Automatica, Vol. 25, No. 1, pp. 147–151, 1989
Printed in Great Britain.

0005–1098/89 \$3.00 + 0.00
Pergamon Press plc

Brief Paper

Consistent Order Estimation for Linear Stochastic Feedback Control Systems (CARMA Model)*

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Key Words—Stochastic systems; discrete time systems; feedback control; adaptive control; convergence analysis; (order estimation).

Abstract—A new criterion CIC is introduced to estimate orders (p_0, q_0, r_0) of the linear stochastic feedback control system with correlated noise described by a CARMA model. It is proved that the estimate is strongly consistent when the upper bounds for p_0, q_0 and r_0 are available, but neither the stability condition nor the ergodicity of the input and output are imposed on the system.

1. *Introduction*

THE ORDER ESTIMATE for an ARMA process is one of the important problems in time series analysis. The estimates (p_n, r_n) for unknown orders (p_0, r_0) are usually given by minimizing some criterion, for example, AIC(p, r) (Akaike, 1969), BIC(p, r) (Akaike, 1977) and ΦIC(p, r) (Hannan and Quinn, 1979). But all these results cannot be applied to the feedback control system, described by the so-called CARMA model, which essentially differs from the ARMA model by additional control terms which are crucial for all real control systems and depend upon the past input and output.

In a recent paper (Chen and Guo, 1987b), having introduced a new criterion to be minimized, the authors obtained consistent estimates for orders of the multidimensional feedback control system with uncorrelated noise. In this paper by introducing a new criterion CIC, we have generalized these results to the correlated noise case, i.e. we have obtained strongly consistent estimates (p_n, q_n, r_n) for orders (p_0, q_0, r_0) of the CARMA process. At the same time, the criterion and conditions used in Chen and Guo (1987b) are simplified.

2. *Statement of problem*

Let the l-input, m-output stochastic control system be described by the following CARMA model

$$\begin{aligned} A(z)y_n &= B(z)u_n + C(z)w_n, \quad n \geq 0; \\ y_n &= w_n = 0, \quad u_n = 0, \quad n < 0 \end{aligned} \tag{1}$$

where w_n is an m-dimensional driven noise, $A(z)$, $B(z)$ and $C(z)$ are matrix polynomials in the shift-back operator z

$$A(z) = I + A_1 z + \cdots + A_{p_0} z^{p_0}, \quad p_0 \geq 0, \tag{2a}$$

$$B(z) = B_1 z + \cdots + B_{q_0} z^{q_0}, \quad q_0 \geq 0, \tag{2b}$$

$$C(z) = I + C_1 z + \cdots + C_{r_0} z^{r_0}, \quad r_0 \geq 0 \tag{2c}$$

with unknown orders p_0, q_0 and r_0 and unknown matrix coefficients A_i, B_j and C_k $(1 \leq i \leq p_0, 1 \leq j \leq q_0, 1 \leq k \leq r_0)$.

* Received 24 April 1987; revised 10 November 1987; received in final form 23 June 1988. The original version of this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Y. Sunahara under the direction of Editor P. C. Parks.

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In the sequel, we denote by $\lambda_{\min}(X)$ the minimum eigenvalue of a matrix X, and by the norm $\|X\|$ we mean the maximum singular value of X.

We make the following assumptions.

(H_1). The driven noise is a martingale difference sequence with respect to a non-decreasing family of σ-algebras $\{\mathscr{F}_n\}$ and such that

$$\sup_n E[\|w_{n+1}\|^\beta \mid \mathscr{F}_n] < \infty, \quad \text{a.s. for some } \beta \geq 2.$$

(H_2). For any $n \geq 1$, u_n is $\mathscr{F}_n$-measurable.

(H_3). The transfer matrix $C^{-1}(z) - \frac{1}{2}I$ is strictly positive real, i.e.

$$C^{-1}(e^{i\theta}) + C^{-\tau}(e^{-i\theta}) - I > 0, \quad \forall \theta \in [0, 2\pi].$$

(H_4). The true orders (p_0, q_0, r_0) belong to a known finite set M:

$$M \triangleq \{(p, q, r): 0 \leq p \leq p^*, 0 \leq q \leq q^*, 0 \leq r \leq r^*\}.$$

(H_5). A sequence of positive numbers $\{a_n\}$ can be found such that

$$\frac{(\log \bar{s}_n^0)(\log\log \bar{s}_n^0)^{\alpha\delta(\beta-2)}}{a_n} \xrightarrow[n\to\infty]{} 0, \quad \text{a.s. for some } \alpha > 1, \tag{3}$$

and

$$\frac{a_n}{\lambda_{\min}^{(p,q,r)}(n)} \xrightarrow[n\to\infty]{} 0, \text{ a.s.} \quad \forall (p, q, r) \in M^* \tag{4}$$

where $\delta(.)$ is the Dirac function

$$\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases},$$

and M^* denotes the set consisting of three points:

$$M^* = \{(p_0, q^*, r^*), (p^*, q_0, r^*), (p^*, q^*, r_0)\} \tag{5}$$

and where $\lambda_{\min}^{(p,q,r)}(n)$ denotes the minimum eigenvalue of

$$\sum_{i=0}^{n-1} \varphi_i^0(p, q, r)\varphi_i^{0\tau}(p, q, r) - \frac{1}{d}I, \quad (d = mp^* + lq^* + mr^*), \tag{6}$$

with

$$\varphi_n^0(p, q, r) \triangleq \{y_n^\tau \ldots y_{n-p+1}^\tau \quad u_n^\tau \ldots u_{n-q+1}^\tau \quad w_n^\tau \ldots w_{n-r+1}^\tau\}^\tau, \tag{7}$$

and

$$\bar{s}_n^0 = \sum_{i=0}^{n-1} \|\varphi_i^0(p^*, q^*, r^*)\|^2 + 1. \tag{8}$$

System (1) under Assumptions (H_1)–(H_5) is, generally speaking, neither stationary nor ergodic because (i) the system input u_n may be an arbitrary $\mathscr{F}_n$-measurable function; (ii) the matrix polynomial $A(z)$ may be unstable, i.e. zeros of $\det A(z)$ may lie outside the closed unit disk and (iii) the system (1) is causal, i.e. the process vanishes for negative time n. Therefore, the usual treatments and criteria developed for estimating orders of stationary ARMA processes (Hannan and Rissanen, 1982; Hannan and Kavalieris, 1984) are not applicable in the present situation.

In Section 3 for a large class of adaptive systems we shall specify $\{a_n\}$ and show that Assumption (H_5) is indeed satisfied.

We introduce the regression vector

$$\bar{\varphi}_n = [y_n^\tau \dots y_{n-p^*+1}^\tau \quad u_n^\tau \dots u_{n-q^*+1}^\tau \quad \hat{w}_n^\tau \dots \hat{w}_{n-r^*+1}^\tau]^\tau, \tag{9}$$

corresponding to the system of largest possible orders, where the estimate $\hat{w}_n$ for w_n is recursively defined as follows:

$$\hat{w}_n = y_n - \bar{\theta}_n^\tau \bar{\varphi}_{n-1}, \quad n \geq 0; \quad \hat{w}_n = 0, \quad n < 0, \tag{10}$$

$$\begin{aligned}\bar{\theta}_{n+1} &= \bar{\theta} + \bar{a}_n \bar{P}_n \bar{\varphi}_n (y_{n+1}^\tau - \bar{\varphi}_n^\tau \bar{\theta}_n),\\ \bar{P}_{n+1} &= \bar{P}_n - \bar{a}_n \bar{P}_n \bar{\varphi}_n \bar{\varphi}_n^\tau \bar{P}_n, \quad \bar{a}_n = (1 + \bar{\varphi}_n^\tau \bar{P}_n \bar{\varphi}_n)^{-1},\end{aligned} \tag{11}$$

with initial value $\bar{\theta}_0$ arbitrarily chosen and $\bar{P}_0 = dI$, where d is given in (6).

For any $(p, q, r) \in M$ set

$$\theta(p, q, r) = [-A_1 \dots -A_p \quad B_1 \dots B_q \quad C_1 \dots C_r]^\tau \tag{12}$$

where by definition

$$A_i = 0,\ B_j = 0,\ C_k = 0 \quad \text{for} \quad i > p_0,\ j > q_0,\ k > r_0. \tag{13}$$

The extended least squares estimate

$$\begin{aligned}&\theta_n(p, q, r)\\ &\quad = [-A_{ln} \dots -A_{pn} \quad B_{ln} \dots B_{qn} \quad C_{ln} \dots C_{rn}]^\tau,\end{aligned} \tag{14}$$

for $\theta(p, q, r)$ at time n is given by

$$\theta_n(p, q, r) = \left(\sum_{i=0}^{n-1} \varphi_i(p, q, r)\varphi_i^\tau(p, q, r) + \frac{1}{d}I\right)^{-1} \sum_{i=0}^{n-1} \varphi_i(p, q, r) y_{i+1}^\tau, \tag{15}$$

where

$$\begin{aligned}&\varphi_n(p, q, r)\\ &\quad \triangleq [y_n^\tau \dots y_{n-p+1}^\tau u_n^\tau \dots u_{n-q+1}^\tau \hat{w}_n^\tau \dots \hat{w}_{n-r+1}^\tau]^\tau.\end{aligned} \tag{16}$$

We introduce a new information criterion CIC(p, q, r), where the first "C" emphasizes that the criterion is designed for feedback control systems:

$$\text{CIC}(p, q, r)_n = \sigma_n(p, q, r) + (p + q + r)a_n, \tag{17}$$

where the subscript n denotes the data size, and where a_n is given in Assumption (H_5) and $\sigma_n(p, q, r)$ is a residual given by

$$\sigma_n(p, q, r) = \sum_{i=0}^{n-1} \|y_{i+1} - \theta_n^\tau(p, q, r)\varphi_i(p, q, r)\|^2. \tag{18}$$

Finally, the estimate (p_n, q_n, r_n) for (p_0, q_0, r_0) is given by minimizing CIC$(p, q, r)_n$,

$$(p_n, q_n, r_n) = \underset{(p,q,r)\in M}{\arg\min}\ \text{CIC}(p, q, r)_n. \tag{19}$$

3. *Strong consistency of order estimates*

Theorem 1. Under conditions (H_1)–(H_5) the order estimate (p_n, q_n, r_n) for (p_0, q_0, r_0) given by (19) is strongly consistent:

$$(p_n, q_n, r_n) \xrightarrow[n\to\infty]{} (p_0, q_0, r_0), \quad \text{a.s.}$$

We first prove three lemmas. Define

$$\xi_n = \hat{w}_n - w_n = y_n - w_n - \bar{\theta}_n^\tau \bar{\varphi}_{n-1}, \tag{20}$$

where $\hat{w}_n$ is defined by (10).

Lemma 1. If conditions (H_1)–(H_4) hold then

$$\sum_{i=0}^{n-1} \|\xi_{i+1}\|^2 = O((\log \bar{s}_n^0)(\log\log \bar{s}_n^0)^{c\delta(\beta-2)}), \quad \text{a.s.} \quad \forall c > 1,$$

where $\bar{s}_n^0$ is defined by (8).

Proof. The estimation established in the proof of Theorem 1 of Chen and Guo (1986a) holds true, and from (29) and (30) of that paper it follows that

$$\begin{aligned}&\sum_{i=0}^{n-1} \|\xi_{i+1}\|^2\\ &\quad = O((\log \bar{s}_n)(\log\log \bar{s}_n)^{c\delta(\beta-2)}), \quad \text{a.s.} \quad \forall c > 1,\end{aligned} \tag{21}$$

where $\bar{s}_n$ is defined by

$$\bar{s}_n = 1 + \sum_{i=0}^{n-1} \|\bar{\varphi}_i\|^2$$

with $\bar{\varphi}_i$ given by (9).

Using (9), (7), (8), (20) and (21) it is easy to show that

$$\bar{s}_n = O(\bar{s}_n^0), \quad \text{a.s.} \tag{22}$$

From this and (21) the desired result follows. □

Lemma 2. Let Assumption (H_1) be satisfied, and let the random vector φ_n be $\mathscr{F}_n$-measurable, $\forall n$. Then as $n \to \infty$,

$$\begin{aligned}&\left\|\left(\sum_{i=0}^{n-1} \varphi_i\varphi_i^\tau + \varepsilon I\right)^{-1/2} \sum_{i=0}^{n-1} \varphi_i w_{i+1}^\tau\right\|^2\\ &\quad = O((\log s_n)(\log\log s_n)^{c\delta(\beta-2)}), \quad \text{a.s.} \quad \forall c > 1,\end{aligned}$$

where $\varepsilon > 0$, and

$$s_n \triangleq 1 + \sum_{i=0}^{n-1} \|\varphi_i\|^2.$$

Proof. Set

$$S_n = \sum_{i=0}^{n-1} \varphi_i w_{i+1}^\tau, \quad R_n = \left(\sum_{i=0}^{n-1} \varphi_i\varphi_i^\tau + \varepsilon I\right)^{-1}. \tag{23}$$

By the matrix inversion formula, it is clear that

$$R_{n+1} = R_n - b_n R_n \varphi_n \varphi_n^\tau R_n, \quad b_n = (1 + \varphi_n^\tau R_n \varphi_n)^{-1}.$$

Hence

$$\begin{aligned}\text{tr}\, S_{n+1}^\tau R_{n+1} S_{n+1} &= \text{tr}\, S_1^\tau R_1 S_1\\ &\quad + \sum_{i=1}^{n} (2b_i w_{i+1}^\tau S_i^\tau R_i \varphi_i + b_i \varphi_i^\tau R_i \varphi_i \|w_{i+1}\|^2\\ &\quad - b_i \|S_i^\tau R_i \varphi_i\|^2).\end{aligned} \tag{24}$$

By the estimate for the martingale difference sequence (Chen and Guo, 1986a; Lai and Wei, 1982) we know that

$$\text{tr}\, S_{n+1}^\tau R_{n+1} S_{n+1} = O(1) + \sum_{i=0}^{n} b_i \varphi_i^\tau R_i \varphi_i \|w_{i+1}\|^2. \tag{25}$$

But in (29) and (30) of Chen and Guo (1986a) we have shown that

$$\begin{aligned}&\sum_{i=0}^{n} b_i \varphi_i^\tau R_i \varphi_i \|w_{i+1}\|^2\\ &\quad = O((\log s_{n+1})(\log\log s_{n+1})^{c\sigma(\beta-2)}), \quad \text{a.s.} \quad \forall c > 1.\end{aligned} \tag{26}$$

Thus, combining (25) with (26) we conclude that

$$\begin{aligned}&\|R_{n+1}^{1/2} S_{n+1}\|^2 \leq \text{tr}\, S_{n+1}^\tau R_{n+1} S_{n+1}\\ &\quad = O((\log s_{n+1})(\log\log s_{n+1})^{c\delta(\beta-2)}), \quad \text{a.s.} \quad \forall c > 1.\end{aligned}$$

This proves the lemma. □

Lemma 3. Under conditions (H_1)–(H_5), CIC$(p, q, r)_n$ defined by (17) satisfies

$$\text{CIC}(p, q, r)_n - \text{CIC}(p_0, q_0, r_0)_n$$

$$\geq \begin{cases} a_n(p + q + r - p_0 - q_0 - r_0 + o(1)), \text{ a.s.} \\ \quad \text{if } (s, t, \lambda) = (p, q, r), & (27)\\ \lambda_{\min}^{(s,t)}(n)(\tfrac{1}{4}\alpha_0 + o(1)), \text{ a.s.} \\ \quad \text{if } (s, t, \lambda) \neq (p, q, r), & (28)\end{cases}$$

for any $(p, q, r) \in M$, where $\alpha_0 = \min\{\|A_{p_0}\|^2, \|B_{q_0}\|^2, \|C_{r_0}\|^2\} > 0$,

$$(s, t, \lambda) \triangleq (p \vee p_0, q \vee q_0, r \vee r_0), \tag{29}$$

and $a \vee b$ denotes $\max(a, b)$.

Proof. By Lemma 1 we have

$$\sum_{i=0}^{n-1} \|\varphi_i(p, q, r) - \varphi_i^0(p, q, r)\|^2 = O((\log \bar{s}_n^0)(\log\log \bar{s}_n^0)^{c\delta(\beta-2)}), \quad \text{a.s.} \quad \forall c > 1, \tag{30}$$

for any $(p, q, r) \in M$, where $\varphi_i(p, q, r)$ is defined by (16). By (18) we have for $(s, t, \lambda) = (p, q, r)$

$$\begin{aligned}\sigma_n(p, q, r) &= \operatorname{tr} \tilde{\theta}_n^\tau(p, q, r) \sum_{i=0}^{n-1} \varphi_i(p, q, r) \\ &\quad \times \varphi_i^\tau(p, q, r)\tilde{\theta}_n(p, q, r) \\ &\quad + 2 \operatorname{tr} \tilde{\theta}_n^\tau(p, q, r) \sum_{i=0}^{n-1} \varphi_i(p, q, r) \\ &\quad \times [-\theta^\tau(p, q, r)\varphi_i^\xi(p, q, r) + w_{i+1}]^\tau \\ &\quad + \sum_{i=0}^{n-1} \|\theta^\tau(p, q, r)\varphi_i^\xi(p, q, r) - w_{i+1}\|^2, \end{aligned} \tag{31}$$

where

$$\tilde{\theta}_n(p, q, r) = \theta(p, q, r) - \theta_n(p, q, r),$$
$$\varphi_n^\xi(p, q, r) = \varphi_n(p, q, r) - \varphi_n^0(p, q, r).$$

By (30) and Schwarz inequality it follows that for any $(p, q, r) \in M$,

$$\begin{aligned}&\left\| \left(\sum_{j=0}^{n-1} \varphi_j(p, q, r)\varphi_j^\tau(p, q, r) + \frac{1}{d}I\right)^{-1/2} \times \sum_{i=0}^{n-1} \varphi_i(p, q, r)\varphi_i^{\xi\tau}(p, q, r)\right\|^2 \\ &\leq \operatorname{tr}\left[\sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1} \varphi_j(p, q, r)_j^\tau(p, q, r) + \frac{1}{d}I\right)^{-1} \times \varphi_i(p, q, r)\varphi_i^\tau(p, q, r)\right] \sum_{i=0}^{n-1} \|\varphi_i^\xi(p, q, r)\|^2 \\ &= O((\log \bar{s}_n^0)(\log\log \bar{s}_n^0)^{c\delta(\beta-2)}), \quad \forall c > 1, \text{ a.s.}\end{aligned} \tag{32}$$

Further, by Lemma 1 and an estimation for martingales (see e.g. Chen and Guo (1986a)) it is easy to see that

$$\sum_{i=0}^{n-1} \|\theta^\tau(p, q, r)\varphi_i^\xi(p, q, r) - w_{i+1}\|^2 = \sum_{i=0}^{n-1} \|w_{i+1}\|^2 + O((\log \bar{s}_n^0)(\log\log \bar{s}_n^0)^{c\delta(\beta-2)}). \tag{33}$$

Hence, by Lemma 2 and by using (22), (31)–(33), it is not difficult to conclude that for all $(s, t, \lambda) = (p, q, r)$

$$\sigma_n(p, q, r) = O((\log \bar{s}_n^0)(\log\log \bar{s}_n^0)^{c\delta(\beta-2)}) + \sum_{i=0}^{n-1} \|w_{i+1}\|^2, \quad \forall c > 1. \tag{34}$$

From this and (17) we get

$$\begin{aligned}&\mathrm{CIC}(p, q, r)_n - \mathrm{CIC}(p_0, q_0, r_0)_n \\ &= a_n\left(p + q + r - p_0 - q_0 - r_0 + O\left(\frac{(\log \bar{s}_n^0)(\log\log \bar{s}_n^0)^{c\delta(\beta-2)}}{a_n}\right)\right), \quad \text{a.s.}\end{aligned} \tag{35}$$

for any $c > 1$ and $(s, t, \lambda) = (p, q, r)$.

Taking $c \in (1, \alpha)$ and using Assumption (H_5), we obtain (27) from (35).

Now we proceed to prove (28).

For any fixed $(p, q, r) \in M$, set as

$$\begin{aligned}\tilde{\theta}_n'(p, q, r) = [&A_1 - A_{1n}' \ldots A_s - A_{sn}' \\ &B_1 - B_{1n}' \ldots B_t - B_{tn}' \\ &C_1 - C_{1n}' \ldots C_\lambda - C_{\lambda n}']^\tau,\end{aligned} \tag{36}$$

where A_i, B_j, C_k, $1 \leq i \leq s$, $1 \leq j \leq t$, $1 \leq k \leq \lambda$ are defined by (2a)–(2c) and (13), and

$$A_{in}' = \begin{cases} A_{in}, & i \leq p \\ 0, & i > p \end{cases}; \quad B_{jn}' = \begin{cases} B_{jn}, & j \leq q \\ 0, & j > q \end{cases}; \quad C_{kn}' = \begin{cases} C_{kn}, & k \leq r \\ 0, & k > r \end{cases}. \tag{37}$$

By a similar method to (31), for any $(p, q, r) \in M$, we have

$$\begin{aligned}\sigma_n(p, q, r) &= \operatorname{tr} \tilde{\theta}_n'^\tau(p, q, r) \sum_{i=0}^{n-1} \varphi_i(s, t, \lambda) \\ &\quad \times \varphi_i^\tau(s, t, \lambda)\tilde{\theta}_n'(p, q, r) \\ &\quad + 2 \operatorname{tr} \tilde{\theta}_n'^\tau(p, q, r) \sum_{i=0}^{n-1} \varphi_i(s, t, \lambda) \\ &\quad \times (-\theta^\tau(s, t, \lambda)\varphi_i^\xi(s, t, \lambda) + w_{i+1})^\tau \\ &\quad + \sum_{i=0}^{n-1} \|w_{i+1} - \theta^\tau(s, t, \lambda)\varphi_i^\xi(s, t, \lambda)\|^2.\end{aligned} \tag{38}$$

By (6) and (7), in the case where $(s, t, \lambda) \neq (p, q, r)$, we have

$$\begin{aligned}\lambda_{\min}^{(s,t,\lambda)}(n) \geq \min\{&\lambda_{\min}^{(p_0,q^*,r^*)}(n), \\ &\lambda_{\min}^{(p^*,q_0,r^*)}(n), \lambda_{\min}^{(p^*,q^*,r_0)}(n)\},\end{aligned} \tag{39}$$

which tends to infinity as $n \to \infty$ by Assumption (H_5).

By an argument completely similar to that used in the proof of Theorem 2 of Chen and Guo (1986a) it can be shown that for sufficiently large n

$$\begin{aligned}&\lambda_{\min}\left(\sum_{i=0}^{n-1} \varphi_i(s, t, \lambda)\varphi_i^\tau(s, t, \lambda)\right) \\ &\geq \tfrac{1}{3}\lambda_{\min}\left(\sum_{i=0}^{n-1} \varphi_i^0(s, t, \lambda)\varphi_i^{0\tau}(s, t)\right) \geq \tfrac{1}{4}\lambda_{\min}^{(s,t,\lambda)}(n),\end{aligned} \tag{40}$$

for $(s, t, \lambda) \neq (p, q, r)$.

Also, in the case where $(s, t, \lambda) \neq (p, q, r)$, it can be seen from (36) that

$$\|\tilde{\theta}_n'(p, q, r)\|^2 \geq \min\{\|A_{p_0}\|^2, \|B_{q_0}\|^2, \|C_{r_0}\|^2\} = \alpha_0 > 0. \tag{41}$$

Hence, by (40) and (41), for the first term on the right-hand side of (38), we have

$$\operatorname{tr} \tilde{\theta}_n'^\tau(p, q, r) \sum_{i=0}^{n-1} \varphi_i(s, t, \lambda)\varphi_i^\tau(s, t, \lambda)\tilde{\theta}_n'(p, q, r) \geq \tfrac{1}{4}\alpha_0\lambda_{\min}^{(s,t,\lambda)}(n). \tag{42}$$

By Lemma 2 and (32), we estimate the second term on the right-hand side of (38) as follows

$$\begin{aligned}&2\left|\operatorname{tr} \tilde{\theta}_n'^\tau(p, q, r) \sum_{i=0}^{n-1} \varphi_i(s, t, \lambda) \times (-\theta^\tau(s, t, \lambda)\varphi_i^\xi(s, t, \lambda) + w_{i+1})^\tau\right| \\ &\leq \left\|\tilde{\theta}_n'^\tau(p, q, r)\left(\sum_{i=0}^{n-1} \varphi_i(s, t, \lambda)\varphi_i^\tau(s, t, \lambda)\right)^{1/2}\right\| \cdot O(\sqrt{(\log \bar{s}_n^0)(\log\log \bar{s}_n^0)^{\alpha\delta(\beta-2)}}).\end{aligned} \tag{43}$$

By a similar consideration to (33), we have

$$\sum_{i=0}^{n-1} \|w_{i+1} - \theta^\tau(s, t, \lambda)\varphi_i^\xi(s, t, \lambda)\|^2 = \sum_{i=0}^{n} \|w_{i+1}\|^2 + O((\log \bar{s}_n^0)(\log\log \bar{s}_n^0)^{\alpha\delta(\beta-2)}). \tag{44}$$

Combining (38), (42), (43) and (44) it follows that

$$\begin{aligned}&\sigma_n(p, q, r) \\ &\geq \frac{\alpha_0}{4}\lambda_{\min}^{(s,t,\lambda)}(n)\left(1 + O\left(\left(\frac{(\log \bar{s}_n^0)(\log\log \bar{s}_n^0)^{\alpha\delta(\beta-2)}}{\lambda_{\min}^{(s,t,\lambda)}(n)}\right)^{1/2}\right)\right) \\ &\quad + \sum_{i=0}^{n-1} \|w_{i+1}\|^2,\end{aligned} \tag{45}$$

for $(s, t, \lambda) \neq (p, q, r)$.