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Decentralized adaptive output-feedback stabilization for large-scale stochastic nonlinear systems☆

Shu-Jun Liu[a,b], Ji-Feng Zhang[a,*], Zhong-Ping Jiang[c]

[a] Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China
[b] Department of Mathematics, Southeast University, Nanjing 210096, China
[c] Department of Electrical and Computer Engineering, Polytechnic University, 5 Metrotech Center, Brooklyn, NY 11201, USA

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Abstract

In this paper, the problem of decentralized adaptive output-feedback stabilization is investigated for large-scale stochastic nonlinear systems with three types of uncertainties, including parametric uncertainties, nonlinear uncertain interactions and stochastic inverse dynamics. Under the assumption that the inverse dynamics of the subsystems are stochastic input-to-state stable, an adaptive output-feedback controller is constructively designed by the backstepping method. It is shown that under some general conditions, the closed-loop system trajectories are bounded in probability and the outputs can be regulated into a small neighborhood of the origin in probability. In addition, the equilibrium of interest is globally stable in probability and the outputs can be regulated to the origin almost surely when the drift and diffusion vector fields vanish at the origin. The contributions of the work are characterized by the following novel features: (1) even for centralized single-input single-output systems, this paper presents a first result in stochastic, nonlinear, adaptive, output-feedback asymptotic stabilization; (2) the methodology previously developed for deterministic large-scale systems is generalized to stochastic ones. At the same time, novel small-gain conditions for small signals are identified in the setting of stochastic systems design; (3) both drift and diffusion vector fields are allowed to be dependent not only on the measurable outputs but some unmeasurable states; (4) parameter update laws are used to counteract the parametric uncertainty existing in both drift and diffusion vector fields, which may appear nonlinearly; (5) the concept of stochastic input-to-state stability and the method of changing supply functions are adapted, for the first time, to deal with stochastic and nonlinear inverse dynamics in the context of decentralized control.

Keywords: Decentralized control; Stochastic nonlinear systems; Stochastic input-to-state stable; Inverse dynamics; Output feedback; Adaptive control

1. Introduction

A large-scale system is often considered as a set of interconnected subsystems, such as power systems, computer and telecommunications networks, economic systems and multi-agent systems. Owing to the complexity of control synthesis and the physical restrictions on information exchange among subsystems, it is often required to design for each subsystem a decentralized controller depending only on local measurements, even if to achieve an objective for the whole large-scale system.

Due to the mixture of differential geometric and analytical methods, much progress has been made recently in the field of nonlinear systems and control (Freeman & Kokotović, 1995; Huang, 2004; Isidori, 1999; Khalil, 2002; Krstić, Kanellakopoulos, & Kokotović, 1995; Marino & Tomei, 1995). Among others, the decentralized control for large-scale nonlinear system has received more and more attention; see, for instance, Arslan and Başar (2003), Chen (1994), Gavel and Šiljak (1989), Guo, Jiang, and Hill (1999), Jain and Khorrami (1997), Jiang (2000), Jiang and Repperger (2001), Jiang, Repperger, and Hill (2001), Liu and Li (2002), Shi and Singh (1993), Šiljak (1991), Spooner and Passino (1996), Spooner,

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* Corresponding author.
E-mail addresses: sjliu@amss.ac.cn (S.-J. Liu), jif@iss.ac.cn (J.-F. Zhang), zjiang@control.poly.edu (Z.-P. Jiang).

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Maggiore, Ordonez, and Passino (2002), Wen (1994), Wen and Soh (1997), Wen and Zhou (2006), Xie and Xie (2000) and Ye and Huang (2003), to name only a few. However, most of these existing papers focus on deterministic large-scale systems, and very few of them are concerned with large-scale, nonlinear, stochastic systems (Arslan & Başar, 2003; Xie & Xie, 2000). The research on decentralized control of large-scale stochastic nonlinear systems is important not only to the development of control theory itself but also to the synthesis of practical control engineering systems. In Xie and Xie (2000), a global decentralized stabilization algorithm is given for a class of large-scale stochastic nonlinear systems with nonlinear interconnections in the drift vector fields. Two classes of decentralized control laws are considered: one is state-feedback control, and the other is output-feedback control. In order to design a decentralized output-feedback stabilizing controller, it is required that the drift and diffusion vector fields of the considered systems depend only on the measurable outputs. In Arslan and Başar (2003), the decentralized risk-sensitive state-feedback control is studied for a class of strict-feedback systems, which interact through their outputs in drift and diffusion vector fields, while the interactions in drift vector fields have uncertain parameters, but the diffusion vector fields are known. Inspired by the recent work on centralized stochastic nonlinear control (Deng, Krstić, & Williams, 2001; Liu & Zhang, 2005; Liu, Zhang, & Jiang, 2006a,b; Pan & Başar, 1999) here we consider the problem of decentralized robust adaptive stabilization for a class of large-scale, nonlinear, stochastic systems with stochastic nonlinear inverse dynamics and uncertain interactions in both the drift and diffusion vector fields. The design procedure is constructive and relies on recursive design tools developed in past literature of constructive nonlinear control, such as backstepping, input-to-state stability (ISS), and nonlinear ISS small-gain. Essentially, the decentralized design scheme developed in this paper involves three parts: the first part is concerned with the estimates of the unmeasurable but observable states; the second one deals with the estimates of the uncertainty functions; and the third and last part is concerned with the design of observer-based, decentralized, adaptive stabilizing controllers. The main contributions of this paper are: (1) the methodology developed in Jiang and Repperger (2001) for deterministic large-scale systems is generalized to stochastic ones; (2) more general systems are considered, in which both the drift and diffusion vector fields are allowed to be dependent on both measurable outputs and some unmeasurable states; (3) parametric uncertainties existing in both drift and diffusion vector fields are allowed and are handled by means of adaptive control techniques; (4) the concept of stochastic ISS and the method of changing supply functions are used to deal with stochastic inverse dynamics which are nonlinear and, moreover, enter subsystems nonadditively; (5) the results of both adaptive practical stabilization and adaptive asymptotic stabilization are obtained.

The remainder of the paper is organized as follows. Section 2 provides some notations and preliminary results. Section 3 describes the problem to be investigated. Section 4 presents the design of decentralized reduced-order observer. The decentralized, adaptive, output-feedback control design procedure is given in Section 5. Stability analysis of the closed-loop system in question is also developed there. Section 6 gives a numerical example to illustrate the efficiency of our results. Section 7 contains some concluding remarks.

2. Notations and preliminary results

The following notations will be used throughout this paper. $\mathbb{R}_+$ denotes the set of all nonnegative real numbers. $\mathbb{R}^n$ denotes the real n-dimensional space; $\mathbb{R}^{n\times r}$ denotes the real $n\times r$ matrix space; for a given vector or matrix X, X^{T} denotes its transpose; $\mathrm{Tr}(X)$ denotes its trace when X is square; $|X|$ denotes the Euclidean norm of a vector X and the corresponding induced norm for matrices is denoted by $\|X\|$; $\|X\|_{\mathrm{F}}$ denotes the Frobenius norm of X defined by $\|X\|_{\mathrm{F}}=\sqrt{\mathrm{Tr}(X^{\mathrm{T}}X)}$ with properties $\|X\|\leqslant\|X\|_{\mathrm{F}}$ and $\|XY\|_{\mathrm{F}}\leqslant\|X\|\|Y\|_{\mathrm{F}}$; $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denotes the minimal eigenvalue and maximal eigenvalue of symmetric real matrix X, respectively. $\mathscr{C}^i$ denotes the set of all functions with continuous ith partial derivatives. $\mathscr{C}^{1,2}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R}_+)$ denotes the family of all nonnegative functions $V(t,x)$ on $\mathbb{R}_+\times\mathbb{R}^n$ which are $\mathscr{C}^1$ in t and $\mathscr{C}^2$ in x; $\mathscr{K}$ denotes the set of all functions: $\mathbb{R}_+\to\mathbb{R}_+$, which are continuous, strictly increasing and vanish at zero; $\mathscr{K}_\infty$ denotes the set of all functions which are of class $\mathscr{K}$ and unbounded; $\mathscr{KL}$ denotes the set of all functions $\beta(s,t):\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, which is of $\mathscr{K}$ for each fixed t, and decreases to zero as $t\to\infty$ for each fixed s.

Consider the following time-varying stochastic system:

$$\mathrm{d}x=(f(t,x)+g(t,x)u)\,\mathrm{d}t+h(t,x)\,\mathrm{d}w, \tag{1}$$

where w is an r-dimensional standard Brownian motion, $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}$ is the control input, $f,g:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ and $h:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^{n\times r}$ are assumed to be $\mathscr{C}^1$ in their arguments.

Definition 1. For any given $V(t,x)\in\mathscr{C}^{1,2}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R}_+)$, associated with the stochastic differential equation (1) we define the differential operator $\mathscr{L}$ as follows:

$$\mathscr{L}V=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x)+\frac{\partial V}{\partial x}g(t,x)u+\frac{1}{2}\mathrm{Tr}\left\{\frac{\partial^2 V}{\partial x^2}hh^{\mathrm{T}}\right\}.$$

Remark 1. The term $\frac{1}{2}\mathrm{Tr}\{(\partial^2V/\partial x^2)hh^{\mathrm{T}}\}$ is called Itô correction term, in which the second-order differential $\partial^2V/\partial x^2$ makes the controller design much more difficult than that of the deterministic case.

For control-free stochastic nonlinear systems of the form

$$\mathrm{d}x=f(t,x)\,\mathrm{d}t+h(t,x)\,\mathrm{d}w, \tag{2}$$

the following stability notions introduced will be used throughout the paper.

Definition 2 (*Has'minskii, 1980*). The solution process $\{x(t),t\geqslant 0\}$ of stochastic system (2) is said to be bounded in probability, if $\lim_{c\to\infty}\sup_{0\leqslant t<\infty}P\{|x(t)|>c\}=0$.

Definition 3 (*Krstić & Deng, 1998*). Consider the system (2) with $f(t,0)\equiv 0, h(t,0)\equiv 0$. The equilibrium $x(t)\equiv 0$ is globally stable in probability if for any $\varepsilon>0$, there exist a class $\mathscr{K}$ function $\gamma(\cdot)$ such that $P\{|x(t)|<\gamma(|x_0|)\}\geqslant 1-\varepsilon$, $\forall t\geqslant 0$, $x_0\in\mathbb{R}^n\backslash\{0\}$.

The following theorem gives sufficient conditions on the boundedness and stability properties. The first two results of the theorem can be proved in a similar way to the proof of Theorem 2.5 of Pan and Başar (1999), by Theorem 4.1 of Chapter 3 of Has'minskii (1980), Theorem 2 of Chapter 3, Section 13 of Gihman and Skorohod (1972). The last result is followed from Theorem 3.2 of Krstić and Deng (1998).

Theorem 1. *Consider the stochastic system* (2) *and assume that* $f(t,x), h(t,x)$ *are* $\mathscr{C}^1$ *in their arguments and* $f(t,0), g(t,0)$ *are bounded uniformly in t. If there exist functions* $V(t,x)\in\mathscr{C}^{1,2}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R}_+)$, $\mu_1(\cdot)$, $\mu_2(\cdot)\in\mathscr{K}_\infty$, *constants* $c_1>0, c_2\geqslant 0$, *and a nonnegative function* $W(t,x)$, *such that*

$$\mu_1(|x|)\leqslant V(t,x)\leqslant\mu_2(|x|),\qquad \mathscr{L}V\leqslant -c_1W(t,x)+c_2,$$

then (a) *for* (2) *there exists an almost surely unique solution on* $[0,\infty)$; (b) *the solution process is bounded in probability, when* $W(t,x)\geqslant cV(t,x)$ *for some constant* $c>0$; (c) *when* $c_2=0$, $f(t,0)\equiv 0$, $h(t,0)\equiv 0$ *and* $W(t,x)=W(x)$ *is continuous, the equilibrium* $x=0$ *is globally stable in probability and the solution* $x(t)$ *satisfies* $P\{\lim_{t\to\infty}W(x(t))=0\}=1$.

Remark 2. It is of interest to note that extensions of Theorem 1 have been obtained in our recent work Liu et al. (2006b), which is available upon request from the contact author. To make this work self-contained, these extensions recalled from Liu et al. (2006b) are stated in Appendix A. As shown in Liu et al. (2006b), they play an instrumental role in the design of global output-feedback stabilizers for stochastic nonlinear systems.

3. Problem formulation

We now give the class of systems to be investigated, each of which is an interconnected large-scale stochastic nonlinear system with N subsystems. The behavior of the ith ($i=1,2,\ldots,N$) subsystem is governed by a set of Itô differential equations of the form

$$\mathrm{d}x_{i0}=f_{i0}(t,x_{i0},y)\,\mathrm{d}t+g_{i0}(t,x_{i0},y)\,\mathrm{d}w_i, \tag{3}$$

$$\begin{aligned}
\mathrm{d}x_{i1}&=(x_{i2}+f_{i1}(t,x_{i0},y))\,\mathrm{d}t+g_{i1}(t,x_{i0},y)\,\mathrm{d}w_i,\\
&\vdots\\
\mathrm{d}x_{i,n_i-1}&=(x_{i,n_i}+f_{i,n_i-1}(t,x_{i0},y))\,\mathrm{d}t\\
&\quad+g_{i,n_i-1}(t,x_{i0},y)\,\mathrm{d}w_i,\\
\mathrm{d}x_{i,n_i}&=(u_i+f_{i,n_i}(t,x_{i0},y))\,\mathrm{d}t+g_{i,n_i}(t,x_{i0},y)\,\mathrm{d}w_i,\\
y_i&=x_{i1},
\end{aligned} \tag{4}$$

where $x_i=[x_{i0}^{\mathrm{T}},x_{i1},\ldots,x_{i,n_i}]^{\mathrm{T}}$, $u_i\in\mathbb{R}$, $y_i\in\mathbb{R}$ represent the state vector, the scalar control input, the scalar output, respectively; $y=[y_1,\ldots,y_N]^{\mathrm{T}}$; $x_{i0}\in\mathbb{R}^{m_i}$ is referred to as the state of the stochastic inverse dynamics and the initial condition $x_i(0)$ is fixed; f_{ij}, g_{ij}, $j=0,1,\ldots,n_i$, $i=1,\ldots,N$, are uncertain $\mathscr{C}^1$ functions in their arguments; w_i, $i=1,\ldots,N$, are r_i-dimensional standard Brownian motions defined on a probability space $(\Omega,\mathscr{F},P)$ with Ω being a sample space, $\mathscr{F}$ being a filtration, and P being a probability measure.

Remark 3. In this paper, it is assumed that the process $w=[w_1,\ldots,w_N]^{\mathrm{T}}$ has independent components, i.e. w is an r-dimensional standard Brownian motion ($r=\sum_{i=1}^N r_i$).

The results of this paper are based on the following assumptions:

A1. For each $i\in\{1,2,\ldots,N\}$ and $j\in\{1,2,\ldots,n_i\}$, there are unknown constants $l_{ij}^*>0, h_{ij}^*>0$, and known smooth functions $\varphi_{ij0}\geqslant 0, \varphi_{ijl}\geqslant 0, \psi_{ij0}\geqslant 0, \psi_{ijl}\geqslant 0$ such that $\forall(t,x_{i0},y)\in\mathbb{R}_+\times\mathbb{R}^{m_i}\times\mathbb{R}^N$,

$$|f_{ij}(t,x_{i0},y)|\leqslant l_{ij}^*\varphi_{ij0}(|x_{i0}|)+l_{ij}^*\sum_{l=1}^N\varphi_{ijl}(|y_l|),$$

$$|g_{ij}(t,x_{i0},y)|\leqslant h_{ij}^*\psi_{ij0}(|x_{i0}|)+h_{ij}^*\sum_{l=1}^N\psi_{ijl}(|y_l|).$$

Without loss of generality, assume that $\varphi_{ij0}(0)=0$ and $\psi_{ij0}(0)=0$.

A2. For each x_{i0}-subsystem (3) with $i\in\{1,2,\ldots,N\}$, there exist $\mathscr{C}^2$ function $V_{i0}(x_{i0})$ and $\mathscr{K}_\infty$ functions $\alpha_{i1},\alpha_{i2},\alpha_{i0}$ and γ_{il}, $l\in\{1,2,\ldots,N\}$, such that $\forall(t,x_{i0},y)\in\mathbb{R}_+\times\mathbb{R}^{m_i}\times\mathbb{R}^N$,

$$\alpha_{i1}(|x_{i0}|)\leqslant V_{i0}(x_{i0})\leqslant\alpha_{i2}(|x_{i0}|), \tag{5}$$

$$\mathscr{L}V_{i0}(x_{i0})\leqslant\sum_{l=1}^N\gamma_{il}(|y_l|)-\alpha_{i0}(|x_{i0}|). \tag{6}$$

Remark 4. By adaptation of the change of coordinates in Krstić et al. (1995, Chapter 7) for deterministic output-feedback systems, it is not hard to transform the class of systems considered in Eqs. (14)–(15) of Xie and Xie (2000) to a special member of the class of systems (4). In addition, in Xie and Xie (2000), $f_{ij}(t,x_{i0},y)=f_{ij}(y)$, $g_{ij}(t,x_{i0},y)=g_{ij}(y_i)$ are only output-dependent nonlinearities without parameter uncertainty, and, moreover, the subsystems interact through the output y only in drift vector fields, while in Arslan and Başar (2003), the diffusion vector field is known and only output-dependent.

Remark 5. For the noise of unknown covariance, if it can be formulated as $\Sigma_i(t)\,\mathrm{d}w_{i1}$ (Deng et al., 2001), where $w_{i1}(t)$ is a standard Brownian motion and $\Sigma_i(t)$ is a bounded deterministic (maybe unknown) function, then we can modify Assumption A1 as follows: $|g_{ij}(t,x_{i0},y)\Sigma_i(t)|\leqslant h_{ij}^*\psi_{ij0}(|x_{i0}|)+$

$h^*_{ij}\sum_{l=1}^N \psi_{ijl}(|y_l|)$. Thus, the main results of Deng and Krstić (2000) can be seen as special case of this paper when $N=1$.

Remark 6. From Assumption A2 and Appendix A, we see that inverse dynamics (3) is stochastic input-to-state stable (SISS) with respect to the inputs $y=[y_1,\ldots,y_N]^{\mathrm{T}}$. The inverse dynamics considered here are nonlinear and enter subsystems nonadditively. There is very few work on the stabilization problem of stochastic systems with such inverse dynamics. The key concepts and tools presented in this paper are stochastic ISS stability and its related theorems; see Liu et al. (2006b) for an application of these tools to the problem of stochastic output-feedback stabilization.

The control objective of this paper is to design a decentralized adaptive output-feedback controller:

$$\dot{\chi}_i=\varpi_i(\chi_i,y_i),$$
$$u_i=\mu_i(\chi_i,y_i), \tag{7}$$

so that (1) the solution process of the closed-loop system is bounded in probability and the outputs $y=[y_1,\ldots,y_N]^{\mathrm{T}}$ can be regulated into a small neighborhood of the origin in probability. In addition, if a prior bound on the unknown constants l^*_{ij}, h^*_{ij} is available, the control law (7) can be tuned to make the neighborhood arbitrarily small; (2) if for every $j=1,\ldots,n_i$, $i,l=1,\ldots,N$, $\varphi_{ijl}(0)=0$, $\psi_{ijl}(0)=0$, the control law (7) can be modified to make the closed-loop equilibrium of interest globally stable in probability. Moreover, the outputs can be regulated to the origin almost surely, i.e. $P\{\lim_{t\to\infty}\sum_{i=1}^N|y_i(t)|=0\}=1$.

4. Design of decentralized reduced-order observers

Let us consider the subsystem (4) for each ith subsystem. First, we introduce a linear reduced-order estimator for each subsystem i $(i=1,\ldots,N)$,

$$\dot{\hat{x}}_{ij}=\hat{x}_{i,j+1}+L_{i,j+1}y_i-L_{ij}(\hat{x}_{i1}+L_{i1}y_i),$$
$$j=1,\ldots,n_i-2,$$
$$\dot{\hat{x}}_{i,n_i-1}=u_i-L_{i,n_i-1}(\hat{x}_{i1}+L_{i1}y_i),$$

where the L_{ij}'s are real numbers such that the following matrix is stable:

$$A_i=\begin{bmatrix}-L_{i1} & \\ -L_{i2} & I_{(n_i-2)\times(n_i-2)}\\ \vdots & \\ -L_{i,n_i-1} & 0\cdots 0\end{bmatrix}.$$

Let $l^*_i=\max\{1,l^*_{ij},h^*_{ij},1\leqslant j\leqslant n_i\}$,

$$F_i(t,x_{i0},y)=[f_{i2}-L_{i1}f_{i1},\ldots,f_{i,n_i}-L_{i,n_i-1}f_{i1}]^{\mathrm{T}},$$

$$G_i(t,x_{i0},y)=[g^{\mathrm{T}}_{i2}-L_{i1}g^{\mathrm{T}}_{i1},\ldots,g^{\mathrm{T}}_{i,n_i}-L_{i,n_i-1}g^{\mathrm{T}}_{i1}]^{\mathrm{T}},$$

and $\tilde{x}_i=[\tilde{x}_{i2},\ldots,\tilde{x}_{i,n_i}]^{\mathrm{T}}$ with the components $\tilde{x}_{ij}$ $(j=2,\ldots,n_i)$ be given as follows:

$$\begin{cases}\tilde{x}_{i2}=\dfrac{1}{l^*_i}(x_{i2}-\hat{x}_{i1}-L_{i1}y_i),\\ \vdots\\ \tilde{x}_{i,n_i}=\dfrac{1}{l^*_i}(x_{i,n_i}-\hat{x}_{i,n_i-1}-L_{i,n_i-1}y_i).\end{cases}$$

Then we get the following error dynamics:

$$\mathrm{d}\tilde{x}_i=\left[A_i\tilde{x}_i+\frac{1}{l^*_i}F_i(t,x_{i0},y)\right]\mathrm{d}t+\frac{1}{l^*_i}G_i(t,x_{i0},y)\,\mathrm{d}w_i. \tag{8}$$

So, the complete form of the ith subsystem is

$$\mathrm{d}x_{i0}=f_{i0}(t,x_{i0},y)\,\mathrm{d}t+g_{i0}(t,x_{i0},y)\,\mathrm{d}w_i,$$
$$\mathrm{d}\tilde{x}_i=\left[A_i\tilde{x}_i+\frac{1}{l^*_i}F_i(t,x_{i0},y)\right]\mathrm{d}t+\frac{1}{l^*_i}G_i(t,x_{i0},y)\,\mathrm{d}w_i,$$
$$\mathrm{d}y_i=[\hat{x}_{i1}+L_{i1}y_i+l^*_i\tilde{x}_{i2}+f_{i1}(t,x_{i0},y)]\,\mathrm{d}t$$
$$+g_{i1}(t,x_{i0},y)\,\mathrm{d}w_i,$$
$$\mathrm{d}\hat{x}_{ij}=[\hat{x}_{i,j+1}+L_{i,j+1}y_i-L_{ij}(\hat{x}_{i1}+L_{i1}y_i)]\,\mathrm{d}t,$$
$$j=1,\ldots,n_i-2,$$
$$\mathrm{d}\hat{x}_{i,n_i-1}=[u_i-L_{i,n_i-1}(\hat{x}_{i1}+L_{i1}y_i)]\,\mathrm{d}t. \tag{9}$$

In (3)–(4), only the output-y_i measurement is available for the ith subsystem. But now, in (9), the measurements of partial-states $[y_i,\hat{x}_{i1},\ldots,\hat{x}_{i,n_i-1}]^{\mathrm{T}}$ are available for the decentralized design. Based on the new decentralized system (9), we will give a recursive controller design method in the following section.

5. Decentralized adaptive control

Note that only the states variables $[y_i,\hat{x}_{i1},\ldots,\hat{x}_{i,n_i-1}]^{\mathrm{T}}$ of the ith subsystem are available to the controller u_i and the decentralized system (9) is made up of three parts: stochastic inverse dynamics, error dynamics, and measurable dynamics. The outline of the control design procedure is as follows. First, iteratively, given $i=1,\ldots,N$, we will take x_{i0} and $y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_N$ as external disturbances of the latter two dynamics, and use the backstepping method to design an adaptive output-feedback controller with a to-be-determined gain function. Then, by introducing a supply function we get a dissipative inequality for the inverse dynamics, by which a proper Lyapunov function and a gain function for the entire closed-loop large-scale system are constructively designed, and the stability analysis of the closed-loop system follows readily.

5.1. Backstepping design

Introduce a new state transformation

$$z_{i1}=y_i,\quad z_{i,j+1}=\hat{x}_{ij}-\phi_{ij}(\bar{\hat{x}}_{ij},\hat{l}_i),$$

where $j=1,\ldots,n_i$, $i=1,\ldots,N$; $\bar{\hat{x}}_{ij}=[y_i,\hat{x}_{i1},\ldots,\hat{x}_{i,j-1}]^{\mathrm{T}}$, and ϕ_{ij} are smooth virtual controls to be designed, $\hat{l}_i(t)$ are parameter estimates. Then, by Itô formula we have

$$\begin{aligned}\mathrm{d}z_{i1}&=(z_{i2}+\phi_{i1}+L_{i1}y_i+l_i^*\tilde{x}_{i2}+f_{i1}(t,x_{i0},y))\,\mathrm{d}t\\&\quad+g_{i1}(t,x_{i0},y)\,\mathrm{d}w_i,\end{aligned}\tag{10}$$

$$\begin{aligned}\mathrm{d}z_{i,j+1}&=(z_{i,j+2}+\phi_{i,j+1}+L_{i,j+1}y_i+\Omega_{i,j+1,1}+\Omega_{i,j+1,2}\\&\quad+\Omega_{i,j+1,3}+\Omega_{i,j+1,4})\,\mathrm{d}t+\Phi_{i,j+1}\,\mathrm{d}w_i,\\&\quad j=1,\ldots,n_i-1,\end{aligned}\tag{11}$$

where $z_{i,n_i+1}=0$, $\phi_{i,n_i}=u_i$, $L_{i,n_i}=0$, $\Omega_{i,j+1,1}=-L_{ij}(\hat{x}_{i1}+L_{i1}y_i)-(\partial\phi_{ij}/\partial y_i)(\hat{x}_{i1}+L_{i1}y_i)-\sum_{k=1}^{j-1}(\partial\phi_{ij}/\partial\hat{x}_{ik})(\hat{x}_{i,k+1}+L_{i,k+1}y_i-L_{ik}(\hat{x}_{i1}+L_{i1}y_i))$, $\Omega_{i,j+1,2}=-(\partial\phi_{ij}/\partial y_i)l_i^*\tilde{x}_{i2}$, $\Omega_{i,j+1,3}=-(\partial\phi_{ij}/\partial y_i)f_{i1}-\frac{1}{2}(\partial^2\phi_{ij}/\partial y_i^2)g_{i1}g_{i1}^{\mathrm{T}}$, $\Omega_{i,j+1,4}=-(\partial\phi_{ij}/\partial\hat{l}_i)\dot{\hat{l}}_i$, $\Phi_{i,j+1}=-(\partial\phi_{ij}/\partial y_i)g_{i1}$, $1\leqslant j\leqslant n_i-1$.

In the following, we will design the local controller step by step for each subsystem.

Step 1: Recall that the parameters L_{ij} are designed such that A_i is stable in Section 4. We know that there exists a positive-definite matrix P_i such that $A_i^{\mathrm{T}}P_i+P_iA_i=-I$. Let

$$V_{i1}=\frac{\delta_{i1}}{2}(\tilde{x}_i^{\mathrm{T}}P_i\tilde{x}_i)^2+\frac{1}{4}y_i^4+\frac{1}{2\lambda_{i0}}(\hat{l}_i-l_i)^2,$$

where $\delta_{i1}>0$, $\lambda_{i0}>0$ are design parameters, l_i is an unknown constant such that $l_i\geqslant\max\{l_i^{*4/3},h_{i1}^{*4}\}$, $\hat{l}_i=\hat{l}_i(t)$ is governed by the update law $\dot{\hat{l}}_i=\varpi_{i,n_i}(\bar{\hat{x}}_{i,n_i},\hat{l}_i)$ and to be designed to counteract the parametric uncertainty.

From (8), (10) and Itô formula, it follows that

$$\begin{aligned}\mathscr{L}V_{i1}=&-\delta_{i1}\tilde{x}_i^{\mathrm{T}}P_i\tilde{x}_i|\tilde{x}_i|^2+\frac{2\delta_{i1}}{l_i^*}\tilde{x}_i^{\mathrm{T}}P_i\tilde{x}_i(F_i^{\mathrm{T}}P_i\tilde{x}_i)\\&+\frac{\delta_{i1}}{l_i^{*2}}\mathrm{Tr}\{(2P_i\tilde{x}_i\tilde{x}_i^{\mathrm{T}}P_i+\tilde{x}_i^{\mathrm{T}}P_i\tilde{x}_iP_i)G_iG_i^{\mathrm{T}}\}\\&+y_i^3(z_{i2}+\phi_{i1}+L_{i1}y_i+l_1^*\tilde{x}_{i2}+f_{i1})\\&+\frac{3}{2}y_i^2g_{i1}g_{i1}^{\mathrm{T}}+\frac{1}{\lambda_{i0}}(\hat{l}_i-l_i)\dot{\hat{l}}_i.\end{aligned}\tag{12}$$

Since $\varphi_{ijl},\psi_{ijl},i,l=1,\ldots,N$, $j=1,\ldots,n_i$, are smooth, there exist smooth nonnegative functions $\kappa_{ijl}^{(1)}(y_l)$, $\kappa_{ijl}^{(2)}(y_l)$ such that

$$\left(\sum_{l=1}^N\varphi_{ijl}(|y_l|)\right)^4\leqslant\sum_{l=1}^N\kappa_{ijl}^{(1)}y_l^4+8\left(\sum_{l=1}^N\varphi_{ijl}(0)\right)^4,\tag{13}$$

$$\left(\sum_{l=1}^N\psi_{ijl}(|y_l|)\right)^4\leqslant\sum_{l=1}^N\kappa_{ijl}^{(2)}y_l^4+8\left(\sum_{l=1}^N\psi_{ijl}(0)\right)^4.\tag{14}$$

For simplicity, here and hereafter the argument y_l of $\kappa_{ijl}^{(1)}(y_l)$ and $\kappa_{ijl}^{(2)}(y_l)$ is omitted.

Thus, using Assumption A1, Young inequality we obtain

$$y_i^3z_{i2}\leqslant\frac{3}{4}\varepsilon_{i1}^{4/3}y_i^4+\frac{1}{4\varepsilon_{i1}^4}z_{i2}^4,$$

$$y_i^3l_i^*\tilde{x}_{i2}\leqslant\frac{3}{4}\varepsilon_{i1}^{4/3}y_i^4l_i^{*4/3}+\frac{1}{4\varepsilon_{i1}^4}\tilde{x}_{i2}^4\leqslant\frac{3}{4}\varepsilon_{i1}^{4/3}y_i^4l_i+\frac{1}{4\varepsilon_{i1}^4}\tilde{x}_{i2}^4,$$

$$\begin{aligned}y_i^3f_{i1}&\leqslant|y_i^3|l_{i1}^*(\varphi_{i10}(|x_{i0}|)+\sum_{l=1}^N\varphi_{i1l}(|y_l|))\\&\leqslant\frac{3}{2}\eta_{i0}^{4/3}l_iy_i^4+\frac{1}{4\eta_{i0}^4}\varphi_{i10}^4+\frac{1}{4\eta_{i0}^4}\sum_{l=1}^N\kappa_{i1l}^{(1)}y_l^4\\&\quad+\frac{2}{\eta_{i0}^4}\left(\sum_{l=1}^N\varphi_{i1l}(0)\right)^4,\end{aligned}$$

$$\begin{aligned}\frac{3}{2}y_i^2g_{i1}g_{i1}^{\mathrm{T}}&\leqslant3|y_i|^2h_{i1}^{*2}\left[\psi_{i10}^2(|x_{i0}|)+\left(\sum_{l=1}^N\psi_{i1l}(|y_i|)\right)^2\right]\\&\leqslant\frac{3}{\eta_{i1}}y_i^4l_i+\frac{3}{2}\eta_{i1}\psi_{i10}^4+\frac{3\eta_{i1}}{2}\sum_{l=1}^N\kappa_{i1l}^{(2)}y_l^4\\&\quad+12\eta_{i1}\left(\sum_{l=1}^N\psi_{i1l}(0)\right)^4,\end{aligned}$$

$$\begin{aligned}&\frac{2\delta_{i1}}{l_i^*}\tilde{x}_i^{\mathrm{T}}P_i\tilde{x}_i(F_i^{\mathrm{T}}P_i\tilde{x}_i)\\&\leqslant2\delta_{i1}\|P_i\|^2|\tilde{x}_i|^3\frac{1}{l_i^*}|F_i|\\&\leqslant2\delta_{i1}\|P_i\|^2\left[\frac{3}{4}\varepsilon_i^{4/3}|\tilde{x}_i|^4+\frac{1}{4\varepsilon_i^4}|\frac{1}{l_i^*}F_i|^4\right]\\&\leqslant\frac{3\delta_{i1}}{2}\|P_i\|^2\varepsilon_i^{4/3}|\tilde{x}_i|^4\\&\quad+C_{2i}\sum_{j=2}^{n_i}(\varphi_{ij0}^4(|x_{i0}|)+L_{i,j-1}^4\varphi_{i10}^4(|x_{i0}|))\\&\quad+C_{2i}\sum_{j=2}^{n_i}\left[\sum_{l=1}^N(\kappa_{ijl}^{(1)}y_l^4+L_{i,j-1}^4\kappa_{i1l}^{(1)}y_l^4)\right.\\&\quad\left.+8\left(\sum_{l=1}^N\varphi_{ijl}(0)\right)^4+8L_{i,j-1}^4\left(\sum_{l=1}^N\varphi_{i1l}(0)\right)^4\right],\end{aligned}$$

$$\frac{\delta_{i1}}{l_i^{*2}}\mathrm{Tr}\{(2P_i\tilde{x}_i\tilde{x}_i^{\mathrm T}P_i+\tilde{x}_i^{\mathrm T}P_i\tilde{x}_iP_i)G_iG_i^{\mathrm T}\}$$

$$=\frac{2\delta_{i1}}{l_i^{*2}}\mathrm{Tr}[G_i^{\mathrm T}P_i\tilde{x}_i(G_i^{\mathrm T}P_i\tilde{x}_i)^{\mathrm T}]$$

$$+\frac{\delta_{i1}}{l_i^{*2}}(\tilde{x}_i^{\mathrm T}P_i\tilde{x}_i)\mathrm{Tr}(G_i^{\mathrm T}P_iG_i)$$

$$\leqslant\frac{2\delta_{i1}}{l_i^{*2}}\|G_i^{\mathrm T}P_i\tilde{x}_i\|_{\mathrm F}^2+\frac{\delta_{i1}}{l_i^{*2}}|\tilde{x}_i^{\mathrm T}P_i\tilde{x}_i|\lambda_{\max}(P_i)\|G_i\|_{\mathrm F}^2$$

$$\leqslant C_{1i}\sum_{j=2}^{n_i}(\psi_{ij0}^4(|x_{i0}|)+L_{i,j-1}^4\psi_{i10}^4(|x_{i0}|))$$

$$+C_{1i}\sum_{j=2}^{n_i}\left[\sum_{l=1}^{N}(\kappa_{ijl}^{(2)}y_l^4+L_{i,j-1}^4\kappa_{i1l}^{(2)}y_l^4)\right.$$

$$\left.+8\left(\sum_{l=1}^{N}\varphi_{ijl}(0)\right)^4+8L_{i,j-1}^4\left(\sum_{l=1}^{N}\varphi_{i1l}(0)\right)^4\right]$$

$$+\frac{2\delta_{i1}\|P_i\|^2+\delta_{i1}\lambda_{\max}^2(P_i)}{2\epsilon_i}|\tilde{x}_i|^4,$$

where $C_{1i}=16(2\delta_{i1}\|P_i\|^2+\delta_{i1}\lambda_{\max}^2(P_i))\epsilon_i(n_i-1)$, $C_{2i}=16\delta_{i1}\|P_i\|^2(n_i-1)/\varepsilon_i^4$, and $\varepsilon_{i1},\eta_{i0},\eta_{i1},\varepsilon_i,\epsilon_i$ $(i=1,\dots,N)$ are positive design constants to be specified. These together with (12) give

$$\mathscr{L}V_{i1}\leqslant y_i^3\left[\phi_{i1}+\left(\frac{3}{4}\varepsilon_{i1}^{4/3}+L_{i1}+\frac{1}{4\eta_{i0}^4}\kappa_{i1i}^{(1)}+\frac{3\eta_{i1}}{2}\kappa_{i1i}^{(2)}\right)y_i\right.$$

$$\left.+\sum_{j=2}^{n_i}\Upsilon_{iji}y_i\right]+l_i\left(\frac{3}{4}\varepsilon_{i1}^{4/3}y_i^4+\frac{3}{2}\eta_{i0}^{4/3}y_i^4+\frac{3}{\eta_{i1}}y_i^4\right)$$

$$+\frac{1}{\lambda_{i0}}(\hat{l}_i-l_i)\dot{\hat{l}}_i+\frac{1}{4\varepsilon_{i1}^4}z_{i2}^4+\frac{1}{4\varepsilon_{i1}^4}\tilde{x}_{i2}^4+\tilde{c}_i|\tilde{x}_i|^4$$

$$-\delta_{i1}\tilde{x}_i^{\mathrm T}P_i\tilde{x}_i|\tilde{x}_i|^2+\Delta_{i1}(|x_{i0}|)+\Psi_{i1}(|x_{i0}|)$$

$$+\Delta_{i2}(y)+\Pi_{i1}(y)+\Delta_{i3}(0)+\Xi_{i1}(0),\tag{15}$$

where $\Upsilon_{ijl}=C_{2i}(\kappa_{ijl}^{(1)}+L_{i,j-1}^4\kappa_{i1l}^{(1)})+C_{1i}(\kappa_{ijl}^{(2)}+L_{i,j-1}^4\kappa_{i1l}^{(2)})$, $\tilde{c}_i=(3\delta_{i1}/2)\|P_i\|^2\varepsilon_i^{4/3}+(2\delta_{i1}\|P_i\|^2+\delta_{i1}\lambda_{\max}^2(P_i))/2\epsilon_i$, $\Delta_{i1}(|x_{i0}|)=C_{2i}\sum_{j=2}^{n_i}(\varphi_{ij0}^4(|x_{i0}|)+L_{i,j-1}^4\varphi_{i10}^4(|x_{i0}|))+C_{1i}\sum_{j=2}^{n_i}(\psi_{ij0}^4(|x_{i0}|)+L_{i,j-1}^4\psi_{i10}^4(|x_{i0}|))$, $\Psi_{i1}(|x_{i0}|)=(1/4\eta_{i0}^4)\varphi_{i10}^4(|x_{i0}|)+\frac{3}{2}\eta_{i1}\psi_{i10}^4(|x_{i0}|)$, $\Delta_{i2}(y)=\sum_{j=2}^{n_i}\sum_{l=1,l\neq i}^{N}\Upsilon_{ijl}y_l^4$, $\Pi_{i1}(y)=\sum_{l=1,l\neq i}^{N}[(1/4\eta_{i0}^4)\kappa_{i1l}^{(1)}+(3\eta_{i1}/2)\kappa_{i1l}^{(2)}]y_l^4$, $\Delta_{i3}(0)=8C_{2i}\sum_{j=2}^{n_i}[(\sum_{l=1}^{N}\varphi_{ijl}(0))^4+L_{i,j-1}^4(\sum_{l=1}^{N}\varphi_{i1l}(0))^4]+8C_{1i}\sum_{j=2}^{n_i}[(\sum_{l=1}^{N}\psi_{ijl}(0))^4+L_{i,j-1}^4(\sum_{l=1}^{N}\psi_{i1l}(0))^4]$, $\Xi_{i1}(0)=(2/\eta_{i0}^4)(\sum_{l=1}^{N}\varphi_{i1l}(0))^4+12\eta_{i1}(\sum_{l=1}^{N}\psi_{i1l}(0))^4$.

Define the virtual parameter update law and the virtual control as follows

$$\varpi_{i1}=-\lambda_{i0}\sigma_{0i}\hat{l}_i+\lambda_{i0}\left(\frac{3}{4}\varepsilon_{i1}^{4/3}y_i^4+\frac{3}{2}\eta_{i0}^{4/3}y_i^4+\frac{3}{\eta_{i1}}y_i^4\right),\tag{16}$$

$$\phi_{i1}(y_i,\hat{l}_i)=-y_i\left\{\beta_{i1}+v_{i1}(y_i^2)+\frac{3}{4}\varepsilon_{i1}^{4/3}+L_{i1}+\frac{1}{4\eta_{i0}}\kappa_{i1i}^{(1)}\right.$$

$$+\frac{3\eta_{i1}}{2}\kappa_{i1i}^{(2)}+\hat{l}_i\left(\frac{3}{4}\varepsilon_{i1}^{4/3}+\frac{3}{2}\eta_{i0}^{4/3}+\frac{3}{\eta_{i1}}\right)$$

$$\left.+\sum_{j=2}^{n_i}\Upsilon_{iji}\right\},\tag{17}$$

where $\beta_{i1}>0$, $\sigma_{0i}>0$ is a design parameter, and $v_{i1}(\cdot)$ is a smooth nonnegative function to be designed later. Obviously, $\phi_{i1}(0,\hat{l}_i)=0$ for all $\hat{l}_i\in\mathbb{R}$.

It follows from (15) to (17) that

$$\mathscr{L}V_{i1}\leqslant-\beta_{i1}y_i^4-v_{i1}(y_i^2)y_i^4+\frac{1}{\lambda_{i0}}(\hat{l}_i-l_i)(\dot{\hat{l}}_i-\varpi_{i1})$$

$$-\sigma_{0i}(\hat{l}_i-l_i)\hat{l}_i+\frac{1}{4\varepsilon_{i1}^4}z_{i2}^4-\delta_{i1}\tilde{x}_i^{\mathrm T}P_i\tilde{x}_i|\tilde{x}_i|^2$$

$$+\tilde{c}_i|\tilde{x}_i|^4+\frac{1}{4\varepsilon_{i1}^4}\tilde{x}_{i2}^4+\Delta_{i1}(|x_{i0}|)+\Psi_{i1}(|x_{i0}|)$$

$$+\Delta_{i2}(y)+\Pi_{i1}(y)+\Delta_{i3}(0)+\Xi_{i1}(0).\tag{18}$$

Step $k(k=2,\dots,n_i)$: At this step, we can obtain similar property to (18) for the ith subsystem. Such a result is presented by the following lemma.

Lemma 1. *For every* $k=1,\dots,n_i$, *there exist smooth functions* $\varpi_{ij},\phi_{ij},(1\leqslant j\leqslant k)$ *and positive constants* β_{ij} *such that* $\phi_{i,j}(0,\hat{l}_i)=0$ *for all* $\hat{l}_i\in\mathbb{R}$ *and that along the solutions of* (9), $V_{ik}=V_{i1}+\frac{1}{4}\sum_{j=2}^{k}z_{ij}^4$ *satisfies*

$$\mathscr{L}V_{ik}\leqslant-\sum_{j=1}^{k}\beta_{ij}z_{ij}^4-v_{i1}(y_i^2)y_i^4+\left[\frac{1}{\lambda_{i0}}(\hat{l}_i-l_i)\right.$$

$$\left.-\sum_{j=2}^{k}z_{ij}^3\frac{\partial\phi_{i,j-1}}{\partial\hat{l}_i}\right](\dot{\hat{l}}_i-\varpi_{ik})-\sigma_{0i}(\hat{l}_i-l_i)\hat{l}_i$$

$$+\frac{1}{4\varepsilon_{ik}^4}z_{i,k+1}^4-\delta_{i1}\tilde{x}_i^{\mathrm T}P_i\tilde{x}_i|\tilde{x}_i|^2+\tilde{c}_i|\tilde{x}_i|^4$$

$$+\sum_{j=1}^{k}\frac{1}{4\varepsilon_{ij}^4}\tilde{x}_{i2}^4+\Delta_{i1}(|x_{i0}|)+\Psi_{ik}(|x_{i0}|)+\Delta_{i2}(y)$$

$$+\Pi_{ik}(y)+\Lambda_{ik}(y_i)+\Delta_{i3}(0)+\Xi_{ik}(0),\tag{19}$$

where $\Psi_{ik}(|x_{i0}|) = (k/4\eta_{i0}^4)\varphi_{i10}^4(|x_{i0}|) + ((3k/2)\eta_{i1} + \sum_{j=2}^k b_{ij}/2)\ \psi_{i10}^4(|x_{i0}|)$, $\Pi_{ik}(y) = \{\sum_{l=1,l\neq i}^N [(k/4\eta_{i0}^4)\kappa_{i1l}^{(1)} + (3k\eta_{i1}/2)\kappa_{i1l}^{(2)}] + \sum_{j=2}^k \sum_{l=1,l\neq i}^N (b_{ij}/2)\kappa_{i1l}^{(2)}\}y_l^4$, $\Lambda_{ik}(y_i) = \{[(k-1)/4\eta_{i0}^4]\kappa_{i1i}^{(1)} + \kappa_{i1i}^{(2)}[3(k-1)\eta_{i1}/2 + \sum_{j=2}^k b_{ij}/2]\}y_i^4$, $\Xi_{ik}(0) = (2k/\eta_{i0})(\sum_{l=1}^N \varphi_{i1l}(0))^4 + 12\eta_{i1}k(\sum_{l=1}^N \psi_{i1l}(0))^4 + \sum_{j=2}^k 4b_{ij}(\sum_{l=1}^N \psi_{i1l}(0))^4$.

This lemma was proved for $k=1$ and will be proved in the Appendix B for $k=2,\ldots,n_i$. At the last step where $k=n_i$, we get the parameter update law and the control law

$$\dot{\hat{l}}_i = \varpi_{i,n_i}(y_i,\ldots,z_{i,n_i-1},\hat{l}_i), \tag{20}$$

$$u_i = \phi_{i,n_i}(y_i,\ldots,z_{i,n_i-1},\hat{l}_i). \tag{21}$$

Thus we obtain the Lyapunov function candidate for subsystem i:

$$V_{i,n_i} = V_{i,n_i-1}(\tilde{x}_i, y_i,\ldots,z_{i,n_i-1},\hat{l}_i) + \tfrac{1}{4} z_{i,n_i}^4, \tag{22}$$

which satisfies

$$\begin{aligned}\mathscr{L}V_{i,n_i} \leqslant & -\sum_{j=1}^{n_i} \beta_{ij} z_{ij}^4 - v_{i1}(y_i^2)y_i^4 - \sigma_{0i}(\hat{l}_i - l_i)\hat{l}_i \\ & - \left[\delta_{i1}\lambda_{\min}(P_i) - \tilde{c}_i - \sum_{j=1}^{n_i}\frac{1}{4\varepsilon_{ij}^4}\right]|\tilde{x}_i|^4 \\ & + \Delta_{i1}(|x_{i0}|) + \Psi_{i,n_i}(|x_{i0}|) + \Delta_{i2}(y) \\ & + \Pi_{i,n_i}(y) + \Lambda_{i,n_i}(y_i) + \Delta_{i3}(0) + \Xi_{i,n_i}(0). \end{aligned} \tag{23}$$

Choose a smooth nonnegative function $v_{i1}(\cdot)$ such that

$$v_{i1}(y_i^2)y_i^4 - \Lambda_{i,n_i}(y_i) \geqslant v_i(y_i^2)y_i^4 \tag{24}$$

with $v_i(\cdot)$ being a smooth nonnegative function to be designed; and choose parameters ϵ_i, ε_{ij}, ε_i, δ_{i1}, $i=1,\ldots,N$, $j=1,\ldots,n_i$, such that $c_{i0} = \delta_{i1}\lambda_{\min}(P_i) - \tilde{c}_i - \sum_{j=1}^{n_i} 1/4\varepsilon_{ij}^4 > 0$ with β_{ij}, λ_{i0}, η_{i0}, η_{i1}, b_{ij} being any positive constants. Notice that $-\sigma_{0i}(\hat{l}_i - l_i)\hat{l}_i \leqslant -(\sigma_{0i}/2)(\hat{l}_i - l_i)^2 + (\sigma_{0i}/2)l_i^2$. Then, it follows from (23) and (24) that

$$\begin{aligned}\mathscr{L}V_{i,n_i} \leqslant & -V_{i,n_i}^0 - v_i(y_i^2)y_i^4 - \frac{\sigma_{0i}}{2}(\hat{l}_i - l_i)^2 + \frac{\sigma_{0i}}{2}l_i^2 \\ & + \delta_i(|x_{i0}|) + \Gamma_i(0) + \Delta_{i2}(y) + \Pi_{i,n_i}(y), \end{aligned} \tag{25}$$

where $\Gamma_i(0) = \Delta_{i3}(0) + \Xi_{i,n_i}(0)$, $V_{i,n_i}^0 = \sum_{j=1}^{n_i}\beta_{ij}z_{ij}^4 + c_{i0}|\tilde{x}_i|^4$, $\delta_i(|x_{i0}|) = \Delta_{i1}(|x_{i0}|) + \Psi_{i,n_i}(|x_{i0}|)$.

We are now in a position to choose the gain function $v_i(\cdot)$ so as to get a desired decentralized control law of the form (20)–(21).

Take the following Lyapunov function candidate for the whole system:

$$V_N = \sum_{i=1}^N V_{i,n_i}. \tag{26}$$

Then, it follows from (25) that

$$\begin{aligned}\mathscr{L}V_N \leqslant & -\sum_{i=1}^N V_{i,n_i}^0 - \sum_{i=1}^N v_i(y_i^2)y_i^4 - \sum_{i=1}^N \frac{\sigma_{0i}}{2}(\hat{l}_i - l_i)^2 \\ & + \sum_{i=1}^N \frac{\sigma_{0i}}{2}l_i^2 + \sum_{i=1}^N \delta_i(|x_{i0}|) + \sum_{i=1}^N \Gamma_i(0) \\ & + \sum_{i=1}^N (\Delta_{i2}(y) + \Pi_{i,n_i}(y)). \end{aligned} \tag{27}$$

To make $\mathscr{L}V_N$ nonpositive, we first choose $v_i(\cdot)(1\leqslant i\leqslant N)$ to ensure the nonpositiveness of the sum of all y_i-related terms on the right-hand side of the above inequality (27). To this end, we choose a smooth nonnegative function $v_i(\cdot)$ such that

$$\begin{aligned} & v_i(y_i^2)y_i^4 - \sum_{l=1,l\neq i}^N \sum_{j=2}^{n_l} \Upsilon_{lji} y_i^4 - \sum_{l=1,l\neq i}^N \sum_{j=2}^{n_l} \frac{b_{lj}}{2}\kappa_{l1i}^{(2)} y_i^4 \\ & - \sum_{l=1,l\neq i}^N \left(\frac{n_l}{4\eta_{l0}^4}\kappa_{l1i}^{(1)} + \frac{3n_l\eta_{l1}}{2}\kappa_{l1i}^{(2)}\right) y_i^4 \geqslant v_{i0}(y_i^2)y_i^4 \end{aligned} \tag{28}$$

with $v_{i0}(\cdot)$ being a smooth nondecreasing function to be designed in the next subsection. Notice that by the definitions of $\Delta_{i2}(\cdot)$ and $\Pi_{i,n_i}(\cdot)$,

$$\begin{aligned} & \sum_{i=1}^N (\Delta_{i2}(y) + \Pi_{i,n_i}(y)) \\ & = \sum_{l=1}^N \left[\sum_{j=2}^{n_l}\sum_{i=1,i\neq l}^N \Upsilon_{lji} y_i^4 + \frac{n_l}{4\eta_{l0}^4}\sum_{i=1,i\neq l}^N \kappa_{l1i}^{(1)} y_i^4 \right. \\ & \quad \left. + \frac{3n_l\eta_{l1}}{2}\sum_{i=1,i\neq l}^N \kappa_{l1i}^{(2)} y_i^4 + \sum_{j=2}^{n_l}\sum_{i=1,i\neq l}^N \frac{b_{lj}}{2}\kappa_{l1i}^{(2)} y_i^4\right] \\ & = \sum_{i=1}^N \sum_{l=1,l\neq i}^N \sum_{j=2}^{n_l} \Upsilon_{lji} y_i^4 + \sum_{i=1}^N \sum_{l=1,l\neq i}^N \sum_{j=2}^{n_l} \frac{b_{lj}}{2}\kappa_{l1i}^{(2)} y_i^4 \\ & \quad + \sum_{i=1}^N \sum_{l=1,l\neq i}^N \left(\frac{n_l}{4\eta_{l0}^4}\kappa_{l1i}^{(1)} + \frac{3n_l\eta_{l1}}{2}\kappa_{l1i}^{(2)}\right) y_i^4. \end{aligned}$$

Then, from (27) and (28) we have

$$\begin{aligned}\mathscr{L}V_N \leqslant & -\sum_{i=1}^N V_{i,n_i}^0 - \sum_{i=1}^N v_{i0}(y_i^2)y_i^4 - \sum_{i=1}^N \frac{\sigma_{0i}}{2}(\hat{l}_i - l_i)^2 \\ & + \sum_{i=1}^N \frac{\sigma_{0i}}{2}l_i^2 + \sum_{i=1}^N \delta_i(|x_{i0}|) + \sum_{i=1}^N \Gamma_i(0). \end{aligned} \tag{29}$$

Remark 7. From the design procedure of the gain function $v_i(\cdot)$, one can see that actually $v_i(\cdot)$ consists of two parts. One is to compensate the effect of the output of ith subsystem on the other subsystems, and the other is to compensate the effect of the output on the inverse dynamics of ith subsystem. The latter will become clear in the next subsection.

5.2. Stability analysis of the closed-loop system

In the sequel, we consider the interconnection between the inverse dynamics and the other dynamics using the method of changing supply functions developed by Sontag and Teel (1995) and adapted to stochastic systems by Liu et al. (2006b).

First, consider the following Assumption:

A3. There exist known smooth nonnegative function ψ_{iz} and ψ_{i0} satisfying $|\partial V_{i0}/\partial x_{i0}| \leqslant \psi_{iz}(|x_{i0}|)$ and $\|g_{i0}(t, x_{i0}, y)\|_{\mathrm{F}} \leqslant \psi_{i0}(|x_{i0}|)$, $i = 1, \ldots, N$.

Remark 8. In the above assumption, $|\partial V_{i0}/\partial x_{i0}| \leqslant \psi_z(|x_{i0}|)$ is a general assumption and easy to satisfy; while $\|g_{i0}(t, x_{i0}, y)\|_{\mathrm{F}} \leqslant \psi_{i0}(|x_{i0}|)$ is a constraint on the diffusion vector field of inverse dynamics, which reflects that the diffusion vector field of inverse dynamics is confined by the dynamics itself, and that the effect of the controlled subsystems (4) can be viewed as bounded. The second assumption is needed to deal with the Itô correction term of Itô formula, which is one of the most important differences between Itô stochastic systems and the deterministic ones.

From Assumptions A2–A3, if for $i = 1, \ldots, N$, $j = 1, \ldots, n_i$,

$$\limsup_{s \to 0+} \frac{\varphi_{ij0}^4(s) + \psi_{ij0}^4(s) + \psi_{iz}^2(s)\psi_{i0}^2(s)}{\alpha_{i0}(s)} < \infty, \tag{30}$$

then we can construct continuous increasing functions $\xi_i(s) \geqslant 0$ and $\zeta_i(s) > 0$ defined on $[0, \infty)$ such that

$$\xi_i(s)\alpha_{i0}(s) \geqslant 4\delta_i(s), \quad \zeta_i(s)\alpha_{i0}(s) \geqslant 2\psi_{iz}^2(s)\psi_{i0}^2(s).$$

In fact, any increasing functions $\xi_i(\cdot) \in \mathscr{C}[0, \infty)$ and $\zeta_i(\cdot) \in \mathscr{C}[0, \infty)$ satisfying $\xi_i(s) \geqslant 4 \sup_{\tau \in (0,s]} \delta_i(\tau)/\alpha_{i0}(\tau)$ and $\zeta_i(s) \geqslant 2 \sup_{\tau \in (0,s]} \psi_{iz}^2(\tau)\psi_{i0}^2(\tau)/\alpha_{i0}(\tau)$, $\forall s > 0$, are good choice.

With these notations we have the following lemma.

Lemma 2. *If* (30) *and*

$$\int_0^\infty [\xi_i(\alpha_{i1}^{-1}(s))]' \exp\left\{-\int_0^s [\zeta_i(\alpha_{i1}^{-1}(\tau))]^{-1}\,\mathrm{d}\tau\right\} \mathrm{d}s < \infty, \tag{31}$$

then there exists a nondecreasing positive function $\rho_i(\cdot) \in \mathscr{C}^1[0, \infty)$ *such that* $\forall x \in \mathbb{R}^{m_i}$,

$$\rho_i(V_{i0}(x))\alpha_{i0}(|x|) \geqslant 2\rho_i'(V_{i0}(x))\psi_{iz}^2(|x|)\psi_{i0}^2(|x|) + 4\delta_i(|x|), \tag{32}$$

where $\delta_i(\cdot)$ *is given by* (25).

Proof. See Appendix C.

Theorem 2. *Consider the system* (3)–(4). *Suppose that the Assumptions* A1–A3, *the conditions of Lemma* 2 *and*

$$\limsup_{s \to 0+} \frac{\gamma_{il}(s)}{s^4} < \infty, \quad 1 \leqslant i,\ l \leqslant N \tag{33}$$

hold. Then, under the controller (20)–(21), *the closed-loop system has an almost surely unique solution on* $[0, \infty)$ *and the solution process is bounded in probability, and moreover, the outputs* $y = [y_1, \ldots, y_N]^{\mathrm{T}}$ *can be regulated into a small neighborhood of the origin in probability. In addition, if a prior bound on the unknown constants* l_{ij}^*, h_{ij}^* *is available, the control law* (20)–(21) *can be tuned to make the neighborhood arbitrarily small.*

Proof. Suppose that $\rho_i(\cdot)$ is the function defined in Lemma 2 such that (32) holds. Let

$$U_i(x_{i0}) = \int_0^{V_{i0}(x_{i0})} \rho_i(t)\,\mathrm{d}t.$$

Then, $U_i(\cdot) \in \mathscr{C}^2(\mathbb{R}^{m_i})$. By Itô formula and Assumptions A2–A3, following the proof of Theorem 1 of Jiang and Repperger (2001), we have

$$\begin{aligned}
\mathscr{L}U_i(x_{i0}) &= \rho_i(V_{i0})\mathscr{L}V_{i0} + \frac{1}{2}\rho_i'(V_{i0})\left\|\left(\frac{\partial V_{i0}}{\partial x_{i0}}\right)^{\mathrm{T}} g_{i0}\right\|_{\mathrm{F}}^2 \\
&\leqslant \rho_i(V_{i0})\left[\sum_{l=1}^N \gamma_{il}(|y_l|) - \alpha_{i0}(|x_{i0}|)\right] \\
&\quad + \frac{1}{2}\rho_i'(V_{i0})\psi_{iz}^2(|x_{i0}|)\psi_{i0}^2(|x_{i0}|) \\
&\leqslant \sum_{l=1}^N \rho_i(\eta_i(|y_l|))\gamma_{il}(|y_l|) - \frac{1}{2}\rho(V_{i0})\alpha_{i0}(|x_{i0}|) \\
&\quad + \frac{1}{2}\rho_i'(V_{i0})\psi_{iz}^2(|x_{i0}|)\psi_{i0}^2(|x_{i0}|),
\end{aligned} \tag{34}$$

where $\eta_i = \alpha_{i2}(\alpha_{i0}^{-1}(2N\gamma_{il}(\cdot))) \in \mathscr{K}_\infty$.

Consider the Lyapunov function candidate for the entire system

$$W = V_N + \sum_{i=1}^N U_i,$$

where V_N is defined by (26). Then, it follows from (29) and (34) that

$$\begin{aligned}
\mathscr{L}W \leqslant & -\sum_{i=1}^N V_{i,n_i}^0 - \sum_{i=1}^N v_{i0}(y_i^2)y_i^4 - \sum_{i=1}^N \frac{\sigma_{0i}}{2}(\hat{l}_i - l_i)^2 \\
& + \sum_{i=1}^N \frac{\sigma_{0i}}{2} l_i^2 + \sum_{i=1}^N \sum_{l=1}^N \rho_i(\eta_i(|y_l|))\gamma_{il}(|y_l|) \\
& + \sum_{i=1}^N \delta_i(|x_{i0}|) + \sum_{i=1}^N \frac{1}{2}\rho_i'(V_{i0})\psi_{iz}^2(|x_{i0}|)\psi_{i0}^2(|x_{i0}|) \\
& - \sum_{i=1}^N \frac{1}{2}\rho_i(V_{i0})\alpha_{i0}(|x_{i0}|) + \sum_{i=1}^N \Gamma_i(0).
\end{aligned} \tag{35}$$

From Lemma 2 we have

$$\frac{1}{4}\rho_i(V_{i0})\alpha_{i0}(|x_{i0}|)-\delta_i(|x_{i0}|)$$
$$\geqslant \frac{1}{2}\rho_i'(V_{i0})\psi_{iz}^2(|x_{i0}|)\psi_{i0}^2(|x_{i0}|). \quad (36)$$

From (33) we can construct a smooth nondecreasing function $v_{i0}(\cdot)$ such that $v_{i0}(y_i^2)y_i^4 \geqslant \sum_{l=1}^N \rho_l(\alpha_{l2}(\alpha_{l0}^{-1}(2N\gamma_{li}(|y_i|))))\gamma_{li}(|y_i|)$. This together with

$$\sum_{i=1}^N\sum_{l=1}^N \rho_i(\alpha_{i2}(\alpha_{i0}^{-1}(2N\gamma_{il}(|y_l|))))\gamma_{il}(|y_l|)$$
$$=\sum_{i=1}^N\sum_{l=1}^N \rho_l(\alpha_{l2}(\alpha_{l0}^{-1}(2N\gamma_{li}(|y_i|))))\gamma_{li}(|y_i|),$$

ensures

$$\sum_{i=1}^N v_{i0}(y_i^2)y_i^4 \geqslant \sum_{i=1}^N\sum_{l=1}^N \rho_i(\eta_i(|y_l|))\gamma_{il}(|y_l|).$$

Thus, by (35) and (36) we have

$$\mathscr{L}W \leqslant -\sum_{i=1}^N \frac{1}{4}\rho_i(0)\alpha_{i0}(|x_{i0}|)-\sum_{i=1}^N V_{i,n_i}^0$$
$$-\sum_{i=1}^N \frac{\sigma_{0i}}{2}(\hat{l}_i-l_i)^2+\sum_{i=1}^N \Gamma_i(0)+\sum_{i=1}^N \frac{\sigma_{0i}}{2}l_i^2.$$

Define

$$W_1(\tilde{x},z,x_0,\hat{l})$$
$$=\sum_{i=1}^N \frac{1}{4}\rho_i(0)\alpha_{i0}(|x_{i0}|)+\sum_{i=1}^N V_{i,n_i}^0+\sum_{i=1}^N \frac{\sigma_{0i}}{2}(\hat{l}_i-l_i)^2,$$

where $\tilde{x}=[\tilde{x}_1^{\mathrm{T}},\ldots,\tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}}$, $z=[z_1^{\mathrm{T}},\ldots,z_N^{\mathrm{T}}]^{\mathrm{T}}$, $x_0=[x_{10}^{\mathrm{T}},\ldots,x_{N0}^{\mathrm{T}}]^{\mathrm{T}}$, $\hat{l}=[\hat{l}_1,\ldots,\hat{l}_N]^{\mathrm{T}}$, $l=[l_1,\ldots,l_N]^{\mathrm{T}}$, $z_i=[z_{i1},\ldots,z_{i,n_i}]^{\mathrm{T}}$. Then, it is easy to see that $W_1(\tilde{x},z,x_0,\hat{l})$ is positive-definite and radially unbounded in its argument $(\tilde{x},z,x_0,\hat{l})$ and satisfies

$$\mathscr{L}W \leqslant -W_1(\tilde{x},z,x_0,\hat{l})+\sum_{i=1}^N \Gamma_i(0)+\sum_{i=1}^N \frac{\sigma_{0i}}{2}l_i^2.$$

By Theorem 1, the closed-loop system has an almost surely unique solution on $[0,\infty)$, and moreover, by Theorem A.2, the solution process of the closed-loop system is bounded in probability and for any given $\varepsilon>0$, there exist a $\mathscr{KL}$ function $\beta(\cdot,\cdot)$ and a $\mathscr{K}$ function $\gamma(\cdot)$ such that $\forall t\geqslant 0$,

$$P\{|(\tilde{x},z,x_0,\hat{l})|<\beta(|(\tilde{x}(0),z(0),x_0(0),\hat{l}(0))|,t)+\gamma(d)\}$$
$$\geqslant 1-\varepsilon,$$

where $(\tilde{x}(0),z(0),x_0(0),\hat{l}(0)) \neq 0$, $d=\sum_{i=1}^N \Gamma_i(0)+\sum_{i=1}^N(\sigma_{0i}/2)l_i^2$. From the definition of $\Gamma_i(\cdot)$, d can be made small if we choose the design parameters $\sigma_{0i},\delta_{i1},\varepsilon_i,\epsilon_i,\eta_{i0},\eta_{i1},b_{ij}$ appropriately. If we know a prior bound on the unknown parameters l_{ij}^*,h_{ij}^*, from the definition of d, it can be made as small as possible by the design parameters chosen appropriately. That is, the outputs y can be steered to an arbitrarily small neighborhood of the origin in probability. □

Corollary 1. *If for every* $j=1,\ldots,n_i,i,l=1,\ldots,N$, $\varphi_{ijl}(0)=0$, $\psi_{ijl}(0)=0$, *and suppose that the conditions of Theorem* 2 *hold. Then, under the controller consisting of* (16), (17), (B.3), (B.4) *and* (20), (21) *with* $\sigma_{0i}=0$ $(i=1,\ldots,N)$ *in* (16), *the equilibrium* $(\tilde{x},z,x_0,\hat{l})=(0,0,0,l)$ *is globally stable in probability, and in addition, the outputs can be regulated to the origin almost surely, more precisely,* $P\{\lim_{t\to\infty}\sum_{i=1}^N(|y_i|+|x_i|)=0\}=1$.

Proof. When $\varphi_{ijl}(0)=0$, $\psi_{ijl}(0)=0$, $(1\leqslant j\leqslant n_i, 1\leqslant i,l\leqslant N)$, we have $\Gamma_i(0)=0$. Thus, by $\sigma_{0i}=0$ one can get $\mathscr{L}W\leqslant -W_2(\tilde{x},z,x_0)$, where $W_2(\tilde{x},z,x_0)=\sum_{i=1}^N \frac{1}{4}\rho_i(0)\alpha_{i0}(|x_{i0}|)+\sum_{i=1}^N V_{i,n_i}^0$. Then, from Theorem 1, it follows that the equilibrium $(\tilde{x},z,x_0,\hat{l})=(0,0,0,l)$ is globally stable in probability and $P\{\lim_{t\to\infty}W_2(\tilde{x},z,x_0)=0\}=1$. □

By the definitions of $W_2(\tilde{x},z,x_0)$ and $\tilde{x},z,x_0$, we can easily get the almost sure result.

Remark 9. For stochastic interconnected nonlinear systems (3)–(4), in order to design a desired adaptive stabilizer, *quartic* (rather than quadratic) Lyapunov function candidates are used in the design procedure to separate inverse dynamics from others and to deal with the Itô correction term of Itô formula. If the diffusion vector fields of the controlled plant only depend on the output y, a locally quadratic Lyapunov-like function can be constructed by combining the methods of this paper and Liu and Zhang (2005, 2006).

Remark 10. To get boundedness in probability of the solution, the small gain condition (30) has been used here, which is different from that in the deterministic case due to the appearance of the second-order differential term of Itô formula. If the inverse dynamics of ith subsystem is to degenerate $g_{i0}(t,x_{i0},y)=0$, then $\psi_{i0}(|x_{i0}|)$ can be simply taken as 0. In this case, (32) becomes $\rho_i(V_{i0}(x))\alpha_{i0}(|x|)\geqslant 4\delta_i(|x|)$, $\forall x\in\mathbb{R}^{m_i}$. To get a ρ_i satisfying the above inequality, the small gain condition (30) on ith subsystem is not needed (see Jiang & Repperger, 2001).

Remark 11. The small gain conditions (30) and (33) are the constraints on the system gains for small signals and are easy to verify. If locally quadratic Lyapunov-like functions are adapted in the special case in Remark 9, the quartic in the small gain conditions may be changed to quadratic accordingly. It is worth pointing out that the condition (31) or the like (Liu et al., 2006b) seems indispensable for asymptotic convergence of the closed-loop signals. Examples of systems with (31) are given in Liu et al. (2006b).

6. Numerical example

In this section, we will give a numerical example to illustrate the efficiency of our results.

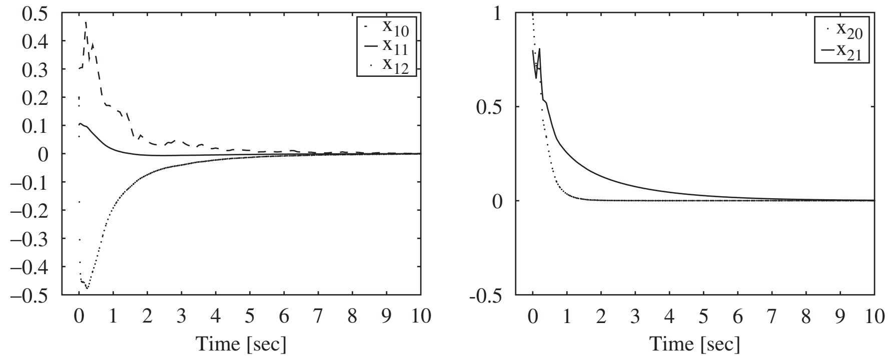

Fig. 1. States of subsystems 1 and 2.

Consider the following interconnected systems:

$$\begin{cases} \mathrm{d}x_{10} = (-3x_{10} + y_1^2 \sin t + y_2)\,\mathrm{d}t + \frac{1}{\sqrt{2}} x_{10} \cos y_1 \,\mathrm{d}w_1, \\ \mathrm{d}x_{11} = (x_{12} + \theta_{11} x_{10} + \frac{\theta_{12}}{2} y_2)\,\mathrm{d}t, \\ \mathrm{d}x_{12} = u_1\,\mathrm{d}t, \\ y_1 = x_{11}, \\ \mathrm{d}x_{20} = (-4x_{20} + \frac{1}{4} y_2^2 + \frac{1}{4} y_1^2)\,\mathrm{d}t + \frac{1}{\sqrt{2}} x_{20} \sin y_2 \,\mathrm{d}w_2, \\ \mathrm{d}x_{21} = (u_2 + y_2^2 y_1)\,\mathrm{d}t + \theta_{21} x_{20} y_2 \,\mathrm{d}w_2, \\ y_2 = x_{21}. \end{cases} \tag{37}$$

With the notations of Assumption A1, we can take $\varphi_{110}(|x_{10}|) = |x_{10}|$, $\varphi_{111}(|y_1|) = 0$, $\varphi_{112}(|y_2|) = |y_2|/2$, $\psi_{110}(|x_{10}|) = 0$, $\psi_{111}(|y_1|) = 0$, $\psi_{112}(|y_2|) = 0$, $\varphi_{120}(|x_{10}|) = 0$, $\varphi_{121}(|y_1|) = 0$, $\varphi_{122}(|y_2|) = 0$, $\psi_{120}(|x_{10}|) = 0$, $\psi_{121}(|y_1|) = 0$, $\psi_{122}(|y_2|) = 0$, $\varphi_{210}(|x_{20}|) = 0$, $\varphi_{211} = \frac{1}{2} y_1^2$, $\varphi_{212}(|y_2|) = \frac{1}{2} y_2^4$, $\psi_{210}(|x_{10}|) = \frac{1}{2}|x_{10}|^2$, $\psi_{211}(|y_1|) = 0$, $\psi_{212}(|y_2|) = \frac{1}{2} y_2^2$, $l_{11}^* = \max\{|\theta_{11}|, |\theta_{12}|\}$, $h_{11}^* = 1$, $l_{12}^* = 1$, $h_{12}^* = 1$, $l_{21}^* = 1$, $h_{21}^* = |\theta_{21}|$. For the subsystem 1, the estimate $\hat{x}_{11}$ of the unmeasurable signal $x_{12} - y_1$ is given by

$$\dot{\hat{x}}_{11} = u_1 - (\hat{x}_{11} + y_1).$$

And the $\tilde{x}_{12}$ in (8) can be expressed as $\tilde{x}_{12} = (1/l_1^*)(x_{12} - \hat{x}_{11} - y_1)$, where $l_1^* = \max\{1, l_{11}^*, l_{12}^*, h_{11}^*, h_{12}^*\}$.

By Theorem 2, the parameter update law and the control can be designed as

$$\begin{cases} \dot{\hat{l}}_1 = \varpi_{12} = \varpi_{11} + 0.01 z_{12}^4 \vartheta_{12}, \\ u_1 = -\frac{1}{2} z_{12} + \hat{x}_{11} + y_1 + \frac{\partial \phi_{11}}{\partial y_1}(\hat{x}_{11} + y_1) \\ \qquad -\frac{1}{4}\frac{1}{2^4} z_{12} - \hat{l}_1 z_{12} \vartheta_{12} + \frac{\partial \phi_{11}}{\partial \hat{l}_1} \varpi_{12}, \end{cases} \tag{38}$$

where $z_{12} = \hat{x}_{11} - \phi_{11}(y_1, \hat{l}_1)$, $\varpi_{11} = 0.01(\frac{3}{4} 2^{4/3} + \frac{9}{2}) y_1^4$, $\phi_{11}(y_1, \hat{l}_1) = -v_{11}(y_1^2) y_1 - (\frac{67}{42} + \frac{3}{4} 2^{4/3}) y_1 - \frac{191}{84} y_1^5 - \hat{l}_1(\frac{3}{4} 2^{4/3} + \frac{153}{100}) y_1$, $\vartheta_{12} = 0.01(\partial^2 \phi_{11}/\partial y_1^2)^2 z_{12}^2 + \frac{3}{2}[(\partial \phi_{11}/\partial y_1)^2 + 1]^{2/3} + \frac{3}{4} 2^{4/3}[(\partial \phi_{11}/\partial y_1)^2 + 1]^{2/3} + \frac{3}{100}(\partial \phi_{11}/\partial y_1)^4$, $v_{11}(y_1^2) = v_1(y_1^2)$.

In the design procedure, let the Lyapunov functions $V_{10} = \frac{1}{4} x_{10}^4$, $V_{12} = \frac{1}{2}(\tilde{x}_1^{\mathrm{T}} \frac{1}{2} \tilde{x}_1)^2 + \frac{1}{4} y_1^4 + \frac{100}{2}(\hat{l}_1 - l_1)^2 + \frac{1}{4} z_{12}^4$, $l_1 \geqslant \max\{l_1^{*4/3}, 1\}$, and the parameters $\eta_{10} = 1$, $\eta_{11} = 100$, $\lambda_{10} = 0.01$, $\varepsilon_{11} = \varepsilon_{12} = 2$, $\delta_{11} = \frac{3}{2}$, $\varepsilon_1 = 4$, $b_{12} = 100$, $\beta_{11} = \beta_{12} = \frac{1}{4}$. Then, we have $\delta_1(s) = \frac{13}{2} s^4$, and

$$\mathscr{L}V_{10} \leqslant -\frac{3}{4} x_{10}^4 + \frac{1}{4} y_1^8 + \frac{1}{4} y_2^4,$$

$$\mathscr{L}V_{12} \leqslant -\frac{1}{2} y_1^4 - \frac{1}{2} z_{12}^4 - \frac{1}{16}|\tilde{x}_1|^4 - v_1(y_1^2) y_1^4 + \frac{13}{2} x_{10}^4 + \frac{13}{32} y_2^4. \tag{39}$$

For the subsystem 2, the state is measurable, and so, no estimator is required. For the present case, quadratic Lyapunov function candidates are suitable. Let $V_{20} = \frac{1}{2} x_{20}^2$, $V_{21} = \frac{1}{2} y_2^2 + \frac{1}{2}(\hat{l}_2 - l_2)^2$, where $l_2 \geqslant \max\{\theta_{21}^4, 1\}$. Then, we have

$$\mathscr{L}V_{20} \leqslant -\frac{7}{2} x_{20}^2 + \frac{1}{8} y_2^4 + \frac{1}{8} y_1^4,$$

$$\mathscr{L}V_{21} \leqslant y_2 u_2 + \frac{3}{4} y_2^4 + l_2 y_2^4 + \frac{1}{4} y_1^4 + \frac{1}{4} x_{20}^4 + (\hat{l}_2 - l_2)\dot{\hat{l}}_2. \tag{40}$$

The parameter update law and control can be designed as

$$\begin{cases} \dot{\hat{l}}_2 = y_2^4, \\ u_2 = -v_2(y_2^2) y_2 - \hat{l}_2 y_2^3 - \frac{1}{2} y_2 - \frac{3}{4} y_2^3. \end{cases} \tag{41}$$

In this case, we have $\mathscr{L}V_{21} \leqslant -\frac{1}{2} y_2^2 - v_2(y_2^2) y_2^2 + \frac{1}{4} y_1^4 + \frac{1}{4} x_{20}^4$. Set $\alpha_{11}(s) = \frac{1}{4} s^4$, $\alpha_{12}(s) = \frac{1}{2} s^4$, $\alpha_{21}(s) = \frac{1}{2} s^2$, $\alpha_{22}(s) = s^2$. From (39) and (40), it follows that $\alpha_{10}(s) = \frac{3}{4} s^4$, $\gamma_{11}(s) = \frac{1}{4} s^8$, $\gamma_{12}(s) = \frac{1}{4} s^4$, $\alpha_{20} = \frac{7}{2} s^2$, $\gamma_{21}(s) = \frac{1}{8} s^4$, $\gamma_{22}(s) = \frac{1}{8} s^4$. Then, by Lemma 2 we can take $\rho_1(s) = 9$, $\rho_2(s) = \frac{4}{3} s + \frac{16}{21}$.

Select $v_{10}(y_1^2) = (\frac{9}{4} + \frac{1}{42}) y_1^4 + \frac{2}{21}$, $v_{20}(y_2^2) = (\frac{9}{4} + \frac{1}{21}) y_2^2 + \frac{1}{42} y_2^6$, $v_1(y_1^2) = \frac{1}{4} + (\frac{9}{4} + \frac{1}{42}) y_1^4 + \frac{2}{21}$, $v_2(y_2^2) = \frac{13}{32} y_2^2 + (\frac{9}{4} + \frac{1}{21}) y_2^2 + \frac{1}{42} y_2^6$. Under the controllers (38) and (41), the outputs of the closed-loop system converge to zero almost surely.

Figs. 1 and 2 present the simulation results with initial conditions $x_{10}(0) = 0.3$, $y_1(0) = x_{11}(0) = 0.1$, $x_{12}(0) = 0.2$, $x_{20}(0) = 1$, $y_2(0) = x_{21}(0) = 0.8$, $\hat{x}_{11}(0) = 0$, $\hat{l}_1(0) = 1$, $\hat{l}_2(0) = 1.5$, $\theta_{11} = 0.1$, $\theta_{12} = 1$, $\theta_{21} = 1$.

7. Concluding remarks

A decentralized robust adaptive controller design is proposed for large-scale stochastic nonlinear systems with stochastic inverse dynamics, parametric and nonlinear uncertainties.

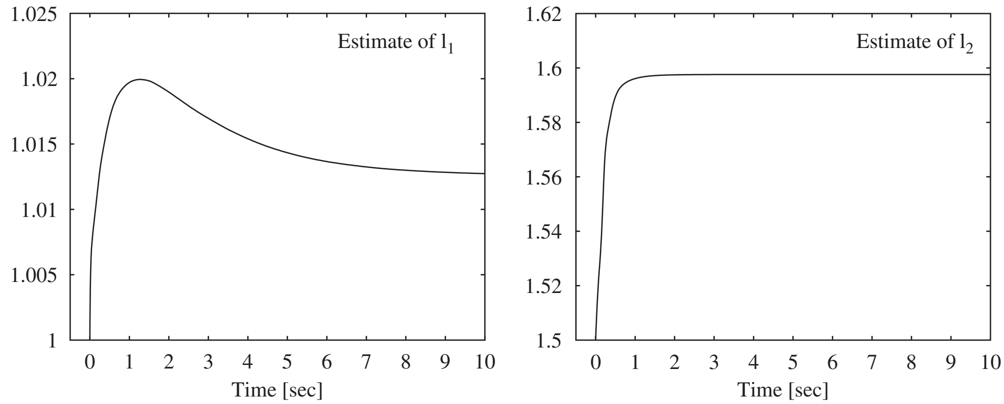

Fig. 2. Parameter estimates of subsystems 1 and 2.

The design procedure is constructive and backstepping-based, and involves three aspects: the design of a decentralized reduced-order observer for the unmeasurable but observable states, the estimates of the uncertain functions, and the design of stabilizing controllers. In sharp contrast to the deterministic counterpart, the use of *nonquadratic* Lyapunov functions seems necessary for the controller design as well as the closed-loop stability analysis. Very often, as shown also in numerous papers of other authors, Lyapunov functions which are quadratic with respect to some transformed states are enough for deterministic systems design. Under some general conditions, the closed-loop system trajectories are shown to be bounded in probability and the output can be regulated into a small neighborhood of the origin in probability. If a prior bound on the unknown parameters is available, then an arbitrarily small neighborhood can be obtained. In addition, the equilibrium of interest is globally stable in probability and the output is regulated to the origin almost surely when the drift and diffusion vector fields vanish at the origin. One numerical example is given to illustrate the efficiency of the design method.

Our future work will be directed at further extending the proposed methodology to a broader class of stochastic nonlinear systems. For example, we are interested in the development of novel small-gain based tools for stochastic nonlinear control design, which further broadens the domain of applicability of deterministic small-gain techniques; see, e.g. Krstić et al. (1995), Jiang (1999), Jiang and Praly (1998), Jiang, Teel, and Praly (1994). Other issues under current investigation include the robustification problem with input unmodeled dynamics (Jiang & Mareels, 1997; Krstić & Kokotović, 1994; Krstić, Sun, & Kokotović, 1996 for related results in the setting of deterministic nonlinear systems) and the problem of stochastic nonlinear output regulation (Byrnes, Priscoli, & Isidori, 1997; Huang, 2004 for recent advances in deterministic output regulation).

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Appendix A. Stochastic ISS

Consider the following system:

$$\mathrm{d}x = f(t, x, v)\,\mathrm{d}t + g(t, x, v)\,\mathrm{d}w, \tag{A.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $v(t) \in \mathbb{R}^m$ is the input, $w(t)$ is an r-dimensional standard Brownian motion, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times r}$ are $\mathscr{C}^1$ in their arguments.

Definition A.1 (*Tang & Başar, 2001*). System (A.1) is SISS if $\forall\varepsilon > 0$, there exist a $\mathscr{KL}$ function $\beta(\cdot,\cdot)$ and a $\mathscr{K}$ function $\gamma(\cdot)$ such that $P\{|x(t)| < \beta(|x_0|, t) + \gamma(\sup_{0\leqslant s\leqslant t}|v(s)|)\} \geqslant 1 - \varepsilon$, $\forall t \geqslant 0$, $x_0 \in \mathbb{R}^n\backslash\{0\}$.

Then, we have the following results, which are an extension of Theorem 1.

Theorem A.1 (*Liu et al., 2006b*). *For the system* (A.1), *if there exist* $V \in \mathscr{C}^2$ *and* $\alpha_i \in \mathscr{K}_\infty$ $(i = 1, 2, 3, 4)$ *such that* $\alpha_1(|x|) \leqslant V(x) \leqslant \alpha_2(|x|)$, $\mathscr{L}V \leqslant -\alpha_3(|x|) + \alpha_4(|v|)$, *then the system* (A.1) *is SISS.*

Theorem A.2 (*Liu et al., 2006b*). *For the system* (2), *if there exist* $V(t, x) \in \mathscr{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$, $\mu_1(\cdot), \mu_2(\cdot) \in \mathscr{K}_\infty$, *positive-definite and radially unbounded function* $W(x)$, *and constant* $c > 0$, *such that* $\mu_1(|x|) \leqslant V(t, x) \leqslant \mu_2(|x|)$, $\mathscr{L}V \leqslant -W(x) + c$, *then the solution process of the system* (2) *is bounded in probability and for any given* $\varepsilon > 0$, *there exist a* $\mathscr{KL}$ *function* $\beta(\cdot,\cdot)$ *and a* $\mathscr{K}$ *function* $\gamma(\cdot)$ *such that* $P\{|x(t)| < \beta(|x_0|, t) + \gamma(c)\} \geqslant 1 - \varepsilon$, $\forall t \geqslant 0$, $x_0 \in \mathbb{R}^n\backslash\{0\}$.

Appendix B. Proof of Lemma 1

As shown at Step 1 of Section 5.1, Lemma 1 holds with $k = 1$. Now, we demonstrate Lemma 1 by induction. Assume that Lemma 1 is true for Step k, we will show that Lemma 1 is still true for Step $k+1$. To this purpose, consider the following function:

$$V_{i,k+1}=V_{ik}(\tilde{x}_i,y_i,\ldots,z_{ik},\hat{l}_i)+\tfrac{1}{4}z_{i,k+1}^4.$$

Then, it follows from (11) that

$$\begin{aligned}\mathscr{L}V_{i,k+1}&=\mathscr{L}V_{ik}+z_{i,k+1}^3[z_{i,k+2}+\phi_{i,k+1}+L_{i,k+1}y_i\\&\quad+\Omega_{i,k+1,1}+\Omega_{i,k+1,2}+\Omega_{i,k+1,3}+\Omega_{i,k+1,4}]\\&\quad+\frac{3}{2}z_{i,k+1}^2\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^2 g_{i1}g_{i1}^{\mathrm{T}}.\end{aligned}\tag{B.1}$$

As in step 1, by Young inequality we have

$$z_{i,k+1}^3 z_{i,k+2}\leqslant\frac{3}{4}\varepsilon_{i,k+1}^{4/3}z_{i,k+1}^4+\frac{1}{4\varepsilon_{i,k+1}^4}z_{i,k+2}^4,$$

$$\begin{aligned}z_{i,k+1}^3\Omega_{i,k+1,2}&\leqslant|z_{i,k+1}^3|\left|\left\|\frac{\partial\phi_{ik}}{\partial y_i}\right\|\right|l_i^*\tilde{x}_{i2}|\\&\leqslant l_i\frac{3}{4}\varepsilon_{i,k+1}^{4/3}\left[\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^2+1\right]^{2/3}z_{i,k+1}^4\\&\quad+\frac{1}{4\varepsilon_{i,k+1}^4}\tilde{x}_{i2}^4,\end{aligned}$$

$$\begin{aligned}&z_{i,k+1}^3\Omega_{i,k+1,3}\\&\quad\leqslant|z_{i,k+1}^3|\left|\left\|\frac{\partial\phi_{ik}}{\partial y_i}\right\|\right|f_{i1}|+\frac{1}{2}|z_{i,k+1}^3|\left|\left\|\frac{\partial^2\phi_{ik}}{\partial y_i^2}\right\|\right|g_{i1}|^2\\&\quad\leqslant l_i\left[\frac{3}{2}\left[\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^2+1\right]^{2/3}\eta_{i0}^{4/3}+z_{i,k+1}^2\left(\frac{\partial^2\phi_{ik}}{\partial y_i^2}\right)^2\frac{1}{b_{i,k+1}}\right]\\&\quad\times z_{i,k+1}^4+\frac{1}{4\eta_{i0}^4}\varphi_{i10}^4+\frac{b_{i,k+1}}{2}\psi_{i10}^4\\&\quad+\frac{1}{4\eta_{i0}^4}\sum_{l=1}^N\kappa_{i1l}^{(1)}y_l^4+\frac{2}{\eta_{i0}^4}\left(\sum_{l=1}^N\varphi_{i1l}(0)\right)^4\\&\quad+\frac{b_{i,k+1}}{2}\sum_{l=1}^N\kappa_{i1l}^{(2)}y_l^4+4b_{i,k+1}\left(\sum_{l=1}^N\psi_{i1l}(0)\right)^4,\end{aligned}$$

$$\begin{aligned}&\frac{3}{2}z_{i,k+1}^2\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^2 g_{i1}g_{i1}^{\mathrm{T}}\\&\quad\leqslant 3z_{i,k+1}^2\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^2 h_{i1}^{*2}\left(\psi_{i10}^2+\left(\sum_{l=1}^N\psi_{i1l}(|y_l|)\right)^2\right)\\&\quad\leqslant l_i\frac{3}{\eta_{i1}}\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^4 z_{i,k+1}^4+\frac{3}{2}\eta_{i1}\psi_{i10}^4\\&\quad+\frac{3}{2}\eta_{i1}\sum_{l=1}^N\kappa_{i1l}^{(2)}y_l^4+12\eta_{i1}\left(\sum_{l=1}^N\psi_{i1l}(0)\right)^4,\end{aligned}$$

where and whereafter, $b_{ij}\,(1\leqslant j\leqslant n_i,\ 1\leqslant i\leqslant N)$ are positive design constants to be specified. These together with (19) and (B.1) leads to

$$\begin{aligned}\mathscr{L}V_{i,k+1}\leqslant&-\sum_{j=1}^k\beta_{ij}z_{ij}^4-v_{i1}(y_i^2)y_i^4+\left[\frac{1}{\lambda_{i0}}(\hat{l}_i-l_i)\right.\\&\left.-\sum_{j=2}^k z_{ij}^3\frac{\partial\phi_{i,j-1}}{\partial\hat{l}_i}\right](\dot{\hat{l}}_i-\varpi_{ik})-\sigma_{0i}(\hat{l}_i-l_i)\hat{l}_i\\&-z_{i,k+1}^3\frac{\partial\phi_{ik}}{\partial\hat{l}_i}\dot{\hat{l}}_i-\delta_{i1}\tilde{x}_i^{\mathrm{T}}P_i\tilde{x}_i|\tilde{x}_i|^2+\tilde{c}_i|\tilde{x}_i|^4\\&+\sum_{j=1}^{k+1}\frac{1}{4\varepsilon_{ij}^4}\tilde{x}_{i2}^4+l_i\left[\frac{1}{b_{i,k+1}}\left(\frac{\partial^2\phi_{ik}}{\partial y_i^2}\right)^2 z_{i,k+1}^6\right.\\&+\frac{3}{2}\eta_{i0}^{4/3}\left[\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^2+1\right]^{2/3}z_{i,k+1}^4\\&+\frac{3}{4}\varepsilon_{i,k+1}^{4/3}\left[\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^2+1\right]^{2/3}z_{i,k+1}^4\\&\left.+\frac{3}{\eta_{i1}}\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^4 z_{i,k+1}^4\right]\\&+z_{i,k+1}^3\left[\phi_{i,k+1}+L_{i,k+1}y_i+\Omega_{i,k+1,1}\right.\\&\left.+\frac{3}{4}\varepsilon_{i,k+1}^{4/3}z_{i,k+1}+\frac{1}{4\varepsilon_{ik}}z_{i,k+1}\right]+\frac{1}{4\varepsilon_{i,k+1}^4}z_{i,k+2}\\&+\Delta_{i1}(|x_{i0}|)+\Psi_{i,k+1}(|x_{i0}|)+\Delta_{i2}(y)\\&+\Pi_{i,k+1}(y)+\Lambda_{i,k+1}(y_i)+\Delta_{i3}(0)\\&+\Xi_{i,k+1}(0).\end{aligned}\tag{B.2}$$

Define the virtual parameter update law and the virtual control as follows:

$$\varpi_{i,k+1}=\varpi_{ik}+\lambda_{i0}z_{i,k+1}^4\vartheta_{i,k+1},\tag{B.3}$$

$$\begin{aligned}&\phi_{i,k+1}(\bar{\hat{x}}_{i,k+1},\hat{l}_i)\\&\quad=-\beta_{i,k+1}z_{i,k+1}-L_{i,k+1}y_i-\Omega_{i,k+1,1}\\&\qquad-\frac{3}{4}\varepsilon_{i,k+1}^{4/3}z_{i,k+1}-\frac{1}{4}\frac{1}{\varepsilon_{ik}^4}z_{i,k+1}+\frac{\partial\phi_{ik}}{\partial\hat{l}_i}\varpi_{i,k+1}\\&\qquad-\left(\hat{l}_i-\lambda_{i0}\sum_{j=2}^k z_{ij}^3\frac{\partial\phi_{i,j-1}}{\partial\hat{l}_i}\right)z_{i,k+1}\vartheta_{i,k+1},\end{aligned}\tag{B.4}$$

where $k=1,\ldots,n_i-2$, $\bar{\hat{x}}_{i,k+1}=[y_i,\hat{x}_{i1},\ldots,\hat{x}_{ik}]^{\mathrm{T}}$, and

$$\vartheta_{i,k+1}=\frac{1}{b_{i,k+1}}\left(\frac{\partial^2\phi_{ik}}{\partial y_i^2}\right)^2 z_{i,k+1}^2+\frac{3}{2}\eta_{i0}^{4/3}\left[\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^2+1\right]^{2/3}+\frac{3}{4}\varepsilon_{i,k+1}^{4/3}\left[\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^2+1\right]^{2/3}+\frac{3}{\eta_{i1}}\left(\frac{\partial\phi_{ik}}{\partial y_i}\right)^4.$$

Obviously, $\phi_{i,k+1}(0,\hat{l}_i)=0$ for all $\hat{l}_i\in\mathbb{R}$.

It follows from (B.2) to (B.4) that for $k=1,\ldots,n_i-2$,

$$\begin{aligned}\mathscr{L}V_{i,k+1}\leqslant&-\sum_{j=1}^{k+1}\beta_{ij}z_{ij}^4-v_{i1}(y_i^2)y_i^4+\left[\frac{1}{\lambda_{i0}}(\hat{l}_i-l_i)-\sum_{j=2}^{k+1}z_{ij}^3\frac{\partial\phi_{i,j-1}}{\partial\hat{l}_i}\right](\dot{\hat{l}}_i-\varpi_{i,k+1})-\sigma_{0i}(\hat{l}_i-l_i)\hat{l}_i\\&+\frac{1}{4\varepsilon_{i,k+1}^4}z_{i,k+2}^4-\delta_{i1}\tilde{x}_i^{\mathrm{T}}P_i\tilde{x}_i|\tilde{x}_i|^2+\tilde{c}_i|\tilde{x}_i|^4\\&+\sum_{j=1}^{k+1}\frac{1}{4\varepsilon_{ij}^4}\tilde{x}_{i2}^4+\varDelta_{i1}(|x_{i0}|)+\varPsi_{i,k+1}(|x_{i0}|)\\&+\varDelta_{i2}(y)+\varPi_{i,k+1}(y)+\varLambda_{i,k+1}(y_i)\\&+\varDelta_{i3}(0)+\varXi_{i,k+1}(0).\end{aligned}$$

Therefore, the proof of Lemma 1 is completed.

Appendix C. Proof of Lemma 2

For notational simplicity, we will omit the subscript i. Let

$$q_1(s)=\frac{1}{\zeta(\alpha_1^{-1}(s))},\quad q_2(s)=\frac{\xi(\alpha_1^{-1}(s))}{\zeta(\alpha_1^{-1}(s))}$$

and

$$\rho(s)=\mathrm{e}^{\int_0^s q_1(\tau)\,\mathrm{d}\tau}\left[\rho(0)-\int_0^s q_2(u)\,\mathrm{e}^{-\int_0^u q_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}u\right]$$

with $\rho(0)$ being any positive number satisfying

$$\rho(0)\geqslant\xi(0)+\int_0^\infty[\xi(\alpha_1^{-1}(s))]'\,\mathrm{e}^{-\int_0^s q_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s.$$

Then, it is easy to see that

$$\rho(s)=\rho'(s)\zeta(\alpha_1^{-1}(s))+\xi(\alpha_1^{-1}(s)),\quad s\geqslant 0. \tag{C.1}$$

Noticing that $\forall s\geqslant 0$,

$$\begin{aligned}&\int_0^s q_2(u)\mathrm{e}^{-\int_0^u q_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}u+\frac{q_2(s)}{q_1(s)}\,\mathrm{e}^{-\int_0^s q_1(\tau)\,\mathrm{d}\tau}\\&\quad=\xi(0)+\int_0^s[\xi(\alpha_1^{-1}(u))]'\mathrm{e}^{-\int_0^u q_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}u\leqslant\rho(0),\end{aligned}$$

we have

$$\rho'(s)=\mathrm{e}^{\int_0^s q_1(\tau)\,\mathrm{d}\tau}\left[\rho(0)-\int_0^s q_2(u)\mathrm{e}^{-\int_0^u q_1(\tau)\,\mathrm{d}\tau}\,\mathrm{d}u-\frac{q_2(s)}{q_1(s)}\mathrm{e}^{-\int_0^s q_1(\tau)\,\mathrm{d}\tau}\right]\geqslant 0\quad\forall s\geqslant 0.$$

This together with (C.1) leads to

$$\begin{aligned}\rho(V_z(x))&=\rho'(V_z(x))\zeta(\alpha_1^{-1}(V_z(x)))+\xi(\alpha_1^{-1}(V_z(x)))\\&\geqslant\rho'(V_z(x))\zeta(|x|)+\xi(|x|)\\&\geqslant\rho'(V_z(x))\frac{2\psi_z^2(|x|)\psi_0^2(|x|)}{\alpha(|x|)}+\frac{4\delta(|x|)}{\alpha(|x|)}.\end{aligned}$$

Multiplying both sides of the above inequality by $\alpha(|x|)$ results in (32).

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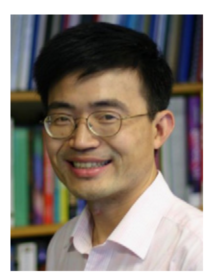
Shu-Jun Liu received her B.S. degree and M.S. degree in Mathematics from Sichuan University in 1999 and 2002, respectively. Since 2002, she has been with Department of Mathematics of Southeast University, Nanjing, China. Now she is a Ph.D. candidate of Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China. Her current research interest is mainly on control design of stochastic nonlinear systems.

Ji-Feng Zhang received his B.S. degree in Mathematics from Shandong University in 1985, and the Ph.D. degree from Institute of Systems Science (ISS), Chinese Academy of Sciences (CAS) in 1991. Since 1985 he has been with ISS, CAS, where he is now a Professor of Academy of Mathematics and Systems Science, the Vice-Director of the ISS. His current research interests include system modeling and identification, adaptive control, stochastic systems, and descriptor systems. He received the Distinguished Young Scholar Fund from National Natural Science Foundation of China in 1997, the First Prize of the Young Scientist Award of CAS in 1995, and now is a Vice-General Secretary of the Chinese Association of Automation (CAA), Vice-Director of the Technical Committee on Control Theory of CAA, Deputy Editor-in-Chief of the journals *Acta Automatica Sinica*, *Journal of Systems Science and Mathematical Sciences*, Associate Editor of other 4 journals, including *IEEE Transactions on Automatic Control*.



Z.P. Jiang received the B.Sc. degree in Mathematics from the University of Wuhan, Wuhan, China, in 1988, the M.Sc. degree in Statistics from the Université de Paris-sud, France, in 1989, and the Ph.D. degree in automatic control and mathematics from the École des Mines de Paris, France, in 1993.

From 1993 to 1998, he held visiting researcher positions with various institutions including INRIA (Sophia-Antipolis), France, the Australian National University, the University of Sydney, and University of California. In January 1999, he joined the Polytechnic University at Brooklyn as an Assistant Professor of Electrical Engineering, and since 2002 he has been an Associate Professor. His main research interests include stability theory, the theory of robust and adaptive nonlinear control, and their applications to underactuated mechanical systems, congestion control, and wireless networks. He is author of 80 journal papers in these areas.

Dr. Jiang currently is a Subject Editor for the *International Journal of Robust and Nonlinear Control,* and has served as an Associate Editor for *Systems and Control Letters*, *IEEE Transactions on Automatic Control* and *European Journal of Control*. Dr. Jiang is a recipient of the prestigious Queen Elizabeth II Fellowship Award from the Australian Research Council, the CAREER Award from the U.S. National Science Foundation, and the JSPS Invitation Fellowship from the Japan Society for the Promotion of Science.