beta(|x_0|, t) + \gamma(\|v\|_\infty) + \gamma_d\} \geq \max\{1 - \varepsilon', 1 - \varepsilon''\} \\ =&1 - \min\{\varepsilon', \varepsilon''\} \triangleq 1 - \varepsilon, \qquad \forall t \geq 0, \quad x_0 \in \mathcal{B}^c \backslash \{0\}. \end{aligned} \tag{A.15}$$

Case 2. $x_0 \in \mathcal{B}\backslash\{0\}$. In this case $\tau_1 = 0$ a.s.

When $t > 0$, $P\{t \in (\tau_1, \infty)\} = P\{t \in (0, \infty)\} = 1$. Following the proof of **Case 1**, we know that (A.14) still holds, and then,

$$\begin{aligned} &P\{|x(t)| < \beta(|x_0|, t) + \gamma(\|v\|_\infty) + \gamma_d\} \\ \geq&P\{|x(t)| < \beta(|x_0|, t) + \alpha_1^{-1}(\delta(\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty) + qd))))\} \\ =&P\{|x(t \vee \tau_1)| < \beta(|x_0|, t) + \alpha_1^{-1}(\delta(\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty) + qd))))\} \\ \geq&P\{\{|x(t \vee \tau_1)| < \alpha_1^{-1}(\delta(\alpha_2(\alpha^{-1}(q\chi(\|v\|_\infty) + qd))))\}\} \geq 1 - \varepsilon''. \end{aligned} \tag{A.16}$$

When $t = 0$, by the definition of the set $\mathcal{B}$ and the definition of the function γ, we obtain

$$P\{|x(0)| < \beta(|x_0|, 0) + \gamma(\|v\|_\infty) + \gamma_d\} \geq P\{|x(0)| < \gamma(\|v\|_\infty) + \gamma_d\} = 1,$$

which implies

$$P\{|x(0)| < \beta(|x_0|, 0) + \gamma(\|v\|_\infty) + \gamma_d\} = 1. \tag{A.17}$$

Thus, by (A.16) and (A.17) we have

$$P\{|x(t)| < \beta(|x_0|, t) + \gamma(\|v\|_\infty) + \gamma_d\} \geq 1 - \varepsilon, \qquad \forall t \geq 0, \quad x_0 \in \mathcal{B}\backslash\{0\}. \tag{A.18}$$

In conclusion, by (A.15) and (A.18) we have

$$P\{|x(t)| < \beta(|x_0|, t) + \gamma(\|v\|_\infty) + \gamma_d\} \geq 1 - \varepsilon, \qquad \forall t \geq 0, \quad x_0 \in \mathbb{R}^n\backslash\{0\}.$$

By causality we get

$$P\big\{|x(t)| < \beta(|x_0|, t) + \gamma(\sup_{0 \leq s \leq t} \|v_s\|) + \gamma_d\big\} \geq 1 - \varepsilon, \qquad \forall t \geq 0, \quad x_0 \in \mathbb{R}^n\backslash\{0\}.$$

The proof is complete.

References

[1] Friedman A. Stochastic differential equations and applications, Volume 1. Academic Press, New York, 1975

[2] Has'minskii R.Z. Stochastic stability of differential equations. Kluwer Academic Publishers, Norwell, Massachusetts, 1980

[3] Imura, J.I., Sugie, T., Yoshikawa T. Global robust stabilization of nonlinear cascaded systems. *IEEE Transactions on Automatic Control*, 39: 1084–1089 (1994)

[4] Isidori, A. Nonlinear Control Systems. Springer-Verlag, London, 1999

[5] Jiang, Z.P., Mareels, I.M.Y., Wang Y. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32: 1211–1215 (1996)

[6] Jiang, Z.P., Praly, L. Design of robust adaptive controller for nonlinear systems with dynamic uncertainties. *Automatica*, 34: 825–840 (1998)

[7] Jiang, Z.P., Teel, A.R., Praly, L. Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals and Systems*, 7: 95–120 (1994)

[8] Khalil, H.K. Nonlinear systems, 3rd Edition. Prentice-Hall, Englewood Cliffs New Jersey, 2002

[9] Kokotović, P.V., Arcak, M. Constructive nonlinear control: a historical perspective. *Automatica*, 37: 637–662 (2001)

[10] Krstić, M., Deng, H. Stabilization of nonlinear uncertain systems. Springer-Verlag, London, 1998

[11] Krstić, M., Kanellakopoulos I., Kokotović P. V.: Nonlinear and Adaptive Control Design, Wiley, New York, 1995
[12] Liu, Y.G., Zhang, J.F. Practical output-feedback risk-sensitive control for stochastic nonlinear systems with stable zero-dynamics. *SIAM J. Control and Optimization*, 45:, 885–926 (2006)
[13] Øksendal, B. Stochastic differential equations: an introduction with applications. Springer-Verlag, Berlin Heidelberg, 1995
[14] Pan, Z., Başar, T. Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost criterion. *SIAM J. on Control and Optimization*, 37: 957–995 (1999)
[15] Sontag, E.D. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34: 435–443 (1989)
[16] Sontag, E.D. Mathematical control theory: deterministic finite dimensional systems. Springer-Verlag, New York, 1998
[17] Sontag, E.D., Teel, A. Changing supply function in input/state stable systems. *IEEE Transactions on Automatic Control*, 40:1476–1478 (1995)
[18] Sontag, E.D., Wang, Y. On characterizations of the input-to-state stability property. *Systems and Control Letters*, 24: 351–359 (1995)
[19] Sontag, E.D., Wang, Y. New characterizations of the input to state stablility property. *IEEE Transactions on Automatic Control*, 41: 1283–1294 (1996)
[20] Tang, C., Başar, T. Stochastic stability of singularly perturbed nonlinear systems, in *Proceedings of 40th IEEE Conference on Decision and Control*. Orlando, Florida, 399–404 (2001)
[21] Tsinias, J. Stochastic input-to-state stability and applications to global feedback stabilization. *Int. J. Control*, 71: 907–930 (1998)
[22] Tsinias, J. The concept of "Exponential input to state stability" for stochastic systems and applications to feedback stabilization. *Systems and Control Letters*, 36: 221–229 (1999)
[23] Wu, Z.J., Xie, X.J., Zhang, S.Y. Adaptive backstepping controller design using stochastic small-gain theorem. *Automatica*, 43: 608–620 (2007)